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By

Arata MINAMIDE and Shota TSUJIMURA

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京都大学　数理解析研究所

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES

KYOTO UNIVERSITY, Kyoto, Japan

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Abstract

Let K be a field. Write G_K for the absolute Galois group of K. In the present paper, we discuss the *slimness* [i.e., the property that every open subgroup is center-free] and the *elasticity* [i.e., the property that every nontrivial topologically finitely generated normal closed subgroup of an open subgroup is open] of G_K. These two group-theoretic properties are closely related to [various versions of] the Grothendieck Conjecture in anabelian geometry. For instance, with regard to the slimness, Mochizuki proved that G_K is slim if K is a subfield of a finitely generated extension of the field of fractions of the Witt ring $W(\overline{\mathbb{F}}_p)$ as a consequence of a [highly nontrivial] Grothendieck Conjecture-type result. In the present paper, we generalize this result to the case where K is a subfield of the field of fractions of an *arbitrary mixed characteristic Noetherian local domain*. Our proof is based on elementary field theories such as Kummer theory. On the other hand, with regard to the elasticity, Mochizuki proved that G_K is elastic if K is a finite extension of the field of p-adic numbers. In the present paper, we generalize this result to the case where K is an *arbitrary mixed characteristic Henselian discrete valuation field*. As a corollary of this generalization, we prove the semi-absoluteness of isomorphisms between the étale fundamental groups of smooth varieties over mixed characteristic Henselian discrete valuation fields. Moreover, we also prove the weak version of the Grothendieck Conjecture for hyperbolic curves of genus 0 over subfields of finitely generated extensions of *mixed characteristic higher local fields*.

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Contents

Introduction

Let p be a prime number; K a field. Write $\mathbb{F}_p$ for the finite field of cardinality p. For any field F, we shall write $\mathrm{char}(F)$ for the characteristic of F; F^{sep} for the separable closure [determined up to isomorphisms] of F; $G_F \stackrel{\mathrm{def}}{=} \mathrm{Gal}(F^{\mathrm{sep}}/F)$. If F is a perfect field, then we shall also write $\overline{F} \stackrel{\mathrm{def}}{=} F^{\mathrm{sep}}$. If $\mathrm{char}(K) \neq p$, then we fix a primitive p-th root of unity $\zeta_p \in K^{\mathrm{sep}}$. For an algebraic variety X [i.e., a separated, of finite type, and geometrically connected scheme] over K, we shall write Π_X for the étale fundamental group of X, relative to a suitable choice of basepoint; $\Delta_X \stackrel{\mathrm{def}}{=} \Pi_{X \times_K K^{\mathrm{sep}}}$.

In anabelian geometry, we often consider

> *whether or not an algebraic variety X may be "reconstructed" from the étale fundamental group* Π_X.

With regard to this inexplicit question, one of the explicit questions in anabelian geometry may be stated as follows:

> Question 1 (Relative version of the Grothendieck Conjecture — (RGC_K)): Let X_1, X_2 be hyperbolic curves over K. Write
>
> $$\mathrm{Isom}_K(X_1, X_2)$$
>
> for the set of K-isomorphisms between the hyperbolic curves X_1 and X_2;
>
> $$\mathrm{Isom}_{G_K}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2})$$
>
> for the set of isomorphisms $\Pi_{X_1} \stackrel{\sim}{\to} \Pi_{X_2}$ [in the category of profinite groups] over G_K, considered up to composition with an inner automorphism arising from Δ_{X_2}. Suppose that $\mathrm{char}(K) = 0$. Then is the natural map
>
> $$\mathrm{Isom}_K(X_1, X_2) \longrightarrow \mathrm{Isom}_{G_K}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Delta_{X_2})$$

bijective? [Strictly speaking, Grothendieck conjectured that this natural map is bijective if K is finitely generated over the field of rational numbers — cf. [8].]

Note that, if $K = \overline{K}$, then $G_K = \{1\}$, hence, in particular, (RGC_K) does not hold. On the other hand, Mochizuki obtained the following remarkable result:

Theorem ([17], Theorem 4.12)**.** *Suppose that K is a generalized sub-p-adic field [i.e., a subfield of a finitely generated extension of the field of fractions of the Witt ring $W(\overline{\mathbb{F}}_p)$ — cf. [17], Definition 4.11]. Then (RGC_K) holds.*

In the authors' knowledge, the above theorem is one of the strongest results for Question 1 so far [cf. see also [16], Theorem A]. Then it is natural to pose the following question:

> Question 2: If K is "sufficiently arithmetic", then do analogous assertions of various theorems in anabelian geometry — including the above theorem — still hold? For instance, since there exist well-established arithmetic theories for *higher local fields* such as higher local class field theory, it would be interesting to consider analogous assertions for *higher local fields* [cf. Definition 1.12; [1], [2]].

Note that Fesenko analyzes that higher class field theory and anabelian geometry are two generalizations of classical class field theory [cf. [4]]. From this viewpoint, our Question 2 may be regarded as a crossover between these two generalizations. With regard to Question 2, as a corollary of [30], Theorem F, we prove the following "weak version" of the Grothendieck Conjecture for hyperbolic curves of genus 0 over subfields of finitely generated extensions of mixed characteristic higher local fields [cf. Corollary 1.16]:

Theorem A. *Suppose that K is a mixed characteristic higher local field such that*

- *the final residue field of K is isomorphic to $\overline{\mathbb{F}}_p$,*
- *the residue characteristic of K is $p > 0$.*

Let L be a subfield of a finitely generated extension of K; U, V hyperbolic curves of genus 0 over L;

$$\phi : \Pi_U \xrightarrow{\sim} \Pi_V$$

an isomorphism of profinite groups such that ϕ lies over the identity automorphism on G_L. Then there exists an isomorphism of L-schemes

$$U \xrightarrow{\sim} V$$

that induces a bijection between the cusps of U and V which is compatible with the bijection between cuspidal inertia subgroups of Π_U and Π_V induced by ϕ.

Theorem A may be regarded as an evidence for the "anabelianity" of higher local fields [cf. Question 2]. On the other hand, we note that the proof of Theorem A does not resort to any highly nontrivial arithmetic theory such as higher local class field theory or p-adic Hodge theory. However, it would be interesting to investigate the extent to which Theorem A may be generalized by making use of such arithmetic theories [cf. Remark 1.16.1; Question 4 below].

Next, we give another evidence for the "anabelianity" of higher local fields. In order to explain this another evidence, let us recall some group-theoretic properties of profinite groups. Let G, Q be profinite groups; $q : G \twoheadrightarrow Q$ an epimorphism [in the category of profinite groups]. Then we shall say that

- G is *slim* if every open subgroup of G is center-free;
- G is *elastic* if every nontrivial topologically finitely generated normal closed subgroup of an open subgroup of G is open in G;
- Q is an *almost pro-p-maximal quotient* of G if there exists a normal open subgroup $N \subseteq G$ such that $\mathrm{Ker}(q)$ coincides with the kernel of the natural surjection $N \twoheadrightarrow N^p$ to the maximal pro-p-quotient of N [cf. Definition 1.5].

With regard to these group-theoretic properties, Mochizuki proved that

- G_K is slim if K is a generalized sub-p-adic field or a Kummer-faithful field [cf. [17], Lemma 4.14; [21], Definition 1.5; [21], Theorem 1.11];
- G_K, as well as any almost pro-p-maximal quotient of G_K, is elastic if K is a finite extension of the field of p-adic numbers $\mathbb{Q}_p$ [cf. [19], Theorem 1.7, (ii)],

and Higashiyama proved that

- G_K^p is slim if K is a generalized sub-p-adic field, and $\zeta_p \in K$ [cf. [10], Lemma 5.3].

The slimness portions of these results are proved by applying highly nontrivial arithmetic theories such as local class field theory or some Grothendieck Conjecture-type results. In fact, the following holds:

> If (RGC_L) holds for every finite extension $K \subseteq L$ $(\subseteq K^{\mathrm{sep}})$, then the absolute Galois group of any subfield of K is slim

[cf. the proof of [10], Lemma 5.3; the proof of [17], Lemma 4.14; the proof of [21], Theorem 1.11; [21], Remark 1.11.2]. On the other hand, the elasticity of the absolute Galois groups of finite extensions of $\mathbb{Q}_p$ are applied to bridge the following important questions [cf. [19], Introduction]:

> Question 3 (Semi-absolute version of the Grothendieck Conjecture): Let K_i be a field of characteristic 0, where $i = 1, 2$; X_i a hyperbolic curve over K_i. Write
>
> $$\mathrm{Isom}(X_1/K_1, X_2/K_2)$$

for the set of isomorphisms $X_1 \xrightarrow{\sim} X_2$ that induce isomorphisms $K_1 \xrightarrow{\sim} K_2$;

$$\mathrm{Isom}(\Pi_{X_1}/G_{K_1},\ \Pi_{X_2}/G_{K_2})/\mathrm{Inn}(\Pi_{X_2})$$

for the set of isomorphisms $\Pi_{X_1} \xrightarrow{\sim} \Pi_{X_2}$ [in the category of profinite groups] that induce isomorphisms $G_{K_1} \xrightarrow{\sim} G_{K_2}$ via the natural surjections $\Pi_{X_1} \twoheadrightarrow G_{K_1}$ and $\Pi_{X_2} \twoheadrightarrow G_{K_2}$, considered up to composition with an inner automorphism arising from Π_{X_2}. Then is the natural map

$$\mathrm{Isom}(X_1/K_1, X_2/K_2) \longrightarrow \mathrm{Isom}(\Pi_{X_1}/G_{K_1},\ \Pi_{X_2}/G_{K_2})/\mathrm{Inn}(\Pi_{X_2})$$

bijective?

Question 4 (Absolute version of the Grothendieck Conjecture): In the notation of Question 3, write

$$\mathrm{Isom}(X_1, X_2)$$

for the set of isomorphisms $X_1 \xrightarrow{\sim} X_2$;

$$\mathrm{Isom}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Pi_{X_2})$$

for the set of isomorphisms $\Pi_{X_1} \xrightarrow{\sim} \Pi_{X_2}$ [in the category of profinite groups], considered up to composition with an inner automorphism arising from Π_{X_2}. Then is the natural map

$$\mathrm{Isom}(X_1, X_2) \longrightarrow \mathrm{Isom}(\Pi_{X_1}, \Pi_{X_2})/\mathrm{Inn}(\Pi_{X_2})$$

bijective [cf. [10], [12], [13], [19], [20], [21], [23]]?

From the viewpoint of Question 2 and Theorem A, it is natural to pose the following question:

Question 5: Suppose that K is a mixed characteristic higher local field of residue characteristic p. Then is G_K, as well as any almost pro-p-maximal quotient of G_K, slim and elastic?

We remark that the absolute Galois groups of Hilbertian fields are slim and elastic. On the other hand, any Henselian discrete valuation field is not Hilbertian [cf. Remark 3.9.2]. In order to state our main results concerning Question 5, for any field F, we shall write

$$F^{\times} \overset{\mathrm{def}}{=} F \setminus \{0\}; \quad \mu_n(F) \overset{\mathrm{def}}{=} \{x \in F^{\times} \mid x^n = 1\}; \quad \mu(F) \overset{\mathrm{def}}{=} \bigcup_{m \geq 1} \mu_m(F);$$

$$\mu_{p^{\infty}}(F) \overset{\mathrm{def}}{=} \bigcup_{m \geq 1} \mu_{p^m}(F); \quad F^{\times p^{\infty}} \overset{\mathrm{def}}{=} \bigcap_{m \geq 1} (F^{\times})^{p^m}; \quad F_{\times p^{\infty}} \overset{\mathrm{def}}{=} F_{\mathrm{prm}}(F^{\times p^{\infty}}) \subseteq F,$$

where $F_{\mathrm{prm}} \subseteq F$ denotes the prime field;

$$F_{p,\mathrm{div}} \stackrel{\mathrm{def}}{=} \bigcup_{F \subseteq E} E_{\times p^\infty} \ (\subseteq F^{\mathrm{sep}}),$$

where $F \subseteq E$ ($\subseteq F^{\mathrm{sep}}$) ranges over the set of finite separable extensions;

$$\widetilde{F}_{p,\mathrm{div}} \stackrel{\mathrm{def}}{=} F_{p,\mathrm{div}}(\mu(F^{\mathrm{sep}})) \ (\subseteq F^{\mathrm{sep}}).$$

We shall say that

- K is *stably p-$\times\mu$-indivisible* if, for every finite extension M of K, $M^{\times p^\infty} \subseteq \mu(M)$ [cf. Definition 1.7, (iv)];
- K is *stably μ_{p^∞}-finite* if, for every finite extension M of K, $\mu_{p^\infty}(M)$ is finite [cf. Definition 1.7, (v)].

