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The Geometry of Hyperbolic Curvoids

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Abstract. — The main purposes of the present paper are to introduce the notion of a hyperbolic curvoid and to study the geometry of hyperbolic curvoids. A hyperbolic curvoid is defined to be a certain profinite group and may be considered to be "group-theoretic abstraction" of the notion of a hyperbolic curve from the viewpoint of anabelian geometry. One typical example of a hyperbolic curvoid is a profinite group isomorphic to the étale fundamental group of a hyperbolic curve either over a number field or over a mixed-characteristic nonarchimedean local field. The first part of the present paper centers around establishments of a construction of the "geometric subgroup" of hyperbolic curvoids and a construction of the "collection of cuspidal inertia subgroups" of hyperbolic curvoids. Moreover, we also consider respective analogues for hyperbolic curvoids of the theory of partial compactifications of hyperbolic curves and the theory of quotient orbicurves of hyperbolic curves by actions of finite groups.

Contents

Introduction

Let K be either an NF [i.e., a number field — cf. the discussion entitled "Numbers" in §0] or an MLF [i.e., a mixed-characteristic nonarchimedean local field — cf. the discussion entitled "Numbers" in §0]. Moreover, let $\overline{K}$ be an algebraic closure of K and X a hyperbolic curve over K [cf. the discussion entitled "Curves" in §0]. Write $\pi_1(X)$, $\pi_1(X \times_K \overline{K})$ for the respective étale fundamental groups of X, $X \times_K \overline{K}$ [relative to some choices of basepoints]. Thus, we have an exact sequence of profinite groups

$$1 \longrightarrow \pi_1(X \times_K \overline{K}) \longrightarrow \pi_1(X) \longrightarrow \mathrm{Gal}(\overline{K}/K) \longrightarrow 1.$$

Now let us recall that it follows from [13, Theorem 2.6, (v), (vi)] that, roughly speaking, one may reconstruct "group-theoretically", from the profinite group $\pi_1(X)$, the *geometric subgroup* of $\pi_1(X)$, i.e., the normal closed subgroup $\pi_1(X \times_k \overline{K}) \subseteq \pi_1(X)$ of $\pi_1(X)$, hence also the above exact sequence of profinite groups. Moreover, it follows immediately from [13, Lemma 4.5, (v)] that, roughly speaking, one may reconstruct "group-theoretically", from the profinite group $\pi_1(X)$, the *collection of inertia subgroups* of $\pi_1(X \times_k \overline{K})$ associated to the cusps of the hyperbolic curve X.

The main purposes of the present paper are to introduce the notion of a *hyperbolic curvoid* and to study the geometry of hyperbolic curvoids. A *hyperbolic curvoid* [cf. Definition 2.1] is defined to be a certain profinite group and may be considered to be "group-theoretic abstraction" of the notion of a hyperbolic curve from the viewpoint of anabelian geometry. One typical example of a hyperbolic curvoid is a profinite group isomorphic to the étale fundamental group [relative to some choice of basepoint] of a hyperbolic curve either over an NF or over an MLF [cf. Corollary 6.7, (ii)]. In the remainder of the present Introduction, let $\square$ be an element of the set $\{\mathrm{MLF}, \mathrm{NF}\}$ and

$$\Pi$$

a *hyperbolic $\square$-curvoid* [cf. Definition 2.1].

The first part of the present paper [cf. §2] centers around establishments of

- a "group-theoretic" construction of the "geometric subgroup" of hyperbolic curvoids [cf. Definition 2.4, (i)] and

- a "group-theoretic" construction of the "collection of cuspidal inertia subgroups" of hyperbolic curvoids [cf. Definition 2.8, (i)].

Put another way, in the first part of the present paper, we give a "group-theoretic" construction of a normal closed subgroup $\Delta(\Pi) \subseteq \Pi$ of Π [cf. Definition 2.4, (i)] such that

- the quotient $G(\Pi) \overset{\text{def}}{=} \Pi/\Delta(\Pi)$ [cf. Definition 2.4, (ii)] of Π by $\Delta(\Pi) \subseteq \Pi$ is isomorphic to the absolute Galois group [relative to some choice of algebraic closure] of an MLF (respectively, NF) whenever $\square = \mathrm{MLF}$ (respectively, $\square = \mathrm{NF}$) [cf. Proposition 2.5, (ii)], and, moreover,

- if one applies this construction of "$\Delta(-)$" to the hyperbolic $\square$-curvoid $\pi_1(X)$ [i.e., obtained by forming the étale fundamental group of the above hyperbolic curve X], then the resulting normal closed subgroup, i.e., $\Delta(\pi_1(X))$, coincides with the *geometric subgroup* $\pi_1(X \times_k \overline{K})$ of $\pi_1(X)$ [cf. Remark 3.4.1; Corollary 6.7, (i)].

In particular, one may associate, to the hyperbolic $\square$-curvoid Π, an exact sequence of profinite groups

$$1 \longrightarrow \Delta(\Pi) \longrightarrow \Pi \longrightarrow G(\Pi) \longrightarrow 1.$$

Moreover, in the first part of the present paper, we also give a "group-theoretic" construction of a collection of closed subgroups of $\Delta(\Pi)$ [cf. Definition 2.8, (i)] such that if one applies this construction to the hyperbolic $\square$-curvoid $\pi_1(X)$ [i.e., obtained by forming the étale fundamental group of the above hyperbolic curve X], then the resulting collection of closed subgroups of $\Delta(\pi_1(X)) = \pi_1(X \times_k \overline{K})$ coincides with the collection of inertia subgroups of $\pi_1(X \times_k \overline{K})$ associated to the cusps of the hyperbolic curve X [cf. Remark 3.7.1; Corollary 6.7, (iii)].

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In §4 of the present paper, we introduce and discuss *partial compactifications* of hyperbolic curvoids [cf. Definition 4.5]. One main result, related to partial compactifications, of the theory of hyperbolic curvoids is as follows [cf. Theorem 4.10].

THEOREM A. — *Let □ be an element of the set* {MLF, NF}, Π *a **hyperbolic □-curvoid** [cf. Definition 2.1], and S a subset of the set of* $\Delta(\Pi)$-*conjugacy classes of cuspidal inertia subgroups of* Π *[cf. Definition 2.8, (i)]. Write*

$$\Pi^{\bullet S}$$

for the quotient of Π *by the normal closed subgroup of* Π *normally topologically generated by the cuspidal inertia subgroups of* Π *that belong to elements of S [cf. Definition 4.5] and*

$$\Delta(\Pi)^{\bullet S}$$

for the image of $\Delta(\Pi)$ *in* $\Pi^{\bullet S}$ *[cf. Definition 4.5]. [So we have a commutative diagram of profinite groups*

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \Delta(\Pi) & \longrightarrow & \Pi & \longrightarrow & G(\Pi) & \longrightarrow & 1 \\
& & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \Delta(\Pi)^{\bullet S} & \longrightarrow & \Pi^{\bullet S} & \longrightarrow & G(\Pi) & \longrightarrow & 1
\end{array}
$$

*— where the horizontal sequences are **exact**, and the vertical arrows are **surjective**.] Then the following three conditions are equivalent:*

(1) *The profinite group* $\Pi^{\bullet S}$ *is a **hyperbolic □-curvoid**.*

(2) *The profinite group* $\Pi^{\bullet S}$ *is a **hyperbolic □-curvoid** such that the **equality** $\Delta(\Pi^{\bullet S}) = \Delta(\Pi)^{\bullet S}$ holds.*

(3) *The profinite group* $\Delta(\Pi)^{\bullet S}$ *is **not abelian**.*

In §5 of the present paper, we prove that a suitable outer continuous action of a finite group on a hyperbolic curvoid gives rise to a hyperbolic orbicurvoid [cf. Definition 3.1] that may be thought of as an analogue [i.e., in the theory of hyperbolic curvoids] of the notion of a *quotient orbicurve*. More precisely, for instance, we prove the following result [cf. Theorem 5.4, (i)].

THEOREM B. — *Let □ be an element of the set* {MLF, NF} *and* Π *a **hyperbolic □-curvoid** [cf. Definition 2.1]. Write* Aut(Π) *for the group of continuous automorphisms of* Π *[cf. the discussion entitled "Profinite Groups" in §0];* Out(Π), Out(G(Π)) *[cf. Definition 2.4, (ii)] for the groups of outer continuous automorphisms of* Π, G(Π), *respectively [cf. the discussion entitled "Profinite Groups" in §0]. Let* $J \subseteq$ Out(Π) *be a **finite** subgroup of* Out(Π). *Write*

$$\Pi[J] \stackrel{\text{def}}{=} \text{Aut}(\Pi) \times_{\text{Out}(\Pi)} J$$

for the fiber product of the natural surjective homomorphism Aut(Π) ↠ Out(Π) *and the natural inclusion* $J \hookrightarrow$ Out(Π) *[cf. Definition 5.3, (ii)]. Suppose that* J *is **contained** in the kernel of the natural homomorphism* Out(Π) → Out(G(Π)) *whenever □ = MLF. Then the profinite group* Π[J] *[cf. Remark 5.3.1, (i)] [fits into an exact sequence of profinite groups*

$$1 \longrightarrow \Pi \longrightarrow \Pi[J] \longrightarrow J \longrightarrow 1$$

— cf. Definition 5.3, (ii) *— and] is a **hyperbolic □-orbicurvoid** [cf. Definition* 3.1].

0. Notations and Conventions

Sets. — If S is a finite set, then we shall write $\sharp S$ for the *cardinality* of S.

Numbers. — We shall write

$$\mathfrak{Primes}$$

for the *set of prime numbers*. We shall refer to a finite extension of the field of rational numbers as a *number field*, or an *NF*, for short. We shall refer to a finite extension of the p-adic completion, for some prime number p, of the field of rational numbers as a *mixed-characteristic nonarchimedean local field*, or an *MLF*, for short.

Profinite Groups. — Let G be a profinite group and $N \subseteq G$ a normal closed subgroup of G. Write $Q \overset{\text{def}}{=} G/N$. Then we shall write

$$\mathrm{Aut}_Q(G)$$

for the group of *continuous automorphisms of G over Q* and

$$\mathrm{Aut}(G)$$

for the "$\mathrm{Aut}_Q(G)$" in the case where we take the "N" to be G, i.e., the group of *continuous automorphisms* of G. Now observe that the image of the homomorphism $N \to \mathrm{Aut}_Q(G)$ by conjugation is *normal*. We shall write

$$\underline{\mathrm{Aut}}_Q(G)$$

for the quotient of $\mathrm{Aut}_Q(G)$ by this image of N and

$$\mathrm{Out}(G)$$

for the "$\underline{\mathrm{Aut}}_Q(G)$" in the case where we take the "N" to be G, i.e., the group of *outer continuous automorphisms* of G.

Let G be a profinite group and $H \subseteq G$ a closed subgroup of G. Then we shall write

$$Z_G(H) \overset{\text{def}}{=} \{\, g \in G \mid gh = hg \text{ for every } h \in H \,\}$$

for the *centralizer* of H in G,

$$C_G(H) \overset{\text{def}}{=} \{\, g \in G \mid H \cap gHg^{-1} \text{ is of finite index both in } H \text{ and in } gHg^{-1} \,\}$$

for the *commensurator* of H in G, and

$$G^{\mathrm{ab}}$$

for the *topological abelianization* of G, i.e., the quotient of G by the normal closed subgroup normally topologically generated by the commutators. We shall say that the closed subgroup $H \subseteq G$ of G is *commensurably terminal* if the inclusion $C_G(H) \subseteq H$, or, alternatively, the equality $C_G(H) = H$, holds. We shall say that the closed subgroup $H \subseteq G$ of G is *characteristic* if the equality $\alpha(H) = H$ holds for an arbitrary continuous automorphism $\alpha \in \mathrm{Aut}(G)$ of G.

CURVES. — Let S be a scheme and (g, r) a pair of nonnegative integers. Then we shall say that a scheme X over S is a *smooth curve of type* (g, r) over S if there exist

- a scheme X^+ over S smooth, proper, geometrically connected, and of relative dimension 1 over S and

- a [possibly empty] closed subscheme $D \subseteq X^+$ of X^+ finite and étale over S

such that

- each geometric fiber of X^+ over S is of genus g,

- the finite étale covering D of S is of degree r, and, moreover,

- the scheme X is isomorphic to $X^+ \setminus D$ over S.

We shall define a *hyperbolic curve of type* (g, r) over S to be a smooth curve of type (g, r) over S such that $2 - 2g - r < 0$. Moreover, we shall define a *smooth curve* (respectively, *hyperbolic curve*) over S to be a smooth curve (respectively, hyperbolic curve) of type (g', r') over S for some pair (g', r') of nonnegative integers.

Let k be a field and X a generically scheme-like algebraic stack over k. Then we shall say that the stack X over k is a *hyperbolic orbicurve* over k if there exist a hyperbolic curve Y over a finite extension of k and a finite étale Galois covering $Y \to X$ over k.

1. SOME PROFINITE GROUP THEORY

In the present §1, we discuss certain aspects of abstract profinite groups, as they relate to the theory of hyperbolic curvoids.

DEFINITION 1.1. — Let G be a profinite group.

(i) We shall say that G is *slim* [cf. the discussion entitled "Topological Groups" in [13, §0]] if the equality $Z_G(H) = \{1\}$ holds for every open subgroup H of G.

(ii) We shall say that G is *elastic* [cf. [13, Definition 1.1, (ii)]] if every closed subgroup of G that is

- nontrivial,

- normal in an open subgroup of G, and

- topologically finitely generated as an abstract profinite group

is of finite index in G.

(iii) We shall say that G is *very elastic* [cf. [13, Definition 1.1, (ii)]] if G is elastic and not topologically finitely generated.

LEMMA 1.2. — *Let G be a profinite group. Suppose that there exists a normal closed subgroup $N \subseteq G$ of G such that both N and G/N are **slim**. Then G is **slim**.*

PROOF. — This assertion follows immediately from the various definitions involved. □

DEFINITION 1.3. — Let G be a profinite group.

(i) We shall say that G is *of MLF-type* [cf. [4, Definition 1.1], [4, Proposition 1.2, (i), (ii)]] (respectively, *of NF-type* [cf. [4, Definition 3.2]]) if G is isomorphic, as an abstract profinite group, to the absolute Galois group [relative to some choice of algebraic closure] of an MLF (respectively, NF).

(ii) Suppose that G is either of MLF-type or of NF-type. Then we shall write

$$\Lambda(G)$$

for the *cyclotome* associated to G [cf. [4, Theorem 1.4, (9)], [4, Proposition 3.7, (4)]].

REMARK 1.3.1.

(i) It is well-known that a profinite group *of MLF-type* is *infinite*. Let us recall that a profinite group *of MLF-type* is also *topologically finitely generated* [cf. [15, Theorem 7.4.1]], *slim* [cf. [13, Theorem 1.7, (ii)]], and *elastic* [cf. [13, Theorem 1.7, (ii)]].

(ii) It is well-known that a profinite group *of NF-type* is *infinite*. Let us recall that a profinite group *of NF-type* is also *slim* [cf. [13, Theorem 1.7, (iii)]] and *very elastic* [cf. [13, Theorem 1.7, (iii)]].

PROPOSITION 1.4. — *Let G be a profinite group **of NF-type**. Then the group $\mathrm{Aut}(G)$ has a natural structure of **profinite group of NF-type**, with respect to which the homomorphism $G \to \mathrm{Aut}(G)$ by conjugation is an **open injective continuous homomorphism**. Let us regard $\mathrm{Aut}(G)$ as a **profinite group of NF-type** by this structure.*

PROOF. — This assertion follows immediately from [16, Theorem], together with the *slimness* portion of Remark 1.3.1, (ii) [cf. also [4, Proposition 5.2, (4)]]. □

DEFINITION 1.5. — Let G be a profinite group and $N \subseteq G$ a normal closed subgroup of G.

(i) We shall say that N is *of co-MLF-type* if G/N is of MLF-type.

(ii) We shall say that N is *pseudo-MLF-geometric* if the following two conditions are satisfied:

(1) The normal closed subgroup N is of co-MLF-type and topologically finitely generated.

(2) For each open subgroup $H \subseteq G$ of G, the maximal H-stable torsion-free quotient of the abelian profinite group $(H \cap N)^{\mathrm{ab}}$ on which the resulting action of H is trivial has a natural structure of [necessarily finitely generated — cf. (1)] free $\widehat{\mathbb{Z}}$-module.

