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Abstract

We focus on a new class of discrete 2-convex functions, which forms a subclass of integrally convex functions. The discrete 2-convexity generalizes existing special integrally convex functions such as the well-established M-/M$^\natural$-convex and L-/L$^\natural$-convex functions by Murota et al., the recently investigated globally/locally discrete midpoint convex functions by Moriguchi, Murota, Tamura, and Tardella, the directed discrete midpoint convex functions by Tamura and Tsurumi, and BS*-convex and UJ-convex functions by one of the authors. We provide a unifying view of all these functions within the class of integrally convex functions having discrete 2-convexity. We also consider discrete 2-convex functions with a locally hereditary orientation property and show parallelogram inequalities, scalability, and proximity results, which extend the results recently established by Moriguchi, Murota, Tamura, and Tardella and Tamura and Tsurumi for special cases of discrete 2-convex functions.

Keywords: Discrete convex functions, integrally convex functions, discrete 2-convexity, parallelogram inequality, scalability, proximity

MSC: 90C27 · 90C25

1. Introduction

Ordinary convexity in $\mathbb{R}^n$ is based on the classical convexity inequality relating the value of a function f at a *single* internal point of the segment joining two endpoints x and y with the values of the function at the two endpoints as

$$f(x) + f(y) \geq 2f(\tfrac{1}{2}(x + y)) \qquad (x, y \in \mathbb{R}^n). \tag{1.1}$$

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Hence it suffices to consider all triples $(x, y, \frac{1}{2}(x+y))$ $(x, y \in \mathbb{R}^n)$ for the definition of ordinary convexity.

For a discrete function f defined on the integer lattice $\mathbb{Z}^n$ the function f is *discrete convex* if its lower envelope $\bar{f}$ is a convex function on $\mathbb{R}^n$ with $\bar{f}(x) = f(x)$ for all $x \in \mathbb{Z}^n$. Triples $(x, y, \frac{1}{2}(x+y))$ $(x, y \in \mathbb{R}^n)$ do not work for discrete functions since x and y are restricted on $\mathbb{Z}^n$ and $\frac{1}{2}(x+y)$ may not be integral. For an *integrally convex function* f, which is a special discrete convex function introduced by Favati and Tardella [1], we need values of $f(z)$ on at most $n+1$ integer points z from an integer neighborhood of $\frac{1}{2}(x+y)$ to get alternative inequalities of (1.1) (its more precise description is given in Section 3).

On the other hand, most notions of discrete convexity proposed in the literature for functions defined on the integer lattice $\mathbb{Z}^n$ are based on what we call a *discrete 2-convexity inequality*, where the values of the function f at $x, y \in \mathbb{Z}^n$ are compared to the values of f at *two* (not necessarily distinct) points $u, v \in \mathbb{Z}^n$ somehow "intermediate" between x and y (see examples given in Section 5).

In this paper we formalize the notion of *discrete 2-convexity* and we show that it extends several notions of discrete convexity over $\mathbb{Z}^n$ including the recent notions of discrete midpoint convexity by Moriguchi, Murota, Tamura, and Tardella [8] and its directed variant developed by Tamura and Tsurumi [13]. Other classes of functions that can be viewed as special cases of discrete 2-convex functions include the well-established M-/M$^\natural$-convex and L-/L$^\natural$-convex functions by Murota et al. (see [2, 9, 10, 11, 12]), and BS*-convex and UJ-convex functions by Fujishige [3].

Discrete 2-convex functions form a subclass of the integrally convex functions introduced by Favati and Tardella [1] in general, and they coincide with this very general class of discrete convex functions under some additional assumptions, which always hold in dimensions smaller than 4 (to be discussed in Section 3). Moreover, we propose a subclass of discrete 2-convex functions, which extends the classes of discrete midpoint convex and directed discrete midpoint convex functions while keeping most of their structural and algorithmic properties such as parallelogram inequalities, scaling, and proximity, which were examined in [8, 13] for special cases.

The present paper is organized as follows. We give definitions of basic concepts in discrete convexity in Section 2. Section 3 deals with integral convexity and discrete midpoint convexity, which leads us to the concept of discrete 2-convexity to be investigated in Section 4. Examples of existing discrete 2-convex functions are shown in Section 5. Under a plausible condition that requires a locally hereditary orientation property, we prove parallelogram inequalities, scalability, and proximity results for discrete 2-convex functions, which extend the results of [8, 13]. Section 7 gives some concluding remarks.

2. Definitions

We denote by $\mathbb{Z}$ the set of integers and by $\mathbb{R}$ the set of reals. Also $\mathbb{Z}_{>0}$ denotes the set of positive integers and $\mathbb{Z}_{\geq 0}$ that of nonnegative integers.

Throughout this paper let n be a positive integer and consider a function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ on the n-dimensional integer lattice $\mathbb{Z}^n$ that has a nonempty *effective domain* $\mathrm{dom}(f) = \{x \in \mathbb{Z}^n \mid f(x) < +\infty\}$. Define $[n] = \{1, 2, \cdots, n\}$. For any $x \in \mathbb{R}^n$ define $\lceil x \rceil$ and $\lfloor x \rfloor$ to be the integer vectors, respectively, obtained by rounding up and down each component $x(i)$ ($i \in [n]$) so that $\lfloor x \rfloor \leq x \leq \lceil x \rceil$. For any $x, y \in \mathbb{Z}^n$ with $x \leq y$ define real and integral intervals, respectively, by

$$[x, y]_{\mathbb{R}} = \{z \in \mathbb{R}^n \mid \forall i \in [n] : x(i) \leq z(i) \leq y(i)\},$$

$$[x, y]_{\mathbb{Z}} = \{z \in \mathbb{Z}^n \mid \forall i \in [n] : x(i) \leq z(i) \leq y(i)\}.$$

For any $x \in \mathbb{R}^n$ define $||x||_\infty = \max\{|x(i)| \mid i \in [n]\}$. For any $A \subseteq [n]$ define $x^A \in \mathbb{R}^n$ to be $x^A(i) = x(i)$ for $i \in A$ and $x^A(i) = 0$ for $i \in [n] \setminus A$. For any $A \subseteq [n]$ denote the characteristic vector, in $\mathbb{R}^n$, of A by χ_A, where $\chi_A(i) = 1$ for $i \in A$ and $\chi_A(i) = 0$ for $i \in [n] \setminus A$. We write $\chi_{\{i\}}$ as χ_i for any singleton $\{i\}$ with $i \in [n]$. Also we define $(+\infty) + (+\infty) = +\infty$ and $+\infty \geq +\infty$.

2.1. Discrete convexity

Denote by $\bar{f} : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ the lower envelope of f, which has the epi-graph $\{(x, \alpha) \mid x \in \mathrm{dom}(\bar{f}), \alpha \in \mathbb{R}, \bar{f}(x) \leq \alpha\}$ that coincides with the convex hull of $\{(x, \alpha) \mid x \in \mathrm{dom}(f), \alpha \in \mathbb{R}, f(x) \leq \alpha\}$. If $\bar{f}(x) = f(x)$ for all $x \in \mathbb{Z}^n$, then f is called a *discrete convex function*.

For any discrete convex function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ and any vector $w \in \mathbb{R}^n$

$$\mathrm{Arg\,min}\{f(x) - \langle w, x \rangle \mid x \in \mathbb{Z}^n\}$$

(the set of all the minimizers of $f(x) - \langle w, x \rangle$ in $x \in \mathbb{Z}^n$) is called an *affinity domain* (or *linearity domain*) of f, where $\langle w, x \rangle = \sum_{i \in [n]} w(i)x(i)$.