Let us note that such fields exist in great abundance [cf. Example 1.14; [30], Lemma D]. For instance, any abelian extension of a generalized sub-p-adic field is stably p-$\times\mu$-indivisible. Then our main results are the following [cf. Theorems 2.4, (ii), (iii), (v); 2.8, (i), (ii); 2.10; 3.9, and Corollary 3.10]:

Theorem B. *Suppose that* $\mathrm{char}(K) \neq p$. *Then the following hold:*

(i) Suppose, moreover, that

$$\widetilde{K}_{p,\mathrm{div}} \subsetneq K^{\mathrm{sep}}.$$

Let L be a finitely generated extension over K. Then G_L is slim. Moreover, if $\zeta_p \in K$, then, for any open subgroup $H \subseteq G_L$, there exists a normal open subgroup $N \subseteq H$ of G_L such that the almost pro-p-maximal quotient associated to N is slim.

(ii) Suppose, moreover, that

- *K is a stably p-$\times\mu$-indivisible field;*
- *if* $\mathrm{char}(K) \neq 0$, *then K is transcendental over K_{prm}.*

Then G_K is slim. Moreover, if $\zeta_p \in K$, then any almost pro-p-maximal quotient of G_K is slim.

(iii) Let A_0 be a mixed characteristic Noetherian local domain of residue characteristic p. Write K_0 for the field of fractions of A_0. Let $K_0 \subseteq L_0$ ($\subseteq K_0^{\mathrm{sep}}$) be a Galois extension such that one of the following conditions hold:

- *$K_0 \subseteq L_0$ ($\subseteq K_0^{\mathrm{sep}}$) is an abelian extension.*
- *L_0 is stably μ_{p^∞}-finite.*

Suppose that K is isomorphic to a subfield of L_0. Then G_K is slim. Moreover, if $\zeta_p \in K$, then any almost pro-p-maximal quotient of G_K is slim.

Theorem C. *Suppose that K is a Henselian discrete valuation field such that the residue field k of K is of characteristic p. Then G_K is slim and elastic. Moreover, the following hold:*

- *G_K is not topologically finitely generated if and only if k is infinite, or* $\mathrm{char}(K) = p$.
- *If k is infinite, and $\zeta_p \in K$ in the case where* $\mathrm{char}(K) = 0$, *then any almost pro-p-maximal quotient of G_K is slim, elastic, and not topologically finitely generated.*
- *If k is finite, then any almost pro-p-maximal quotient of G_K is slim and elastic.*

In particular, the absolute Galois groups of higher local fields of residue characteristic p are slim and elastic. Thus, Theorem C may be regarded as another evidence for the "anabelianity" of higher local fields. Here, we note that the proof of Theorem B consists of some elementary observations on p-divisible elements of the multiplicative groups of fields. This allows us to obtain the above generalizations. Next, we remark that

- with regard to the positive characteristic portions of Theorem C, the key ingredients of our proof are Theorem B, (iii), and the theory of fields of norms.

It seems interesting to the authors that an "anabelian question" in the world of characteristic p may be reduced to an "anabelian question" in the world of characteristic 0 via the theory of fields of norms. We also remark that

- since abelian extensions of generalized sub-p-adic fields are stably p-$\times\mu$-indivisible, Theorem B, (ii) [also Theorem B, (iii)] may be regarded as a generalization of [10], Lemma 5.3; [17], Lemma 4.14, which are corollaries of a [highly nontrivial] Grothendieck Conjecture-type result;
- the elasticity portion of Theorem C is a solution of the elasticity portion of the question in [15], Remark 2.5 in a quite general situation.

Furthermore, it would be interesting to investigate the extent to which the assumptions of Theorems B, C may be weakened [cf., e.g., Remarks 2.4.1, 2.8.1, 3.9.1].

Finally, as a corollary of Theorem C, we also prove the semi-absoluteness [cf. Definition 4.5, (i)] of isomorphisms between the étale fundamental groups of smooth varieties [i.e., smooth, of finite type, separated, and geometrically connected schemes] over mixed characteristic Henselian discrete valuation fields, which may be regarded as a generalization of [19], Corollary 2.8 [cf. Corollary 4.6]:

Corollary D. *Let K_i be a mixed characteristic Henselian discrete valuation field, where $i = 1, 2$; X_i a smooth variety over K_i. Note that we have an exact sequence of profinite groups*

$$1 \longrightarrow \Delta_{X_i} \longrightarrow \Pi_{X_i} \longrightarrow G_{K_i} \longrightarrow 1.$$

Suppose that we are given an isomorphism of profinite groups

$$\phi : \Pi_{X_1} \xrightarrow{\sim} \Pi_{X_2}.$$

Then ϕ *induces an isomorphism of profinite groups* $\Delta_{X_1} \xrightarrow{\sim} \Delta_{X_2}$.

In particular, Corollary D implies that Question 3 is equivalent to Question 4 for the smooth varieties over mixed characteristic Henselian discrete valuation fields [cf. [29], Lemma 4.2]. We remark that there exists a research of the semi-absoluteness of isomorphisms between the étale fundamental groups of algebraic varieties [satisfying certain conditions] over real closed fields [cf. [14]].

The present paper is organized as follows. In §1, we define and recall some notions on profinite groups and fields [including higher local fields], and give basic properties. Then, by applying these properties, we prove the weak version of the Grothendieck Conjecture for hyperbolic curves of genus 0 over subfields of finitely generated extensions of mixed characteristic higher local fields [cf. Theorem A]. In §2, we first discuss properties of the subgroups of p-divisible elements of the multiplicative groups of fields. Next, by applying these properties, we prove the slimness of the absolute Galois groups of various fields such that the subgroups of p-divisible elements of the multiplicative groups are relatively small [cf. Theorem B]. In §3, we first give a general criterion of the elasticity of profinite groups. Next, by applying this criterion, we prove the elasticity of the absolute Galois groups of Henselian discrete valuation fields [cf. Theorem C]. In §4, we recall the definition of the semi-absoluteness of isomorphisms between the étale fundamental groups of smooth varieties over fields of characteristic 0. Then, by applying Theorem C, we prove the semi-absoluteness in the case where the base fields are mixed characteristic Henselian discrete valuation fields [cf. Corollary D].

Notations and Conventions

Numbers: The notation $\mathbb{Z}$ will be used to denote the additive group of integers. The notation $\mathbb{Z}_{\geq 1}$ will be used to denote the set of positive integers. The notation $\widehat{\mathbb{Z}}$ will be used to denote the profinite completion of $\mathbb{Z}$. If p is a prime number, then the notation $\mathbb{Z}_p$ will be used to denote the maximal pro-p-quotient of $\widehat{\mathbb{Z}}$; the notation $\mathbb{F}_p$ will be used to denote the finite field of cardinality p. We shall refer to a finite extension field of the field of p-adic numbers $\mathbb{Q}_p$ as a *p-adic local field*.

Fields: Let F be a field. Then we shall write F^{sep} for the separable closure [determined up to isomorphisms] of F; $F_{\mathrm{prm}} \subseteq F$ for the prime field; $G_F \stackrel{\mathrm{def}}{=} \mathrm{Gal}(F^{\mathrm{sep}}/F)$; $\mathrm{char}(F)$ for the characteristic of F; $F((t))$ for the one parameter formal power series field over F. If p is a prime number, and $\mathrm{char}(F) \neq p$, then we shall fix a primitive p-th root of unity $\zeta_p \in F^{\mathrm{sep}}$.

Profinite groups: Let p be a prime number; G a profinite group. Then we shall write G^p for the maximal pro-p quotient of G; $\mathrm{Aut}(G)$ for the group of automorphisms of G [in the category of profinite groups].

Fundamental groups: For a connected locally Noetherian scheme S, we shall write Π_S for the étale fundamental group of S, relative to a suitable choice of basepoint. [Note that, for any field F, $\Pi_{\mathrm{Spec}(F)} \cong G_F$.]

1 Weak version of the Grothendieck Conjecture for hyperbolic curves of genus 0 over mixed characteristic higher local fields

In this section, we define some notions concerning profinite groups and fields and give some basic properties. Moreover, by combining these properties with [30], Theorem F, we prove the weak version of the Grothendieck Conjecture for hyperbolic curves of genus 0 over subfields of finitely generated extensions of higher local fields whose final residue fields [cf. Definition 1.12, (iii)] are isomorphic to an algebraic closure of a finite field [cf. Corollary 1.16].

In the present section, let p be a prime number.

Definition 1.1 ([19], Notations and Conventions; [19], Definition 1.1, (ii))**.** Let G be a profinite group; $H \subseteq G$ a closed subgroup of G.

- (i) We shall write $Z_G(H)$ for the *centralizer* of H in G, i.e., the closed subgroup $\{g \in G \mid ghg^{-1} = h \text{ for any } h \in H\}$. We shall refer to $Z(G) \overset{\mathrm{def}}{=} Z_G(G)$ as the *center* of G.
- (ii) We shall say that G is *slim* if $Z_G(U) = \{1\}$ for every open subgroup U of G.
- (iii) We shall say that G is *elastic* if every nontrivial topologically finitely generated normal closed subgroup of an open subgroup of G is open in G. If G is elastic, but not topologically finitely generated, then we shall say that G is *very elastic*.

Proposition 1.2. *Let G be a nontrivial profinite group. Then the following hold:*

- *(i) G is slim if and only if, for every open subgroup $U \subseteq G$, $Z(U) = \{1\}$.*
- *(ii) G is very elastic if and only if every topologically finitely generated normal closed subgroup of G is trivial.*

Proof. First, we verify assertion (i). *Necessity* is immediate. Let us verify *sufficiency*. Let $H \subseteq G$ be an open subgroup; $\sigma \in Z_G(H)$. Write $U \subseteq G$ for the open subgroup generated by H and σ. Then since $\sigma \in Z(U)$, it follows from our assumption that $Z(U) = \{1\}$ that $\sigma = 1$. This completes the proof of *sufficiency*, hence of assertion (i).

Next, we verify assertion (ii). *Necessity* is immediate. Let us verify *sufficiency*. Note that since G is nontrivial, our assumption implies that G is not topologically finitely generated. Let $H \subseteq G$ be an open subgroup; $F \subseteq H$ a topologically finitely generated normal closed subgroup of H. Our goal is to prove that $F = \{1\}$. Write

$$F_g \stackrel{\text{def}}{=} g^{-1} \cdot F \cdot g \subseteq G$$

for each $g \in G$; $N \subseteq G$ for the closed subgroup topologically generated by the subgroups F_g ($g \in G$). Then since $H \subseteq G$ is an open subgroup, it follows immediately that N is a topologically finitely generated normal closed subgroup of G. Thus, we conclude from our assumption that $N = \{1\}$, hence that $F = \{1\}$. This completes the proof of assertion (ii), hence of Proposition 1.2. $\square$

Remark 1.2.1. Write $H \stackrel{\text{def}}{=} \mathbb{Z}_p \oplus \mathbb{Z}_p$; $i_1, i_2 \in \operatorname{Aut}(H)$ for the automorphisms of order 2 that map $(x, y) \in \mathbb{Z}_p \oplus \mathbb{Z}_p$ to $(-x, y), (y, x) \in \mathbb{Z}_p \oplus \mathbb{Z}_p$, respectively; $D \subseteq \operatorname{Aut}(H)$ for the subgroup generated by i_1, i_2 [which is a dihedral group of order 8]; $G \stackrel{\text{def}}{=} H \rtimes D$. Then it follows immediately that there exists a nontrivial topologically finitely generated normal closed subgroup of H that is not open in H, i.e., G is not elastic. However,

every nontrivial normal closed subgroup of G is open in G.

Indeed, let $F \subseteq G$ be a normal closed subgroup of infinite index. Then since D is finite, $F \cap H \subseteq G$ is a normal closed subgroup of infinite index. In particular, $F \cap H$ is a $\mathbb{Z}_p$-submodule of H of rank 0 or 1. On the other hand, it follows immediately from a direct computation that there is no $\mathbb{Z}_p$-submodule of H of rank 1 that is preserved by the action of D. Thus, we conclude that $F \cap H = \{1\}$, hence that we have a natural injection $F \hookrightarrow D$. Then since $F \cap H \subseteq G$ is a normal subgroup, we conclude that

$$[F, H] \ \subseteq \ F \cap H = \{1\},$$

where $[F, H]$ denotes the commutator subgroup of F and H. Therefore, since the natural composite $F \hookrightarrow D \subseteq \operatorname{Aut}(H)$ is injective, and $[F, H] = \{1\}$, it follows immediately that $F = \{1\}$.

Lemma 1.3 ([19], §0, Topological Groups)**.** *Let G be a slim profinite group; $F \subseteq G$ a finite normal subgroup. Then $F = \{1\}$.*

Proof. Write $\phi : G \to \mathrm{Aut}(F)$ for the natural [continuous] homomorphism determined by taking conjugates. Since $\mathrm{Aut}(F)$ is a finite group, $\mathrm{Ker}(\phi)$ is an open subgroup of G. Thus, the slimness of G implies that $F = \{1\}$. $\square$

Lemma 1.4 ([19], Proposition 1.3, (i))**.** *Let G be a slim profinite group; $H \subseteq G$ an open subgroup. Suppose that H is elastic (respectively, very elastic). Then G is elastic (respectively, very elastic).*

Proof. Since $H \subseteq G$ is an open subgroup, to verify Lemma 1.4, it suffices to verify the elasticity portion. Let $G_1 \subseteq G$ be an open subgroup; $F \subseteq G_1$ a nontrivial topologically finitely generated normal closed subgroup. Our goal is to prove that $F \subseteq G$ is an open subgroup. By replacing G by G_1, we may assume without loss of generality that $G = G_1$. Then it follows immediately from Lemma 1.3 that $F \cap H \subseteq H$ is a nontrivial topologically finitely generated normal closed subgroup. Thus, since H is elastic, we conclude that $F \cap H \subseteq H$ is an open subgroup, hence that $F \subseteq G$ is an open subgroup. This completes the proof of Lemma 1.4. $\square$

Definition 1.5 ([19], Definition 1.1, (iii))**.** Let G, Q be profinite groups; $q : G \twoheadrightarrow Q$ an epimorphism [in the category of profinite groups]. Then we shall say that Q is an *almost pro-p-maximal quotient* of G if there exists a normal open subgroup $N \subseteq G$ such that $\mathrm{Ker}(q)$ coincides with the kernel of the natural surjection $N \twoheadrightarrow N^p$.