LEMMA 1.6. — *Let G be a profinite group. Then the following assertions hold:*

(i)   *Let $H \subseteq G$ be an open subgroup of G and $N \subseteq G$ a normal closed subgroup of G. Suppose that N is **of co-MLF-type** (respectively, **pseudo-MLF-geometric**). Then the normal closed subgroup $H \cap N$ of H is **of co-MLF-type** (respectively, **pseudo-MLF-geometric**).*

(ii)   *Suppose that G has a **pseudo-MLF-geometric** normal closed subgroup. Then G is **topologically finitely generated**.*

(iii)   *Suppose that G has a **pseudo-MLF-geometric** normal closed subgroup. For each open subgroup $H \subseteq G$ of G, write*

$$\zeta(H) \overset{\text{def}}{=} \sup\{\, \dim_{\mathbb{Q}_l}(H^{\text{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q}_l) - \dim_{\mathbb{Q}_{l'}}(H^{\text{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q}_{l'}) \mid l,\, l' \in \mathfrak{Primes} \,\} \quad (< \infty)$$

[cf. (i), (ii)];

$$\Delta(G) \subseteq G$$

for the normal closed subgroup of G obtained by forming the intersection of the normal open subgroups $H \subseteq G$ of G such that the equality $\zeta(H) = \zeta(G) \cdot [G : H]$ holds. [Note that the equality "$\zeta(H) = \zeta(G) \cdot [G : H]$" holds if one takes the "H" to be G.] Then, for a normal closed subgroup of G, the following two conditions are equivalent:

(1)   *The normal closed subgroup **coincides** with $\Delta(G)$.*

(2)   *The normal closed subgroup is **pseudo-MLF-geometric**.*

(iv)   *The set of **pseudo-MLF-geometric** normal closed subgroups of G is **of cardinality ≤ 1**.*

PROOF. — Assertion (i) follows immediately from [4, Remark 1.2.1], together with the various definitions involved. Assertion (ii) follows from Remark 1.3.1, (i), and condition (1) of Definition 1.5, (ii). Next, we verify assertion (iii). Let $N \subseteq G$ be a *pseudo-MLF-geometric* normal closed subgroup of G and $H \subseteq G$ an open subgroup of G. Write $Q_H \overset{\text{def}}{=} H/(H \cap N) \subseteq Q_G \overset{\text{def}}{=} G/N$. Thus, since [it follows from condition (1) of Definition 1.5, (ii), and assertion (i) that] Q_H is *of MLF-type*, it follows immediately from [5, Lemma 1.2, (i)], [5, Lemma 1.7], and [5, Proposition 3.6] that,

(a)   for each prime number l, the equality

$$\dim_{\mathbb{Q}_l}(Q_H^{\text{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q}_l) = \begin{cases} 1 & \text{if } l \neq p(Q_H) \\ d(Q_H) + 1 & \text{if } l = p(Q_H) \end{cases}$$

[cf. [5, Definition 3.5, (i), (ii)]] holds.

Next, let us observe that it follows immediately from [15, Theorem 7.2.6] that, for each prime number l, the Leray spectral sequence of the group extension $1 \to H \cap N \to H \to Q_H \to 1$ yields an exact sequence

$$0 \longrightarrow H^1(Q_H, \mathbb{Q}_l/\mathbb{Z}_l) \longrightarrow H^1(H, \mathbb{Q}_l/\mathbb{Z}_l) \longrightarrow H^1(H \cap N, \mathbb{Q}_l/\mathbb{Z}_l)^{Q_H} \longrightarrow 0.$$

— where we write $H^1(H \cap N, \mathbb{Q}_l/\mathbb{Z}_l)^{Q_H}$ for the submodule of $H^1(H \cap N, \mathbb{Q}_l/\mathbb{Z}_l)$ of Q_H-*invariants*. In particular, for each prime number l, the natural continuous homomorphisms $H \cap N \hookrightarrow H \twoheadrightarrow Q_H$ determine an exact sequence of $\mathbb{Q}_l$-vector spaces

$$0 \longrightarrow \big((H \cap N)^{\text{ab}}\big)_{Q_H} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q}_l \longrightarrow H^{\text{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q}_l \longrightarrow Q_H^{\text{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q}_l \longrightarrow 0$$

— where we write $((H \cap N)^{\mathrm{ab}})_{Q_H}$ for the maximal Q_H-stable quotient of $(H \cap N)^{\mathrm{ab}}$ on which the resulting action of Q_H is *trivial* — which thus implies that

(b) the equality

$$\dim_{\mathbb{Q}_l}(H^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q}_l) = \dim_{\mathbb{Q}_l}\left(\left((H \cap N)^{\mathrm{ab}}\right)_{Q_H} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q}_l\right) + \dim_{\mathbb{Q}_l}(Q_H^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q}_l).$$

holds.

Next, let us observe that it follows from condition (2) of Definition 1.5, (ii), that the dimension "$\dim_{\mathbb{Q}_l}(((H \cap N)^{\mathrm{ab}})_{Q_H} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q}_l)$" does *not depend* on the choice of the prime number "l". Thus, it follows from (a), (b) that

$$\zeta(H) = d(Q_H).$$

In particular, since $d(Q_H) = d(Q_G) \cdot [Q_G : Q_H]$ [cf. [5, Proposition 3.6]], it holds that the equality $\zeta(H) = \zeta(G) \cdot [G : H]$ holds if and only if H *contains* N. Thus, we conclude that $N = \Delta(G)$, as desired. This completes the proof of assertion (iii). Assertion (iv) follows from assertion (iii). This completes the proof of Lemma 1.6. $\qquad\square$

DEFINITION 1.7. — Let G be a profinite group and $N \subseteq G$ a normal closed subgroup of G.

(i) We shall say that N is *of co-NF-type* if G/N is of NF-type.

(ii) We shall say that N is *pseudo-NF-geometric* if N is of co-NF-type and topologically finitely generated.

LEMMA 1.8. — *Let G be a profinite group. Then the following assertions hold:*

(i) *Let $H \subseteq G$ be an open subgroup of G and $N \subseteq G$ a normal closed subgroup of G. Suppose that N is **of co-NF-type** (respectively, **pseudo-NF-geometric**). Then the normal closed subgroup $H \cap N$ of H is **of co-NF-type** (respectively, **pseudo-NF-geometric**).*

(ii) *Suppose that G has a **pseudo-NF-geometric** normal closed subgroup. Then a **pseudo-NF-geometric** normal closed subgroup of G is the **uniquely determined minimal** normal closed subgroup **of co-NF-type** of G.*

(iii) *The set of **pseudo-NF-geometric** normal closed subgroups of G is **of cardinality ≤ 1**.*

PROOF. — Assertion (i) follows immediately from [4, Remark 3.2.1, (i)], together with the various definitions involved. Next, we verify assertion (ii). Let $N_1 \subseteq G$ be a *pseudo-NF-geometric* normal closed subgroup of G and $N_2 \subseteq G$ a normal closed subgroup *of co-NF-type* of G. Then since N_1 is *topologically finitely generated* and *normal* in Π, and Π/N_2 is *very elastic* [cf. Remark 1.3.1, (ii)], it follows immediately that the image of N_1 in Π/N_2 is *trivial*, i.e., that $N_1 \subseteq N_2$, as desired. This completes the proof of assertion (ii). Assertion (iii) follows from assertion (ii). This completes the proof of Lemma 1.8. $\qquad\square$

REMARK 1.8.1. — Lemma 1.8, (ii), may lead us to a consideration of the validity of the following assertion:

(∗): Let G be a profinite group. Suppose that G has a *pseudo-MLF-geometric* normal closed subgroup. Then a *pseudo-MLF-geometric* normal closed subgroup of G is the *uniquely determined minimal* normal closed subgroup *of co-MLF-type* of G.

On the other hand, this assertion (∗) does *hot hold* in general. A counter-example may be obtained as follows: Let Q be a profinite group *of MLF-type* and F a *finitely generated discrete free* group *of rank* ≥ 3. Write $\widehat{F}$ for the profinite completion of F and $G \overset{\text{def}}{=} \widehat{F} \times Q$. [So it is immediate from Remark 1.3.1, (i), that G is naturally identified with the profinite completion of $F \times Q$.] Then since the absolute Galois group [relative to some choice of algebraic closure] of the 2-adic completion $\mathbb{Q}_2$ of the field of rational numbers is *topologically generated by 3 elements* [cf. [15, Theorem 7.4.1]], there exists a surjective continuous homomorphism from G to the absolute Galois group of $\mathbb{Q}_2$ that *factors* through the first projection $G \twoheadrightarrow \widehat{F}$. In particular, since the absolute Galois group of $\mathbb{Q}_2$ is *nontrivial* [cf. Remark 1.3.1, (i)], to verify that the present situation yields a counter-example of the assertion (∗), it suffices to verify that the normal closed subgroup $\widehat{F} \times \{1\} \subseteq \widehat{F} \times Q = G$ of G is *pseudo-MLF-geometric*.

To this end, let us observe that it is immediate that the normal closed subgroup $\widehat{F} \times \{1\} \subseteq G$ satisfies condition (1) of Definition 1.5, (ii). To verify the assertion that the normal closed subgroup $\widehat{F} \times \{1\} \subseteq G$ satisfies condition (2) of Definition 1.5, (ii), let us observe that it is immediate that an arbitrary open subgroup of G may be *naturally identified* with the profinite completion of a subgroup of $F \times Q$ of finite index. Thus, the desired assertion follows immediately from the [well-known] *flatness* of $\widehat{\mathbb{Z}}$ over $\mathbb{Z}$.

DEFINITION 1.9. — Let G be a profinite group and $N \subseteq G$ a normal closed subgroup of G. Then we shall say that a normal closed subgroup $J \subseteq G$ of G is a *co-elastic hull* of N if J contains N as an open subgroup, and, moreover, the quotient G/J is infinite and elastic.

LEMMA 1.10. — *Let G be a profinite group and $N \subseteq G$ a normal closed subgroup of G. Then the set of* **co-elastic hulls** *of N in G is* **of cardinality ≤ 1**.

PROOF. — Let J_1, $J_2 \subseteq G$ be *co-elastic hulls* of N. Now let us observe that one verifies immediately that, to verify $J_1 = J_2$, we may assume without loss of generality, by replacing G by G/N, that $N = \{1\}$, which thus implies that both J_1 and J_2 are *finite*. Thus, since G/J_1 is *infinite* and *elastic*, the image of J_2 in G/J_1 is *trivial*, i.e., $J_2 \subseteq J_1$. Moreover, it follows from a similar argument to this argument that $J_1 \subseteq J_2$. In particular, the equality $J_1 = J_2$ holds, as desired. This completes the proof of Lemma 1.10. $\square$

2. Hyperbolic Curvoids

In the present §2, we introduce and discuss the notion of a *hyperbolic curvoid* [cf. Definition 2.1 below].

DEFINITION 2.1. — Let Π be a profinite group. Then we shall say that Π is a *hyperbolic MLF-curvoid* (respectively, *hyperbolic NF-curvoid*) if there exist

(a) a normal closed subgroup $N \subseteq \Pi$ of Π,

(b) a semi-graph $\mathcal{G}$ of anabelioids of pro-$\mathfrak{Primes}$ PSC-type [cf. [12, Definition 1.1, (i)]] — whose PSC-fundamental group [cf. [12, Definition 1.1, (ii)]] we denote by $\Pi_{\mathcal{G}}$ — and

(c) an outer continuous isomorphism $N \overset{\sim}{\to} \Pi_{\mathcal{G}}$

that satisfy the following four conditions:

(1) The normal closed subgroup $N \subseteq \Pi$ of Π is of co-MLF-type (respectively, of co-NF-type).

(2) There exists a normal open subgroup $H \subseteq \Pi$ of Π such that H contains N, and, moreover, N is pseudo-MLF-geometric (respectively, pseudo-NF-geometric) as a normal closed subgroup of H.

(3) The composite

$$\Pi/N \longrightarrow \mathrm{Out}(N) \longrightarrow \mathrm{Out}(\Pi_{\mathcal{G}})$$

— where the first arrow is the outer continuous action by conjugation, and the second arrow is the isomorphism obtained by conjugation by the outer continuous isomorphism $N \overset{\sim}{\to} \Pi_{\mathcal{G}}$ of (c) — factors through the closed subgroup $\mathrm{Aut}(\mathcal{G}) \subseteq \mathrm{Out}(\Pi_{\mathcal{G}})$ of $\mathrm{Out}(\Pi_{\mathcal{G}})$ discussed at the beginning of [12, §2].

(4) For each prime number l, there exists an open subgroup $U_l \subseteq \Pi/N$ of Π/N such that

  • the restriction to $U_l \subseteq \Pi/N$ of the continuous character $\Pi/N \to \mathbb{Z}_l^{\times}$ obtained by forming the composite of the resulting homomorphism $\Pi/N \to \mathrm{Aut}(\mathcal{G})$ [cf. (3)] and the pro-l cyclotomic character $\mathrm{Aut}(\mathcal{G}) \to \mathbb{Z}_l^{\times}$ [cf. [12, Lemma 2.1]]

coincides with

  • the restriction to $U_l \subseteq \Pi/N$ of the continuous character $\Pi/N \to \mathbb{Z}_l^{\times}$ determined by the maximal pro-l quotient of the cyclotome $\Lambda(\Pi/N)$ associated to Π/N [cf. (1); Definition 1.3, (ii)].

REMARK 2.1.1. — We will give some examples of hyperbolic curvoids that arise from *scheme theory* in Theorem 6.5, (i), below and Theorem 6.6, (i), below.

In the remainder of the present §2, let $\square$ be an element of the set $\{\mathrm{MLF}, \mathrm{NF}\}$ and

$$\Pi$$

a *hyperbolic $\square$-curvoid*.

PROPOSITION 2.2. — *Every open subgroup of a **hyperbolic MLF-curvoid** (respectively, **hyperbolic NF-curvoid**) is a **hyperbolic MLF-curvoid** (respectively, **hyperbolic NF-curvoid**).*

PROOF. — Suppose that we are in the situation of Definition 2.1, and that $\square = \mathrm{MLF}$ (respectively, $\square = \mathrm{NF}$). Let $U \subseteq \Pi$ be an open subgroup of Π. Then, to verify the open subgroup U is a hyperbolic MLF-curvoid (respectively, hyperbolic NF-curvoid), let us observe that it follows from Lemma 1.6, (i) (respectively, Lemma 1.8, (i)), that the normal closed subgroup $U \cap N \subseteq U$ of U satisfies conditions (1), (2) of Definition 2.1.

Fix a continuous isomorphism $N \xrightarrow{\sim} \Pi_{\mathcal{G}}$ that lifts the outer continuous isomorphism of (c) of Definition 2.1. Write

  • $\mathcal{H} \to \mathcal{G}$ for the connected finite étale covering of $\mathcal{G}$ that corresponds to the open subgroup of $\Pi_{\mathcal{G}}$ obtained by forming the image of $U \cap N \subseteq N$ by the fixed continuous isomorphism $N \xrightarrow{\sim} \Pi_{\mathcal{G}}$ and

  • $\Pi_{\mathcal{H}}$ ($\subseteq \Pi_{\mathcal{G}}$) for the PSC-fundamental group of $\mathcal{H}$.

Then it follows immediately from [4, Theorem 1.4, (iv)] (respectively, [4, Theorem 3.8, (i)]) and the final portion of [12, Lemma 2.1], together with the various definitions involved, that the collection of data consisting of

  • the normal closed subgroup $U \cap N \subseteq U$ of U,

  • the semi-graph $\mathcal{H}$ of anabelioids of pro-$\mathfrak{Primes}$ PSC-type, and

  • the outer continuous isomorphism $U \cap N \xrightarrow{\sim} \Pi_{\mathcal{H}}$ determined by the fixed continuous isomorphism $N \xrightarrow{\sim} \Pi_{\mathcal{G}}$

satisfies conditions (3), (4) of Definition 2.1. This completes the proof of Proposition 2.2.
□

LEMMA 2.3. — *Suppose that Π is a **hyperbolic MLF-curvoid** (respectively, **hyperbolic NF-curvoid**). Then the following assertions hold:*

(i) *For a normal closed subgroup $N_0 \subseteq \Pi$ of Π, the following two conditions are equivalent:*

  (1) *The normal closed subgroup N_0 **satisfies** conditions (1), (2) of Definition 2.1 [i.e., imposed on "N"].*

  (2) *There exist a normal open subgroup $J \subseteq \Pi$ of Π and a normal closed subgroup $\Delta \subseteq \Pi$ of Π such that*

    • *the **inclusions** $\Delta \subseteq N_0 \subseteq J$ hold,*

    • *Δ is **pseudo-MLF-geometric** (respectively, **pseudo-NF-geometric**) as a normal closed subgroup of J, and, moreover,*

    • *N_0 is a **co-elastic hull** of Δ in Π.*

(ii) *The set of normal closed subgroups of Π that **satisfy** conditions (1), (2) of Definition 2.1 is **of cardinality 1**.*

PROOF. — First, we verify the implication (1) ⇒ (2) of assertion (i). Suppose that condition (1) is satisfied. Then one verifies easily from Remark 1.3.1, (i) (respectively, Remark 1.3.1, (ii)), and condition (1) of Definition 2.1 that N_0 is a *co-elastic hull* of N_0. Thus, since N_0 satisfies condition (2) of Definition 2.1, we conclude that the normal closed subgroup $N_0 \subseteq \Pi$ of Π satisfies condition (2), as desired. This completes the proof of the implication (1) ⇒ (2) of assertion (i).