2.2. Integral convexity

For a discrete convex function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$, if the restriction of $\bar{f}$ on every unit hypercube $[z, z + \mathbf{1}]_{\mathbb{R}}$ for $z \in \mathbb{Z}^n$ coincides with the lower envelope of the restriction of f on $[z, z + \mathbf{1}]_{\mathbb{Z}}$, then we call f an *integrally convex function* ([1]). Another equivalent description of integrally convex function is given in Section 3.1.

3. Integral Convexity and Discrete Midpoint Convexity

3.1. Weak discrete midpoint convexity

For any $x \in \mathbb{R}^n$ the *integer neighborhood* $N(x)$ of x is defined by

$$N(x) = \{z \in \mathbb{Z}^n \mid \forall i \in [n] : |z(i) - x(i)| < 1\}. \tag{3.1}$$

For any $x \in \mathbb{R}^n$ denote by $f_{N(x)}$ the restriction of $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ on $N(x)$. (Here we allow $\mathrm{dom}(f_{N(x)}) = \emptyset$.) Also let $\overline{f_{N(x)}}$ be the lower envelope of $f_{N(x)}$. (When $\mathrm{dom}(f_{N(x)}) = \emptyset$, we have $\overline{f_{N(x)}}(y) = +\infty$ for all $y \in \mathbb{R}^n$.) Moreover, define

$$\tilde{f}(x) = \overline{f_{N(x)}}(x) \qquad (\forall x \in \mathbb{R}^n). \tag{3.2}$$

If $\tilde{f}$ is convex, then f is *integrally convex* ([1]) as defined in Section 2.

For any function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ with $\mathrm{dom}(f) \neq \emptyset$, if f satisfies

$$f(x) + f(y) \geq 2\overline{f_{N(z)}}(z) \tag{3.3}$$

for all $x, y \in \mathbb{Z}^n$ with $z = \frac{1}{2}(x + y)$, f is said to satisfy *weak discrete midpoint convexity* ([8, 13]).

A nonempty set $Q \subseteq \mathbb{Z}^n$ is called *integrally convex* if its indicator function $\mathbf{1}_Q$, defined by $\mathbf{1}_Q(x) = 0$ for $x \in Q$ and $= +\infty$ for $x \in \mathbb{Z}^n \setminus Q$, is integrally convex.

The following two facts due to [1, Proposition 3.3], [8, Theorem A.1], and [7, Theorem 2.4] are fundamental.

Proposition 3.1 ([1, 8]): *For an arbitrary function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ with $\mathrm{dom}(f) \neq \emptyset$, f is integrally convex if and only if f satisfies the weak midpoint convexity* (3.3) *for all $x, y \in \mathbb{Z}^n$ and $z = \frac{1}{2}(x + y)$.*

Proposition 3.2 ([1, 7]): *For an arbitrary function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$, if $\mathrm{dom}(f)$ is a nonempty integrally convex set, then f is integrally convex if and only if f satisfies the weak midpoint convexity* (3.3) *for all $x, y \in \mathbb{Z}^n$ with $||x - y||_\infty = 2$ and $z = \frac{1}{2}(x + y)$.*

It should be noted that the inequality (3.3) always holds for any $x, y \in \mathbb{Z}^n$ with $||x - y||_\infty \leq 1$. Hence it suffices to consider $x, y \in \mathbb{Z}^n$ with $||x - y||_\infty \geq 2$.

We now show that, under appropriate assumptions, integral convexity can be characterized by means of inequalities involving only the values of f at suitable quadruples of points in $\mathbb{Z}^n$.

Theorem 3.3: *For an arbitrary discrete function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ with $\mathrm{dom}(f) \neq \emptyset$ let $\bar{f}$ be the lower envelope of f and suppose that for any half-integer point $z \in \frac{1}{2}\mathbb{Z}^n \cap$*

$\mathrm{dom}(\bar{f})$ *there exists an affinity domain of* $\bar{f}$ *containing two opposite vertices of the hypercube* $N(z)$.

Then, f *is integrally convex if and only if it satisfies*

$$f(x) + f(y) \geq \min\{f(w_1) + f(w_2) \mid w_1, w_2 \in N(\tfrac{1}{2}(x+y)), w_1 + w_2 = x + y\} \quad (3.4)$$

for all $x, y \in \mathbb{Z}^n$.

Moreover, if $\mathrm{dom}(f)$ *is a nonempty integrally convex set, then* f *is integrally convex if and only if it satisfies inequality* (3.4) *for all* $x, y \in \mathbb{Z}^n$ *with* $||x - y||_\infty = 2$.

Proof. The "if" part follows easily by Propositions 3.1 and 3.2 by observing that $\bar{f}(\frac{1}{2}(x+y))$ is bounded above by the right-hand side of inequality (3.4). For the "only if" part, take any $x, y \in \mathbb{Z}^n$ and let w_1, w_2 be two opposite points in the hypercube $N(\frac{1}{2}(x+y))$ that belong to an affinity domain of $\bar{f}$. Then clearly $\frac{1}{2}(x+y) = \frac{1}{2}(w_1 + w_2)$ also belongs to the same affinity domain of $\bar{f}$. Hence,

$$\bar{f}(\tfrac{1}{2}(x+y)) = \tfrac{1}{2}(\bar{f}(w_1) + \bar{f}(w_2)) = \tfrac{1}{2}(f(w_1) + f(w_2)),$$

so that integral convexity of f implies condition (3.4).

Moreover, suppose that $\mathrm{dom}(f)$ is a nonempty integrally convex set. Note that we have

$$f(w_1) + f(w_2) \geq 2\overline{f_{N(z)}}(z) \quad (3.5)$$

for any w_1, w_2 appearing in (3.4) and $z = \frac{1}{2}(x+y)$, and the minimum of the left-hand side of (3.5) is equal to the right-hand side of (3.5) because of the assumption that for any half-integer point $z \in \frac{1}{2}\mathbb{Z}^n \cap \mathrm{dom}(\bar{f})$ there exists an affinity domain of $\bar{f}$ containing two opposite vertices of the hypercube $N(z)$. Hence it follows from Proposition 3.2 that f is integrally convex if and only if it satisfies inequality (3.4) for all $x, y \in \mathbb{Z}^n$ with $||x - y||_\infty = 2$. □

In the case of small dimensions, it can easily be seen that condition (3.4) becomes equivalent to integral convexity.

Corollary 3.4: *Condition* (3.4) *is equivalent to integral convexity of* f *when* $n = 2$ *or when* $n = 3$ *and* $\mathrm{dom}(f)$ *is nonempty and integrally convex.*

Proof. Note that $N(\frac{1}{2}(x+y))$ always has dimension at most 2 when $x, y \in \mathbb{Z}^2$ or when $x, y \in \mathbb{Z}^3$ and $||x - y||_\infty = 2$. In this case, the affinity domains of $\bar{f}$ on $N(\frac{1}{2}(x+y))$ always contain two opposite vertices and thus the conclusion follows from Theorem 3.3. □

The above theorem, Theorem 3.3, seems to be new to the authors' knowledge. In Example 1 given below we show an integrally convex function that does not satisfy (3.4) for all $x, y \in \mathbb{Z}^n$.

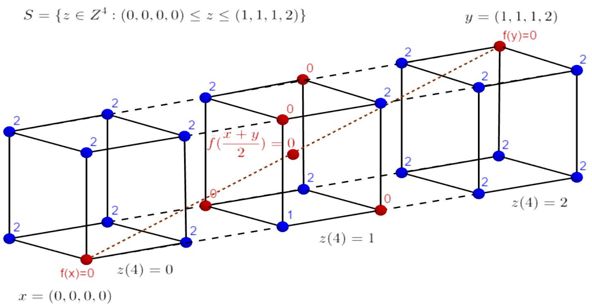

Figure 1: An integrally convex but not discrete 2-convex function.