Remark 1.5.1. It follows immediately from the various definitions involved that the maximal pro-p quotient of a profinite group is an *almost pro-p-maximal quotient*.

Lemma 1.6. *Let G be a profinite group. Suppose that, for each open subgroup $H \subseteq G$, there exists a normal open subgroup $N \subseteq H$ of G such that the almost pro-p-maximal quotient of G associated to N is slim (respectively, very elastic). Then G is slim (respectively, very elastic).*

Proof. Lemma 1.6 follows immediately from the fact that profinite groups are Hausdorff, together with the definition of almost pro-p-maximal quotients. $\square$

Definition 1.7. Let K be a field; $n \in \mathbb{Z}_{\geq 1}$.

(i) We shall write

$$K^\times \stackrel{\mathrm{def}}{=} K \setminus \{0\}; \quad \mu_n(K) \stackrel{\mathrm{def}}{=} \{x \in K^\times \mid x^n = 1\}; \quad \mu(K) \stackrel{\mathrm{def}}{=} \bigcup_{m \geq 1} \mu_m(K);$$

$$\mu_{p^\infty}(K) \stackrel{\mathrm{def}}{=} \bigcup_{m \geq 1} \mu_{p^m}(K); \quad K^{\times p^\infty} \stackrel{\mathrm{def}}{=} \bigcap_{m \geq 1} (K^\times)^{p^m};$$

(ii) We shall write

$$K^{\mathrm{cyc}} \stackrel{\mathrm{def}}{=} K(\mu(K^{\mathrm{sep}})) \ (\subseteq K^{\mathrm{sep}}); \quad K_{\times p^\infty} \stackrel{\mathrm{def}}{=} K_{\mathrm{prm}}(K^{\times p^\infty}) \subseteq K.$$

(iii) We shall say that K is *torally Kummer-faithful* if $\mathrm{char}(K) = 0$, and, for every finite extension L of K,

$$L^{\times\infty} = \{1\}$$

[cf. [21], Definition 1.5].

(iv) We shall say that K is *stably* p-$\times\mu$ (respectively, *stably* $\times\mu$)*-indivisible* if, for every finite extension L of K,

$$L^{\times p^\infty} \subseteq \mu(L) \quad \text{(respectively, } L^{\times\infty} \subseteq \mu(L)\text{)}.$$

(v) We shall say that K is *stably* μ_{p^∞} (respectively, *stably* μ)*-finite* if, for every finite extension L of K, $\mu_{p^\infty}(L)$ (respectively, $\mu(L)$) is a finite group.

(vi) For each separable algebraic extension $K \subseteq M$ $(\subseteq K^{\mathrm{sep}})$, we shall write

$$K_{p,\mathrm{div},M} \stackrel{\mathrm{def}}{=} \bigcup_{K\subseteq L} L_{\times p^\infty} \subseteq M; \quad \widetilde{K}_{p,\mathrm{div},M} \stackrel{\mathrm{def}}{=} K_{p,\mathrm{div},M}(\mu(M)) \ (\subseteq M),$$

where $K \subseteq L$ ranges over the set of finite separable extensions $\subseteq M$. If $M = K^{\mathrm{sep}}$, then we shall write $K_{p,\mathrm{div}} \stackrel{\mathrm{def}}{=} K_{p,\mathrm{div},M}$; $\widetilde{K}_{p,\mathrm{div}} \stackrel{\mathrm{def}}{=} \widetilde{K}_{p,\mathrm{div},M}$.

Remark 1.7.1. It follows immediately from the various definitions involved that torally Kummer-faithful fields are stably $\times\mu$-indivisible fields.

Proposition 1.8. *Let K be a field; L a finitely generated extension over K. Write $K^\dagger$ $(\subseteq L)$ for the algebraic closure of K in L. Then $L^{\times p^\infty} = (K^\dagger)^{\times p^\infty}$ (respectively, $L^{\times\infty} = (K^\dagger)^{\times\infty}$). In particular,*

- *if L is separably generated over K, then $L_{p,\mathrm{div}} = K_{p,\mathrm{div}}$;*
- *if K is a stably p-$\times\mu$ (respectively, $\times\mu$)-indivisible field, then L is a stably p-$\times\mu$ (respectively, $\times\mu$)-indivisible field.*

Proof. The inclusion $L^{\times p^\infty} \supseteq (K^\dagger)^{\times p^\infty}$ (respectively, $L^{\times\infty} \supseteq (K^\dagger)^{\times\infty}$) is immediate. Thus, it suffices to prove that $L^{\times p^\infty} \subseteq (K^\dagger)^{\times p^\infty}$ (respectively, $L^{\times\infty} \subseteq (K^\dagger)^{\times\infty}$). Let X be a connected proper normal scheme over K such that the function field of X is L. [Note that L is a finite extension of a purely transcendental extension M of K. Let P be a projective space over K such that the function field of P is M. Then the existence of such a scheme follows immediately by taking the normalization of P in L.] Write $\mathcal{O}_X$ for the structure sheaf of X. Note that, since X is proper integral over K, $\mathcal{O}_X(X)$ is a finite extension of K. In particular, we have $\mathcal{O}_X(X) \subseteq K^\dagger$. Let $x \in X$ be a point such that the Zariski closure $\overline{\{x\}} \subseteq X$ is codimension 1; v_x a discrete valuation on L associated to x; $f \in L^{\times p^\infty}$ (respectively, $f \in L^{\times\infty}$). Then it follows immediately that $v_x(f) = 0$. Thus, since X is normal, we conclude that $f \in \mathcal{O}_X(X)$. Moreover, since $\mathcal{O}_X(X)$ is algebraically closed in L [cf. the fact that X is normal], we have $f \in (K^\dagger)^{\times p^\infty}$ (respectively, $f \in (K^\dagger)^{\times\infty}$). This completes the proof of Proposition 1.8. $\square$

Next, we recall the following well-known lemma:

Lemma 1.9. *Let A be a Noetherian local domain. Write K for the quotient field of A; $\mathfrak{m}$ for the maximal ideal of A; $k \stackrel{\text{def}}{=} A/\mathfrak{m}$. Then there exists a discrete valuation ring A' $(\subseteq K)$ such that*

- *A' dominates A, and*
- *the residue field extension $k \hookrightarrow k'$ is finitely generated, where k' denotes the residue field of A'.*

Proof. Lemma 1.9 follows immediately from the usual construction of A' [cf. [9], Chapter II, Exercise 4.11, (a)], together with [24], Theorem 33.2, i.e., Krull-Akizuki's theorem. $\square$

Proposition 1.10. *In the notation of Lemma 1.9, suppose that the residue field k is a stably p-$\times\mu$-indivisible field of characteristic p. Then K is stably p-$\times\mu$-indivisible.*

Proof. First, by applying Proposition 1.8 and Lemma 1.9, we may assume without loss of generality that K is a discrete valuation field. Moreover, by replacing K by the completion of K, we may also assume without loss of generality that K is a complete discrete valuation field. Then since every finite extension of K is a complete discrete valuation field, it suffices to prove that $K^{\times p^\infty} \subseteq \mu(K)$.

Let $x \in K^{\times p^\infty}$ be an element. Write $A^\triangleright \stackrel{\text{def}}{=} A \setminus \{0\}$. Then since $x \in K^{\times p^\infty}$, x is a unit $\in A$. In particular, we have

$$x \in \bigcap_{m \geq 1} (A^\triangleright)^{p^m}.$$

Write $\overline{x} \in k$ for the image of x via the natural surjection $A \twoheadrightarrow k$. Then our assumption that k is *stably p-$\times\mu$-indivisible* implies that $\overline{x} \in \mu(k)$. In particular, since A is complete, we have

$$x \in (1+\mathfrak{m}) \times \mu'(K) \subseteq A^\triangleright,$$

where

$$\mu'(K) \stackrel{\text{def}}{=} \bigcup_{m \geq 1,\ p \nmid m} \mu_m(K).$$

Since $\mathrm{char}(k) = p$, it holds that $p \in \mathfrak{m}$, hence that $(1 + \mathfrak{m}^i)^p \subseteq 1 + \mathfrak{m}^{i+1}$ for each $i \in \mathbb{Z}_{\geq 1}$. Thus, we conclude that

$$x \in \big(\bigcap_{i \geq 1}(1 + \mathfrak{m}^i)\big) \times \mu'(K).$$

On the other hand, since A is a Noetherian local ring, it follows from Krull's intersection theorem that $\bigcap_{i \geq 1}(1+\mathfrak{m}^i) = \{1\}$. In particular, we have $x \in \mu'(K)$. This completes the proof of Proposition 1.10. □

Remark 1.10.1. Let K be a field of characteristic 0. Then the one parameter formal power series field $K((t))$ over K is not stably $\times\mu$-indivisible. Indeed, write $K[[t]]$ $(\subseteq K((t)))$ for the one parameter formal power series ring. Then it follows immediately by a direct calculation that any element $\in 1 + t \cdot K[[t]]$ is divisible.

Lemma 1.11. *In the notation of Lemma 1.9, suppose that k is stably μ_{p^∞} (respectively, stably μ)-finite. Then K is stably μ_{p^∞} (respectively, stably μ)-finite.*

Proof. First, by applying Lemma 1.9, we may assume without loss of generality that K is a discrete valuation field. Moreover, by replacing K by the completion of K, we may also assume without loss of generality that K is a complete discrete valuation field. Then since every finite extension of K is a complete discrete valuation field, it suffices to prove that $\mu_{p^\infty}(K)$ (respectively, $\mu(K)$) is a finite group.

Let l be a prime number such that $\mathrm{char}(k) \neq l$. Then, since K is complete, we have a natural isomorphism $\mu_{l^\infty}(K) \xrightarrow{\sim} \mu_{l^\infty}(k)$. Thus, it suffices to prove that, if $\mathrm{char}(k) = p$, then $\mu_{p^\infty}(K)$ is a finite group. However, this follows immediately from our assumption that K is a discrete valuation field, together with the fact that $p \in \mathfrak{m}$. This completes the proof of Lemma 1.11. □

Definition 1.12 ([5], Chapter I, §1.1)**.** Let K be a field; $d \in \mathbb{Z}_{\geq 1}$.

(i) A structure of *local field of dimension d* on K is a sequence of complete discrete valuation fields $K^{(d)} \stackrel{\text{def}}{=} K, K^{(d-1)}, \ldots, K^{(0)}$ such that

- $K^{(0)}$ is a perfect field;
- for each integer $0 \leq i \leq d-1$, $K^{(i)}$ is the residue field of the complete discrete valuation field $K^{(i+1)}$.

(ii) We shall say that K is a *higher local field* if K admits a structure of local field of some positive dimension. In the remainder of the present paper, for each higher local field, we fix a structure of local field of some positive dimension.

(iii) Suppose that K is a higher local field of dimension d. We shall refer to $K^{(0)}$ as the *final residue field* of K. We shall say that K is a *mixed* (respectively, *positive*) *characteristic higher local field* if $\mathrm{char}(K) = 0$ and $\mathrm{char}(K^{(d-1)}) > 0$ (respectively, $\mathrm{char}(K) > 0$).

Remark 1.12.1. For each complete discrete valuation field F with a discrete valuation v_F, write

$$F\{\{t\}\} \stackrel{\mathrm{def}}{=} \Big\{ \sum_{i=-\infty}^{\infty} a_i t^i \mid \inf v_F(a_i) > -\infty, \lim_{i \to -\infty} v_F(a_i) = \infty \Big\}.$$

We note that $F\{\{t\}\}$ is a complete discrete valuation field via the discrete valuation $\sum_{i=-\infty}^{\infty} a_i t^i \mapsto \inf v_F(a_i)$. Let $d \in \mathbb{Z}_{\geq 1}$; K a higher local field of dimension d. Then it follows immediately from Cohen's structure theorem, together with [6], Chapter II, Proposition 5.6, that the following hold:

(i) Suppose that $\mathrm{char}(K) = p > 0$. Then K is isomorphic to $K^{(0)}((t_1)) \cdots ((t_d))$.

(ii) Suppose that $\mathrm{char}(K^{(d-1)}) = 0$. Then K is isomorphic to $K^{(d-1)}((t))$.

(iii) Suppose that K is a mixed characteristic higher local field. Write M_0 for the field of fractions of the Witt ring associated to $K^{(0)}$. Then K is isomorphic to a finite extension of $M_0\{\{t_1\}\} \cdots \{\{t_{d-1}\}\}$.