Next, we verify the implication (2) ⇒ (1) of assertion (i) and assertion (ii). Let $J \subseteq \Pi$ be a normal open subgroup of Π and $\Delta \subseteq \Pi$ a normal closed subgroup of Π such that

  • J contains Δ,

  • Δ is *pseudo-MLF-geometric* (respectively, *pseudo-NF-geometric*) as a normal closed subgroup of J, and, moreover,

- Δ has a *co-elastic hull* $C \subseteq \Pi$ of Δ in Π.

Suppose that we are in the situation of Definition 2.1. Then let us observe that one verifies immediately [cf. the implication (1) $\Rightarrow$ (2) of assertion (i) already verified] that, to verify the implication (2) $\Rightarrow$ (1) of assertion (i) and assertion (ii), it suffices to verify that $C = N$.

Next, to verify $C = N$, let us observe that it is immediate that

(a) C is a *co-elastic hull* of $H \cap \Delta$ in Π.

Next, let us observe that it follows from Lemma 1.6, (i) (respectively, Lemma 1.8, (i)), together with condition (2) of Definition 2.1, that both $H \cap \Delta$ and $J \cap N$ are *pseudo-MLF-geometric* (respectively, *pseudo-NF-geometric*) normal closed subgroups of $J \cap H$. Thus, it follows from Lemma 1.6, (iv) (respectively, Lemma 1.8, (iii)), that $H \cap \Delta = J \cap N$. In particular, it follows from condition (1) of Definition 2.1, together with Remark 1.3.1, (i) (respectively, Remark 1.3.1, (ii)), that

(b) N is a *co-elastic hull* of $H \cap \Delta$ in Π.

Thus, it follows from (a), (b), together with Lemma 1.10, that $C = N$, as desired. This completes the proofs of the implication (2) $\Rightarrow$ (1) of assertion (i) and assertion (ii). $\square$

DEFINITION 2.4.

(i) We shall write

$$\Delta(\Pi) \subseteq \Pi$$

for the uniquely determined [cf. Lemma 2.3, (ii)] normal closed subgroup of Π that satisfies conditions (1), (2) of Definition 2.1 [i.e., the uniquely determined normal closed subgroup of Π that satisfies condition (2) of Lemma 2.3, (i), imposed on "N_0" — cf. Lemma 2.3, (i), (ii)] and refer to $\Delta(\Pi)$ as the *geometric subgroup* of Π.

(ii) We shall write

$$G(\Pi) \overset{\text{def}}{=} \Pi/\Delta(\Pi)$$

and refer to $G(\Pi)$ as the *arithmetic quotient* of Π.

Thus, we have an exact sequence of profinite groups

$$1 \longrightarrow \Delta(\Pi) \longrightarrow \Pi \longrightarrow G(\Pi) \longrightarrow 1.$$

PROPOSITION 2.5. — *The following assertions hold:*

(i) *The geometric subgroup* $\Delta(\Pi)$ *of* Π *is* **topologically finitely generated**, **slim**, *and* **elastic**.

(ii) *If the profinite group* Π *is a* **hyperbolic MLF-curvoid** (*respectively,* **hyperbolic NF-curvoid**)*, then the arithmetic quotient* $G(\Pi)$ *of* Π *is* **of MLF-type** (*respectively,* **of NF-type**)*.*

(iii) *Let* $H \subseteq \Pi$ *be an open subgroup of* Π*. [So* H *is a* **hyperbolic $\square$-curvoid** — *cf. Proposition 2.2]. Then the* **geometric subgroup** *of* H*, i.e.,* $\Delta(H) \subseteq H$*, is given by* $H \cap \Delta(\Pi)$*. In particular, the natural inclusion* $H \hookrightarrow \Pi$ *fits into a commutative diagram*

of profinite groups

$$1 \longrightarrow \Delta(H) \longrightarrow H \longrightarrow G(H) \longrightarrow 1$$

$$1 \longrightarrow \Delta(\Pi) \longrightarrow \Pi \longrightarrow G(\Pi) \longrightarrow 1$$

*— where the horizontal sequences are **exact**, and the vertical arrows are **open injective**.*

PROOF. — First, we verify assertion (i). Let us first observe that one verifies immediately from the existence of the outer continuous isomorphism of (c) of Definition 2.1 and [12, Remark 1.1.3] that $\Delta(\Pi)$ is *isomorphic*, as an abstract profinite group, to the étale fundamental group of some hyperbolic curve over an algebraically closed field of characteristic zero. Thus, assertion (i) follows from [13, Proposition 2.2] and [13, Proposition 2.3, (i)]. This completes the proof of assertion (i). Assertion (ii) follows from condition (1) of Definition 2.1. Assertion (iii) follows immediately from the proof of Proposition 2.2. This completes the proof of Proposition 2.5. □

DEFINITION 2.6. — Let l be a prime number, G a profinite group, M a $\mathbb{Q}_l$-vector space of finite dimension equipped with a continuous action of G, and $\chi \colon G \to \mathbb{Z}_l^\times$ a continuous character.

(i) We shall say that M is *quasi-trivial* if the action of G on M factors through a finite quotient of G.

(ii) We shall write

$$\tau(M)$$

for the sum of the $\mathbb{Q}_l$-dimensions of the quasi-trivial subquotients "M_i/M_{i+1}" by a composition series $\{0\} = M_n \subseteq \ldots \subseteq M_1 \subseteq M_0 = M$ of the $\mathbb{Q}_l$-vector space M equipped with a continuous action of G. Note that one verifies easily that this sum does not depend on the choice of the composition series "$\{0\} = M_n \subseteq \ldots \subseteq M_1 \subseteq M_0 = M$" of M.

(iii) We shall write

$$d_\chi(M) \overset{\text{def}}{=} \tau\big(M(\chi^{-1})\big) - \tau\big(\mathrm{Hom}_{\mathbb{Q}_l}(M, \mathbb{Q}_l)\big)$$

— where $M(\chi^{-1})$ denotes the result of "twisting" M by the character $\chi^{-1} \colon G \to \mathbb{Z}_l^\times$.

LEMMA 2.7. — *Suppose that the profinite group $\Delta(\Pi)$ is **free**. Let l be a prime number. For each open subgroup $U \subseteq G(\Pi)$ of $G(\Pi)$, write $\chi_U^{l\text{-cyc}} \colon U \to \mathbb{Z}_l^\times$ for the continuous character obtained by forming the restriction to $U \subseteq G(\Pi)$ of the continuous character $G(\Pi) \to \mathbb{Z}_l^\times$ determined by the maximal pro-l quotient of the cyclotome $\Lambda(G(\Pi))$ associated to $G(\Pi)$ [cf. Proposition 2.5, (ii)]. Moreover, for each open subgroup $H \subseteq \Delta(\Pi)$ of $\Delta(\Pi)$, write $H^{(l)}$ for the maximal pro-l quotient of H. Then the following assertions hold:*

(i) *Let $H \subseteq \Delta(\Pi)$ be a characteristic open subgroup of $\Delta(\Pi)$. Write*

$$\widetilde{\mathcal{C}}_l(H)$$

*for the set of **maximal** closed subgroups $I \subseteq H^{(l)}$ of $H^{(l)}$ that satisfy the following two conditions:*

- The profinite group I is **isomorphic**, as an abstract profinite group, to $\mathbb{Z}_l$. Write $I^l \subseteq I$ for the uniquely determined open subgroup of I of index l.

- Let $J \subseteq H^{(l)}$ be a characteristic open subgroup of $H^{(l)}$; $\widetilde{I \cdot J}$, $\widetilde{I^l \cdot J} \subseteq \Pi$ open subgroups of Π such that the geometric subgroups $\Delta(\widetilde{I \cdot J})$, $\Delta(\widetilde{I^l \cdot J})$ [cf. Proposition 2.2] are given by the inverse images of $I \cdot J$, $I^l \cdot J \subseteq H^{(l)}$ by the natural surjective continuous homomorphism $H \twoheadrightarrow H^{(l)}$, respectively [cf. Proposition 2.5, (iii)]. Then the **inequality**

$$d_{\chi^{l\text{-cyc}}_{G(\widetilde{I^l \cdot J})}} \left((I^l \cdot J)^{\mathrm{ab}} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \right) + 1 < l \cdot \left(d_{\chi^{l\text{-cyc}}_{G(\widetilde{I \cdot J})}} \left((I \cdot J)^{\mathrm{ab}} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \right) + 1 \right)$$

holds [cf. Proposition 2.5, (i), (iii)].

Then an arbitrary $\Delta(\Pi)$-conjugate of an element of $\widetilde{\mathcal{C}}_l(H)$ is an element of $\widetilde{\mathcal{C}}_l(H)$.

(ii)   In the situation of (i), the quotient

$$\mathcal{C}_l(H)$$

of $\widetilde{\mathcal{C}}_l(H)$ by the action of H, i.e., by conjugation [cf. (i)], is **finite**.

(iii)   Let $H_1 \subseteq H_2 \subseteq \Delta(\Pi)$ be characteristic open subgroups of $\Delta(\Pi)$. Then the assignment "$I \mapsto C_{H_2}(\mathrm{Im}(I))$", where we write $\mathrm{Im}(I) \subseteq H_2^{(l)}$ for the image of $I \subseteq H_1^{(l)}$ in $H_2^{(l)}$, determines a **$\Delta(\Pi)$-equivariant** [cf. (i)] map

$$\widetilde{\mathcal{C}}_l(H_1) \longrightarrow \widetilde{\mathcal{C}}_l(H_2),$$

which thus determines a **$\Delta(\Pi)$-equivariant** map

$$\mathcal{C}_l(H_1) \longrightarrow \mathcal{C}_l(H_2)$$

[cf. (ii)].

(iv)   Write

$$\mathcal{I}_l$$

for the set of subgroups of $\Delta(\Pi)$ obtained by forming the stabilizers of elements of the profinite set [cf. (ii)]

$$\varprojlim_H \mathcal{C}_l(H)$$

— where the projective limit is taken over the characteristic open subgroups $H \subseteq \Delta(\Pi)$ of $\Delta(\Pi)$ [cf. (iii)] — i.e., with respect to the action of $\Delta(\Pi)$ on the profinite set [cf. (iii)]. Then, for a closed subgroup of $\Delta(\Pi)$, the following two conditions are equivalent:

(1)   The closed subgroup is an element of $\mathcal{I}_l$.

(2)   In the situation of Definition 2.1, the image of the closed subgroup of $\Delta(\Pi)$ by some [or, alternatively, an arbitrary] continuous isomorphism $\Delta(\Pi) = N \xrightarrow{\sim} \Pi_{\mathcal{G}}$ that lifts the outer continuous isomorphism of (c) is a **cuspidal** subgroup of $\Pi_{\mathcal{G}}$ [cf. [12, Definition 1.1, (ii)]].

In particular, the set $\mathcal{I}_l$ does **not depend** on the choice of the prime number l.

PROOF. — These assertions follow immediately — in light of conditions (3), (4) of Definition 2.1, [4, Theorem 1.4, (iv)], and [4, Theorem 3.8, (i)] — from a similar argument to the argument applied in the proof of [13, Lemma 4.5, (iv)] [cf. also [14, Remark 1.2.2, (ii)]]. $\square$

DEFINITION 2.8.

(i) If the profinite group $\Delta(\Pi)$ is not free, then we shall define the set of *cuspidal inertia subgroups* of Π to be the empty set. If the profinite group $\Delta(\Pi)$ is free, then we shall say that a closed subgroup of $\Delta(\Pi)$ is a *cuspidal inertia subgroup* of Π if the closed subgroup satisfies condition (1) of Lemma 2.7, (iv), for some [or, alternatively — cf. the final portion of Lemma 2.7, (iv) — an arbitrary] prime number l.

(ii) We shall say that a closed subgroup of Π is a *cuspidal decomposition subgroup* of Π if the closed subgroup is obtained by forming the commensurator in Π of a cuspidal inertia subgroup of Π.

(iii) We shall write

$$\mathrm{Cusp}(\Pi)$$

for the set of $\Delta(\Pi)$-conjugacy classes [cf. condition (1) of Lemma 2.7, (iv)] of cuspidal inertia subgroups of Π.

REMARK 2.8.1. — It follows from the existence of the outer continuous isomorphism of (c) of Definition 2.1 and [12, Remark 1.1.3] that the *geometric subgroup* of a hyperbolic $\square$-curvoid is *isomorphic*, as an abstract profinite group, to the étale fundamental group of some hyperbolic curve over an algebraically closed field of characteristic zero [or, alternatively, to the profinite completion of the topological fundamental group of some hyperbolic Riemann surface of finite type]. Moreover, it follows immediately from Lemma 2.7, (iv), that one may take such a continuous isomorphism so as to induce a *bijective* map between the set of *cuspidal inertia subgroups* and the set of *inertia subgroups* associated to cusps of the hyperbolic curve.

PROPOSITION 2.9. — *The following assertions hold:*

(i) *Every cuspidal inertia (respectively, decomposition) subgroup of Π is **commensurably terminal** in $\Delta(\Pi)$ (respectively, Π). In particular, the intersection of the geometric subgroup and a cuspidal decomposition subgroup is a cuspidal inertia subgroup.*

(ii) *The set $\mathrm{Cusp}(\Pi)$ is **finite**. In particular, the image in $G(\Pi)$ of every cuspidal decomposition subgroup of Π is **open**.*

(iii) *Let $H \subseteq \Pi$ be an open subgroup of Π. [So H is a **hyperbolic $\square$-curvoid** — cf. Proposition 2.2.] Then the assignments "$I \mapsto H \cap I$", "$J \mapsto C_{\Delta(\Pi)}(J)$" determine a **bijective** map between the set of cuspidal inertia subgroups of Π and the set of cuspidal inertia subgroups of H. In particular, the second assignment determines a **surjective** map*

$$\mathrm{Cusp}(H) \longrightarrow\!\!\!\!\!\rightarrow \mathrm{Cusp}(\Pi).$$

PROOF. — First, we verify assertion (i). The commensurable terminality of a cuspidal inertia subgroup in $\Delta(\Pi)$, hence also the final portion of assertion (i), follows from Lemma 2.7, (iv), and [12, Proposition 1.2, (ii)]. The commensurable terminality of a cuspidal decomposition subgroup in Π follows immediately from the final portion of assertion (i) already verified, together with Lemma 2.7, (iv), and [12, Proposition 1.2, (i)]. This completes the proof of assertion (i).

Next, we verify assertion (ii). The first portion of assertion (ii) follows, in light of Remark 2.8.1, from the well-known structure of the étale fundamental groups of hyperbolic

curves over algebraically closed fields of characteristic zero. Thus, since [one verifies easily that] the action of Π on $\mathrm{Cusp}(\Pi)$, i.e., by conjugation, is *continuous*, the final portion of assertion (ii) follows. This completes the proof of assertion (ii). Assertion (iii) follows, in light of Remark 2.8.1, from the well-known structure of the étale fundamental groups of hyperbolic curves over algebraically closed fields of characteristic zero. This completes the proof of Proposition 2.9. $\qquad\square$

DEFINITION 2.10.

(i)   Let $H \subseteq \Pi$ be an open subgroup of Π. [So H is a hyperbolic $\square$-curvoid — cf. Proposition 2.2.] Then we shall define a [necessarily connected] semi-graph [cf. the discussion at the beginning of [11, §1]]

$$\mathbb{G}(H)$$

as follows: The set of vertices of $\mathbb{G}(H)$ is defined to be the set [necessarily of cardinality 1] consisting of the profinite group H. The set of closed edges of $\mathbb{G}(H)$ is defined to be the empty set. The set of open edges of $\mathbb{G}(H)$ is defined to be the set $\mathrm{Cusp}(H)$. Every open edge of $\mathbb{G}(H)$ abuts to the unique vertex H.

(ii)   Let $H_1 \subseteq H_2 \subseteq \Pi$ be open subgroups of Π. [So H_1 and H_2 are hyperbolic $\square$-curvoids — cf. Proposition 2.2.] Then the map $\mathrm{Cusp}(H_1) \twoheadrightarrow \mathrm{Cusp}(H_2)$ obtained by applying the final portion of Proposition 2.9, (iii), naturally determines a morphism of semi-graphs [cf. the discussion at the beginning of [11, §1]]

$$\mathbb{G}(H_1) \longrightarrow \mathbb{G}(H_2).$$

We shall write

$$\widetilde{\mathbb{G}}(\Pi) \stackrel{\mathrm{def}}{=} \big(\mathbb{G}(H)\big)_{H \subseteq \Pi}$$

for the projective system of semi-graphs consisting of the various $\mathbb{G}(H)$'s — where H ranges over the open subgroups of Π.