Example 1: In Figure 1 we show the values of an integrally convex function $f : \mathbb{Z}^4 \to \mathbb{R} \cup \{+\infty\}$ with effective domain $S = \{z \in \mathbb{Z}^4 \mid (0,0,0,0) \le z \le (1,1,1,2)\}$. The values are represented on the three 3-dimensional cubes corresponding to the intersections of S with the hyperplanes $z(4) = 0$, $z(4) = 1$, and $z(4) = 2$. We can see that f is integrally convex, which can also be seen by a characterization given in [4, Theorem 2.2]. For this function and for the points $x = (0,0,0,0)$ and $y = (1,1,1,2)$ we observe that $0 = f(x) + f(y) = 2\bar{f}(\frac{x+y}{2})$, but $f(w_1) + f(w_2) > 0$ for all $w_1, w_2 \in N(\frac{x+y}{2})$ such that $w_1 + w_2 = x + y$. Observe that the affinity domain that includes $\frac{1}{2}(x+y)$ is given by the simplex formed by the convex hull of four points $(0,0,1,1), (0,1,0,1), (1,0,0,1), (1,1,1,1)$ lying on the hyperplane $z(4) = 1$.

The class of functions f satisfying (3.4) for all $x, y \in \mathbb{Z}^n$ is thus a proper subclass of the class of integrally convex functions when $n \ge 4$. In the next section we name them *discrete 2-convex functions* and, in Section 5, we show that they provide a common generalization of several well-known and recently introduced classes of discrete convex functions (see, e.g., [7, 8, 13]).

4. Discrete 2-convex Functions

We use the following notation. For any $\alpha \in \mathbb{R}$ define its *signed* upper and lower rounding to integers as follows.

$$\lceil \alpha \rceil^+ = \lfloor \alpha \rfloor^- = \lceil \alpha \rceil, \qquad \lfloor \alpha \rfloor^+ = \lceil \alpha \rceil^- = \lfloor \alpha \rfloor. \tag{4.1}$$

Also for any $z \in \mathbb{Z}^n$ and any *sign vector* (or *orientation*) $\sigma \in \{+,-\}^n$ define $\lceil z \rceil^\sigma, \lfloor z \rfloor^\sigma \in \mathbb{Z}^n$ by

$$\lceil z \rceil^\sigma = (\lceil z(i) \rceil^{\sigma(i)} \mid i \in [n]), \qquad \lfloor z \rfloor^\sigma = (\lfloor z(i) \rfloor^{\sigma(i)} \mid i \in [n]). \tag{4.2}$$

We describe here three new classes of discrete convex functions with decreasing levels of generality ranging from integrally convex to discrete midpoint convex functions: the class $\mathfrak{F}_{D2}$ of *discrete 2-convex* functions, the class $\mathfrak{F}_{RD2}$ of *regular discrete 2-convex* functions, and the class $\mathfrak{F}_{OD2}$ of *oriented discrete 2-convex* functions. All classes are defined by means of the following *discrete 2-convexity inequality*

$$f(x) + f(y) \geq f(\lceil \tfrac{1}{2}(x+y) \rceil^{\sigma(x,y)}) + f(\lfloor \tfrac{1}{2}(x+y) \rfloor^{\sigma(x,y)}) \tag{4.3}$$

with respect to a sign vector $\sigma_{(x,y)} \in \{+,-\}^n$. The difference among the classes is based on how the sign vector (or orientation) $\sigma_{(x,y)}$ depends on the points x and y and on the function f.

(D2) For all $f \in \mathfrak{F}_{D2}$ and $x, y \in \mathbb{Z}^n$ with $||x-y||_\infty \geq 2$, there exist $\sigma \in \{+,-\}^n$ such that (4.3) holds.

(RD2) For all $x, y \in \mathbb{Z}^n$ with $||x-y||_\infty \geq 2$, there exist $\sigma \in \{+,-\}^n$ such that for all $f \in \mathfrak{F}_{RD2}$ (4.3) holds.

(OD2) There exist $\sigma \in \{+,-\}^n$ such that for all $x, y \in \mathbb{Z}^n$ with $||x-y||_\infty \geq 2$, and for all $f \in \mathfrak{F}_{OD2}$ (4.3) holds.

We now show that discrete 2-convexity is equivalent to a seemingly much more general condition where the values of f at x and y are compared to those at a pair of (not necessarily distinct) points u and v belonging to the smallest box containing x and y and satisfying $u + v = x + y$ and $\{u, v\} \cap \{x, y\} = \emptyset$. More precisely, we consider the box

$$B^\circ(x,y) = \{z \in \mathbb{Z}^n \setminus \{x,y\} \mid \forall i \in [n] : x(i) \leq z(i) \leq y(i) \text{ or } x(i) \geq z(i) \geq y(i)\} \tag{4.4}$$

and we obtain the following result.

Theorem 4.1: *A function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ with $\mathrm{dom}(f) \neq \emptyset$ is discrete 2-convex if and only if for all $x, y \in \mathbb{Z}^n$ with $||x-y||_\infty \geq 2$ there exist $u, v \in B^\circ(x,y)$ such that*

$$x + y = u + v \quad \textit{and} \quad f(x) + f(y) \geq f(u) + f(v). \tag{4.5}$$

Proof. If f is discrete 2-convex, then for any $x, y \in \mathbb{Z}^n$ with $||x-y||_\infty \geq 2$ there exists a sign vector $\sigma \in \{+,-\}^n$ such that (4.3) holds. Hence (4.5) holds with $u = \lceil \frac{1}{2}(x+y) \rceil^\sigma$ and $v = \lfloor \frac{1}{2}(x+y) \rfloor^\sigma$. Conversely, suppose that for all $x, y \in \mathbb{Z}^n$ with $||x-y||_\infty \geq 2$ there exist $u, v \in B^\circ(x,y)$ such that (4.5) holds. Then, repeating $x \leftarrow u$ and $y \leftarrow v$ for such u, v from x, y, we eventually obtain u, v such that $||u-v||_\infty = 1$ and (4.5) holds. The obtained u and v can be expressed as $u = \lceil \frac{1}{2}(x+y) \rceil^\sigma$ and $v = \lfloor \frac{1}{2}(x+y) \rfloor^\sigma$ with an appropriate sign vector σ. □

In the next section we will show that the class $\mathfrak{F}_{RD2}$ of regular 2-convex functions contains the class of directed discrete midpoint convex functions of Tamura and Tsurumi, while the class $\mathfrak{F}_{OD2}$ of oriented discrete 2-convex functions contains the class of discrete midpoint convex functions of Moriguchi, Murota, Tamura, and Tardella. More precisely, the class $\mathfrak{F}_{OD2}$ coincides with the class of those functions f such that $f_\sigma(x) = f(\sigma(1)x(1), \cdots, \sigma(n)x(n))$ is discrete midpoint convex for some $\sigma \in \{+, -\}^n$.

5. Special Cases of Discrete 2-convex Functions

We present here an overview of known classes of discrete convex functions from the literature that are discrete 2-convex.

5.1. Discrete midpoint-convex functions [1, 9]

If f satisfies

$$f(x) + f(y) \geq f(\lceil \tfrac{1}{2}(x+y) \rceil) + f(\lfloor \tfrac{1}{2}(x+y) \rfloor) \tag{5.1}$$

for all $x, y \in \mathbb{Z}^n$, then f is said to satisfy *discrete midpoint convexity* ([1, 9]). These functions satisfying (5.1) for all $x, y \in \mathbb{Z}^n$ are called *submodular integrally convex functions* [1] and *L$^\natural$-convex functions* [9]. Since both $\lceil \frac{1}{2}(x+y) \rceil, \lfloor \frac{1}{2}(x+y) \rfloor \in N(\frac{1}{2}(x+y))$ and $\lceil \frac{1}{2}(x+y) \rceil + \lfloor \frac{1}{2}(x+y) \rfloor = x + y$, (5.1) implies (3.3), i.e., discrete midpoint convexity implies weak discrete midpoint convexity.