Lemma 1.13. *Let K be a higher local field. Suppose that $K^{(0)}$ is a stably μ_{p^∞}-finite field. Then K is also a stably μ_{p^∞}-finite field. In particular, if $\mathrm{char}(K) \neq p$, then the p-adic cyclotomic character $G_K \to \mathbb{Z}_p^\times$ is open.*

Proof. Since K is a higher local field, Lemma 1.13 follows immediately by applying Lemma 1.11 inductively. □

Next, we give examples of stably p-$\times\mu$-indivisible fields that are not given in [30], Remark 3.4.1.

Example 1.14. Let $\overline{\mathbb{F}}_p$ be an algebraic closure of $\mathbb{F}_p$.

(i) Let K be a higher local field such that

- $K^{(0)}$ is isomorphic to a subfield of $\overline{\mathbb{F}}_p$,
- the residue characteristic of K is $p > 0$.

Then it follows immediately by applying Proposition 1.10 inductively that K is *stably* p-$\times\mu$-*indivisible*. Moreover, if $\mathrm{char}(K) = 0$, then it follows from Lemma 1.13, together with [30], Lemma D, (iv), that any abelian extension of K is *stably* p-$\times\mu$-*indivisible*.

(ii) Let X be a normal scheme of finite type over Spec $\overline{\mathbb{F}}_p$; $x \in X$ a point. Write $\widehat{\mathcal{O}}_{X,x}$ for the completion of the stalk $\mathcal{O}_{X,x}$ at x; K_x for the quotient field of $\widehat{\mathcal{O}}_{X,x}$. Then K_x is *stably* p-$\times\mu$-*indivisible*. Indeed, write k_x for the residue field of $\mathcal{O}_{X,x}$. Since k_x is a finitely generated extension over $\overline{\mathbb{F}}_p$, k_x is *stably* p-$\times\mu$-*indivisible* [cf. Proposition 1.8]. Thus, since $\widehat{\mathcal{O}}_{X,x}$ is Noetherian local domain, it follows from Proposition 1.10 that K_x is *stably* p-$\times\mu$-*indivisible*.

In particular, since any subfield of a stably p-$\times\mu$-indivisible field is stably p-$\times\mu$-indivisible [cf. [30], Lemma D, (ii)], Example 1.14 implies that many [arithmetic geometric] examples [including, for example, K'_x and K_y appeared in [3], §1.1] are stably p-$\times\mu$-indivisible.

Definition 1.15. Let $\overline{\mathbb{F}}_p$ be an algebraic closure of $\mathbb{F}_p$; L a field of characteristic 0. Then we shall say that L is an *absolute higher sub-local field* if there exists a higher local field K such that

- $K^{(0)}$ is isomorphic to $\overline{\mathbb{F}}_p$,
- the residue characteristic of K is $p > 0$, and
- L is isomorphic to a subfield of a finitely generated extension of K.

Corollary 1.16. *Let L be an absolute higher sub-local field of residue characteristic p; U and V be hyperbolic curves of genus* 0 *over L;*

$$\phi : \Pi_U \xrightarrow{\sim} \Pi_V$$

an isomorphism of profinite groups such that ϕ lies over the identity automorphism on G_L. Then there exists an isomorphism of L-schemes

$$U \xrightarrow{\sim} V$$

that induces a bijection between the cusps of U and V which is compatible with the bijection between cuspidal inertia subgroups of Π_U and Π_V induced by ϕ.

Proof. First, it follows immediately from Lemma 1.13, together with [18], Corollary 2.7, (i), that ϕ induces a bijection between the set of cuspidal inertia subgroups of Π_U and the set of cuspidal inertia subgroups of Π_V. On the other hand, it follows immediately from Proposition 1.8, together with Example 1.14, (i), that L is a stably p-$\times\mu$-indivisible field of characteristic 0. Thus, Corollary 1.16 follows immediately from [30], Theorem F. □

Remark 1.16.1. In the notation of Corollary 1.16, at the time of writing the present paper, the authors do not know whether there exists an isomorphism of L-schemes

$$U \xrightarrow{\sim} V$$

that induces ϕ. The authors hope to be able to address such an issue [i.e., the Grothendieck Conjecture for hyperbolic curves over higher local fields] in the future paper.

2 Slimness of (almost pro-p-maximal quotients of) the absolute Galois groups of discrete valuation fields

In this section, we prove that the absolute Galois groups of subfields of mixed characteristic discrete valuation fields are slim. Moreover, we also prove that the absolute Galois groups of positive characteristic complete [hence, Henselian — cf. Lemma 3.1] discrete valuation fields are slim.

In the present section, let p be a prime number.

Lemma 2.1. *Let L be a field. Write*

$$(L^{\times p^\infty} \subseteq)\ S \stackrel{\text{def}}{=} \{a \in L^\times \mid \exists n \in \mathbb{Z}_{\geq 1} \text{ such that } a^n \in L^{\times p^\infty}\}$$

for the saturation of $L^{\times p^\infty}$ in $L^\times$. Then the following hold:

(i) Suppose that $\mu_{p^\infty}(L)$ is finite. Then $S = \mu_{p^\infty}(L) \cdot L^{\times p^\infty}$. In particular, if $L^\times/L^{\times p^\infty}$ is a torsion group, then $L^\times = \mu_{p^\infty}(L) \cdot L^{\times p^\infty}$.

(ii) Suppose that $\mu_{p^\infty}(L)$ is infinite. Then $S = L^{\times p^\infty}$. In particular, if $L^\times/L^{\times p^\infty}$ is a torsion group, then $L^\times = L^{\times p^\infty}$.

Proof. Let $a \in S$ be an element. Then there exists $s \in \mathbb{Z}_{\geq 1}$ such that $a^s \in L^{\times p^\infty}$. Let us note that, for each $(d, i) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$ such that d is coprime to p, the d-th power map on [the $\mathbb{Z}/p^i\mathbb{Z}$-module] $L^\times/(L^\times)^{p^i}$ is bijective, hence, in particular, the d-th power map on $L^\times/L^{\times p^\infty}$ is injective. Thus, we may assume without loss of generality that $s = p^t$, where $t \in \mathbb{Z}_{\geq 1}$. Then, for each $n \in \mathbb{Z}_{\geq 1}$, there exists $b_n \in L^\times$ such that $(b_n)^{p^{t+n}} = a^{p^t}$. In particular, we have $(b_n)^{p^n} \cdot a^{-1} \in \mu_{p^\infty}(L)$. Note that we have $b_n \in S$.

First, we verify assertion (i). Write p^m for the cardinality of $\mu_{p^\infty}(L)$. Then it follows that $(b_n)^{p^{m+n}} = a^{p^m}$. Thus, it follows that $S^{p^m} \subseteq L^{\times p^\infty}$. Moreover, since $(b_m)^{p^{2m}} = a^{p^m}$, we conclude that

$$a \in (b_m)^{p^m} \cdot \mu_{p^\infty}(L) \subseteq \mu_{p^\infty}(L) \cdot S^{p^m} \subseteq \mu_{p^\infty}(L) \cdot L^{\times p^\infty}.$$

This completes the proof of assertion (i).

Next, we verify assertion (ii). Let us observe that, since $\mu_{p^\infty}(L)$ is infinite,

$$\mu_{p^\infty}(L) = \mu_{p^\infty}(L^{\mathrm{sep}}) \subseteq L.$$

Then this observation immediately implies that, for each $n \in \mathbb{Z}_{\geq 1}$, there exists $z_n \in \mu_{p^\infty}(L)$ such that $(z_n \cdot b_n)^{p^n} = a$. Thus, we conclude that $S = L^{\times p^\infty}$. This completes the proof of assertion (ii), hence of Lemma 2.1. $\square$

Lemma 2.2. *Let L be a field such that* $\mathrm{char}(L) \neq 2$, *and* $\sqrt{-1} \in L$; $\sigma \in \mathrm{Aut}(L)$ *a field automorphism such that* $\sigma^2 = 1$, *and* $(\sqrt{-1})^\sigma = -\sqrt{-1}$. *Write* $\overline{\sigma} \in \mathrm{Aut}(L^\times / L^{\times p^\infty})$ *for the group automorphism induced by* σ. *Suppose that*

$$\overline{\sigma}(\overline{x}) = \overline{x}^{-1} \ (\overline{x} \in L^\times / L^{\times p^\infty}).$$

Then $L = L_{\times p^\infty}(\sqrt{-1})$.

Proof. Our assumption that $\overline{\sigma}(\overline{x}) = \overline{x}^{-1}$ $(\overline{x} \in L^\times / L^{\times p^\infty})$ implies that, for each $x \in L \setminus \{0, 1\}$, it holds that

$$x \cdot x^\sigma \in L^{\times p^\infty}, \quad (1-x)(1-x^\sigma) \in L^{\times p^\infty}.$$

In particular, we have $x + x^\sigma \in L_{\times p^\infty}$. Write L^σ $(\subseteq L)$ for the subfield fixed by σ. Then since $\mathrm{char}(L) \neq 2$, we conclude that $L^\sigma \subseteq L_{\times p^\infty}$ $(\subseteq L)$. On the other hand, our assumptions concerning σ imply that $[L : L^\sigma] = 2$, and $\sqrt{-1} \notin L^\sigma$. Thus, we conclude that $L = L_{\times p^\infty}(\sqrt{-1})$. This completes the proof of Lemma 2.2. $\square$

Lemma 2.3. *Let L be a field such that* $\mathrm{char}(L) \neq p$; $L \subseteq M$ $(\subseteq L^{\mathrm{sep}})$ *a Galois extension;* $\sigma \in Z(\mathrm{Gal}(M/L))$ $(\subseteq \mathrm{Gal}(M/L))$. *Suppose that,*

- $\zeta_p \in L$;
- $M^\times = M^{\times p^\infty}$.

Write

$$\chi_p : \mathrm{Gal}(M/L) \to \mathbb{Z}_p^\times$$

for the p-adic cyclotomic character. [Note that since $\zeta_p \in L$, *and* $M^\times = M^{\times p^\infty}$, *we have* $\mu_{p^\infty}(M) = \mu_{p^\infty}(L^{\mathrm{sep}})$.*] Then the following hold:*

(i) Suppose, moreover, that

  - *if* $p = 2$, *then* $\sqrt{-1} \in L$;
  - *there exists a finite Galois extension* $L \subseteq L^\dagger$ $(\subseteq M)$ *such that the quotient* $(L^\dagger)^\times / (L^\dagger)^{\times p^\infty}$ *is not a torsion group.*

  Then $\chi_p(\sigma) = 1$.

(ii) *Suppose, moreover, that* $\chi_p(\sigma) = 1$. *Then, for each finite Galois extension* $L \subseteq L^\dagger$ $(\subseteq M)$ *such that* $(L^\dagger)_{\times p^\infty} \subsetneq L^\dagger$, σ *acts trivially on* $L^\dagger$.

Proof. For each finite Galois extension $L \subseteq L^\dagger$ $(\subseteq M)$, write

$$\kappa_{L^\dagger} : (L^\dagger)^\times \twoheadrightarrow (L^\dagger)^\times/(L^\dagger)^{\times p^\infty} \hookrightarrow H^1(\mathrm{Gal}(M/L^\dagger), \mathbb{Z}_p(1))$$

for the Kummer map, where "(1)" denotes the Tate twist.

First, we verify assertion (i). Let $L \subseteq L^\dagger$ $(\subseteq M)$ be a finite Galois extension such that $(L^\dagger)^\times/(L^\dagger)^{\times p^\infty}$ is not a torsion group. Write e for the cardinality of $\mathrm{Gal}(L^\dagger/L)$. Note that we have natural actions of $\sigma^e \in \mathrm{Gal}(M/L)$ on $(L^\dagger)^\times$ and $H^1(\mathrm{Gal}(M/L^\dagger), \mathbb{Z}_p(1))$ compatible with $\kappa_{L^\dagger}$. Let us note that σ^e acts trivially on $(L^\dagger)^\times$. Then since $(L^\dagger)^\times/(L^\dagger)^{\times p^\infty}$ contains a torsion-free element, and $\sigma^e \in Z(\mathrm{Gal}(M/L^\dagger))$, it follows that $\chi_p(\sigma^e) = 1$. Here, we observe that since $\zeta_p \in L$ (respectively, $\sqrt{-1} \in L$), the image of χ_p is torsion-free. Thus, we conclude that $\chi_p(\sigma) = 1$. This completes the proof of assertion (i).