(iii)   One verifies easily that the profinite group Π acts on the projective system $\widetilde{\mathbb{G}}(\Pi)$ by conjugation. Moreover, one also verifies immediately from Remark 2.8.1, together with the various definitions involved, that the projective system $\widetilde{\mathbb{G}}(\Pi)$ of semi-graphs and the [restriction to $\Delta(\Pi) \subseteq \Pi$ of the] resulting action of Π on $\widetilde{\mathbb{G}}(\Pi)$ naturally determine

(1)   a semi-graph of anabelioids of pro-$\mathfrak{Primes}$ PSC-type that has no node

$$\mathcal{G}(\Pi)$$

and

(2)   an outer continuous isomorphism

$$\Delta(\Pi) \stackrel{\sim}{\longrightarrow} \Pi_{\mathcal{G}(\Pi)}$$

— where we write $\Pi_{\mathcal{G}(\Pi)}$ for the PSC-fundamental group of $\mathcal{G}(\Pi)$ —

such that

(a)   the collection of data consisting of

•   the normal closed subgroup $\Delta(\Pi) \subseteq \Pi$ of Π,

•   the semi-graph $\mathcal{G}(\Pi)$ of anabelioids of pro-$\mathfrak{Primes}$ PSC-type of (1), and

- the outer continuous isomorphism $\Delta(\Pi) \xrightarrow{\sim} \Pi_{\mathcal{G}(\Pi)}$ of (2)

satisfies the four conditions (1), (2), (3), (4) of Definition 2.1 [i.e., imposed on the collection of data consisting of (a), (b), (c) of Definition 2.1],

(b) the restriction to $\Delta(\Pi) \subseteq \Pi$ of the action of Π on $\widetilde{\mathbb{G}}(\Pi)$ determines an identification between

- the set of cuspidal inertia subgroups of Π and

- the set of stabilizers [i.e., with respect to the action of $\Delta(\Pi)$ on $\widetilde{\mathbb{G}}(\Pi)$] of compatible systems of open edges in $\widetilde{\mathbb{G}}(\Pi)$, and, moreover,

(c) the outer continuous isomorphism of (2) determines a $\Pi_{\mathcal{G}(\Pi)}$-orbit of bijective maps between

- the set of cuspidal inertia subgroups of Π and

- the set of cuspidal subgroups of $\Pi_{\mathcal{G}(\Pi)}$,

hence also a bijective map $\mathrm{Cusp}(\Pi) \xrightarrow{\sim} \mathrm{Cusp}(\mathcal{G}(\Pi))$ [cf. [6, Definition 1.1, (i)]], by means of which let us identify $\mathrm{Cusp}(\Pi)$ with $\mathrm{Cusp}(\mathcal{G}(\Pi))$:

$$\mathrm{Cusp}(\Pi) = \mathrm{Cusp}\big(\mathcal{G}(\Pi)\big).$$

(iv) We shall write

$$\Lambda(\Pi) \overset{\mathrm{def}}{=} \Lambda_{\mathcal{G}(\Pi)}$$

[cf. [7, Definition 3.8, (i)]] and refer to $\Lambda(\Pi)$ as the *geometric cyclotome* associated to Π.

(v) Let $I \subseteq \Delta(\Pi)$ be a cuspidal inertia subgroup of Π. Then it follows from [7, Corollary 3.9, (v)] [cf. also (c) of (iii)] that we have a natural isomorphism "$\mathfrak{syn}_b$" of I with $\Lambda(\Pi)$ functorial with respect to isomorphisms of the pair "(Π, I)". We shall write

$$\mathfrak{syn}_I \colon I \xrightarrow{\;\sim\;} \Lambda(\Pi)$$

for this isomorphism.

REMARK 2.10.1.

(i) It follows from [7, Definition 3.8, (i)] that the geometric cyclotome associated to a hyperbolic $\square$-curvoid is *isomorphic*, as an abstract $\widehat{\mathbb{Z}}$-module, to $\widehat{\mathbb{Z}}$.

(ii) Let l be a prime number. Then it follows — in light of condition (a) of Definition 2.10, (iii) — from condition (4) of Definition 2.1 and [7, Remark 3.8.1] that there exists an open subgroup $U_l \subseteq G(\Pi)$ of $G(\Pi)$ such that

- the restriction to $U_l \subseteq G(\Pi)$ of the continuous character $G(\Pi) \to \mathbb{Z}_l^\times$ determined by the maximal pro-l quotient of the *geometric cyclotome* $\Lambda(\Pi)$ associated to Π [cf. (i)]

coincides with

- the restriction to $U_l \subseteq G(\Pi)$ of the continuous character $G(\Pi) \to \mathbb{Z}_l^\times$ determined by the maximal pro-l quotient of the cyclotome $\Lambda(G(\Pi))$ associated to $G(\Pi)$.

LEMMA 2.11. — *Let $H \subseteq \Pi$ be an open subgroup of Π. [So H is a **hyperbolic $\square$-curvoid** — cf. Proposition 2.2.] Then the homomorphism*

$$\Lambda(\Pi) \longrightarrow \Lambda(H)$$

*induced by the natural inclusion $H \hookrightarrow \Pi$ [cf. Proposition 2.5, (iii); Proposition 2.9, (iii)] is an **injective** homomorphism whose image is given by*

$$[\Delta(\Pi) : \Delta(H)] \cdot \Lambda(H) \subseteq \Lambda(H).$$

PROOF. — This assertion follows — in light of Remark 2.10.1, (i) — from [7, Theorem 3.7, (v)], together with the various definitions involved. $\square$

DEFINITION 2.12. — Suppose that Π is a hyperbolic NF-curvoid. Let D be an element of $\widetilde{\mathcal{V}}(G(\Pi))$ [cf. Proposition 2.5, (ii); [4, Proposition 3.5, (1)]].

(i) We shall write

$$\Pi|_D \overset{\text{def}}{=} \Pi \times_{G(\Pi)} D$$

for the fiber product of the natural surjective homomorphism $\Pi \twoheadrightarrow G(\Pi)$ and the natural inclusion $D \hookrightarrow G(\Pi)$ and refer to $\Pi|_D$ as the *localization* of Π at D. Thus, we have a commutative diagram of profinite groups

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \Delta(\Pi) & \longrightarrow & \Pi|_D & \longrightarrow & D & \longrightarrow & 1 \\
& & \| & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \Delta(\Pi) & \longrightarrow & \Pi & \longrightarrow & G(\Pi) & \longrightarrow & 1
\end{array}
$$

— where the horizontal sequences are exact, and the vertical arrows are injective.

(ii) We shall say that $D \in \widetilde{\mathcal{V}}(G(\Pi))$ is *curvoidal* if the localization $\Pi|_D$ at D is a hyperbolic MLF-curvoid whose geometric subgroup is given by $\Delta(\Pi) \subseteq \Pi|_D$ [cf. the diagram of (i)].

PROPOSITION 2.13. — *Suppose that Π is a **hyperbolic NF-curvoid**. Let D be a **curvoidal** element of $\widetilde{\mathcal{V}}(G(\Pi))$. Then the natural inclusion $\Pi|_D \hookrightarrow \Pi$ determines an **isomorphism** of semi-graphs of anabelioids [cf. [11, Definition 2.1]]*

$$\mathcal{G}(\Pi|_D) \overset{\sim}{\longrightarrow} \mathcal{G}(\Pi).$$

In particular, the natural inclusion $\Pi|_D \hookrightarrow \Pi$ determines

- *a **D-equivariant isomorphism***

$$\Lambda(\Pi|_D) \overset{\sim}{\longrightarrow} \Lambda(\Pi)$$

and

- *a **bijective** map between the set of cuspidal inertia subgroups of $\Pi|_D$ and the set of cuspidal inertia subgroups of Π, which thus gives*

- *a **bijective** map*

$$\mathrm{Cusp}(\Pi|_D) \overset{\sim}{\longrightarrow} \mathrm{Cusp}(\Pi).$$

PROOF. — Let us observe that it follows immediately from condition (a) of Definition 2.10, (iii), and [4, Theorem 3.8, (ii)] that conditions (3), (4) of Definition 2.1 in the case where one takes the collection "$(\Pi, N, N \overset{\sim}{\to} \Pi_{\mathcal{G}})$" of data of Definition 2.1 to be the collection of data consisting of

- the hyperbolic MLF-curvoid $\Pi|_D$,

- the geometric subgroup $\Delta(\Pi|_D)$ $(= \Delta(\Pi))$ of $\Pi|_D$, and

- the outer continuous isomorphism $\Delta(\Pi|_D) \overset{\sim}{\to} \Pi_{\mathcal{G}(\Pi)}$ obtained by forming the composite of the outer continuous isomorphism $\Delta(\Pi|_D) \overset{\sim}{\to} \Delta(\Pi)$ determined by the natural identification $\Delta(\Pi|_D) = \Delta(\Pi)$ and the outer continuous isomorphism $\Delta(\Pi) \overset{\sim}{\to} \Pi_{\mathcal{G}(\Pi)}$ of (2) of Definition 2.10, (iii),

are satisfied. Thus, Proposition 2.13 follows immediately from Lemma 2.7, (iv), and [12, Proposition 1.5, (ii)]. This completes the proof of Proposition 2.13. □

3. Hyperbolic Orbicurvoids

In the present §3, we introduce and discuss the notion of a *hyperbolic orbicurvoid* [cf. Definition 3.1 below].

DEFINITION 3.1. — Let Π be a profinite group. Then we shall say that Π is a *hyperbolic MLF-orbicurvoid* (respectively, *hyperbolic NF-orbicurvoid*) if there exist a normal closed subgroup $N \subseteq \Pi$ of Π and a normal open subgroup $H \subseteq \Pi$ of Π that satisfy the following two conditions:

(1) The normal closed subgroup N is slim and of co-MLF-type (respectively, of co-NF-type).

(2) The normal open subgroup H is a hyperbolic MLF-curvoid (respectively, hyperbolic NF-curvoid) whose geometric subgroup is given by $H \cap N$.

REMARK 3.1.1. — We will give some examples of hyperbolic orbicurvoids that arise from *scheme theory* in Corollary 6.7, (i), below.

In the remainder of the present §3, let $\square$ be an element of the set $\{\mathrm{MLF}, \mathrm{NF}\}$ and

$$\Pi$$

a *hyperbolic $\square$-orbicurvoid*.

PROPOSITION 3.2. — *The following assertions hold:*

(i) *A **hyperbolic MLF-curvoid** (respectively, **hyperbolic NF-curvoid**) is a **hyperbolic MLF-orbicurvoid** (respectively, **hyperbolic NF-orbicurvoid**).*

(ii) *Suppose that Π is a **hyperbolic MLF-orbicurvoid** (respectively, **hyperbolic NF-orbicurvoid**). Then every open subgroup of Π is a **hyperbolic MLF-orbicurvoid** (respectively, **hyperbolic NF-orbicurvoid**).*

(iii) *The profinite group* Π *is* **slim**.

(iv) *The following three conditions are equivalent:*

    (1) *The profinite group* Π *is a* **hyperbolic MLF-orbicurvoid**, *i.e.,* $\square = \mathrm{MLF}$.

    (2) *The profinite group* Π *is not a* **hyperbolic NF-orbicurvoid**, *i.e.,* $\square \neq \mathrm{NF}$.

    (3) *The profinite group* Π *is* **topologically finitely generated**.

PROOF. — Assertion (i) follows from Proposition 2.5, (i), (ii). Assertion (ii) follows from Lemma 1.6, (i) (respectively, Lemma 1.8, (i)), Proposition 2.2, and Proposition 2.5, (iii). Assertion (iii) follows, in light of condition (1) of Definition 3.1, from Lemma 1.2 and Remark 1.3.1, (i) (respectively, Remark 1.3.1, (ii)).

Finally, we verify assertion (iv). The implication (2) $\Rightarrow$ (1) is immediate. The implication (1) $\Rightarrow$ (3) follows, in light of condition (2) of Definition 2.1 and condition (2) of Definition 3.1, from Lemma 1.6, (ii). The implication (3) $\Rightarrow$ (2) follows, in light of condition (1) of Definition 3.1, from Remark 1.3.1, (ii). This completes the proof of assertion (iv), hence also of Proposition 3.2. $\qquad\square$

LEMMA 3.3. — *Suppose that* Π *is a hyperbolic MLF-orbicurvoid* (*respectively, hyperbolic NF-orbicurvoid*). *Then the following assertions hold:*

(i) *For a normal closed subgroup* $N \subseteq \Pi$ *of* Π, *the following two conditions are equivalent:*

    (1) *The normal closed subgroup* N *is* **slim** *and* **of co-MLF-type** (*respectively,* **of co-NF-type**), *and, moreover, there exists a normal open subgroup* $H \subseteq \Pi$ *of* Π *such that* H *is a* **hyperbolic MLF-curvoid** (*respectively,* **hyperbolic NF-curvoid**) *whose geometric subgroup is given by* $H \cap N$.

    (2) *There exists a normal open subgroup* $J \subseteq \Pi$ *of* Π *such that* J *is a* **hyperbolic MLF-curvoid** (*respectively,* **hyperbolic NF-curvoid**), *and, moreover,* N *is a* **co-elastic hull** *of* $\Delta(J)$ *in* Π. [*Note that one verifies easily that the closed subgroup* $\Delta(J) \subseteq \Pi$ *of* Π *is* **normal**.]

(ii) *The set of normal closed subgroups of* Π *that satisfy* (1) *of* (i) *is* **of cardinality 1**.

PROOF. — First, we verify the implication (1) $\Rightarrow$ (2) of assertion (i). Suppose that condition (1) is satisfied. Then it follows from Remark 1.3.1, (i) (respectively, Remark 1.3.1, (ii)), that N is a *co-elastic hull* of $H \cap N$ in Π, as desired. This completes the proof of the implication (1) $\Rightarrow$ (2) of assertion (i).

Next, we verify the implication (2) $\Rightarrow$ (1) of assertion (i) and assertion (ii). Let $J \subseteq \Pi$ be a normal open subgroup of Π such that J is a *hyperbolic MLF-curvoid* (respectively, *hyperbolic NF-curvoid*), and, moreover, the geometric subgroup $\Delta(J)$ of J has a *co-elastic hull* $C \subseteq \Pi$ in Π. Moreover, let $N \subseteq \Pi$ be a normal closed subgroup of Π that satisfies condition (1) of assertion (i) [cf. conditions (1), (2) of Definition 3.1], which thus implies that there exists a normal open subgroup $H \subseteq \Pi$ of Π such that H is a *hyperbolic MLF-curvoid* (respectively, *hyperbolic NF-curvoid*) whose geometric subgroup is given by $H \cap N$. Then let us observe that one verifies immediately [cf. the implication (1) $\Rightarrow$ (2) of assertion (i) already verified] that, to verify the implication (2) $\Rightarrow$ (1) of assertion (i) and assertion (ii), it suffices to verify that $C = N$.

Next, to verify $C = N$, let us observe that it is immediate that

(a)   the normal closed subgroup C is a *co-elastic hull* of $H \cap \Delta(J)$ in Π.

Next, let us observe that it follows from Proposition 2.2 and Proposition 2.5, (iii), that $H \cap \Delta(J) = \Delta(J \cap H) = J \cap H \cap N$. In particular, it follows from Remark 1.3.1, (i) (respectively, Remark 1.3.1, (ii)), that

(b)   the normal closed subgroup N is a *co-elastic hull* of $H \cap \Delta(J)$ in Π.

Thus, it follows from (a), (b), together with Lemma 1.10, that $C = N$, as desired. This completes the proofs of the implication $(2) \Rightarrow (1)$ of assertion (i) and assertion (ii). $\square$

DEFINITION 3.4.

(i)   We shall write

$$\Delta(\Pi) \subseteq \Pi$$

for the uniquely determined [cf. Lemma 3.3, (ii)] normal closed subgroup of Π that satisfies condition (1) of Lemma 3.3, (i) [i.e., the uniquely determined normal closed subgroup of Π that satisfies condition (2) of Lemma 3.3, (i) — cf. Lemma 3.3, (i), (ii)], and refer to $\Delta(\Pi)$ as the *geometric subgroup* of Π.

(ii)   We shall write

$$G(\Pi) \stackrel{\text{def}}{=} \Pi/\Delta(\Pi)$$

and refer to $G(\Pi)$ as the *arithmetic quotient* of Π. Thus, we have an exact sequence of profinite groups

$$1 \longrightarrow \Delta(\Pi) \longrightarrow \Pi \longrightarrow G(\Pi) \longrightarrow 1.$$

(iii)   Let Π_1 and Π_2 be hyperbolic MLF-orbicurvoids (respectively, hyperbolic NF-orbicurvoids). Then we shall say that an open continuous homomorphism $\Pi_1 \to \Pi_2$ is an *arithmetic equivalence* if the open continuous homomorphism maps $\Delta(\Pi_1)$ to $\Delta(\Pi_2)$, and, moreover, the [necessarily open continuous] induced homomorphism $G(\Pi_1) \to G(\Pi_2)$ is a continuous isomorphism.