Moriguchi, Murota, Tamura, and Tardella [8] further investigated discrete convex functions satisfying (5.1) for all $x, y \in \mathbb{Z}^n$ with (a) $||x - y||_\infty = 2$ and (b) $||x - y||_\infty \geq 2$. The discrete midpoint convexity with (a) is called *local discrete midpoint convexity*, and the latter with (b) is called *global discrete midpoint convexity*. Note that globally discrete midpoint-convex functions are discrete 2-convex functions, while this may not be the case for locally discrete midpoint-convex functions (see [8]). Also note that local discrete midpoint convexity and global discrete midpoint convexity do not require the inequality (5.1) for $x, y \in \mathbb{Z}^n$ with $||x - y||_\infty = 1$, so that they lose the underlying submodularity structure that L$^\natural$-convex functions have. In [8] it is shown that the classes of L$^\natural$-convex functions, of globally midpoint-convex functions, of locally midpoint-convex functions, and of integrally convex functions strictly expand in this order.

5.2. Directed discrete midpoint-convex functions [13]

Very recently, Tamura and Tsurumi [13] have analyzed the concept of *directed discrete midpoint convexity*[1] defined as follows. For any ordered pair (x, y) of $x, y \in \mathbb{Z}^n$ define

[1]It is mentioned in [13] that the concept of directed discrete midpoint convexity was suggested by Fabio Tardella.

$\mu(x, y) \in \mathbb{Z}^n$ by

$$\mu(x,y)(i) = \begin{cases} \lceil \frac{1}{2}(x+y)(i) \rceil & \text{if } x(i) \geq y(i) \\ \lfloor \frac{1}{2}(x+y)(i) \rfloor & \text{if } x(i) < y(i) \end{cases} \qquad (\forall i \in [n]). \tag{5.2}$$

We say that f satisfies *directed discrete midpoint convexity* [13] if

$$f(x) + f(y) \geq f(\mu(x, y)) + f(\mu(y, x)) \tag{5.3}$$

for all $x, y \in \mathbb{Z}^n$. Note that $\mu(x, y), \mu(y, x) \in N(\frac{1}{2}(x+y))$ and $\mu(x, y)+\mu(y, x) = x+y$. Hence f is a discrete 2-convex function. Note that (5.3) holds with equality for any $x, y \in \mathbb{Z}^n$ with $||x - y||_\infty = 1$ since $\{x, y\} = \{\mu(x, y), \mu(y, x)\}$.

Tamura and Tsurumi [13] investigated directed discrete midpoint-convex functions and revealed that they share nice properties with globally or locally discrete midpoint-convex functions such as proximity and scaling properties (see [8, 13]).

5.3. M-/M$^\natural$-convex functions [9, 11]

By definition ([11]) an M$^\natural$-convex function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ satisfies the condition that for any distinct $x, y \in \mathbb{Z}^n$ the following (i) or (ii) holds:

(i) For any $i \in [n]$ with $x(i) < y(i)$ there exist $j \in [n]$ such that $x(j) > y(j)$ and $f(x) + f(y) \geq f(x + \chi_i - \chi_j) + f(y - \chi_i + \chi_j)$.

(ii) There exists $i \in [n]$ such that $f(x) + f(y) \geq f(x + \chi_i) + f(y - \chi_i)$.

M-convex functions satisfy the condition with (i) alone (without (ii)) (see [9, 10, 11, 12] for more details).

We easily see that M-/M$^\natural$-convex functions satisfy the characterization of discrete 2-convex functions shown by Theorem 4.1.

5.4. BS*-convex functions and UJ-convex functions [3]

One of the authors [3] investigated a class of discrete convex functions related to bisubmodular functions.

For the unit hypercube $[0, 1]^n$ a *Freudenthal cell* is defined as follows. Let $\lambda = (v_1, \cdots, v_n)$ be a permutation of $[n]$. For each $i = 0, 1, \cdots, n$ denote by S_i the set of the first i elements of λ. Then the simplex formed by χ_{S_i} $(i = 0, 1, \cdots, n)$ is a *Freudenthal cell*. The collection of the $n!$ such Freudenthal cells corresponding to the permutations of $[n]$ gives us the (*standard*) *Freudenthal simplicial division* of the unit hypercube $[0, 1]^n$. For each integer lattice point $z \in \mathbb{Z}^n$ consider the simplicial division of the unit hypercube $\{z\} + [0, 1]^n$ by translation of the standard Freudenthal simplicial division of $[0, 1]^n$. Then this gives us a simplicial division of $\mathbb{R}^n$, called the *Freudenthal simplicial division* of $\mathbb{R}^n$. Note that L$^\natural$-convex functions on $\mathbb{Z}^n$ are exactly those functions whose lower envelopes are convex extensions on the Freudenthal simplicial division of $\mathbb{R}^n$ (see [2, 6]).

For each $z \in \mathbb{Z}^n$ denote by I_z the integral unit hypercube $\{z\} + [0, 1]^n$. Suppose that for each $z \in \mathbb{Z}^n$ we are given a subset $T_z \subseteq [n]$. Consider the reflection of the Freudenthal simplicial division of the unit hypercube $\{z\} + [0, 1]^n$ by a subset $T_z \subseteq [n]$, which is obtained by making points $z+\chi_X$ correspond to points $z+\chi_{(X\setminus T_z)\cup(T_z\setminus X)}$ for all $X \subseteq [n]$. Also suppose that the collection of such simplicial divisions of the reflections of $\{z\} + [0, 1]^n$ with subsets $T_z \subseteq [n]$ for all $z \in \mathbb{Z}^n$ forms a simplicial division $\mathcal{S}$ of $\mathbb{R}^n$. Then, a discrete convex function f on $\mathbb{Z}^n$ is called a *BS***-convex function* with respect to the simplicial division $\mathcal{S}$ if the extension of f on the simplicial division $\mathcal{S}$ is convex in $\mathbb{R}^n$. The discrete conjugate convex function of a BS*-convex function is called a *BS-convex function* (see [3]). *UJ-convex functions* are BS*-convex functions whose underlying simplicial divisions bring up the image of Union Jack (Union Flag) when $n = 2$ (see [3, Fig. 3]).

For any BS*-convex function f on $\mathbb{Z}^n$, we have for any $x, y \in \mathbb{Z}^n$

$$f(x) + f(y) \geq f(\mu_1(x, y)) + f(\mu_2(x, y)), \tag{5.4}$$

where putting $z = \frac{1}{2}(x + y)$, $z_- = \lfloor z \rfloor$, and $z_+ = \lceil z \rceil$, $\mu_1(x, y)$ and $\mu_2(x, y)$ are given by

$$\mu_1(x, y)(i) = \begin{cases} z_+(i) & \text{if } i \in [n] \setminus T_{z_-} \\ z_-(i) & \text{if } i \in T_{z_-} \end{cases} \qquad (i \in [n]), \tag{5.5}$$

$$\mu_2(x, y)(i) = \begin{cases} z_-(i) & \text{if } i \in [n] \setminus T_{z_-} \\ z_+(i) & \text{if } i \in T_{z_-} \end{cases} \qquad (i \in [n]). \tag{5.6}$$

Like L$^\natural$-convex functions, for any $x, y \in \mathbb{Z}^n$ the *exact* function value of the lower envelope $\bar{f}$ at the midpoint $\frac{1}{2}(x + y)$ is given by

$$\bar{f}(\tfrac{1}{2}(x + y)) = \tfrac{1}{2}\{f(\mu_1(x, y)) + f(\mu_2(x, y))\}. \tag{5.7}$$

Hence (5.4) characterizes BS*-convex functions, which belong to the class of discrete 2-convex functions that give exact values of (5.7).