Next, we verify assertion (ii). Let $a \in L^\dagger \setminus (L^\dagger)_{\times p^\infty}$ be an element [so, $1 - a \in L^\dagger \setminus (L^\dagger)_{\times p^\infty}$]. Note that we have natural actions of $\sigma \in \mathrm{Gal}(M/L)$ on $(L^\dagger)^\times$ and $H^1(\mathrm{Gal}(M/L^\dagger), \mathbb{Z}_p(1))$ compatible with $\kappa_{L^\dagger}$. Thus, since $\sigma \in Z(\mathrm{Gal}(M/L))$ $(\subseteq \mathrm{Gal}(M/L))$, and $\chi_p(\sigma) = 1$, we conclude that there exist $s, t \in (L^\dagger)_{\times p^\infty}$ such that

$$a^\sigma = s \cdot a, \quad 1 - a^\sigma = (1-a)^\sigma = t \cdot (1-a).$$

If $a \neq a^\sigma$, then it follows immediately that

$$s \neq 1, \quad t \neq 1, \quad s \neq t, \quad a = \frac{1-t}{s-t} \in (L^\dagger)_{\times p^\infty}.$$

This is a contradiction. Then we have $a = a^\sigma$. On the other hand, we note that, for each $x \in (L^\dagger)_{\times p^\infty} \subseteq L^\dagger$,

$$x^\sigma = (a+x)^\sigma - a^\sigma = (a+x) - a = x$$

[$a + x \in L \setminus (L^\dagger)_{\times p^\infty}$]. Thus, we conclude that σ acts trivially on L. This completes the proof of assertion (ii), hence of Lemma 2.3. $\square$

Theorem 2.4. *Let* K *be a field such that* $\mathrm{char}(K) \neq p$; $K \subseteq M$ $(\subseteq K^{\mathrm{sep}})$ *a Galois extension. Then the following hold:*

(i) *Suppose that,*

- $\zeta_p \in K$;
- $M^\times = M^{\times p^\infty}$;
- $\widetilde{K}_{p,\mathrm{div},M} \subsetneq M$.

Then $\mathrm{Gal}(M/K)$ *is slim.*

(ii) Suppose that

$$\widetilde{K}_{p,\mathrm{div}} \subsetneq K^{\mathrm{sep}}.$$

Let L be a finitely generated extension over K. Then the absolute Galois group G_L is slim.

(iii) Suppose that,

- $\zeta_p \in K$;
- $\widetilde{K}_{p,\mathrm{div}} \subsetneq K^{\mathrm{sep}}$.

Let L be a finitely generated extension over K. Then, for each open subgroup $H \subseteq G_L$, there exists a normal open subgroup $N \subseteq H$ of G_L such that the almost pro-p-maximal quotient associated to N is slim.

(iv) Let L be a finitely generated transcendental extension over K. Then G_L is slim. Moreover, if $\zeta_p \in L$ [where we fix an embedding $K^{\mathrm{sep}} \subseteq L^{\mathrm{sep}}$], then any almost pro-$p$-maximal quotient of G_L is slim.

(v) Suppose that

- *K is a stably p-$\times\mu$-indivisible field [cf. Definition 1.7, (iv)];*
- *if $\mathrm{char}(K) \neq 0$, then K is transcendental over K_{prm}.*

Then the absolute Galois group G_K is slim. Moreover, if $\zeta_p \in K$, then any almost pro-p-maximal quotient of G_K is slim.

Proof. First, we verify assertion (i). Let us note that, for every finite separable extension $K \subseteq K^\dagger$ ($\subseteq M$), $K_{p,\mathrm{div},M} = K^\dagger_{p,\mathrm{div},M}$. Thus, it suffices to prove that $\mathrm{Gal}(M/K)$ is center-free [cf. Proposition 1.2, (i)].

Let $\sigma \in Z(\mathrm{Gal}(M/K))$ ($\subseteq \mathrm{Gal}(M/K)$) be an element. Write

$$\chi_p : \mathrm{Gal}(M/K) \to \mathbb{Z}_p^\times$$

for the p-adic cyclotomic character. [Note that since $\zeta_p \in K$, and $M^\times = M^{\times p^\infty}$, we have $\mu_{p^\infty}(M) = \mu_{p^\infty}(K^{\mathrm{sep}})$.] First, it follows formally from Lemma 2.1, together with our assumption that $\widetilde{K}_{p,\mathrm{div},M} \subsetneq M$, that there exists a finite Galois extension $K \subseteq K^\dagger$ ($\subseteq M$) such that $(K^\dagger)^\times/(K^\dagger)^{\times p^\infty}$ contains a torsion-free element.

Suppose that $p \neq 2$, or $\sqrt{-1} \in K$. Then it follows immediately from Lemma 2.3, (i), that $\chi_p(\sigma) = \{1\}$. On the other hand, we note that, for every finite Galois extension $K \subseteq K^\dagger$ ($\subseteq M$), there exists a finite Galois extension $K \subseteq K^\ddagger$ ($\subseteq M$) such that $K^\dagger \subseteq K^\ddagger$, and $K^\ddagger \not\subseteq K_{p,\mathrm{div},M}$. Thus, we conclude from Lemma 2.3, (ii), that $\sigma = 1$.

Finally, we consider the case where $p = 2$, and $\sqrt{-1} \notin K$. Note that since $M^\times = M^{\times p^\infty}$, we have $\sqrt{-1} \in M$. Then it follows immediately from the above discussion that $Z(\mathrm{Gal}(M/K(\sqrt{-1}))) = \{1\}$. Write $M^\sigma \subseteq M$ for the subfield fixed by σ. Suppose that $\sigma \neq 1$. Then since $Z(\mathrm{Gal}(M/K(\sqrt{-1}))) = \{1\}$, we have $\chi_p(\sigma) \neq 1$. Now observe that

$$\sigma^2 = 1, \quad \mathrm{char}(K) \neq 2, \quad M = M^\sigma(\sqrt{-1}), \quad \sqrt{-1} \notin M^\sigma.$$

Thus, since $\sigma^2 = 1$, and $\chi_p(\sigma) \neq 1$, we have $\chi_p(\sigma) = -1$. For each finite Galois extension $K \subseteq K^\dagger$ ($\subseteq M$), let us consider natural actions of $\sigma \in G_K$ on $(K^\dagger)^\times$ and $H^1(G_{K^\dagger}, \mathbb{Z}_p(1))$, which are compatible with the Kummer map

$$(K^\dagger)^\times \twoheadrightarrow (K^\dagger)^\times/(K^\dagger)^{\times p^\infty} \hookrightarrow H^1(G_{K^\dagger}, \mathbb{Z}_p(1)).$$

Then, by applying Lemma 2.2 to various finite Galois extensions $K^\dagger$ such that $\sqrt{-1} \in K^\dagger$, we obtain $K_{p,\mathrm{div},M}(\sqrt{-1}) = M$. This contradicts our assumption that $\widetilde{K}_{p,\mathrm{div},M} \subsetneq M$. Thus, we conclude that $\sigma = 1$. This completes the proof of assertion (i).

Next, we verify assertion (ii). Since every purely inseparable extension does not change the absolute Galois group, we may assume without loss of generality that L is separably generated over K. Then, by applying Proposition 1.8, we observe that

$$\widetilde{L}_{p,\mathrm{div}} = \widetilde{K}_{p,\mathrm{div}} \subsetneq K^{\mathrm{sep}} \subseteq L^{\mathrm{sep}},$$

where we fix an embedding $K^{\mathrm{sep}} \subseteq L^{\mathrm{sep}}$. Thus, we may assume without loss of generality that $L = K$. Let us note that, for every finite separable extension $K \subseteq K^\dagger$ ($\subseteq K^{\mathrm{sep}}$), $K_{p,\mathrm{div}} = K^\dagger_{p,\mathrm{div}}$. Thus, it suffices to prove that G_K is center-free [cf. Proposition 1.2, (i)].

Let $\sigma \in Z(G_K)$ ($\subseteq G_K$) be an element. First, we observe that $Z(G_{K(\zeta_p)}) = \{1\}$ [cf. (i)]. In particular, it holds that σ is a torsion element. Write $\chi_p : G_K \to \mathbb{Z}_p^\times$ for the p-adic cyclotomic character; $(K^{\mathrm{sep}})^\sigma \subseteq K^{\mathrm{sep}}$ for the subfield fixed by σ. Suppose that $\sigma \neq 1$. Then since $Z(G_{K(\zeta_p)}) = \{1\}$, we have $\chi_p(\sigma) \neq 1$. Now observe that

$$\sigma^2 = 1, \quad \mathrm{char}(K) = 0, \quad K^{\mathrm{sep}} = (K^{\mathrm{sep}})^\sigma(\sqrt{-1}), \quad \sqrt{-1} \notin (K^{\mathrm{sep}})^\sigma$$

[cf. Artin-Schreier theorem]. Thus, we conclude from Lemma 2.2, together with a similar argument to the argument applied in the final part of the proof of assertion (i), that $\sigma = 1$. This completes the proof of assertion (ii).

Next, we verify assertion (iii). By a similar argument to the argument applied in the beginning part of the proof of assertion (ii), we may assume without loss of generality that $L = K$. For each open subgroup $H \subseteq G_K$, write $K_H \subseteq K^{\mathrm{sep}}$ for the finite separable extension of K associated to H; $K_H^p \subseteq K^{\mathrm{sep}}$ for the maximal pro-p extension of K_H. Let $H \subseteq G_K$ be an open subgroup. Then it follows immediately from the various definitions involved that, if every normal open subgroup $N \subseteq H$ of G_K satisfies $\widetilde{K}_{p,\mathrm{div},K_N^p} = K_N^p$, then $\widetilde{K}_{p,\mathrm{div}} = K^{\mathrm{sep}}$. This contradicts our assumption that $\widetilde{K}_{p,\mathrm{div}} \subsetneq K^{\mathrm{sep}}$. Thus, we conclude that there exists a normal open subgroup $N \subseteq H$ of G_K such that

$$\widetilde{K}_{p,\mathrm{div},K_N^p} \subsetneq K_N^p.$$

Here, since $\zeta_p \in K_N^p$, we have $(K_N^p)^\times = (K_N^p)^{\times p^\infty}$. Then it follows immediately from assertion (i) that $\mathrm{Gal}(K_N^p/K)$ is slim. This completes the proof of assertion (iii).

Next, we verify assertion (iv). Since every purely inseparable extension does not change the absolute Galois group, we may assume without loss of generality that L is separably generated over K. Then since L is transcendental over K, by applying Proposition 1.8, we observe that

$$\widetilde{L}_{p,\mathrm{div}} = \widetilde{K}_{p,\mathrm{div}} \subseteq K^{\mathrm{sep}} \subsetneq L^{\mathrm{sep}},$$

where we fix an embedding $K^{\mathrm{sep}} \subseteq L^{\mathrm{sep}}$. Thus, we conclude from assertion (ii) that G_L is slim. Next, we suppose that $\zeta_p \in L$. Let $N \subseteq G_L$ be a normal open subgroup. Write $L_N \subseteq L^{\mathrm{sep}}$ for the finite Galois extension of L associated to N; $L_N^p \subseteq L^{\mathrm{sep}}$ for the maximal pro-p extension of L_N. Again, by applying Proposition 1.8, we observe that

$$\widetilde{L}_{p,\mathrm{div},L_N^p} \subseteq K^{\mathrm{sep}} \cap L_N^p \subsetneq L_N^p.$$

Then it follows immediately from assertion (i) that $\mathrm{Gal}(L_N^p/L)$ is slim. This completes the proof of assertion (iv).

Next, we verify assertion (v). The slimness of G_K follows immediately from assertion (ii). Suppose that $\zeta_p \in K$. Let $N \subseteq G_K$ be a normal open subgroup. Then it suffices to prove that

$$\widetilde{K}_{p,\mathrm{div},K_N^p} \subsetneq K_N^p$$

[cf. (i)]. Since K is a stably p-$\times\mu$-indivisible field, it follows immediately that $\widetilde{K}_{p,\mathrm{div},K_N^p}$ is a cyclotomic extension of K_{prm}, hence a stably p-$\times\mu$-indivisible field [cf. [30], Lemma D, (iv)]. Recall our assumption that, if $\mathrm{char}(K) \neq 0$, then K is transcendental over K_{prm}. Thus, since $(K_N^p)^\times = (K_N^p)^{\times p^\infty}$, we conclude that $\widetilde{K}_{p,\mathrm{div},K_N^p} \subsetneq K_N^p$. This completes the proof of assertion (v), hence of Theorem 2.4. $\square$

Remark 2.4.1. Note that stably p-$\times\mu$-indivisible fields are stably $\times\mu$-indivisible fields. Then it is natural to pose the following questions:

> Question 1: Is the absolute Galois group of any torally Kummer-faithful field slim [cf. [12], Proposition 1.5, (i)]?
>
> Question 2: More generally [cf. Remark 1.7.1], is the absolute Galois group of any stably $\times\mu$-indivisible field of characteristic 0 slim?

However, at the time of writing the present paper, the authors do not know whether these questions are affirmative or not.