REMARK 3.4.1. — One verifies easily from Proposition 2.5, (i), (ii), that if Π is a hyperbolic $\square$-curvoid, hence also a hyperbolic $\square$-orbicurvoid [cf. Proposition 3.2, (i)], then the notions of the *geometric subgroup*, *arithmetic quotient* of Π in the sense of Definition 2.4, (i), (ii), *coincide* with the notions of the *geometric subgroup*, *arithmetic quotient* of Π in the sense of Definition 3.4, (i), (ii), respectively.

PROPOSITION 3.5. — *The following assertions hold:*

(i)   *The geometric subgroup $\Delta(\Pi)$ of Π is* ***topologically finitely generated***, ***slim***, *and* ***elastic***.

(ii)   *If the profinite group Π is a* ***hyperbolic MLF-orbicurvoid*** (*respectively,* ***hyperbolic NF-orbicurvoid***)*, then the arithmetic quotient $G(\Pi)$ of Π is* ***of MLF-type*** (*respectively,* ***of NF-type***)*.

(iii)   *Let $H \subseteq \Pi$ be an open subgroup of Π. [So H is a* ***hyperbolic $\square$-orbicurvoid*** *— cf. Proposition 3.2, (ii).] Then the* ***geometric subgroup*** *of H, i.e., $\Delta(H) \subseteq H$, is*

given by $H \cap \Delta(\Pi)$. In particular, the natural inclusion $H \hookrightarrow \Pi$ fits into a commutative diagram of profinite groups

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \Delta(H) & \longrightarrow & H & \longrightarrow & G(H) & \longrightarrow & 1 \\
& & \cap\downarrow & & \cap\downarrow & & \cap\downarrow & & \\
1 & \longrightarrow & \Delta(\Pi) & \longrightarrow & \Pi & \longrightarrow & G(\Pi) & \longrightarrow & 1
\end{array}
$$

*— where the horizontal sequences are **exact**, and the vertical arrows are **open injective**.*

PROOF. — First, we verify assertion (i). It follows from Proposition 2.5, (i), that $\Delta(\Pi)$ is *topologically finitely generated.* Moreover, it follows from condition (1) of Lemma 3.3, (i), that $\Delta(\Pi)$ is *slim.* Thus, it follows from Proposition 2.5, (i), and [13, Proposition 1.3, (i)] that $\Delta(\Pi)$ is *elastic.* This completes the proof of assertion (i). Assertion (ii) follows from condition (1) of Lemma 3.3, (i). Assertion (iii) follows immediately — in light of Lemma 1.6, (i), and Lemma 1.8, (i) — from Proposition 2.5, (iii). This completes the proof of Proposition 3.5. □

LEMMA 3.6. — *Let $I \subseteq \Delta(\Pi)$ be a closed subgroup of $\Delta(\Pi)$. Then the following two conditions are equivalent:*

(1) *For every open subgroup $H \subseteq \Pi$ of Π that is a **hyperbolic □-curvoid**, the intersection $\Delta(H) \cap I$ is a **cuspidal inertia subgroup** of H, and, moreover, the **equality** $I = C_{\Delta(\Pi)}(\Delta(H) \cap I)$ holds.*

(2) *There exist an open subgroup $H \subseteq \Pi$ of Π that is a **hyperbolic □-curvoid** and a **cuspidal inertia subgroup** $J \subseteq \Delta(H)$ of H such that the **equality** $I = C_{\Delta(\Pi)}(J)$ holds.*

PROOF. — The implication $(1) \Rightarrow (2)$ is immediate. The implication $(2) \Rightarrow (1)$ follows immediately — in light of Proposition 2.2 and Proposition 2.5, (iii) — from Proposition 2.9, (iii). □

DEFINITION 3.7.

(i) We shall say that a closed subgroup of $\Delta(\Pi)$ is a *cuspidal inertia subgroup* of Π if the closed subgroup satisfies condition (1) of Lemma 3.6.

(ii) We shall say that a closed subgroup of Π is a *cuspidal decomposition subgroup* of Π if the closed subgroup is obtained by forming the commensurator in Π of a cuspidal inertia subgroup of Π.

(iii) We shall write

$$\mathrm{Cusp}(\Pi)$$

for the set of $\Delta(\Pi)$-conjugacy classes [cf. condition (1) of Lemma 2.7, (iv)] of cuspidal inertia subgroups of Π.

REMARK 3.7.1. — One verifies easily from Proposition 2.9, (iii), and Remark 3.4.1 that if Π is a hyperbolic □-curvoid, hence also a hyperbolic □-orbicurvoid [cf. Proposition 3.2, (i)], then the notions of a *cuspidal inertia subgroup, cuspidal decomposition subgroup* of Π in the sense of Definition 2.8, (i), (ii), *coincide* with the notions of a *cuspidal inertia*

subgroup, *cuspidal decomposition subgroup* of Π in the sense of Definition 3.7, (i), (ii), respectively. In particular, it follows from Remark 3.4.1 that the set $\mathrm{Cusp}(\Pi)$ in the sense of Definition 2.8, (iii), may be naturally *identified* with the set $\mathrm{Cusp}(\Pi)$ in the sense of Definition 3.7, (iii).

PROPOSITION 3.8. — *The following assertions hold:*

(i) *Every cuspidal inertia (respectively, decomposition) subgroup of Π is **commensurably terminal** in $\Delta(\Pi)$ (respectively, Π). In particular, the intersection of the geometric subgroup and a cuspidal decomposition subgroup is a cuspidal inertia subgroup.*

(ii) *The set $\mathrm{Cusp}(\Pi)$ is **finite**. In particular, the image in $G(\Pi)$ of every cuspidal decomposition subgroup of Π is **open**.*

(iii) *Let $H \subseteq \Pi$ be an open subgroup of Π. [So H is a **hyperbolic $\square$-orbicurvoid** — cf. Proposition 3.2, (ii).] Then the assignments "$I \mapsto H \cap I$", "$J \mapsto C_{\Delta(\Pi)}(J)$" determine a **bijective** map between the set of cuspidal inertia subgroups of Π and the set of cuspidal inertia subgroups of H. In particular, the second assignment determines a **surjective** map*

$$\mathrm{Cusp}(H) \longrightarrow\!\!\!\!\rightarrow \mathrm{Cusp}(\Pi).$$

PROOF. — First, we verify assertion (i). The commensurable terminality of a cuspidal inertia subgroup in $\Delta(\Pi)$, hence also the final portion of assertion (i), follows from the definition of the notion of a cuspidal inertia subgroup. The commensurable terminality of a cuspidal decomposition subgroup in Π follows immediately from the final portion of assertion (i) already verified, together with Lemma 2.7, (iv), and [12, Proposition 1.2, (i)]. This completes the proof of assertion (i). Assertion (ii) is a formal consequence of Proposition 2.9, (ii). Assertion (iii) follows immediately from assertion (i). This completes the proof of Proposition 3.8. □

DEFINITION 3.9. — We shall write

$$\Lambda(\Pi) \stackrel{\mathrm{def}}{=} [\Delta(\Pi) : \Delta(H)] \cdot \Lambda(H)$$

[cf. Definition 2.10, (iv)] for some open subgroup $H \subseteq \Pi$ of Π such that H is a hyperbolic $\square$-curvoid [cf. condition (2) of Definition 3.1] and refer to $\Lambda(\Pi)$ as the *geometric cyclotome* associated to Π. Note that it follows from Lemma 2.11 that $\Lambda(\Pi)$ does not depend on the choice of the open subgroup "H".

REMARK 3.9.1. — One verifies easily from the various definitions involved that if Π is a hyperbolic $\square$-curvoid, hence also a hyperbolic $\square$-orbicurvoid [cf. Proposition 3.2, (i)], then the *geometric cyclotome* associated to Π in the sense of Definition 2.10, (iv), may be naturally *identified* with the *geometric cyclotome* associated to Π in the sense of Definition 3.9.

REMARK 3.9.2. — It follows from Remark 2.10.1, (i), that the geometric cyclotome associated to a hyperbolic $\square$-orbicurvoid is *isomorphic*, as an abstract $\widehat{\mathbb{Z}}$-module, to $\widehat{\mathbb{Z}}$.

DEFINITION 3.10. — Suppose that Π is a hyperbolic NF-orbicurvoid. Let D be an element of $\widetilde{\mathcal{V}}(G(\Pi))$ [cf. Proposition 3.5, (ii); [4, Proposition 3.5, (1)]].

(i) We shall write

$$\Pi|_D \overset{\text{def}}{=} \Pi \times_{G(\Pi)} D$$

for the fiber product of the natural surjective continuous homomorphism $\Pi \twoheadrightarrow G(\Pi)$ and the natural inclusion $D \hookrightarrow G(\Pi)$ and refer to $\Pi|_D$ as the *localization* of Π at D. Thus, we have a commutative diagram of profinite groups

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \Delta(\Pi) & \longrightarrow & \Pi|_D & \longrightarrow & D & \longrightarrow & 1 \\
 & & \| & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \Delta(\Pi) & \longrightarrow & \Pi & \longrightarrow & G(\Pi) & \longrightarrow & 1
\end{array}
$$

— where the horizontal sequences are exact, and the vertical arrows are injective.

(ii) We shall say that $D \in \widetilde{\mathcal{V}}(G(\Pi))$ is *orbicurvoidal* if the localization $\Pi|_D$ at D is a hyperbolic MLF-orbicurvoid whose geometric subgroup is given by $\Delta(\Pi) \subseteq \Pi|_D$ [cf. the diagram of (i)].

PROPOSITION 3.11. — *Suppose that Π is a **hyperbolic NF-orbicurvoid**. Let D be an **orbicurvoidal** element of $\widetilde{\mathcal{V}}(G(\Pi))$. Then the natural inclusion $\Pi|_D \hookrightarrow \Pi$ determines*

- *a **D-equivariant isomorphism***

$$\Lambda(\Pi|_D) \overset{\sim}{\longrightarrow} \Lambda(\Pi)$$

and

- *a **bijective** map between the set of cuspidal inertia subgroups of $\Pi|_D$ and the set of cuspidal inertia subgroups of Π, which thus gives*

- *a **bijective** map*

$$\text{Cusp}(\Pi|_D) \overset{\sim}{\longrightarrow} \text{Cusp}(\Pi).$$

PROOF. — This assertion follows immediately, in light of Proposition 2.2, from Proposition 2.13 and Proposition 3.8, (iii). □

DEFINITION 3.12. — We shall say that the hyperbolic $\square$-orbicurvoid Π is *relatively corelike* if, for an arbitrary open subgroup $H \subseteq \Pi$ of Π and an arbitrary open injective continuous homomorphism $\phi \colon H \hookrightarrow \Pi$ over $G(\Pi)$ [i.e., such that

- the composite of the open injective continuous homomorphism $\phi \colon H \hookrightarrow \Pi$ and the natural surjective continuous homomorphism $\Pi \twoheadrightarrow G(\Pi)$

coincides with

- the composite of the natural inclusion $H \hookrightarrow \Pi$ and the natural surjective continuous homomorphism $\Pi \twoheadrightarrow G(\Pi)$],

the following condition is satisfied:

- The restriction $\phi|_{\Delta(H)} \colon \Delta(H) \hookrightarrow \Delta(\Pi)$ [cf. Proposition 3.2, (ii)] of ϕ to $\Delta(H) \subseteq H$ [cf. Proposition 3.5, (iii)]

coincides with

- some $\Delta(\Pi)$-conjugate of the restriction $\Delta(H) \hookrightarrow \Delta(\Pi)$ of the natural inclusion $H \hookrightarrow \Pi$ to $\Delta(H) \subseteq H$ [cf. Proposition 3.5, (iii)].

PROPOSITION 3.13. — *Suppose that one of the following two conditions is satisfied:*

(1) *There exists an open subgroup $H \subseteq \Pi$ of Π such that the hyperbolic $\square$-orbicurvoid H [cf. Proposition 3.2, (ii)] is **relatively core-like**, and, moreover, the natural inclusion $H \hookrightarrow \Pi$ restricts to a continuous **isomorphism** $\Delta(H) \xrightarrow{\sim} \Delta(\Pi)$.*

(2) *The profinite group Π is a **hyperbolic NF-orbicurvoid**, and, moreover, there exists an **orbicurvoidal** element D of $\widetilde{\mathcal{V}}(G(\Pi))$ such that the hyperbolic MLF-orbicurvoid $\Pi|_D$ is **relatively core-like**.*

*Then the hyperbolic $\square$-orbicurvoid Π is **relatively core-like**.*

PROOF. — This assertion follows immediately from Proposition 3.5, (iii), together with the various definitions involved. $\qquad\square$

DEFINITION 3.14. — Suppose that Π is a hyperbolic NF-orbicurvoid. Then, by applying the functorial "group-theoretic" algorithm established in [4] [cf. [4, Theorem A]] to the profinite group $G(\Pi)$ of NF-type [cf. Proposition 3.5, (ii)], we obtain an algebraically closed field

$$\overline{F}(\Pi) \overset{\text{def}}{=} \widetilde{F}\big(G(\Pi)\big)$$

[cf. [4, Theorem A]] equipped with a continuous action of $G(\Pi)$ such that

- the subfield

$$F(\Pi) \overset{\text{def}}{=} \overline{F}(\Pi)^{G(\Pi)}$$

of $\overline{F}(\Pi)$ consisting of $G(\Pi)$-invariants is an NF, and, moreover,

- the continuous action of $G(\Pi)$ on $\overline{F}(\Pi)$ determines a continuous isomorphism

$$G(\Pi) \xrightarrow{\ \sim\ } \mathrm{Gal}\big(\overline{F}(\Pi)/F(\Pi)\big).$$

4. PARTIAL COMPACTIFICATIONS

In the present §4, we introduce and discuss *partial compactifications* of hyperbolic curvoids [cf. Definition 4.5 below, Theorem 4.10 below, and Theorem 4.11 below]. In the present §4, let $\square$ be an element of the set $\{\mathrm{MLF}, \mathrm{NF}\}$ and

$$\Pi$$

a *hyperbolic $\square$-curvoid*.

DEFINITION 4.1.

(i) We shall write

$$\Delta(\Pi)^{\mathrm{ab/cusp}}$$

for the quotient of the abelian profinite group $\Delta(\Pi)^{\text{ab}}$ by the [necessarily normal] closed subgroup topologically generated by the images of the cuspidal inertia subgroups of Π.

(ii) It follows from Remark 2.8.1, together with the well-known structure of the étale fundamental groups of hyperbolic curves over algebraically closed fields of characteristic zero, that the abelian profinite group $\Delta(\Pi)^{\text{ab/cusp}}$ of (i) has a natural structure of free $\widehat{\mathbb{Z}}$-module of even rank. We shall write

$$g(\Pi) \overset{\text{def}}{=} \text{rank}_{\widehat{\mathbb{Z}}}\big(\Delta(\Pi)^{\text{ab/cusp}}\big)/2.$$

Thus, $g(\Pi)$ is a nonnegative integer.

(iii) Let us recall from Proposition 2.9, (ii), that the set $\text{Cusp}(\Pi)$ is finite. We shall write

$$r(\Pi) \overset{\text{def}}{=} \sharp\text{Cusp}(\Pi).$$

Thus, $r(\Pi)$ is a nonnegative integer.

PROPOSITION 4.2. — *The following assertions hold:*

(i) *The **inequality** $2 - 2g(\Pi) - r(\Pi) < 0$ holds.*

(ii) *Suppose that Π is a **hyperbolic NF-curvoid**. Let D be a **curvoidal** element of $\widetilde{\mathcal{V}}(G(\Pi))$. Then the **equality** $(g(\Pi), r(\Pi)) = (g(\Pi|_D), r(\Pi|_D))$ holds.*

PROOF. — Assertion (i) follows from Remark 2.8.1, together with the well-known structure of the étale fundamental groups of hyperbolic curves over algebraically closed fields of characteristic zero. Assertion (ii) follows from Proposition 2.13. □

LEMMA 4.3. — *Let $J_1 \subseteq J_2 \subseteq \Delta(\Pi)^{\text{ab}}$ be closed subgroups of $\Delta(\Pi)^{\text{ab}}$. Suppose that the following four conditions are satisfied:*

(1) *The closed subgroups $J_1 \subseteq J_2$ are **contained** in the kernel of the natural surjective continuous homomorphism $\Delta(\Pi)^{\text{ab}} \twoheadrightarrow \Delta(\Pi)^{\text{ab/cusp}}$.*

(2) *The quotient J_2/J_1 is **torsion-free**.*

(3) *The continuous action of Π by conjugation on $\Delta(\Pi)^{\text{ab}}$ **preserves** the closed subgroups $J_1 \subseteq J_2 \subseteq \Delta(\Pi)^{\text{ab}}$.*

(4) *The resulting [cf. (3)] continuous action of Π on J_2/J_1 is **trivial**.*

*Then the **equality** $J_1 = J_2$ holds.*

PROOF. — This assertion follows immediately — in light of the existence of the isomorphism "$\mathfrak{syn}_I$" of Definition 2.10, (v), and Remark 2.10.1, (ii) — from [4, Theorem 1.4, (iv)] and [4, Proposition 3.7, (iii)]. □

DEFINITION 4.4.