It should be noted that for BS*-convex functions $\mu_1(x, y)$ and $\mu_2(x, y)$ both depend only on $\frac{1}{2}(x + y)$ (under a given family of T_z $(z \in \mathbb{Z}^n)$ for reflections), which is not the case for directed midpoint-convex functions of Tamura and Tsurumi [13].

6. Discrete 2-convex Functions with Locally Hereditary Orientation Property

6.1. Locally hereditary orientation

For any ordered pair (x, y) of distinct $x, y \in \mathrm{dom}(f)$ define

$$S_{(x,y)} = \{i \in [n] \mid x(i) > y(i)\}, \qquad T_{(x,y)} = \{i \in [n] \mid x(i) < y(i)\}. \tag{6.1}$$

We are given an orientation $\sigma_{(x,y)} : [n] \to \{+, -\}$. For simplicity we often write S, T, and σ without the suffix (x, y). Also define

$$\begin{aligned} S^+ &= \{i \in S \mid \sigma(i) = +\},\ \ S^- = \{i \in S \mid \sigma(i) = -\}, \\ T^+ &= \{i \in T \mid \sigma(i) = +\},\ \ T^- = \{i \in T \mid \sigma(i) = -\}, \end{aligned} \tag{6.2}$$

where we omitted the suffix (x, y). Signed integer roundings of any half-integer $\alpha \in \frac{1}{2}\mathbb{Z}$ are given as follows:

$$\lceil \alpha \rceil^+ = \lfloor \alpha \rfloor^- = \lceil \alpha \rceil, \qquad \lfloor \alpha \rfloor^+ = \lceil \alpha \rceil^- = \lfloor \alpha \rfloor, \tag{6.3}$$

and we note that $\frac{1}{2}(\lceil \alpha \rceil^\tau + \lfloor \alpha \rfloor^\tau) = \alpha$ for $\tau \in \{+, -\}$. For any half-integral vector $u \in (\frac{1}{2}\mathbb{Z})^n$ recall that

$$\lceil u \rceil^\sigma = (\lceil u(i) \rceil^{\sigma(i)} \mid i \in [n]), \quad \lfloor u \rfloor^\sigma = (\lfloor u(i) \rfloor^{\sigma(i)} \mid i \in [n]) \tag{6.4}$$

(see (4.1) and (4.2)). It should be noted that

(a) when $S^-_{(x,y)} = T^-_{(x,y)} = \emptyset$ for all $x, y \in \mathrm{dom}(f)$, we have the rounding for oriented discrete 2-convex functions in $\mathfrak{F}_{OD2}$ and

(b) when $S^-_{(x,y)} = T^+_{(x,y)} = \emptyset$ for all $x, y \in \mathrm{dom}(f)$, we have the rounding for directed discrete midpoint convex functions in $\mathfrak{F}_{RD2}$.

Let us consider a discrete function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ that satisfies

$$f(x) + f(y) \geq f(\lceil \tfrac{1}{2}(x + y) \rceil^{\sigma(x,y)}) + f(\lfloor \tfrac{1}{2}(x + y \rfloor^{\sigma(x,y)}) \tag{6.5}$$

for all $x, y \in \mathbb{Z}^n$. As in Section 4 such a function f is called a *regular discrete 2-convex function*.

For any $x, y \in \mathrm{dom}(f)$ we have a box $B(x, y)$ defined by

$$B(x, y) = \{z \in \mathbb{Z}^n \mid \forall i \in [n] : x(i) \leq z(i) \leq y(i) \text{ or } x(i) \geq z(i) \geq y(i)\}. \tag{6.6}$$

For any ordered pair (w, z) of $w, z \in \mathrm{dom}(f)$ we write $(w, z) \preceq (x, y)$ if $w, z \in B(x, y)$ and ($0 \leq (w - z)(i) \leq (x - y)(i)$ or $0 \geq (w - z)(i) \geq (x - y)(i)$) for all $i \in [n]$.

Suppose that $||x - y||_\infty = m \geq 1$. The difference $x - y$ can be expressed in terms of $\{0, \pm 1\}$-vectors $d_i = \chi_{A_i} - \chi_{B_i}$ $(i \in [m])$ as

$$x - y = \sum_{i \in [m]} d_i = \sum_{i \in [m]} (\chi_{A_i} - \chi_{B_i}) \tag{6.7}$$

in such a way that

$$A_1 \cap S^+ \supseteq A_2 \cap S^+ \supseteq \cdots \supseteq A_m \cap S^+, \tag{6.8}$$

$$A_1 \cap S^- \subseteq A_2 \cap S^- \subseteq \cdots \subseteq A_m \cap S^-, \tag{6.9}$$

$$B_1 \cap T^- \supseteq B_2 \cap T^- \supseteq \cdots \supseteq B_m \cap T^-, \tag{6.10}$$

$$B_1 \cap T^+ \subseteq B_2 \cap T^+ \subseteq \cdots \subseteq B_m \cap T^+. \tag{6.11}$$

Under the condition (6.8)–(6.11) the expression (6.7) is unique, where some sets are possibly empty but at least one of the four sequences (regarded as multisets) consist of m non-empty sets. Put

$$d_1 = \chi_{A_1 \cap S^+} + \chi_{A_1 \cap S^-} - \chi_{B_1 \cap T^-} - \chi_{B_1 \cap T^+} (= \chi_{A_1 \cap S} - \chi_{B_1 \cap T}) \tag{6.12}$$

and denote d_1 by $\eta(x, y)$. Note that d_1 is a non-zero $\{0, \pm 1\}$-vector. We consider $\eta(x, y)$ as a mapping from an ordered pair (x, y) of distinct $x, y \in \mathrm{dom}(f)$ to a non-zero $\{0, \pm 1\}$-vector.

It should be noted that we have

$$d_i = \eta(x, y + d_1 + \cdots + d_{i-1}) = \chi_{A_i \cap S} - \chi_{B_i \cap T} \quad (i = 1, \cdots, m) \tag{6.13}$$

and

$$x = y + d_1 + \cdots + d_m. \tag{6.14}$$

Let us denote by $D_{(x,y)}(x - y)$ the family $(d_i \mid i \in [m])$ of non-zero $\{0, \pm 1\}$-vectors d_i $(i \in [m])$ defined by (6.13).

Moreover, we consider a locally hereditary condition on the orientation σ described as follows.

(H) Given $x, y \in \mathrm{dom}(f)$, for any $w, z \in B(x, y)$ such that $(w, z) \preceq (x, y)$ we have $\sigma_{(w,z)} = \sigma_{(x,y)}$.

(Here we can slightly relax the condition **(H)** in such a way that instead of $\sigma_{(w,z)} = \sigma_{(x,y)}$ we impose $\sigma_{(w,z)}(i) = \sigma_{(x,y)}(i)$ for all $i \in S_{\sigma_{(w,z)}} \cup T_{\sigma_{(w,z)}}$.)

The following lemma is crucial in the arguments about parallelogram inequalities to be examined in the next subsection.