Lemma 2.5. *Let K be a stably μ_{p^∞}-finite field such that* $\mathrm{char}(K) \neq p$*; $K \subseteq L$ $(\subseteq K^{\mathrm{sep}})$ a Galois extension such that one of the following conditions hold:*

- *$K \subseteq L$ $(\subseteq K^{\mathrm{sep}})$ is an abelian extension.*
- *L is stably μ_{p^∞}-finite.*

Then

$$L^{\times p^\infty} \subseteq \bigcup_{K \subseteq K^\dagger} (K^\dagger)^{\times p^\infty} \cdot \mu_{p^\infty}(K^{\mathrm{sep}}) \ (\subseteq K^{\mathrm{sep}}),$$

where $K \subseteq K^\dagger$ $(\subseteq K^{\mathrm{sep}})$ ranges over the set of finite separable extensions $\subseteq K^{\mathrm{sep}}$. In particular, we have

$$\widetilde{K}_{p,\mathrm{div}} = \widetilde{L}_{p,\mathrm{div}} \ (\subseteq K^{\mathrm{sep}}).$$

Proof. Lemma 2.5 follows from a similar argument to the argument given in the proof of [30], Lemma 3.4, (iv), (v), together with Lemma 2.1. $\square$

Lemma 2.6. *Let A be a complete discrete valuation ring such that the residue field k is of characteristic p. Write K for the quotient field of A. Then $K^{\times p^\infty}$ coincides with the image of Teichmüller character $k^{\times p^\infty} \hookrightarrow A$.*

Proof. Since A is a discrete valuation ring, we have $K^{\times p^\infty} \subseteq A$. Thus, Lemma 2.6 follows immediately from [the proof of] [28], Chapter II, Proposition 8. $\square$

Lemma 2.7. *Let A be a mixed characteristic discrete valuation ring such that the residue field k is of characteristic p. Write K for the quotient field of A. For each separable algebraic extension $K \subseteq M$ $(\subseteq K^{\mathrm{sep}})$, write $A_M \subseteq M$ for the integral closure of A in M; $A_M^\times \subseteq A_M$ for the subgroup of units. Let $K \subseteq L$ $(\subseteq K^{\mathrm{sep}})$ be a Galois extension such that one of the following conditions hold:*

- *$K \subseteq L$ $(\subseteq K^{\mathrm{sep}})$ is an abelian extension.*
- *L is stably μ_{p^∞}-finite.*

Then the following hold:

(i) $L^{\times p^\infty} \subseteq A_L^\times$.

(ii) $(\widetilde{L}_{p,\mathrm{div}})^{\times p^\infty} \subseteq A_{K^{\mathrm{sep}}}^\times$.

(iii) Let $F \subseteq L$ be a subfield; $F \subseteq M_F$ $(\subseteq F^{\mathrm{sep}})$ a separable algebraic extension such that $p \in (M_F)^{\times p^\infty}$. Then it holds that $\widetilde{F}_{p,\mathrm{div},M_F} \subsetneq M_F$.

Proof. First, we verify assertion (i). Let us observe that

$$L^{\times p^\infty} \subseteq \bigcup_{K \subseteq K^\dagger} (K^\dagger)^{\times p^\infty} \cdot \mu_{p^\infty}(K^{\mathrm{sep}}),$$

where $K \subseteq K^\dagger$ ($\subseteq K^{\mathrm{sep}}$) ranges over the set of finite separable extensions $\subseteq K^{\mathrm{sep}}$ [cf. Lemmas 1.11, 2.5]. Note that, for each finite separable extension $K \subseteq K^\dagger$ ($\subseteq K^{\mathrm{sep}}$), it follows that $A_{K^\dagger}$ is normal, hence that $(K^\dagger)^{\times p^\infty} \subseteq A^\times_{K^\dagger}$. Thus, we conclude that

$$L^{\times p^\infty} \subseteq A^\times_{K^{\mathrm{sep}}} \bigcap L^\times = A^\times_L.$$

This completes the proof of assertion (i).

Next, we verify assertion (ii). By applying Lemmas 1.11, 2.5, we may assume without loss of generality that $K = L$. Write $\widehat{K}$ for the completion of K; $\widehat{K} \subseteq \widehat{K}^{\mathrm{ur}}$ ($\subseteq (\widehat{K})^{\mathrm{sep}}$) for the maximal unramified extension; $A_{\widehat{K}^{\mathrm{ur}}}$ for the [discrete] valuation ring of $\widehat{K}^{\mathrm{ur}}$. Fix an embedding $K^{\mathrm{sep}} \subseteq (\widehat{K})^{\mathrm{sep}}$ over K. Let us note that any finite extension of $\widehat{K}$ is also a complete discrete valuation field. Then it follows immediately from Lemma 2.6, together with the various definitions involved, that $\widetilde{K}_{p,\mathrm{div}} \subseteq (\widehat{K}^{\mathrm{ur}})^{\mathrm{cyc}}$. Thus, since $\widehat{K}^{\mathrm{ur}}$ is a mixed characteristic discrete valuation field of residue characteristic p, and $\widehat{K}^{\mathrm{ur}} \subseteq (\widehat{K}^{\mathrm{ur}})^{\mathrm{cyc}}$ is an abelian extension, we conclude from assertion (i) that

$$(\widetilde{K}_{p,\mathrm{div}})^{\times p^\infty} \subseteq ((\widehat{K}^{\mathrm{ur}})^{\mathrm{cyc}})^{\times p^\infty} \subseteq A^\times_{(\widehat{K}^{\mathrm{ur}})^{\mathrm{cyc}}},$$

where $A^\times_{(\widehat{K}^{\mathrm{ur}})^{\mathrm{cyc}}}$ denotes the group of units of the integral closure of $A_{\widehat{K}^{\mathrm{ur}}}$ in $(\widehat{K}^{\mathrm{ur}})^{\mathrm{cyc}}$. Then, by varying embeddings $K^{\mathrm{sep}} \subseteq (\widehat{K})^{\mathrm{sep}}$, we obtain $(\widetilde{K}_{p,\mathrm{div}})^{\times p^\infty} \subseteq A^\times_{K^{\mathrm{sep}}}$. This completes the proof of assertion (ii).

Assertion (iii) follows immediately from assertion (ii). This completes the proof of Lemma 2.7. □

Theorem 2.8. *Let A_0 be a mixed characteristic Noetherian local domain of residue characteristic p. Write K_0 for the field of fractions of A_0. Let $K_0 \subseteq L_0$ ($\subseteq K_0^{\mathrm{sep}}$) be a Galois extension such that one of the following conditions hold:*

- *$K_0 \subseteq L_0$ ($\subseteq K_0^{\mathrm{sep}}$) is an abelian extension.*
- *L_0 is stably μ_{p^∞}-finite.*

Let K be a subfield of L_0. Then the following hold:

(i) The absolute Galois group G_K is slim.

(ii) Suppose that $\zeta_p \in K$. Then any almost pro-p-maximal quotient of G_K is slim.

Proof. Let us recall that, since A_0 is a Noetherian local domain, A_0 is dominated by a discrete valuation ring [whose field of fractions is K_0]. Thus, assertion (i) (respectively, (ii)) follows immediately from Lemma 2.7, (iii), together with Theorem 2.4, (ii) (respectively, Theorem 2.4, (i)). This completes the proof of Theorem 2.8. $\square$

Remark 2.8.1. It is natural to pose the following question:

> Question: In the notation of Theorem 2.8, can the assumption that $\zeta_p \in K$ be dropped?

However, at the time of writing the present paper, the authors do not know whether this question is affirmative or not.

Now we recall the following well-known fact [cf. [6], Chapter III, §5; [31]]:

Theorem 2.9. *Let k be a perfect field of characteristic p. Write K for the quotient field of the Witt ring associated to k. Then the field of norms*

$$N(K(\mu_{p^\infty}(K^{\mathrm{sep}}))/K)$$

is isomorphic to $k((t))$. Moreover, the absolute Galois group $G_{K(\mu_{p^\infty}(K^{\mathrm{sep}}))}$ is isomorphic to the absolute Galois group $G_{k((t))}$.

Theorem 2.10. *Let K be a Henselian discrete valuation field of characteristic p. Then any almost pro-p-maximal quotient of the absolute Galois group G_K is slim. In particular, G_K is slim [cf. Lemma 1.6].*

Proof. First, by replacing K by $\widehat{K}$, we may assume without loss of generality that K is a complete discrete valuation field [cf. Lemma 3.1 below]. Write k for the residue field of K. Recall from Cohen's structure theorem that K is isomorphic to $k((t))$ [cf. [9], Chapter I, Theorem 5.5A]. Moreover, by replacing k by the perfection of k, if necessary, we may assume without loss of generality that k is perfect. Thus, Theorem 2.10 follows immediately from Theorems 2.8, (ii); 2.9. $\square$

Corollary 2.11. *Let K be a higher local field. Write k for the residue field of K. Then the following hold:*

(i) Suppose that $\mathrm{char}(K) = p$. Then the absolute Galois group G_K is slim. Moreover, any almost pro-p-maximal quotient of G_K is slim.

(ii) Suppose that $(\mathrm{char}(K), \mathrm{char}(k)) = (0, p)$. Then the absolute Galois group G_K is slim. Moreover, if $\zeta_p \in K$, then any almost pro-p-maximal quotient of the absolute Galois group G_K is slim.

(iii) *Suppose that* $\mathrm{char}(K^{(0)}) \neq 0$*, and* $K^{(0)}$ *is a stably* μ_{l^∞}*-finite field for any prime number* l*. Then the absolute Galois group* G_K *is slim. In particular, if* $K^{(0)}$ *is finite, then* G_K *is slim.*

Proof. Assertion (i) follows immediately from Theorem 2.10. Assertion (ii) follows immediately from Theorem 2.8, (i), (ii).

Next, we verify assertion (iii). In light of assertions (i), (ii), we may assume without loss of generality that $(\mathrm{char}(K), \mathrm{char}(k)) = (0, 0)$. We prove the slimness of G_K by induction on the dimension of K. Note that $K \xrightarrow{\sim} k((t))$ [cf. Remark 1.12.1]. Then we have an exact sequence of profinite groups

$$1 \longrightarrow \widehat{\mathbb{Z}}(1) \longrightarrow G_K \longrightarrow G_k \longrightarrow 1.$$

Now it follows from induction hypothesis, together with assertion (ii), that the absolute Galois group G_k is slim. Note that since any finite extension of K is also a higher local field [of residue characteristic 0], to verify that G_K is slim, it suffices to prove that $Z(G_K) = \{1\}$ [cf. Proposition 1.2]. Next, since $Z(G_k) = \{1\}$, we *observe* that $Z(G_K) \subseteq \widehat{\mathbb{Z}}(1)$. On the other hand, since $\mathrm{char}(k) = 0$, it follows from our assumption on $K^{(0)}$ that, for any prime number l, the l-adic cyclotomic character $G_k \to \mathbb{Z}_l^\times$ is open [cf. Lemma 1.13]. Note that the cyclotomic character $G_k \to \widehat{\mathbb{Z}}^\times$ coincides with the natural homomorphism determined by the conjugation action of G_K on $\widehat{\mathbb{Z}}(1)$. Thus, we conclude from the above *observation* that $Z(G_K) = \{1\}$, hence that G_K is slim. This completes the proof of assertion (iii), hence of Corollary 2.11. □

3 Elasticity of (almost pro-p-maximal quotients of) the absolute Galois groups of Henselian discrete valuation fields

In this section, we prove that the absolute Galois groups of Henselian discrete valuation fields with positive characteristic residue fields are elastic.

Let p be a prime number; A a Henselian discrete valuation ring of residue characteristic p. Write K for the quotient field of A; $\mathfrak{m}$ for the maximal ideal of A; $k \stackrel{\mathrm{def}}{=} A/\mathfrak{m}$; $\widehat{K}$ for the completion of K.

First, we begin by recalling the following well-known facts:

Lemma 3.1. *Write* $f : G_{\widehat{K}} \to G_K$ *for the natural outer homomorphism determined by the natural injection* $K \hookrightarrow \widehat{K}$*. Then* f *is bijective.*

Proof. The injectivity of f follows immediately from Krasner's lemma [cf. [26], Lemma 8.1.6]. On the other hand, the surjectivity of f follows immediately from the uniqueness of the extension of the valuation on K to finite extensions of K [cf. [25], Chapter II, Theorem 6.2]. This completes the proof of Lemma 3.1. □

Lemma 3.2 ([25], Chapter II, Theorem 6.2)**.** *Let L be an algebraic extension of K. Write B ($\subseteq L$) for the integral closure of A in L. Then B is a Henselian valuation ring.*

Next, we give a general criterion of the elasticity of profinite groups.

Proposition 3.3. *Let G be a profinite group. Suppose that, for each open subgroup $H \subseteq G$, there exists a normal open subgroup $N \subseteq H$ of G such that*

- *the almost pro-p-maximal quotient $G_N \stackrel{\text{def}}{=} G/\mathrm{Ker}(N \twoheadrightarrow N^p)$ associated to N is slim;*
- *N^p is not topologically finitely generated;*
- *$H^2(N, \mathbb{F}_p) = \{0\}$.*

Then G is very elastic.