(i) We shall write

$$\underline{\text{Cusp}}(\Pi)$$

for the set of Π-conjugacy classes of cuspidal inertia subgroups of Π. Thus, we have a natural surjective map

$$\mathrm{Cusp}(\Pi) \twoheadrightarrow \underline{\mathrm{Cusp}}(\Pi).$$

(ii) We shall say that a subset S of $\mathrm{Cusp}(\Pi)$ is Π-*stable* if S is preserved by the action of Π on $\mathrm{Cusp}(\Pi)$ by conjugation, or, alternatively, the inverse image by the natural surjective map of (i) of the image by the natural surjective map of (i) of S coincides with S.

REMARK 4.4.1. — One verifies easily from Proposition 2.9, (i), that there exists a natural *bijective* map between the set $\underline{\mathrm{Cusp}}(\Pi)$ and the set of Π-conjugacy classes of *cuspidal decomposition subgroups* of Π.

DEFINITION 4.5. — Let S be a subset of $\mathrm{Cusp}(\Pi)$. Then we shall write

$$\Pi^{\bullet S}$$

for the quotient of Π by the normal closed subgroup normally topologically generated by the cuspidal inertia subgroups of Π that belong to elements of S and refer to $\Pi^{\bullet S}$ as the *partial compactification* of Π with respect to S. Moreover, we shall write

$$\Delta(\Pi)^{\bullet S}$$

for the image of $\Delta(\Pi)$ in $\Pi^{\bullet S}$. Thus, we have a commutative diagram of profinite groups

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \Delta(\Pi) & \longrightarrow & \Pi & \longrightarrow & G(\Pi) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \Delta(\Pi)^{\bullet S} & \longrightarrow & \Pi^{\bullet S} & \longrightarrow & G(\Pi) & \longrightarrow & 1
\end{array}
$$

— where the horizontal sequences are exact, and the vertical arrows are surjective.

REMARK 4.5.1. — In the situation of Definition 4.5, one verifies easily that if we write $T \subseteq \mathrm{Cusp}(\Pi)$ for the [uniquely determined] minimal Π-*stable* subset that contains S, then $\Pi^{\bullet S} = \Pi^{\bullet T}$.

PROPOSITION 4.6. — *Suppose that* Π *is a* **hyperbolic NF-curvoid**. *Let* S *be a* **Π-stable** *subset of* $\mathrm{Cusp}(\Pi)$ *and* D *a* **curvoidal** *element of* $\widetilde{\mathcal{V}}(G(\Pi))$. *Write* $S_D \subseteq \mathrm{Cusp}(\Pi|_D)$ *for the subset of* $\mathrm{Cusp}(\Pi|_D)$ *obtained by forming the inverse image of* $S \subseteq \mathrm{Cusp}(\Pi)$ *by the bijective map* $\mathrm{Cusp}(\Pi|_D) \overset{\sim}{\to} \mathrm{Cusp}(\Pi)$ *of Proposition 2.13. Then the natural inclusion* $\Pi|_D \hookrightarrow \Pi$ *determines a continuous* **isomorphism** *over* D

$$(\Pi|_D)^{\bullet S_D} \overset{\sim}{\longrightarrow} \Pi^{\bullet S} \times_{G(\Pi)} D,$$

that restricts to a continuous **isomorphism**

$$\Delta(\Pi|_D)^{\bullet S_D} \longrightarrow \Delta(\Pi)^{\bullet S}.$$

Let us identify $(\Pi|_D)^{\bullet S_D}$, $\Delta(\Pi|_D)^{\bullet S_D}$ with $\Pi^{\bullet S} \times_{G(\Pi)} D$, $\Delta(\Pi)^{\bullet S}$ by means of these continuous isomorphisms determined by the natural inclusion $\Pi|_D \hookrightarrow \Pi$, respectively:

$$(\Pi|_D)^{\bullet S_D} = \Pi^{\bullet S} \times_{G(\Pi)} D, \qquad \Delta(\Pi|_D)^{\bullet S_D} = \Delta(\Pi)^{\bullet S}.$$

PROOF. — This assertion follows immediately from the various definitions involved. □

LEMMA 4.7. — *Let S be a subset of $\mathrm{Cusp}(\Pi)$ and $H \subseteq \Pi^{\bullet S}$ an open subgroup of $\Pi^{\bullet S}$. Write $\widetilde{H} \subseteq \Pi$ for the inverse image of $H \subseteq \Pi^{\bullet S}$ by the natural surjective continuous homomorphism $\Pi \twoheadrightarrow \Pi^{\bullet S}$. [Thus, $\widetilde{H}$ is an open subgroup of Π, hence — cf. Proposition 2.2 — also a **hyperbolic □-curvoid**.] Write, moreover, $T \subseteq \mathrm{Cusp}(\widetilde{H})$ for the subset of $\mathrm{Cusp}(\widetilde{H})$ obtained by forming the inverse image, by the surjective map $\mathrm{Cusp}(\widetilde{H}) \twoheadrightarrow \mathrm{Cusp}(\Pi)$ of Proposition 2.9, (iii), of the minimal Π-stable subset of $\mathrm{Cusp}(\Pi)$ that contains S. Then the natural surjective continuous homomorphism $\widetilde{H} \twoheadrightarrow H$ determines a continuous **isomorphism** $\widetilde{H}^{\bullet T} \xrightarrow{\sim} H$. Let us identify $\widetilde{H}^{\bullet T}$ with H by means of this continuous isomorphism determined by the natural surjective continuous homomorphism $\widetilde{H} \twoheadrightarrow H$:*

$$\widetilde{H}^{\bullet T} = H.$$

PROOF. — This assertion follows immediately, in light of Remark 2.8.1, from the well-known structure of the étale fundamental groups of smooth curves over algebraically closed fields of characteristic zero. □

LEMMA 4.8. — *Let d be a positive integer and x an element of $\underline{\mathrm{Cusp}}(\Pi)$. Then the following three conditions are equivalent:*

(1) *The fiber of the natural surjective map $\mathrm{Cusp}(\Pi) \twoheadrightarrow \underline{\mathrm{Cusp}}(\Pi)$ at x is **of cardinality d**.*

(2) *The image in $G(\Pi)$ of every cuspidal decomposition subgroup of Π that belongs to the Π-conjugacy class of cuspidal decomposition subgroups corresponding to $x \in \underline{\mathrm{Cusp}}(\Pi)$ [cf. Remark 4.4.1] is **of index d**.*

(3) *The image in $G(\Pi)$ of some cuspidal decomposition subgroup of Π that belongs to the Π-conjugacy class of cuspidal decomposition subgroups corresponding to $x \in \underline{\mathrm{Cusp}}(\Pi)$ is **of index d**.*

PROOF. — This assertion follows immediately from the various definitions involved. □

DEFINITION 4.9. — Let d be a positive integer.

(i) We shall say that an element of $\underline{\mathrm{Cusp}}(\Pi)$ is *of degree d* if the positive integer d and the element of $\underline{\mathrm{Cusp}}(\Pi)$ satisfy condition (1) of Lemma 4.8 [i.e., imposed on "(d, x)"].

(ii) We shall say that a subset of $\underline{\mathrm{Cusp}}(\Pi)$ is *of degree d* if the sum of the degrees of the elements of the subset is equal to d.

(iii) We shall say that a subset of $\mathrm{Cusp}(\Pi)$ is *of degree d* if the subset of $\underline{\mathrm{Cusp}}(\Pi)$ determined by the subset is of degree d.

THEOREM 4.10. — *Let*

$$\Pi$$

be a **hyperbolic MLF-curvoid** (*respectively,* **hyperbolic NF-curvoid**), *d a positive integer, and S a subset of* $\mathrm{Cusp}(\Pi)$ *of degree d. Then the following five conditions are equivalent:*

(1) *The profinite group* $\Pi^{\bullet S}$ *is a* **hyperbolic MLF-curvoid** (*respectively,* **hyperbolic NF-curvoid**).

(2) *The profinite group* $\Pi^{\bullet S}$ *is a* **hyperbolic MLF-curvoid** (*respectively,* **hyperbolic NF-curvoid**) *whose* **geometric subgroup** *is given by* $\Delta(\Pi)^{\bullet S} \subseteq \Pi^{\bullet S}$.

(3) *The profinite group* $\Delta(\Pi)^{\bullet S}$ *is* **not abelian**.

(4) *The profinite group* $\Delta(\Pi)^{\bullet S}$ *is* **slim**.

(5) *The* **inequality** $2 - 2g(\Pi) - r(\Pi) < -d$ *holds.*

PROOF. — First, to verify the equivalences

$$(3) \Longleftrightarrow (4) \Longleftrightarrow (5),$$

let us observe that one verifies easily from Remark 2.8.1, together with the well-known structure of the étale fundamental groups of smooth curves over algebraically closed fields of characteristic zero, that the profinite group $\Delta(\Pi)^{\bullet S}$ is *isomorphic*, as an abstract profinite group, to the étale fundamental group of a smooth curve *of type* $(g(\Pi), r(\Pi) - d)$ over an algebraically closed field of characteristic zero [cf. also Remark 4.5.1]. Thus, the equivalences (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) follow from the well-known structure of the étale fundamental groups of smooth curves over algebraically closed fields of characteristic zero [cf. also [13, Proposition 2.3, (i)]].

The implication

$$(2) \Longrightarrow (1)$$

is immediate. Next, we verify the implication

$$(1) \Longrightarrow (2).$$

Suppose that condition (1) is satisfied. Then it follows from the diagram of Definition 4.5 that $\Delta(\Pi)^{\bullet S} \subseteq \Pi^{\bullet S}$ satisfies condition (1) of Definition 2.1. Next, to verify the assertion that $\Delta(\Pi)^{\bullet S} \subseteq \Pi^{\bullet S}$ satisfies condition (2) of Definition 2.1, let us observe that one verifies immediately [cf. also Proposition 2.2 and Proposition 2.5, (iii)] that we may assume without loss of generality, by replacing Π by "H" as in condition (2) of Definition 2.1 [i.e., with respect to $\Delta(\Pi) \subseteq \Pi$] and S by a suitable subset of $\mathrm{Cusp}(\Pi)$ that contains S, that $\Delta(\Pi) \subseteq \Pi$ is *pseudo-MLF-geometric* (respectively, *pseudo-NF-geometric*). Then if Π is a *hyperbolic NF-curvoid*, then [since $\Delta(\Pi) \subseteq \Pi$ is *pseudo-NF-geometric*] one verifies easily that $\Delta(\Pi)^{\bullet S} \subseteq \Pi^{\bullet S}$ is *pseudo-NF-geometric*, hence also satisfies condition (2) of Definition 2.1, as desired. In particular, to verify the desired assertion, it suffices to verify that if Π is a *hyperbolic MLF-curvoid*, then $\Delta(\Pi)^{\bullet S} \subseteq \Pi^{\bullet S}$ is *pseudo-MLF-geometric*. On the other hand, since [we have assumed that] $\Delta(\Pi) \subseteq \Pi$ is *pseudo-MLF-geometric*, one verifies easily that $\Delta(\Pi)^{\bullet S} \subseteq \Pi^{\bullet S}$ satisfies condition (1) of Definition 1.5, (ii). Thus, to verify the desired assertion, it suffices to verify that $\Delta(\Pi)^{\bullet S} \subseteq \Pi^{\bullet S}$ satisfies condition (2) of Definition 1.5, (ii). In particular, it follows immediately from Lemma 4.7, together with Lemma 1.6, (i), and Proposition 2.5, (iii), that, to verify the desired assertion, it suffices to verify the following assertion:

(a)   The maximal $G(\Pi)$-stable torsion-free quotient of the abelian profinite group $(\Delta(\Pi)^{\bullet S})^{\mathrm{ab}}$ on which the resulting action of $G(\Pi)$ is trivial has a natural structure of *free* $\widehat{\mathbb{Z}}$-module.

To verify (a), let us observe that it is immediate that the kernel of the surjective continuous homomorphism $\Delta(\Pi)^{\mathrm{ab}} \twoheadrightarrow (\Delta(\Pi)^{\bullet S})^{\mathrm{ab}}$ induced by the natural surjective continuous homomorphism $\Delta(\Pi) \twoheadrightarrow \Delta(\Pi)^{\bullet S}$ is *contained* in the kernel of the natural surjective continuous homomorphism $\Delta(\Pi)^{\mathrm{ab}} \twoheadrightarrow \Delta(\Pi)^{\mathrm{ab/cusp}}$. Thus, it follows from Lemma 4.3 that the natural surjective continuous homomorphism from $\Delta(\Pi)^{\mathrm{ab}}$ onto the maximal $G(\Pi)$-stable torsion-free quotient of the abelian profinite group $\Delta(\Pi)^{\mathrm{ab}}$ on which the resulting action of $G(\Pi)$ is trivial *factors* through the surjective continuous homomorphism $\Delta(\Pi)^{\mathrm{ab}} \twoheadrightarrow (\Delta(\Pi)^{\bullet S})^{\mathrm{ab}}$. Thus, since [we have assumed that] $\Delta(\Pi) \subseteq \Pi$ is *pseudo-MLF-geometric*, we conclude that (a) holds, as desired. This completes the proof of (a), hence also of the implication $(1) \Rightarrow (2)$.

The implication

$$(2) \Longrightarrow (4)$$

follows from Proposition 2.5, (i). Finally, we verify the implication

$$(5) \Longrightarrow (1).$$

Suppose that condition (5) is satisfied. Let us first observe that one verifies immediately from a similar argument to the argument applied in the proof of the implication $(1) \Rightarrow (2)$ that $\Delta(\Pi)^{\bullet S} \subseteq \Pi^{\bullet S}$ satisfies conditions (1), (2) of Definition 2.1. Write $T \subseteq \mathrm{Cusp}(\mathcal{G}(\Pi))$ for the subset of $\mathrm{Cusp}(\mathcal{G}(\Pi))$ that corresponds [cf. the final portion of Definition 2.10, (iii)] to the minimal Π-*stable* subset of $\mathrm{Cusp}(\Pi)$ that contains S. Then one verifies immediately [cf. also Remark 4.5.1] that condition (5) implies that this subset $T \subseteq \mathrm{Cusp}(\mathcal{G}(\Pi))$ is *omittable* [cf. [7, Definition 2.4, (i)]]. Moreover, one also verifies easily [cf. condition (c) of Definition 2.10, (iii)] that the outer continuous isomorphism $\Delta(\Pi) \overset{\sim}{\to} \Pi_{\mathcal{G}(\Pi)}$ of (2) of Definition 2.10, (iii), fits into a commutative diagram of profinite groups

$$
\begin{array}{ccc}
\Delta(\Pi) & \overset{\sim}{\longrightarrow} & \Pi_{\mathcal{G}(\Pi)} \\
\downarrow & & \downarrow \\
\Delta(\Pi)^{\bullet S} & \overset{\sim}{\longrightarrow} & \Pi_{\mathcal{G}(\Pi)_{\bullet T}}
\end{array}
$$

— where we write $\Pi_{\mathcal{G}(\Pi)_{\bullet T}}$ for the PSC-fundamental group of $\mathcal{G}(\Pi)_{\bullet T}$ [cf. [7, Definition 2.4, (ii)]]; the left-hand vertical arrow is the outer surjective continuous homomorphism determined by the natural surjective continuous homomorphism $\Delta(\Pi) \twoheadrightarrow \Delta(\Pi)^{\bullet S}$; the right-hand vertical arrow is the outer surjective continuous homomorphism of the final portion of [7, Definition 2.4, (ii)]; the lower horizontal arrow is an outer continuous isomorphism. In particular, it follows immediately from condition (a) of Definition 2.10, (iii), and [7, Corollary 3.9, (ii)] that the collection of data consisting of

- the normal closed subgroup $\Delta(\Pi)^{\bullet S} \subseteq \Pi^{\bullet S}$,

- the semi-graph $\mathcal{G}(\Pi)_{\bullet T}$ of anabelioids of pro-$\mathfrak{Primes}$ PSC-type, and

- the lower horizontal arrow $\Delta(\Pi)^{\bullet S} \overset{\sim}{\to} \Pi_{\mathcal{G}(\Pi)_{\bullet T}}$ of the above commutative diagram

satisfies conditions (3), (4) of Definition 2.1. This completes the proof of the implication $(5) \Rightarrow (1)$, hence also of Theorem 4.10. $\qquad\square$

THEOREM 4.11. — *In the situation of Theorem* 4.10, *suppose that the five conditions* (1), (2), (3), (4), (5) *of Theorem* 4.10 *are satisfied. Then the following assertions hold:*

(i) *The natural surjective continuous homomorphism* $\Pi \twoheadrightarrow \Pi^{\bullet S}$ *is an* **arithmetic equivalence** *and determines a continuous* **isomorphism** $\Delta(\Pi)^{\bullet S} \overset{\sim}{\to} \Delta(\Pi^{\bullet S})$. *Let us identify* $\Delta(\Pi)^{\bullet S}$, $G(\Pi)$ *with* $\Delta(\Pi^{\bullet S})$, $G(\Pi^{\bullet S})$ *by means of the continuous isomorphisms determined by the natural surjective continuous homomorphism* $\Pi \twoheadrightarrow \Pi^{\bullet S}$, *respectively:*

$$\Delta(\Pi)^{\bullet S} = \Delta(\Pi^{\bullet S}), \qquad G(\Pi) = G(\Pi^{\bullet S}).$$

(ii) *For a closed subgroup of* $\Delta(\Pi)^{\bullet S} = \Delta(\Pi^{\bullet S})$ [*cf.* (i)], *the following two conditions are equivalent:*

• *The closed subgroup is a* **cuspidal inertia subgroup** *of* $\Pi^{\bullet S}$.