Lemma 6.1: *Suppose* **(H)**. *Then, for any* $x, y \in \mathrm{dom}(f)$ *with* $D_{(x,y)}(x-y) = (d_i \mid i \in [m])$ *satisfying* (6.8)–(6.11) *and for any* $w, z \in B(x,y)$ *such that* $w = y + \sum_{i \in J_1} d_i$ *and* $z = y + \sum_{i \in J_2} d_i$ *for some* $J_1, J_2 \subseteq [m]$ *with* $J_1 \supseteq J_2$*, we have* $D_{(w,z)}(w-z) = (d_i \mid i \in J_1 \setminus J_2) \subseteq D_{(x,y)}(x-y)$ *as a multiset inclusion.*

Proof. The present lemma follows from the hereditary assumption **(H)** and the definition of the mapping η. □

Also we have the following lemma.

Lemma 6.2: *Suppose* **(H)**. *Then, for any distinct* $j, k \in [m]$ *we have as multisets*

$$\{\lceil \tfrac{1}{2}(d_j + d_k)\rceil^{\sigma_{(x,y)}}, \lfloor \tfrac{1}{2}(d_j + d_k)\rfloor^{\sigma_{(x,y)}}\} = \{d_j, d_k\}. \tag{6.15}$$

Proof. Let $w = y + d_j + d_k$ and $z = y$. Then we have $w, z \in B(x,y)$ and $w - z = d_j + d_k \preceq x - y$, which implies $\sigma_{(w,z)} = \sigma_{(x,y)}$ under assumption **(H)** and hence (6.15) holds. □

We call a regular discrete 2-convex function with a locally hereditary σ a *hereditary regular discrete 2-convex function* or *HRD2-convex function* with respect to σ. Lemmas 6.1 and 6.2 are used explicitly or implicitly in the following arguments. Note that directed discrete midpoint convex functions [13] and L$^\natural$-convex functions are HRD2-convex functions while M$^\natural$-convex functions and BS*-convex functions are not in general.

6.2. Parallelogram inequalities for HRD2-convex functions

Let $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ be an HRD2-convex function with respect to an orientation σ. Choose distinct arbitrary $x, y \in \mathrm{dom}(f)$ with $||x - y||_\infty = m \geq 1$ and let $D(x-y) = (d_i \mid i \in [m])$ which satisfies (6.7)–(6.11). For any $J \subseteq [m]$ define $d_J = \sum_{i \in J} d_j$.

We first show the following lemma. Because of **(H)** every σ appearing below should be regarded as $\sigma_{(x,y)}$.

Lemma 6.3: *For any* $J \subseteq [m]$ *we have* $x - d_J,\ y + d_J \in \mathrm{dom}(f)$.

Proof. Our proof consists of the following three steps (I), (II), and (III).

(I) Put $z = y$ and repeat

$$z \leftarrow \lfloor \tfrac{1}{2}(x + z)\rfloor^{\sigma} \tag{6.16}$$

until we get $||x - z||_\infty = 1$, where we see that for all z computed during the execution we keep

$$z \in \mathrm{dom}(f), \qquad d_1 \in D(x - z) \tag{6.17}$$

because of HRD2-convexity and of the definition of the d_is. Hence the finally obtained z satisfies $z = x - d_1 \in \mathrm{dom}(f)$. Moreover, put $x^{(1)} = x - d_1$ and $z = y$, and repeat

$$z \leftarrow \lfloor \tfrac{1}{2}(x^{(1)} + z) \rfloor^{\sigma} \tag{6.18}$$

until we get $||x^{(1)} - z||_{\infty} = 1$. Then, the finally obtained z satisfies $z = x^{(1)} - d_2 = x - d_1 - d_2 \in \mathrm{dom}(f)$. Repeating this argument yields

$$x - d_1 - \cdots - d_i \in \mathrm{dom}(f) \qquad (\forall i = 1, \cdots, m-1). \tag{6.19}$$

Similarly by adapting the arguments in (6.16)–(6.19) we can also show the following:

$$y + d_1 + \cdots + d_i \in \mathrm{dom}(f) \qquad (\forall i = 1, \cdots, m-1). \tag{6.20}$$

(II) Next, for any $k \in [m]$ put $z = y + d_1 + \cdots + d_{k-1}$. Then repeat

$$z \leftarrow \lfloor \tfrac{1}{2}(x + z) \rfloor^{\sigma} \tag{6.21}$$

until we get $||x - z||_{\infty} = 1$, where any z computed during this process satisfies

$$d_k \in D(x - z). \tag{6.22}$$

Then, the finally obtained z satisfies $z = x - d_k \in \mathrm{dom}(f)$. Similarly, we can also show $y + d_k \in \mathrm{dom}(f)$.

(III) Now, for any $J \subseteq [m]$ suppose that $J = \{j_i \mid i = 1, \cdots, \ell\}$ with $j_1 < \cdots < j_\ell$. From the arguments in (I) and (II) we have $x - d_{j_1} \in \mathrm{dom}(f)$ and hence, starting from $x - d_{j_1}$ and y, we further obtain $x - d_{j_1} - d_{j_2} \in \mathrm{dom}(f)$. Repeating this argument, we get $x - d_{j_1} - \cdots - d_{j_\ell} = x - d_J \in \mathrm{dom}(f)$. Similarly, we can also show $y + d_J \in \mathrm{dom}(f)$. □

Finally we show the following theorem. The proof given below is a straightforward adaptation of the one in [8] (also see [13]).

Theorem 6.4 (Parallelogram Inequality): *For any $J \subseteq [m]$ we have*

$$f(x) + f(y) \geq f(x - d_J) + f(y + d_J). \tag{6.23}$$

Proof. Choose any $J \subseteq [m]$. Suppose that $J = \{j_1, \cdots, j_p\}$ with $j_1 < \cdots < j_p$ and $K \equiv [m] \setminus J = \{k_1, \cdots, k_q\}$ with $k_1 < \cdots < k_q$. If $J = [m]$ or $J = \emptyset$, then (6.23) trivially holds with equality. Hence we assume $p, q \geq 1$ (and $p + q = m$). Note that (6.23) can be rewritten as

$$f(x) + f(x - d_J - d_K) \geq f(x - d_J) + f(x - d_K). \tag{6.24}$$

Now, adapting the proof in [8, Theorem7], we show (6.24). For all $s \in \{0\} \cup [p]$ and $t \in \{0\} \cup [q]$ define

$$x_{(s,t)} = x - \sum_{i=1}^{s} d_{j_i} - \sum_{i=1}^{t} d_{k_i}, \tag{6.25}$$

which belongs to $\mathrm{dom}(f)$ due to Lemma 6.3. We first show

$$f(x_{(s,t)}) + f(x_{(s-1,t-1)}) \geq f(x_{(s,t-1)}) + f(x_{(s-1,t)}) \qquad (s \in [p],\ t \in [q]). \tag{6.26}$$

Because of the definition (6.25), of HRD2-convexity, and of Lemma 6.2 we have

$$\begin{aligned}
& f(x_{(s,t)}) + f(x_{(s-1,t-1)}) \\
&= f(x_{(s-1,t-1)} - d_{j_s} - d_{k_t}) + f(x_{(s-1,t-1)}) \\
&\geq f(x_{(s-1,t-1)} - \lceil \tfrac{1}{2}(d_{j_s} + d_{k_t}) \rceil^{\sigma}) + f(x_{(s-1,t-1)} - \lfloor \tfrac{1}{2}(d_{j_s} + d_{k_t}) \rfloor^{\sigma}) \\
&= f(x_{(s-1,t-1)} - d_{j_s}) + f(x_{(s-1,t-1)} - d_{k_t}) \\
&= f(x_{(s,t-1)}) + f(x_{(s-1,t)}),
\end{aligned} \tag{6.27}$$

which validates (6.26).