Proof. Let $H \subseteq G$ be an open subgroup; $N \subseteq H$ a normal open subgroup of G satisfying the above three conditions. Then we have an exact sequence

$$1 \longrightarrow \mathrm{Ker}(N \twoheadrightarrow N^p) \longrightarrow N \longrightarrow N^p \longrightarrow 1.$$

The Hochschild-Serre spectral sequence associated to the above exact sequence induces an exact sequence

$$\mathrm{Hom}(\mathrm{Ker}(N \twoheadrightarrow N^p), \mathbb{F}_p)^{N^p} \longrightarrow H^2(N^p, \mathbb{F}_p) \longrightarrow H^2(N, \mathbb{F}_p) = \{0\}.$$

Note that $\mathrm{Hom}(\mathrm{Ker}(N \twoheadrightarrow N^p), \mathbb{F}_p) = \{0\}$, hence that $H^2(N^p, \mathbb{F}_p) = \{0\}$. Thus, we conclude that N^p is a free pro-p group that is not topologically finitely generated, hence that N^p is very elastic [cf. [27], Theorem 8.6.6]. Then since G_N is slim, it follows from Lemma 1.4 that G_N is very elastic. Thus, by varying open subgroups $H \subseteq G$, we conclude from Lemma 1.6 that G is very elastic. This completes the proof of Proposition 3.3. $\square$

Theorem 3.4. *Suppose that* $\mathrm{char}(K) = p$. *Then the absolute Galois group G_K, as well as any almost pro-p-maximal quotient of G_K, is very elastic.*

Proof. First, by replacing K by $\widehat{K}$, we may assume without loss of generality that K is a complete discrete valuation field [cf. Lemma 3.1]. Recall from Cohen's structure theorem that K is isomorphic to $k((t))$ [cf. [9], Chapter I, Theorem 5.5A]. Then Theorem 3.4 follows immediately from Theorem 2.10, Proposition 3.3, together with [26], Corollary 6.1.2; [26], Proposition 6.1.7. $\square$

Lemma 3.5. *Let $M \subseteq K^{\mathrm{sep}}$ be a Galois extension of K such that* $\mathrm{Gal}(M/K)$ *is topologically finitely generated. Suppose that* $\mathrm{char}(K) = 0$, *$\zeta_p \in K$, and k is infinite. Then G_K^p and G_M^p are not topologically finitely generated.*

Proof. First, we have a right exact sequence

$$G_M^p \longrightarrow G_K^p \longrightarrow \mathrm{Gal}(M/K)^p \longrightarrow 1.$$

Since $\mathrm{Gal}(M/K)^p$ is topologically finitely generated, it suffices to verify that G_K^p is not topologically finitely generated. On the other hand, since $\zeta_p \in K$,

$$H^1(G_K^p, \mathbb{F}_p) = H^1(G_K, \mathbb{F}_p) = H^1(G_K, \mu_p(K)) \xleftarrow{\sim} K^\times/(K^\times)^p.$$

Thus, it suffices to verify that $K^\times/(K^\times)^p$ is an infinite group. Next, since K is a discrete valuation field, and $\mathrm{char}(k) = p$, we have a natural injection

$$(1+\mathfrak{m})/(1+\mathfrak{m})^p \hookrightarrow K^\times/(K^\times)^p.$$

Moreover, since $p \in \mathfrak{m}$, we have $(1+\mathfrak{m})^p \subseteq 1+\mathfrak{m}^2$. Then we obtain a natural surjection

$$(1+\mathfrak{m})/(1+\mathfrak{m})^p \twoheadrightarrow (1+\mathfrak{m})/(1+\mathfrak{m}^2) \ (\cong k).$$

Thus, since k is an infinite field, we conclude that $K^\times/(K^\times)^p$ is an infinite group. This completes the proof of Lemma 3.5. $\square$

Next, we recall the following well-known properties of the Brauer groups of complete discrete valuation fields.

Proposition 3.6 ([28], Chapter XII, §3, Theorem 2; [28], Chapter XII, §3, Exercise 2)**.** *Suppose that K is complete, and k is perfect. Write $B_K \stackrel{\mathrm{def}}{=} H^2(G_K, (K^{\mathrm{sep}})^\times)$; $B_k \stackrel{\mathrm{def}}{=} H^2(G_k, (k^{\mathrm{sep}})^\times)$. Then the following hold:*

(i) We have a natural exact sequence

$$0 \longrightarrow B_k \longrightarrow B_K \longrightarrow \mathrm{Hom}(G_k, \mathbb{Q}/\mathbb{Z}) \longrightarrow 0.$$

(ii) Let $L \subseteq K^{\mathrm{sep}}$ be a finite separable extension of K. Write $k_L \subseteq k^{\mathrm{sep}}$ for the residue field of L; e_L for the ramification index of the extension L/K; $B_L \stackrel{\mathrm{def}}{=} H^2(G_L, (K^{\mathrm{sep}})^\times)$; $B_{k_L} \stackrel{\mathrm{def}}{=} H^2(G_{k_L}, (k^{\mathrm{sep}})^\times)$. Then we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & B_k \arrow[r] \arrow[d] & B_K \arrow[r] \arrow[d] & \mathrm{Hom}(G_k, \mathbb{Q}/\mathbb{Z}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B_{k_L} \arrow[r] & B_L \arrow[r] & \mathrm{Hom}(G_{k_L}, \mathbb{Q}/\mathbb{Z}) \arrow[r] & 0,
\end{tikzcd}$$

where the horizontal sequences are the exact sequences of assertion (i); the left-hand (respectively, the middle) vertical arrow is the restriction homomorphism induced by the natural inclusion $G_{k_L} \subseteq G_k$ (respectively, $G_L \subseteq G_K$); the right-hand vertical arrow is the homomorphism induced by the natural inclusion $G_{k_L} \subseteq G_k$ and the homomorphism $\mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ determined by the multiplication by e_L.

Proposition 3.7. *Suppose that* $\mathrm{char}(K) = 0$*, and* k *is perfect. Then the following hold:*

(i) Suppose that k *is infinite. Let* $M \subseteq K^{\mathrm{sep}}$ *be an abelian extension of* K *such that* $\mathrm{Gal}(M/K)$ *is topologically finitely generated, and the ramification index is divisible by* p^∞*. Then the absolute Galois group* G_M *is very elastic. Moreover, if* $\zeta_p \in K$*, then any almost pro-*p*-maximal quotient of* G_M *is very elastic.*

(ii) Suppose that k *is a* p*-closed field [i.e., a field that has no Galois extensions of degree* p*], and* $\zeta_p \in K$*. Then the absolute Galois group* G_K*, as well as any almost pro-*p*-maximal quotient of* G_K*, is very elastic.*

Proof. First, by replacing K, M by $\widehat{K}$, the composite field of $\widehat{K}$ and M [in a separably closed field], respectively, we may assume without loss of generality that K is a complete discrete valuation field [cf. Lemma 3.1]. Moreover, it follows immediately from Lemma 1.4; Theorem 2.8, (i), that we may assume without loss of generality that $\zeta_p \in K$.

Next, we verify assertion (i). It suffices to prove that any almost pro-p-maximal quotient of G_M is very elastic [cf. Lemma 1.6]. Let $M^\dagger \subseteq K^{\mathrm{sep}}$ be a finite separable extension of M; $L \subseteq M^\dagger$ a finite separable extension of K. Since k is perfect, the multiplication by p on $H^2(G_{k_L}, (k^{\mathrm{sep}})^\times)$ is an isomorphism. Then it follows immediately from Proposition 3.6, (i), together with Hilbert's theorem 90, that

$$H^2(G_L, \mathbb{F}_p) \xrightarrow{\sim} \mathrm{Hom}(G_{k_L}, \mathbb{F}_p).$$

Thus, since the ramification index of the extension $K \subseteq M^\dagger$ is divisible by p^∞, it follows formally from Proposition 3.6, (ii), that

$$H^2(G_{M^\dagger}, \mathbb{F}_p) \xrightarrow{\sim} \varinjlim_{K \subseteq L \subseteq M^\dagger} H^2(G_L, \mathbb{F}_p) = \{0\},$$

where $K \subseteq L$ ($\subseteq M^\dagger$) ranges over the set of finite separable extensions of K. In particular, this isomorphism implies that $H^2(G^p_{M^\dagger}, \mathbb{F}_p) = \{0\}$. On the other hand, it follows from Lemma 3.5 that $G^p_{M^\dagger}$ is not topologically finitely generated. Then assertion (i) follows immediately from Theorem 2.8, (ii); Proposition 3.3.

Next, we verify assertion (ii). It suffices to prove that any almost pro-p-maximal quotient of G_K is very elastic [cf. Lemma 1.6]. Since k is p-closed, it follows immediately from Proposition 3.6, (i), together with Hilbert's theorem 90, that $H^2(G_K, \mathbb{F}_p) = \{0\}$, hence that $H^2(G^p_K, \mathbb{F}_p) = \{0\}$. Then G^p_K is a free pro-p group that is not topologically finitely generated [cf. Lemma 3.5]. Thus, we conclude that G^p_K is very elastic [cf. [27], Theorem 8.6.6]. Let $(G_K \twoheadrightarrow)\ Q$ be an almost pro-p-maximal quotient; $F \subseteq Q$ a topologically finitely generated normal closed subgroup. Since G^p_K is very elastic, we have $F \subseteq \mathrm{Ker}(Q \twoheadrightarrow G^p_K)$. In particular, it follows from our assumption that $\zeta_p \in K$ that

$$F \subseteq \mathrm{Gal}(K_Q/K_{p^\infty}),$$

where $K_Q \subseteq K^{\mathrm{sep}}$ denotes the subfield fixed by Q; $K_{p^\infty} \stackrel{\mathrm{def}}{=} K(\mu_{p^\infty}(K^{\mathrm{sep}}))$ ($\subseteq K^{\mathrm{sep}}$). Thus, we conclude from assertion (i) that $F = \{1\}$, hence that Q is very elastic. This completes the proof of assertion (ii), hence of Proposition 3.7. $\square$

Theorem 3.8. *Suppose that k is a perfect infinite field. Then the absolute Galois group G_K is very elastic. Moreover, if $\zeta_p \in K$ in the case where* $\mathrm{char}(K) = 0$*, then any almost pro-p-maximal quotient of G_K is very elastic.*

Proof. First, by replacing K by the completion of K, we may assume without loss of generality that K is a complete discrete valuation field [cf. Lemma 3.1]. On the other hand, it follows immediately from Lemma 1.4; Theorems 2.8, (i); 3.4, that we may assume without loss of generality that

$$\mathrm{char}(K) = 0, \quad \zeta_p \in K.$$

Then it suffices to prove that any almost pro-p-maximal quotient of G_K is very elastic [cf. Lemma 1.6]. Moreover, it follows immediately from Lemma 1.4; Theorem 2.8, (ii), that it suffices to prove that G_K^p is very elastic.

Let $F \subseteq G_K^p$ be a topologically finitely generated normal closed subgroup. Let us note that, since $\mathrm{char}(k) = p$, G_k^p is a free pro-p group [cf. [26], Theorem 6.1.4]. In particular, G_k^p is elastic [cf. [27], Theorem 8.6.6]. Write

$$F_k \subseteq G_k^p$$

for the image of F via the natural composite $G_K^p \twoheadrightarrow G_k^p$. If $F_k = \{1\}$, then it follows from Proposition 3.7, (ii), that $F = \{1\}$ [cf. Lemma 3.2]. If $F_k \neq \{1\}$, then since G_k^p is elastic, $F_k \subseteq G_k^p$ is an open subgroup.

Next, since F is topologically finitely generated, and G_K^p is not topologically finitely generated [cf. Lemma 3.5], there exists a normal closed subgroup $Q \subseteq G_K^p$ of infinite index such that F is a normal closed subgroup of Q of infinite index. Write

$$K \subseteq K_Q \ (\subseteq K^{\mathrm{sep}})$$

for the pro-p extension of K associated to Q. Note that since $F_k \subseteq G_k^p$ is an open subgroup, the ramification index of the extension $K \subseteq K_Q$ is divisible by p^∞. Then it follows from a similar argument to the argument applied in the proof of Proposition 3.7, (i), that $H^2(Q, \mathbb{F}_p) = \{0\}$. In particular, Q is a free pro-p group. Thus, since $F \subseteq Q$ is a topologically finitely generated normal closed subgroup of infinite index, we conclude that $F = \{1\}$ [cf. [27], Theorem 8.6.6]. This completes the proof of Theorem 3.8. $\square$

Finally, we strengthen Theorem 3.8 by dropping the assumption that the residue field k is perfect as follows:

Theorem 3.9. *Suppose that k is infinite. Then the absolute Galois group G_K is very elastic. Moreover, if $\zeta_p \in K$ in the case where* $\mathrm{char}(K) = 0$*, then any almost pro-p-maximal quotient of G_K is very elastic.*

Proof. First, by applying Lemma 1.4; Theorems 2.8, (i); 3.4, we may assume without loss of generality that

$$\mathrm{char}(K) = 0, \quad \zeta_p \in K.$$

Then it suffices to prove that any almost pro-p-maximal quotient of G_K is very elastic [cf. Lemma 1.6]. Moreover, it follows immediately from Lemma 1.4, Theorem 2.8, (ii), that it suffices to prove that G_K^p is very elastic.

Let $F \subseteq G_K^p$ be a topologically finitely generated normal closed subgroup. Write $K \subseteq K_F$ ($\subseteq K^{\mathrm{sep}}$) for the pro-$p$ extension of K associated to F; $A_F \subseteq K_F$ for the integral closure of A in K_F; k_F for the residue field of the Henselian valuation ring A_F [cf. Lemma 3.2]; $\overline{k}$ for the residue field of the integral closure of A in K^{sep} [where $\overline{k}$ is an algebraic closure of k]. Then since $\zeta_p \in K \subseteq K_F$, we have

$$K_F^\times/(K_F^\times)^p \xrightarrow{\sim} \mathrm{Hom}(F, \mathbb{F}_p).$$

Note that since F is topologically finitely generated, $\mathrm{Hom}(F, \mathbb{F}_p)$ is finite [so $K_F^\times/(K_F^\times)^p$ is finite].