• *The closed subgroup is obtained by forming the image of a* **cuspidal inertia subgroup** *of* Π.

(iii) *The* **equality**

$$\big(g(\Pi^{\bullet S}), r(\Pi^{\bullet S})\big) = \big(g(\Pi), r(\Pi) - d\big)$$

holds.

PROOF. — First, we verify assertion (i). It follows from condition (2) of Theorem 4.10 that the natural surjective continuous homomorphism $\Pi \twoheadrightarrow \Pi^{\bullet S}$ determines a continuous *isomorphism* $\Delta(\Pi)^{\bullet S} \overset{\sim}{\to} \Delta(\Pi^{\bullet S})$. Thus, it follows from the diagram of Definition 4.5 that the natural surjective continuous homomorphism $\Pi \twoheadrightarrow \Pi^{\bullet S}$ is an *arithmetic equivalence*. This completes the proof of assertion (i).

Next, we verify assertion (ii). Let us first observe that it follows immediately from condition (c) of Definition 2.10, (iii), together with the proof of the implication (5) $\Rightarrow$ (1) of Theorem 4.10, that, to verify assertion (ii), it suffices to verify that, in the situation of the proof of the implication (5) $\Rightarrow$ (1) of Theorem 4.10, for a closed subgroup of $\Pi_{\mathcal{G}_{\bullet T}}$, the following two conditions are equivalent:

• The closed subgroup is a *cuspidal subgroup* of $\Pi_{\mathcal{G}(\Pi)_{\bullet T}}$.

• The closed subgroup is obtained by forming the image, by some lifting of the outer surjective continuous homomorphism $\Pi_{\mathcal{G}(\Pi)} \twoheadrightarrow \Pi_{\mathcal{G}(\Pi)_{\bullet T}}$ of the final portion of [7, Definition 2.4, (ii)], of a *cuspidal subgroup* of $\Pi_{\mathcal{G}(\Pi)}$.

On the other hand, this equivalence follows from the definition of the semi-graph $\mathcal{G}(\Pi)_{\bullet T}$ of anabelioids and the definition of the outer surjective continuous homomorphism $\Pi_{\mathcal{G}(\Pi)} \twoheadrightarrow \Pi_{\mathcal{G}(\Pi)_{\bullet T}}$ of the final portion of [7, Definition 2.4, (ii)]. This completes the proof of assertion (ii). Assertion (iii) follows immediately from assertions (i), (ii), together with the various definitions involved [cf. also Remark 4.5.1]. This completes the proof of Theorem 4.11. $\qquad\square$

5. QUOTIENT ORBICURVOIDS BY OUTER ACTIONS OF FINITE GROUPS

In the present §5, we prove that a suitable outer continuous action of a finite group on a hyperbolic orbicurvoid gives rise to a hyperbolic orbicurvoid that may be thought of as an analogue [i.e., in the theory of hyperbolic curvoids] of the notion of a *quotient*

orbicurve [cf. Theorem 5.4 below and Corollary 5.5 below]. In the present §5, let $\square$ be an element of the set $\{\mathrm{MLF}, \mathrm{NF}\}$ and

$$\Pi$$

a *hyperbolic $\square$-orbicurvoid*.

LEMMA 5.1. — *The natural homomorphism* $\mathrm{Aut}(\Pi) \to \mathrm{Aut}(G(\Pi))$ *and the exact sequence of Definition* 3.4, (ii), *fit into a **commutative diagram** of groups*

$$
\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Delta(\Pi) & \longrightarrow & \Pi & \longrightarrow & G(\Pi) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Aut}_{G(\Pi)}(\Pi) & \longrightarrow & \mathrm{Aut}(\Pi) & \longrightarrow & \mathrm{Aut}\big(G(\Pi)\big) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \underline{\mathrm{Aut}}_{G(\Pi)}(\Pi) & \longrightarrow & \mathrm{Out}(\Pi) & \longrightarrow & \mathrm{Out}\big(G(\Pi)\big) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}
$$

— *where the upper vertical arrows are the continuous actions by conjugation, and the sequences are **exact**. By means of the left-hand lower horizontal arrow of this diagram, let us regard* $\underline{\mathrm{Aut}}_{G(\Pi)}(\Pi)$ *as a **subgroup** of* $\mathrm{Out}(\Pi)$:

$$\underline{\mathrm{Aut}}_{G(\Pi)}(\Pi) \subseteq \mathrm{Out}(\Pi).$$

PROOF. — This assertion follows immediately from the *slimness* of $\Delta(\Pi)$ [cf. Proposition 3.5, (i)], the *slimness* of Π [cf. Proposition 3.2, (iii)], and the *slimness* of $G(\Pi)$ [cf. Remark 1.3.1, (i), (ii); Proposition 3.5, (ii)]. $\square$

REMARK 5.1.1.

(i) Suppose that Π is a *hyperbolic MLF-orbicurvoid*, which thus [cf. Proposition 3.2, (iv)] implies that Π is *topologically finitely generated*. Thus, the topology of Π admits a basis of *characteristic* open subgroups, which thus induces a *profinite topology* on $\mathrm{Aut}(\Pi)$, hence also a *profinite topology* on $\mathrm{Out}(\Pi)$. Let us regard $\mathrm{Aut}(\Pi)$, $\mathrm{Out}(\Pi)$ as *profinite groups* by means of these profinite topologies, respectively. Thus, it follows from Proposition 3.2, (iii), that we have an *exact sequence of profinite groups*

$$1 \longrightarrow \Pi \longrightarrow \mathrm{Aut}(\Pi) \longrightarrow \mathrm{Out}(\Pi) \longrightarrow 1.$$

(ii) Suppose that Π is a *hyperbolic NF-orbicurvoid*, and that $\underline{\mathrm{Aut}}_{G(\Pi)}(\Pi)$ is *finite*. Then it follows from Proposition 1.4 and Proposition 3.5, (ii), that $\mathrm{Out}(G(\Pi))$, hence [cf. Lemma 5.1] also $\mathrm{Out}(\Pi)$, is *finite*. Thus, it follows from Lemma 5.1 that the homomorphism $\Pi \to \mathrm{Aut}(\Pi)$ by conjugation is an *injective homomorphism whose image is of finite index*, which thus implies that the profinite topology of Π determines a *profinite topology* on $\mathrm{Aut}(\Pi)$, hence also a *profinite topology* on $\mathrm{Out}(\Pi)$. Let us regard $\mathrm{Aut}(\Pi)$,

Out(Π) as *profinite groups* by means of these profinite topologies, respectively. Thus, it follows from Proposition 3.2, (iii), that we have an *exact sequence of profinite groups*

$$1 \longrightarrow \Pi \longrightarrow \mathrm{Aut}(\Pi) \longrightarrow \mathrm{Out}(\Pi) \longrightarrow 1.$$

PROPOSITION 5.2. — *The following assertions hold:*

(i) *The natural homomorphism*

$$\mathrm{Aut}_{G(\Pi)}(\Pi) \longrightarrow \mathrm{Aut}\big(\Delta(\Pi)\big),$$

hence also the natural homomorphism

$$\underline{\mathrm{Aut}}_{G(\Pi)}(\Pi) \longrightarrow \mathrm{Out}\big(\Delta(\Pi)\big),$$

*is **injective**.*

(ii) *Suppose that Π is a **hyperbolic NF-orbicurvoid**. Let D be a(n) [not necessarily orbicurvoidal] element of $\widetilde{\mathcal{V}}(G(\Pi))$. Then the natural homomorphism*

$$\underline{\mathrm{Aut}}_{G(\Pi)}(\Pi) \longrightarrow \underline{\mathrm{Aut}}_D(\Pi|_D)$$

*is **injective**.*

(iii) *Suppose that Π is **relatively core-like**. Then the group $\underline{\mathrm{Aut}}_{G(\Pi)}(\Pi)$ is **trivial**.*

PROOF. — Assertion (i) follows from the *slimness* of $\Delta(\Pi)$ [cf. Proposition 3.5, (i)] and [1, Lemma 4.10]. Next, we verify assertion (ii). Let us observe that it is immediate that the homomorphism of the second display of assertion (i) *factors* through the homomorphism under consideration. Thus, assertion (ii) follows from assertion (i). This completes the proof of assertion (ii). Assertion (iii) follows immediately from assertion (i), together with the various definitions involved. This completes the proof of Proposition 5.2. □

DEFINITION 5.3. — Let $J \subseteq \mathrm{Out}(\Pi)$ be a finite subgroup of $\mathrm{Out}(\Pi)$.

(i) We shall write

$$\Delta(J) \overset{\text{def}}{=} J \cap \underline{\mathrm{Aut}}_{G(\Pi)}(\Pi), \qquad G(J) \overset{\text{def}}{=} J/\Delta(J) \subseteq \mathrm{Out}\big(G(\Pi)\big)$$

[cf. Lemma 5.1].

(ii) We shall write

$$\Pi[J] \overset{\text{def}}{=} \mathrm{Aut}(\Pi) \times_{\mathrm{Out}(\Pi)} J$$

for the fiber product of the natural surjective homomorphism $\mathrm{Aut}(\Pi) \twoheadrightarrow \mathrm{Out}(\Pi)$ and the natural inclusion $J \hookrightarrow \mathrm{Out}(\Pi)$,

$$\Delta(\Pi) \overset{\text{out}}{\rtimes} \Delta(J) \overset{\text{def}}{=} \mathrm{Aut}_{G(\Pi)}(\Pi) \times_{\underline{\mathrm{Aut}}_{G(\Pi)}(\Pi)} \Delta(J)$$

for the fiber product of the natural surjective homomorphism $\mathrm{Aut}_{G(\Pi)}(\Pi) \twoheadrightarrow \underline{\mathrm{Aut}}_{G(\Pi)}(\Pi)$ and the natural inclusion $\Delta(J) \hookrightarrow \underline{\mathrm{Aut}}_{G(\Pi)}(\Pi)$, and

$$G(\Pi) \overset{\text{out}}{\rtimes} G(J) \overset{\text{def}}{=} \mathrm{Aut}\big(G(\Pi)\big) \times_{\mathrm{Out}\big(G(\Pi)\big)} G(J)$$

for the fiber product of the natural surjective homomorphism $\mathrm{Aut}(G(\Pi)) \twoheadrightarrow \mathrm{Out}(G(\Pi))$ and the natural inclusion $G(J) \hookrightarrow \mathrm{Out}(G(\Pi))$. Thus, it follows from Remark 1.3.1, (i), (ii); Proposition 3.2, (iii); Proposition 3.5, (i), (ii), that we have exact sequences of groups

$$1 \longrightarrow \Pi \longrightarrow \Pi[J] \longrightarrow J \longrightarrow 1,$$

$$1 \longrightarrow \Delta(\Pi) \longrightarrow \Delta(\Pi) \overset{\mathrm{out}}{\rtimes} \Delta(J) \longrightarrow \Delta(J) \longrightarrow 1,$$

$$1 \longrightarrow G(\Pi) \longrightarrow G(\Pi) \overset{\mathrm{out}}{\rtimes} G(J) \longrightarrow G(J) \longrightarrow 1.$$

REMARK 5.3.1. — Let $J \subseteq \mathrm{Out}(\Pi)$ be a finite subgroup of $\mathrm{Out}(\Pi)$.

(i) Since J is a *finite* group, it is immediate that Π (respectively, $\Delta(\Pi)$; $G(\Pi)$) is *of finite index* in $\Pi[J]$ (respectively, $\Delta(\Pi) \overset{\mathrm{out}}{\rtimes} \Delta(J)$; $G(\Pi) \overset{\mathrm{out}}{\rtimes} G(J)$) [cf. the exact sequences of the final display of Definition 5.3, (ii)]. Thus, the natural inclusion $\Pi \hookrightarrow \Pi[J]$ (respectively, $\Delta(\Pi) \hookrightarrow \Delta(\Pi) \overset{\mathrm{out}}{\rtimes} \Delta(J)$; $G(\Pi) \hookrightarrow G(\Pi) \overset{\mathrm{out}}{\rtimes} G(J)$) and the profinite topology of Π (respectively, $\Delta(\Pi)$; $G(\Pi)$) determine a *profinite topology* on $\Pi[J]$ (respectively, $\Delta(\Pi) \overset{\mathrm{out}}{\rtimes} \Delta(J)$; $G(\Pi) \overset{\mathrm{out}}{\rtimes} G(J)$), with respect to which the first (respectively, second; third) exact sequence of the final display of Definition 5.3, (ii), is an *exact sequence of profinite groups*. Let us regard $\Pi[J]$ (respectively, $\Delta(\Pi) \overset{\mathrm{out}}{\rtimes} \Delta(J)$; $G(\Pi) \overset{\mathrm{out}}{\rtimes} G(J)$) as a *profinite group* by means of this profinite topology.

(ii) Let us recall that it follows immediately from the various definitions involved that we have a commutative diagram of groups

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \Pi & \longrightarrow & \Pi[J] & \longrightarrow & J & \longrightarrow & 1 \\
 & & \| & & \cap\downarrow & & \cap\downarrow & & \\
1 & \longrightarrow & \Pi & \longrightarrow & \mathrm{Aut}(\Pi) & \longrightarrow & \mathrm{Out}(\Pi) & \longrightarrow & 1
\end{array}
$$

— where the horizontal sequences are *exact*, and the right-hand square is *cartesian* — which thus determines a commutative diagram of groups

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & G(\Pi) & \longrightarrow & \Pi[J]/\Delta(\Pi) & \longrightarrow & J & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \cap\downarrow & & \\
1 & \longrightarrow & G(\Pi) & \longrightarrow & \mathrm{Aut}(\Pi)/\mathrm{Inn}\big(\Delta(\Pi)\big) & \longrightarrow & \mathrm{Out}(\Pi) & \longrightarrow & 1
\end{array}
$$

— where we write $\mathrm{Inn}(\Delta(\Pi))$ for the group of inner automorphisms of Π determined by elements of $\Delta(\Pi) \subseteq \Pi$, the horizontal sequences are *exact*, and the right-hand square is *cartesian*. Thus, the natural inclusion $\Delta(J) \hookrightarrow (\underline{\mathrm{Aut}}_{G(\Pi)}(\Pi) \subseteq) \mathrm{Aut}(\Pi)/\mathrm{Inn}(\Delta(\Pi))$ determines a splitting of the right-hand upper horizontal arrow $\Pi[J]/\Delta(\Pi) \twoheadrightarrow J$ on the subgroup $\Delta(J) \subseteq J$. Now let us observe that one verifies immediately from the various definitions involved [cf. also (i)] that the natural inclusion $\mathrm{Aut}_{G(\Pi)}(\Pi) \hookrightarrow \mathrm{Aut}(\Pi)$ restricts to a continuous *isomorphism* of

- the subgroup $\Delta(\Pi) \overset{\mathrm{out}}{\rtimes} \Delta(J) \subseteq \mathrm{Aut}_{G(\Pi)}(\Pi)$ of $\mathrm{Aut}_{G(\Pi)}(\Pi)$ with

- the inverse image of the image of this splitting $\Delta(J) \hookrightarrow \Pi[J]/\Delta(\Pi)$ by the natural surjective continuous homomorphism $\Pi[J] \twoheadrightarrow \Pi[J]/\Delta(\Pi)$.