Next, by summing up (6.26) for all $s \in [p]$ and $t \in [q]$ we obtain (6.24). □

Remark: Defining $F(s,t) = f(x_{(s,t)})$ for all $s \in \{0\} \cup [p]$ and $t \in \{0\} \cup [q]$, $F : (\{0\} \cup [p]) \times (\{0\} \cup [q]) \to \mathbb{R}$ is an ordinary submodular function in the orthant $(\{1\}, \{2\})$ (or orientation $(\sigma(1), \sigma(2)) = (+, -)$). The proof technique to show (6.24) from (6.26) is a well-known technique to show the submodularity of set functions from their local submodularity.

6.3. Scalability of HRD2-convex functions

Here we provide a proof of the scalability of HRD2-convex functions based on two lemmas given below. Let $x \in (\mathbb{Z}_{\geq 0})^n$ be an arbitrary nonnegative vector, which is uniquely expressed as

$$x = \sum_{i=1}^{\ell} p_i \chi_{A_i} \tag{6.28}$$

with positive integers ℓ and p_i $(i \in [\ell])$ and sets A_i $(i \in [\ell])$ satisfying

$$[n] \supseteq A_1 \supset A_2 \supset \cdots \supset A_\ell \neq \emptyset. \tag{6.29}$$

Now let us consider the upper-rounding $\lceil \frac{1}{2}x \rceil$ of the half-integral vector $\frac{1}{2}x$. Defining

$$q_j = \left\lceil \frac{1}{2} \sum_{i=1}^{j} p_i \right\rceil - \left\lceil \frac{1}{2} \sum_{i=1}^{j-1} p_i \right\rceil \quad (j \in [\ell]), \tag{6.30}$$

the upper-rounding of the half-integral vector $\frac{1}{2}x$ can be rewritten as

$$\lceil \tfrac{1}{2}x \rceil = \sum_{i=1}^{\ell} q_i \chi_{A_i}. \tag{6.31}$$

It follows from (6.30) that integers q_i $(i \in [\ell])$ satisfy

$$0 \leq q_i \leq p_i \qquad (\forall i \in [\ell]). \tag{6.32}$$

Similarly, we can show that the lower-rounding $\lfloor \frac{1}{2}x \rfloor$ is expressed as

$$\lfloor \tfrac{1}{2}x \rfloor = \sum_{i=1}^{\ell} r_i \chi_{A_i} \tag{6.33}$$

with integers r_i $(i \in [\ell])$ satisfying

$$0 \leq r_i \leq p_i \qquad (\forall i \in [\ell]). \tag{6.34}$$

Since $\lceil \frac{1}{2}x \rceil + \lfloor \frac{1}{2}x \rfloor = x$, we have

$$q_i + r_i = p_i \qquad (\forall i \in [\ell]). \tag{6.35}$$

Summing up, we have the following lemma.

Lemma 6.5: *For any nonzero vector $x \in (\mathbb{Z}_{\geq 0})^n$ expressed as*

$$x = \sum_{i=1}^{\ell} p_i \chi_{A_i}$$

with positive integers p_i $(i \in [\ell])$ and sets A_i $(i \in [\ell])$ satisfying

$$[n] \supseteq A_1 \supset A_2 \supset \cdots \supset A_\ell \neq \emptyset,$$

the upper-rounding $\lceil \frac{1}{2}x \rceil$ and the lower-rounding $\lfloor \frac{1}{2}x \rfloor$ are, respectively, expressed as

$$\lceil \tfrac{1}{2}x \rceil = \sum_{i=1}^{\ell} q_i \chi_{A_i}, \qquad \lfloor \tfrac{1}{2}x \rfloor = \sum_{i=1}^{\ell} r_i \chi_{A_i}$$

with integers q_i $(i \in [\ell])$ and r_i $(i \in [\ell])$ satisfying

$$0 \leq q_i \leq p_i, \quad 0 \leq r_i \leq p_i, \quad q_i + r_i = p_i \qquad (\forall i \in [\ell]).$$

More generally, for any non-zero vector $x \in \mathbb{Z}^n$ and an orientation σ, vector x can be expressed as

$$x = \sum_{i=1}^{\ell} p_i(\chi_{A_i} - \chi_{B_i}) \tag{6.36}$$

for a positive integer ℓ in such a way that, putting $S = \{i \in [n] \mid x(i) > 0\}$, $T = \{i \in [n] \mid x(i) < 0\}$, $S^{\pm} = \{i \in S \mid \sigma(i) = \pm\}$, and $T^{\pm} = \{i \in T \mid \sigma(i) = \pm\}$, we have

$$A_1 \cap S^+ \supseteq A_2 \cap S^+ \supseteq \cdots \supseteq A_\ell \cap S^+, \tag{6.37}$$
$$A_1 \cap S^- \subseteq A_2 \cap S^- \subseteq \cdots \subseteq A_\ell \cap S^-, \tag{6.38}$$
$$B_1 \cap T^- \supseteq B_2 \cap T^- \supseteq \cdots \supseteq B_\ell \cap T^-, \tag{6.39}$$
$$B_1 \cap T^+ \subseteq B_2 \cap T^+ \subseteq \cdots \subseteq B_\ell \cap T^+, \tag{6.40}$$

and for each $i \in [\ell - 1]$ at least one of the following four relations hold with strict inclusion:

$$\begin{aligned} &A_i \cap S^+ \supseteq A_{i+1} \cap S^+, \qquad &&A_i \cap S^- \subseteq A_{i+1} \cap S^-, \\ &B_i \cap T^- \supseteq B_{i+1} \cap T^-, \qquad &&B_i \cap T^+ \subseteq B_{i+1} \cap T^+. \end{aligned} \tag{6.41}$$

We can now show the following lemma that generalizes Lemma 6.5.

Lemma 6.6: *For any non-zero vector $z \in \mathbb{Z}^n$ and an orientation σ, z is uniquely expressed for some positive integers p_i $(i \in [\ell])$ as*

$$z = \sum_{i=1}^{\ell} p_i(\chi_{A_i} - \chi_{B_i}) \tag{6.42}$$

with sets A_i and B_i $(i \in [\ell])$ satisfying the conditions described above ((6.37)–(6.41)). *Furthermore, the signed upper-rounding $\lceil \frac{1}{2} z \rceil^{\sigma}$ is expressed as*

$$\lceil \tfrac{1}{2} z \rceil^{\sigma} = \sum_{i=1}^{\ell} q_i(\chi_{A_i \cap S} - \chi_{B_i \cap T}) \tag{6.43}$$

with integers q_i $(i \in [m])$ such that $0 \le q_i \le p_i$. Also, the signed lower-rounding $\lfloor \frac{1}{2} z \rfloor^{\sigma}$ is expressed as

$$\lfloor \tfrac{1}{2} z \rfloor^{\sigma} = \sum_{i=1}^{\ell} r_i(\chi_{A_i \cap S} - \chi_{B_i \cap T}) \tag{6.44}$$

with integers r_i $(i \in [m])$ such that $0 \le r_i \le p_i$. Moreover, we have $q_i + r_i = p_i$ $(\forall i \in [\ell])$.