Next, we verify the following assertion:

> Claim 3.9.A : Let $k \subseteq k_1$ ($\subseteq \overline{k}$) be a purely inseparable extension of degree p. Then there exists a finite extension $K \subseteq K_1$ ($\subseteq K^{\mathrm{sep}}$) of degree p such that the residue field of K_1 is k_1, and $K_1 \subseteq K_F$. [Note that the extension $K \subseteq K_1$ is weakly unramified.]

Let $T_1 \in k_1 \setminus k$ be an element. Write $k^p \stackrel{\mathrm{def}}{=} \{a^p \mid a \in k\} \subseteq k$; $T \stackrel{\mathrm{def}}{=} T_1^p \in k \setminus k^p$. Let $\widetilde{T} \in A^\times \subseteq K$ be a lifting of T; for each $x \in (k^p)^\times$, $\tilde{x} \in A^\times$ a lifting of x. Now we consider the subset

$$S \stackrel{\mathrm{def}}{=} \left\{1 + \tilde{x}\widetilde{T} \in A^\times \mid x \in (k^p)^\times\right\} \subseteq A^\times.$$

Note that since k is infinite, k^p is also infinite. In particular, S is infinite. Then since $K_F^\times/(K_F^\times)^p$ is finite, there exist distinct elements $x_1, x_2 \in (k^p)^\times$, and $b \in A_F^\times$ such that

$$b^p = \frac{1 + \tilde{x}_1\widetilde{T}}{1 + \tilde{x}_2\widetilde{T}} \in A^\times.$$

Write $K_1 \stackrel{\mathrm{def}}{=} K(b) \subseteq K_F$; $y_i \in k^\times$ for the element such that $y_i^p = x_i \in (k^p)^\times$, where $i = 1, 2$ [so $y_1 \neq y_2$]. Then the image of $b \in A_F^\times$ via the natural surjection $A_F^\times \twoheadrightarrow k_F^\times$ is

$$z \stackrel{\mathrm{def}}{=} \frac{1 + y_1 T_1}{1 + y_2 T_1} \in k_1^\times.$$

Thus, since $T_1 \in k_1 \setminus k$, and $y_1 \neq y_2$, it follows that $z \in k_1 \setminus k$, hence that $k_1 = k(z)$. Therefore, we conclude that the extension $K \subseteq K_1$ is of degree p, and the residue field of K_1 is k_1. This completes the proof of Claim 3.9.A.

Let $\{t_i \ (i \in I)\}$ be a p-basis of k; for each $(i, j) \in I \times \mathbb{Z}_{\geq 1}$,

$$K_{i,j-1} \subseteq K_{i,j} \ (\subseteq K_F)$$

a weakly unramified extension of degree p such that the residue field of $K_{i,j}$ is generated by the p^j-th root $\in \overline{k}$ of t_i over k, where $K_{i,0} \stackrel{\mathrm{def}}{=} K$ [cf. Claim 3.9.A]. Write

$$L \ (\subseteq K_F)$$

for the composite field of the fields $\{K_{i,j} \mid (i,j) \in I \times \mathbb{Z}_{\geq 1}\}$. Then we observe that L is a Henselian discrete valuation field with a perfect residue field [cf. Lemma 3.2]. Thus, we conclude from Theorem 3.8 that $F = \{1\}$, hence that G_K^{p} is very elastic. This completes the proof of Theorem 3.9. $\square$

Remark 3.9.1. It is natural to pose the following questions:

> Question 1: Is the absolute Galois group of any discrete valuation field with a positive characteristic residue field elastic?
>
> Question 2: More generally, is the absolute Galois group of any subfield of a discrete valuation field with a positive characteristic residue field elastic?
>
> Question 3: In the notation of Theorem 3.9, can the assumption that $\zeta_p \in K$ [in the case where $\mathrm{char}(K) = 0$] be dropped?

However, at the time of writing the present paper, the authors do not know whether these questions are affirmative or not.

Remark 3.9.2. Let M be a Hilbertian field. Then the slimness and very elasticity of the absolute Galois group G_M is well-known [cf. [15], Theorem 2.1]. On the other hand, we note that any Henselian discrete valuation field is not Hilbertian [cf. [7], Lemma 15.5.4].

Corollary 3.10. *Suppose that K is a higher local field. [Recall that* $\mathrm{char}(k) =$ *p.] Then the absolute Galois group G_K is elastic. Moreover,*

- *G_K is very elastic if and only if k is infinite, or* $\mathrm{char}(K) = p$*;*
- *if k is infinite, and $\zeta_p \in K$ in the case where* $\mathrm{char}(K) = 0$*, then any almost pro-p-maximal quotient of G_K is very elastic;*
- *if k is finite, then any almost pro-p-maximal quotient of G_K is elastic.*

Proof. Corollary 3.10 follows immediately from Theorems 3.4, 3.9, together with [19], Theorem 1.7, (ii). $\square$

Remark 3.10.1. Let M be a field such that $\mathrm{char}(M) = 0$, and the absolute Galois group G_M is not finite. Then the absolute Galois group $G_{M((t))}$ is not elastic. Indeed, we have an exact sequence of profinite groups

$$1 \longrightarrow \widehat{\mathbb{Z}}(1) \longrightarrow G_{M((t))} \longrightarrow G_M \longrightarrow 1.$$

4 Application to absolute anabelian geometry over mixed characteristic Henselian discrete valuation fields

In this section, as a corollary of the results obtained in §3, we prove the semi-absoluteness [cf. Definition 4.5, (i)] of isomorphisms between the étale fundamental groups of smooth varieties [i.e., smooth, of finite type, separated, and geometrically connected schemes] over mixed characteristic Henselian discrete valuation fields, which may be regarded as a generalization of [19], Corollary 2.8 [cf. Corollary 4.6]. This semi-absoluteness, together with its proof, implies that "absolute anabelian geometry" is equivalent to "semi-absolute anabelian geometry" for the smooth varieties over mixed characteristic Henselian discrete valuation fields [cf. Corollary 4.4; Remark 4.4.1; [19], Introduction].

Definition 4.1 (A special case of [19], Definition 2.1, (ii))**.** Let Σ be a nonempty set of prime numbers; K a field of characteristic 0; X a smooth variety over K. Write Δ_X for the maximal pro-Σ quotient of $\Pi_{X\times_K K^{\mathrm{sep}}}$;

$$\Pi_X^{(\Sigma)} \stackrel{\mathrm{def}}{=} \Pi_X/\mathrm{Ker}(\Pi_{X\times_K K^{\mathrm{sep}}} \twoheadrightarrow \Delta_X).$$

Then we have an exact sequence of profinite groups

$$1 \longrightarrow \Delta_X \longrightarrow \Pi_X^{(\Sigma)} \longrightarrow G_K \longrightarrow 1.$$

We shall refer to any extension

$$1 \longrightarrow \Delta \longrightarrow \Pi \longrightarrow G \longrightarrow 1$$

of profinite groups which is isomorphic to the above exact sequence as an *extension of [geometrically pro-Σ] AFG-type* [where "AFG" is to be understood as an abbreviation for "*arithmetic fundamental group*"].

Remark 4.1.1. In the notation of Definition 4.1 in the case where $K = K^{\mathrm{sep}}$, it follows from Hironaka's resolution of singularities [cf. [11]] that there exists a smooth compactification $\overline{X}$ of X such that $\overline{X} \setminus X \subseteq \overline{X}$ is a normal crossing divisor.

Lemma 4.2. *Let*

$$1 \longrightarrow \Delta \longrightarrow \Pi \longrightarrow G \longrightarrow 1$$

be an extension of AFG-type. Then Δ is topologically finitely generated.

Proof. Lemma 4.2 follows immediately from Remark 4.1.1; [19], Proposition 2.2. □

Proposition 4.3. *In the notation of Definition 4.1, suppose that*

- *K is a subfield of an abelian extension of the field of fractions of a mixed characteristic Noetherian local domain;*
- *X is a configuration space [cf. [22], Definition 2.3] associated to a hyperbolic curve over K;*
- *if $\dim X \geq 2$, then Σ consists of all prime numbers or a single element.*

Then the following hold:

(i) Δ_X *is slim.*

(ii) $\Pi_X^{(\Sigma)}$ *is slim, but not elastic.*

Proof. Assertion (i) follows immediately from [22], Proposition 2.2, (ii). The slimness portion of assertion (ii) follows immediately from assertion (i), together with Theorem 2.8, (i); [15], Proposition 1.8, (i). The elasticity portion of assertion (ii) follows immediately from Lemma 4.2, together with the easily verified fact that G_K is an infinite group [cf. Lemma 1.3]. This completes the proof of Proposition 4.3. $\square$

Corollary 4.4. *Let*

$$1 \longrightarrow \Delta \longrightarrow \Pi \longrightarrow G \longrightarrow 1$$

be an extension of AFG-type. Suppose that G is isomorphic to the absolute Galois group of a mixed characteristic Henselian discrete valuation field, and Π is not topologically finitely generated. Then the subgroup $\Delta \subseteq \Pi$ may be characterized as the maximal topologically finitely generated normal closed subgroup of Π.

Proof. Note that since Δ is topologically finitely generated [cf. Lemma 4.2], Π is topologically finitely generated if and only if G is topologically finitely generated. Then it follows immediately from Theorem 3.9, Corollary 3.10, together with our assumption that Π is not topologically finitely generated, that G is very elastic. Thus, we conclude that the subgroup $\Delta \subseteq \Pi$ coincides with the maximal topologically finitely generated normal closed subgroup of Π. This completes the proof of Corollary 4.4. $\square$

Remark 4.4.1. We maintain the notation and the assumption on G of Corollary 4.4. In the case where Π is topologically finitely generated, Mochizuki obtained a group-theoretic characterization of the subgroup $\Delta \subseteq \Pi$ [cf. Lemma 3.1; Theorem 3.9; [19], Theorem 2.6, (v)].

Definition 4.5 (A special case of [19], Definition 2.4). For $i = 1, 2$, let

$$1 \longrightarrow \Delta_i \longrightarrow \Pi_i \longrightarrow G_i \longrightarrow 1$$

be an extension of AFG-type. Let

$$\phi : \Pi_1 \xrightarrow{\sim} \Pi_2$$

be an isomorphism of profinite groups. Then:

(i) We shall say that ϕ is *semi-absolute* if $\phi(\Delta_1) \subseteq \Delta_2$.

(ii) We shall say that ϕ is *strictly semi-absolute* if ϕ is semi-absolute, and $\phi(\Delta_1) \subseteq \Delta_2$ is an open subgroup.

Remark 4.5.1. In the notation of Definition 4.5, if ϕ and ϕ^{-1} are semi-absolute, then $\phi(\Delta_1) = \Delta_2$. In particular, ϕ is strictly semi-absolute.

Remark 4.5.2. In the notation of Definition 4.5, suppose that G_1 is slim. Then it follows immediately from Lemma 1.3 that ϕ is strictly semi-absolute if and only if $\phi(\Delta_1) = \Delta_2$.

Remark 4.5.3. In the notation of Definition 4.5, suppose that G_2 is very elastic. Then it follows immediately from the fact that Δ_1 is topologically finitely generated [cf. Lemma 4.2] that ϕ is semi-absolute.

Corollary 4.6. *In the notation of Definition 4.5, suppose that G_1 (respectively, G_2) is isomorphic to the absolute Galois group of a mixed characteristic Henselian discrete valuation field K_1 (respectively, K_2). Then $\phi(\Delta_1) = \Delta_2$. In particular, ϕ is strictly semi-absolute.*

Proof. If Π_1 is not topologically finitely generated, then Corollary 4.6 follows immediately from Corollary 4.4. Suppose that Π_1 is topologically finitely generated. Then G_1 is also topologically finitely generated. Thus, by applying Lemma 3.1, Theorem 3.9, we may assume without loss of generality that K_i is a p_i-adic local field for some prime number p_i, where $i = 1, 2$. In this case, the equality $\phi(\Delta_1) = \Delta_2$ follows from [19], Corollary 2.8, (ii). This completes the proof of Corollary 4.6. $\square$

Remark 4.6.1. Let p be a prime number. Then, as a consequence of Corollary 4.6, it seems to the authors that similar results to the main results of [13] over *strictly generalized sub-p-adic* fields [cf. Definition 4.7 below] hold without any intrinsic change of the arguments in [13]. The authors hope to be able to address a further generalization including this generalization in the future paper.

Definition 4.7. Let p be a prime number; K a field. Then we shall say that K is a *strictly generalized sub-p-adic* field if K is a generalized sub-p-adic field [cf. [17], Definition 4.11] and contains a p-adic local field as a subfield.

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(Arata Minamide) Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan
Email address: minamide@kurims.kyoto-u.ac.jp

(Shota Tsujimura) Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan
Email address: stsuji@kurims.kyoto-u.ac.jp