Proof. We see that the signed upper-rounding $\lceil \frac{1}{2} z(i) \rceil^{\sigma(i)}$ becomes $\lceil \frac{1}{2} z(i) \rceil$ for $i \in$

$S^+ \cup T^+$ with the decreasing vector sequences $(\chi_{A_i \cap S^+} \mid i = 1, \cdots, \ell)$ in (6.37) and $(-\chi_{B_i \cap T^+} \mid i = 1, \cdots, \ell)$ in (6.39), and becomes $\lfloor \frac{1}{2} z(i) \rfloor$ for $i \in S^- \cup T^-$ with the increasing vector sequences $(\chi_{A_i \cap S^-} \mid i = 1, \cdots, \ell)$ in (6.38) and $(-\chi_{B_i \cap T^-} \mid i = 1, \cdots, \ell)$ in (6.40). Similarly, the signed lower-rounding $\lfloor \frac{1}{2} z(i) \rfloor^{\sigma(i)}$ becomes $\lfloor \frac{1}{2} z(i) \rfloor$ for $i \in S^+ \cup T^+$ with the decreasing vector sequences $(\chi_{A_i \cap S^+} \mid i = 1, \cdots, \ell)$ in (6.37) and $(-\chi_{B_i \cap T^+} \mid i = 1. \cdots, \ell)$ in (6.39), and becomes $\lceil \frac{1}{2} z(i) \rceil$ for $i \in S^- \cup T^-$ with the increasing vector sequences $(\chi_{A_i \cap S^-} \mid i = 1. \cdots, \ell)$ in (6.38) and $(-\chi_{B_i \cap T^-} \mid i = 1. \cdots, \ell)$ in (6.40). Hence the present lemma follows from Lemma 6.5. □

Now suppose that we are given an HRD2-convex function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ and an integer $k \geq 2$ such that $\mathrm{dom}(f) \cap (k\mathbb{Z})^n \neq \emptyset$. Define

$$f^k(x) = f(kx) \qquad (\forall x \in \mathbb{Z}^n). \tag{6.45}$$

The function f^k is called a *k-scaled function* of f.

Theorem 6.7 (Scalability): *For any HRD2-convex function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ with respect to an orientation σ and for any integer $k \geq 2$ such that $\mathrm{dom}(f) \cap (k\mathbb{Z})^n \neq \emptyset$, the k-scaled function f^k is again an HRD2-convex function with respect to the orientation σ^k given by $\sigma^k_{(x,y)} = \sigma_{(kx,ky)}$.*

Proof. For any distinct $x, y \in \mathbb{Z}^n$ put $z = x - y$. Because of Lemma 6.6, for $z \in \mathbb{Z}^n$ we have expressions (6.43) and (6.44) for a positive integer ℓ. Hence we have

$$k(x - y) = \sum_{i=1}^{\ell} k p_i (\chi_{A_i} - \chi_{B_i}), \tag{6.46}$$

$$k \lceil \tfrac{1}{2}(x - y) \rceil^{\sigma} = \sum_{i=1}^{\ell} k q_i (\chi_{A_i \cap S} - \chi_{B_i \cap T}), \tag{6.47}$$

$$k \lfloor \tfrac{1}{2}(x - y) \rfloor^{\sigma} = \sum_{i=1}^{\ell} k r_i (\chi_{A_i \cap S} - \chi_{B_i \cap T}) \tag{6.48}$$

with $q_i + r_i = p_i$ $(i \in [\ell])$. Moreover, under the condition **(H)** we have that for any $x, y \in \mathrm{dom}(f^k)$

$$f^k(\lceil \tfrac{1}{2}(x + y) \rceil^{\sigma}) = f(k \lceil \tfrac{1}{2}(x + y) \rceil^{\sigma}), \quad f^k(\lfloor \tfrac{1}{2}(x + y) \rfloor^{\sigma}) = f(k \lfloor \tfrac{1}{2}(x + y) \rfloor^{\sigma}), \tag{6.49}$$

and

$$\lceil \tfrac{1}{2}(x + y) \rceil^{\sigma} = y + d \quad \lfloor \tfrac{1}{2}(x + y) \rfloor^{\sigma} = x - d, \tag{6.50}$$

where

$$d = (\lceil \tfrac{1}{2}(x(i) - y(i)) \rceil \mid i \in S^+ \cup T^+) \oplus (\lfloor \tfrac{1}{2}(x(i) - y(i)) \rfloor \mid i \in S^- \cup T^-). \tag{6.51}$$

Hence, from (6.46)–(6.51) and Theorem 6.4 we obtain

$$f^k(x) + f^k(y) \geq f^k(\lceil \tfrac{1}{2}(x+y) \rceil^\sigma) + f^k(\lfloor \tfrac{1}{2}(x+y) \rfloor^\sigma) \quad (\forall x, y \in \mathrm{dom}(f^k)). \tag{6.52}$$

□

It should be noted that the HRD2-convexity inequality (6.52) for f^k is an instance of the parallelogram inequalities for f.

6.4. Proximity results for HRD2-convex functions

We show a proximity theorem for HRD2-convex functions. The proof given below is an adaptation of the one in [13].

Theorem 6.8 (Proximity): *For any HRD2-convex function* $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ *with respect to an orientation* σ *and for any integer* $k \geq 2$ *such that* $\mathrm{dom}(f) \cap (k\mathbb{Z})^n \neq \emptyset$*, let* $\hat{x}$ *be a vector in* $\mathrm{dom}(f^k)(= \mathrm{dom}(f) \cap (k\mathbb{Z})^n)$ *such that*

$$f^k(\hat{x}) \leq f^k(\hat{x} + d) \qquad (\forall d \in \{0, \pm 1\}^n). \tag{6.53}$$

Then there exists a minimizer x^* *of* f *such that* $||k\hat{x} - x^*||_\infty \leq n(k-1)$.

Proof. Let y be any vector in $\mathrm{dom}(f)$ such that $||k\hat{x} - y||_\infty = m > n(k-1)$. Note that the difference $k\hat{x} - y$ is expressed as (6.42) with z replaced by $k\hat{x} - y$, where

$$\ell \leq n, \qquad \sum_{i=1}^{\ell} p_i = m > n(k-1). \tag{6.54}$$

Hence $p_{i^*} > k$ for some $i^* \in [\ell]$ and let $d^* = k(\chi_{A_{i^*}} - \chi_{B_{i^*}})$ from (6.42). It follows from the parallelogram inequality (Theorem 6.4) that we have

$$f(k\hat{x}) + f(y) \geq f(k\hat{x} - d^*) + f(y + d^*) \geq f(k\hat{x}) + f(y + d^*). \tag{6.55}$$

Hence, $f(y) \geq f(y + d^*)$ and $||k\hat{x} - (y + d^*)||_\infty = ||k\hat{x} - y||_\infty - k$. Put $y_0 = y$ and repeat the above arguments by putting $y \leftarrow y + d^*$ until we obtain a vector $y \in \mathrm{dom}(f)$ such that $f(y_0) \geq f(y)$ and $||k\hat{x} - y||_\infty \leq n(k-1)$. It follows that there exists a minimizer x^* of f such that $||k\hat{x} - x^*||_\infty \leq n(k-1)$. □

We have thus shown that HRD2-convex functions satisfy a proximity result with the same proximity bound as the more restricted classes of L$^\natural$-convex, discrete midpoint convex, and directed discrete midpoint convex functions (see [9, 8, 13]).

7. Concluding Remarks

We have drawn the readers' attention to the class of *discrete 2-convex functions*, which is a subclass of integrally convex functions. We have examined discrete 2-convex functions that have nice combinatorial structures under a plausible condition, i.e., a *locally hereditary orientation* property. We have shown parallelogram inequalities, scalability, and proximity results for discrete 2-convex functions with the locally hereditary orientation property, which extend the known results for special cases in [8, Theorem 7] and [13, Theorem 5] (also [7]). Such extensions naturally lead to simple extensions of the minimization algorithms developed in [8] and [13] for discrete midpoint convex and for directed discrete midpoint convex functions to the class of HRD2-convex functions.

Our proofs in the unifying framework are simple and will lead us to further deeper understanding of discrete 2-convex functions even in the special cases in the literature. The class of discrete 2-convex functions we have focused on is worth further investigation.

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