Let us identify $\Delta(\Pi) \overset{\text{out}}{\rtimes} \Delta(J)$ with the inverse image of the image of this splitting $\Delta(J) \hookrightarrow \Pi[J]/\Delta(\Pi)$ by the natural surjective continuous homomorphism $\Pi[J] \twoheadrightarrow \Pi[J]/\Delta(\Pi)$ by means of the resulting continuous isomorphism. In particular, we have an *injective* continuous homomorphism [cf. (i)]

$$\Delta(\Pi) \overset{\text{out}}{\rtimes} \Delta(J) \lhook\joinrel\longrightarrow \Pi[J].$$

(iii) Next, let us also observe that one verifies immediately from the various definitions involved that the diagram of Lemma 5.1 and the injective continuous homomorphism $\Delta(\Pi) \overset{\text{out}}{\rtimes} \Delta(J) \hookrightarrow \Pi[J]$ of the final display of (ii) determine a commutative diagram of groups

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \Delta(\Pi) & \longrightarrow & \Pi & \longrightarrow & G(\Pi) & \longrightarrow & 1 \\
& & \cap\downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Delta(\Pi) \overset{\text{out}}{\rtimes} \Delta(J) & \longrightarrow & \Pi[J] & \longrightarrow & G(\Pi) \overset{\text{out}}{\rtimes} G(J) & \longrightarrow & 1 \\
& & \cap\uparrow & & \cap\uparrow & & \cap\uparrow & & \\
1 & \longrightarrow & \mathrm{Aut}_{G(\Pi)}(\Pi) & \longrightarrow & \mathrm{Aut}(\Pi) & \longrightarrow & \mathrm{Aut}\big(G(\Pi)\big) & &
\end{array}
$$

— where the horizontal sequences are *exact*, the lower vertical arrows are *injective*, and the upper vertical arrows are *open injective* [cf. (i)]; in particular, the two left-hand squares are *cartesian*.

(iv) Suppose that $\underline{\mathrm{Aut}}_{G(\Pi)}(\Pi)$ is *finite* if Π is a *hyperbolic NF-orbicurvoid*. Then one verifies immediately from the various definitions involved that each of the middle vertical arrows $\Pi \hookrightarrow \Pi[J] \hookrightarrow \mathrm{Aut}(\Pi)$ of the diagram of (iii) is *continuous* [cf. (i); Remark 5.1.1, (i), (ii)].

THEOREM 5.4. — *Let*

$$\Pi$$

*be a **hyperbolic MLF-orbicurvoid** (respectively, **hyperbolic NF-orbicurvoid**) and $J \subseteq \mathrm{Out}(\Pi)$ a **finite** subgroup of $\mathrm{Out}(\Pi)$. Suppose that J is **contained** in the subgroup $\underline{\mathrm{Aut}}_{G(\Pi)}(\Pi) \subseteq \mathrm{Out}(\Pi)$ [cf. Lemma 5.1] whenever Π is a **hyperbolic MLF-orbicurvoid**. Then the following assertions hold:*

(i) *The profinite group $\Pi[J]$ [cf. Remark 5.3.1, (i)] is a **hyperbolic MLF-orbicurvoid** (respectively, **hyperbolic NF-orbicurvoid**) whose **geometric subgroup** is given by $\Delta(\Pi) \overset{\text{out}}{\rtimes} \Delta(J) \subseteq \Pi[J]$ [cf. Remark 5.3.1, (ii)]:*

$$\Delta(\Pi[J]) = \Delta(\Pi) \overset{\text{out}}{\rtimes} \Delta(J).$$

(ii) *It holds that the natural inclusion $\Pi \hookrightarrow \Pi[J]$ is an **arithmetic equivalence** [cf. (i)] if and only if J is **contained** in $\underline{\mathrm{Aut}}_{G(\Pi)}(\Pi)$.*

(iii) *Suppose that Π is a **hyperbolic NF-orbicurvoid**, and that J is **contained** in $\underline{\mathrm{Aut}}_{G(\Pi)}(\Pi)$. Let D be an **orbicurvoidal** element of $\widetilde{\mathcal{V}}(G(\Pi))$. Write $J_D \subseteq \underline{\mathrm{Aut}}_D(\Pi|_D)$ for the image of $J \subseteq \underline{\mathrm{Aut}}_{G(\Pi)}(\Pi)$ by the injective homomorphism $\underline{\mathrm{Aut}}_{G(\Pi)}(\Pi) \hookrightarrow \underline{\mathrm{Aut}}_D(\Pi|_D)$*

of Proposition 5.2, (ii). *Then the natural inclusion* $\Pi|_D \hookrightarrow \Pi$ *determines a continuous* ***isomorphism*** $\Pi|_D[J_D] \xrightarrow{\sim} \Pi[J]|_D$ *over* D [*cf.* (i), (ii)]. *Let us identify* $\Pi|_D[J_D]$ *with* $\Pi[J]|_D$ *by means of this continuous isomorphism induced by the natural inclusion* $\Pi|_D \hookrightarrow \Pi$:

$$\Pi|_D[J_D] = \Pi[J]|_D.$$

(iv) *Suppose that* Π *is* ***relatively core-like***. *Then the natural inclusion* $\Pi \hookrightarrow \Pi[J]$ *restricts to a continuous* ***isomorphism*** $\Delta(\Pi) \xrightarrow{\sim} \Delta(\Pi[J])$, *and, moreover, the hyperbolic MLF-orbicurvoid* (*respectively, hyperbolic NF-orbicurvoid*) $\Pi[J]$ [*cf.* (i)] *is* ***relatively core-like***.

PROOF. — First, we verify assertion (i). Let $H \subseteq \Pi$ be a normal open subgroup of Π as in Definition 3.1. Now it follows from Proposition 2.2 and Proposition 2.5, (iii), that we may assume without loss of generality, by replacing H by the intersection of the $\Pi[J]$-conjugates of H, that H is *normal* in $\Pi[J]$. Then since the left-hand upper square of the diagram of Remark 5.3.1, (iii), is *cartesian*, we conclude that the pair consisting of the normal closed subgroup $\Delta(\Pi) \overset{\text{out}}{\rtimes} \Delta(J) \subseteq \Pi[J]$ of $\Pi[J]$ and the normal open subgroup $H \subseteq \Pi[J]$ of $\Pi[J]$ satisfies condition (2) of Definition 3.1. Thus, to verify assertion (i), it suffices to verify that the normal closed subgroup $\Delta(\Pi) \overset{\text{out}}{\rtimes} \Delta(J) \subseteq \Pi[J]$ of $\Pi[J]$ satisfies condition (1) of Definition 3.1, i.e., that $\Delta(\Pi) \overset{\text{out}}{\rtimes} \Delta(J)$ is *slim* and *of co-MLF-type* (respectively, *of co-NF-type*).

The *slimness* of $\Delta(\Pi) \overset{\text{out}}{\rtimes} \Delta(J)$ follows immediately from the *slimness* of $\Delta(\Pi)$ [cf. Proposition 3.5, (i)] and Proposition 5.2, (i), together with the definition of $\Delta(\Pi) \overset{\text{out}}{\rtimes} \Delta(J)$.

Next, we verify the assertion that $\Delta(\Pi) \overset{\text{out}}{\rtimes} \Delta(J)$ is *of co-MLF-type* if Π is a *hyperbolic MLF-orbicurvoid*. Suppose that Π is a *hyperbolic MLF-orbicurvoid*. Then since [we have assumed that] $J \subseteq \underline{\mathrm{Aut}}_{G(\Pi)}(\Pi)$ [i.e., $J = \Delta(J)$], the right-hand upper vertical arrow $G(\Pi) \hookrightarrow G(\Pi) \overset{\text{out}}{\rtimes} G(J)$ of the diagram of Remark 5.3.1, (iii), is a continuous *isomorphism*. Thus, one verifies easily from Proposition 3.5, (ii), that $\Delta(\Pi) \overset{\text{out}}{\rtimes} \Delta(J)$ is *of co-MLF-type*, as desired. This completes the proof of the assertion that $\Delta(\Pi) \overset{\text{out}}{\rtimes} \Delta(J)$ is *of co-MLF-type* if Π is a *hyperbolic MLF-orbicurvoid*.

Next, we verify the assertion that $\Delta(\Pi) \overset{\text{out}}{\rtimes} \Delta(J)$ is *of co-NF-type* if Π is a *hyperbolic NF-orbicurvoid*. Suppose that Π is a *hyperbolic NF-orbicurvoid*. Then since $G(\Pi)$ is *of NF-type* [cf. Proposition 3.5, (ii)], it follows immediately from Proposition 1.4 that $\mathrm{Aut}(G(\Pi))$ is *of NF-type*, and the composite $G(\Pi) \hookrightarrow G(\Pi) \overset{\text{out}}{\rtimes} G(J) \hookrightarrow \mathrm{Aut}(G(\Pi))$ of the right-hand vertical arrows of the diagram of Remark 5.3.1, (iii) — hence also the right-hand lower vertical arrow $G(\Pi) \overset{\text{out}}{\rtimes} G(J) \hookrightarrow \mathrm{Aut}(G(\Pi))$ of the diagram of Remark 5.3.1, (iii) — is an *open injective* continuous homomorphism. Thus, we conclude from [4, Remark 3.2.1, (i)] that $G(\Pi) \overset{\text{out}}{\rtimes} G(J)$ is *of NF-type*, which thus implies that $\Delta(\Pi) \overset{\text{out}}{\rtimes} \Delta(J)$ is *of co-NF-type*, as desired. This completes the proof of the assertion that $\Delta(\Pi) \overset{\text{out}}{\rtimes} \Delta(J)$ is *of co-NF-type* if Π is a *hyperbolic NF-orbicurvoid*, hence also of assertion (i).

Assertions (ii), (iii) follow immediately from assertion (i) and the diagram of Remark 5.3.1, (iii). Finally, we verify assertion (iv). It follows immediately — in light

of Proposition 3.5, (iii) — from assertion (i) and Proposition 5.2, (iii), that the natural inclusion $\Pi \hookrightarrow \Pi[J]$ restricts to a continuous *isomorphism* $\Delta(\Pi) \overset{\sim}{\to} \Delta(\Pi[J])$. Thus, it follows from Proposition 3.13 that $\Pi[J]$ is *relatively core-like*, as desired. This completes the proof of assertion (iv), hence also of Theorem 5.4. $\qquad\square$

COROLLARY 5.5. — *Let* Π *be a **relatively core-like hyperbolic NF-orbicurvoid**. Then the profinite group* $\mathrm{Aut}(\Pi)$ *[cf. Remark 5.1.1, (ii); Proposition 5.2, (iii)] is a **relatively core-like hyperbolic NF-orbicurvoid**. Moreover, the natural inclusion* $\Pi \hookrightarrow \mathrm{Aut}(\Pi)$ *by conjugation [cf. Proposition 3.2, (iii)] fits into a **commutative diagram** of profinite groups*

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \Delta(\Pi) & \longrightarrow & \Pi & \longrightarrow & G(\Pi) & \longrightarrow & 1 \\
& & \downarrow{\wr} & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \Delta\big(\mathrm{Aut}(\Pi)\big) & \longrightarrow & \mathrm{Aut}(\Pi) & \longrightarrow & G\big(\mathrm{Aut}(\Pi)\big) & \longrightarrow & 1
\end{array}
$$

*— where the horizontal sequences are **exact**, the vertical arrows are **open injective**, and the left-hand vertical arrow is a continuous **isomorphism**.*

PROOF. — Let us first observe that it follows from Proposition 1.4 and Proposition 3.5, (ii), that $\mathrm{Out}(G(\Pi))$, hence [cf. Lemma 5.1 and Proposition 5.2, (iii)] also $\mathrm{Out}(\Pi)$, is *finite*. Thus, Corollary 5.5 follows — in light of Remark 5.3.1, (iv) — from Theorem 5.4, (i), (iv), i.e., in the case where we take the "J" to be $\mathrm{Out}(\Pi)$. This completes the proof of Corollary 5.5. $\qquad\square$

6. EXAMPLES FROM SCHEME THEORY

In the present §6, we give some examples of hyperbolic curvoids that arise from *scheme theory* [cf. Theorem 6.5 below, Theorem 6.6 below, and Corollary 6.7 below].

DEFINITION 6.1. — Let k be an MLF, $\overline{k}$ an algebraic closure of k, and X a hyperbolic curve over k.

(i) We shall write

$$
\pi_1(X), \qquad \pi_1(X \times_k \overline{k})
$$

for the respective *étale fundamental groups* of X, $X \times_k \overline{k}$ [relative to some choices of basepoints]. Thus, we have an exact sequence of profinite groups

$$
1 \longrightarrow \pi_1(X \times_k \overline{k}) \longrightarrow \pi_1(X) \longrightarrow \mathrm{Gal}(\overline{k}/k) \longrightarrow 1.
$$

(ii) We shall write

$$
\mathbb{G}_X
$$

for the *dual semi-graph* of the special fiber of the stable model [i.e., over the ring of integers of $\overline{k}$] of the hyperbolic curve $X \times_k \overline{k}$,

$$
\pi_1^{\mathrm{top}}(\mathbb{G}_X)
$$

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for the *topological fundamental group* of $\mathbb{G}_X$ [relative to some choice of basepoint], and

$$\pi_1^{\mathrm{top}}(\mathbb{G}_X)^\wedge$$

for the *profinite completion* of $\pi_1^{\mathrm{top}}(\mathbb{G}_X)$. Thus, we have a natural outer surjective continuous homomorphism

$$\pi_1(X \times_k \overline{k}) \longrightarrow\!\!\!\!\!\rightarrow \pi_1^{\mathrm{top}}(\mathbb{G}_X)^\wedge.$$

LEMMA 6.2. — *Suppose that we are in the situation of Definition 6.1. Then the natural surjective continuous homomorphism from $\pi_1(X \times_k \overline{k})^{\mathrm{ab}}$ onto the **maximal $\mathrm{Gal}(\overline{k}/k)$-stable torsion-free quotient** of the abelian profinite group $\pi_1(X \times_k \overline{k})^{\mathrm{ab}}$ on which the resulting action of $\mathrm{Gal}(\overline{k}/k)$ is **trivial factors** through the surjective continuous homomorphism $\pi_1(X \times_k \overline{k})^{\mathrm{ab}} \twoheadrightarrow (\pi_1^{\mathrm{top}}(\mathbb{G}_X)^\wedge)^{\mathrm{ab}}$ induced by the outer surjective continuous homomorphism of the final display of Definition 6.1, (ii).*

PROOF. — This assertion follows immediately from [13, Lemma 2.7, (ii)] and the proof of [13, Theorem 2.11, (iii)]. □

DEFINITION 6.3. — Let k be an MLF and $\overline{k}$ an algebraic closure of k. Write $\mathbb{K} \stackrel{\mathrm{def}}{=} k((t))$. Let $\overline{\mathbb{K}}$ be an algebraic closure of $\mathbb{K}$ that contains $\overline{k}$ and X a hyperbolic curve over $\mathbb{K}$. Write $\mathcal{O} \subseteq \overline{\mathbb{K}}$ for the integral closure in $\overline{\mathbb{K}}$ of the subring $k[[t]] \subseteq k((t)) = \mathbb{K}$ of $\mathbb{K}$. [So the residue field of $\mathcal{O}$ may be naturally identified with $\overline{k}$.]

(i) We shall write

$$\pi_1(X), \qquad \pi_1(X \times_{\mathbb{K}} \overline{\mathbb{K}})$$

for the respective *étale fundamental groups* of X, $X \times_{\mathbb{K}} \overline{\mathbb{K}}$ [relative to some choices of basepoints] and

$$\mathbb{I}_{\mathbb{K}} \subseteq \mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K})$$

for the kernel of the natural surjective continuous homomorphism $\mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K}) \twoheadrightarrow \mathrm{Gal}(\overline{k}/k)$, i.e., induced by the natural inclusion $\overline{k} \hookrightarrow \overline{\mathbb{K}}$. [So it is well-known that the profinite group $\mathbb{I}_{\mathbb{K}}$ is abelian, and there exists a natural isomorphism $\mathbb{I}_{\mathbb{K}} \stackrel{\sim}{\to} \widehat{\mathbb{Z}}(1)$ of $\mathrm{Gal}(\overline{k}/k)$-modules — where the "(1)" denotes a "Tate twist".] Thus, we have exact sequences of profinite groups

$$1 \longrightarrow \pi_1(X \times_{\mathbb{K}} \overline{\mathbb{K}}) \longrightarrow \pi_1(X) \longrightarrow \mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K}) \longrightarrow 1,$$

$$1 \longrightarrow \mathbb{I}_{\mathbb{K}} \longrightarrow \mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K}) \longrightarrow \mathrm{Gal}(\overline{k}/k) \longrightarrow 1.$$

(ii) We shall write

$$\mathcal{G}_X$$

for the *semi-graph of anabelioids of pro-𝕻rimes PSC-type* determined by the special fiber of the stable model [i.e., over $\mathcal{O}$] of the hyperbolic curve $X \times_{\mathbb{K}} \overline{\mathbb{K}}$ [cf. [12, Definition 1.1, (i)]],

$$\Pi_{\mathcal{G}_X}$$

for the *PSC-fundamental group* of $\mathcal{G}_X$,

$$\mathbb{G}_X$$

for the *underlying semi-graph* of $\mathcal{G}_X$,

$$\pi_1^{\mathrm{top}}(\mathbb{G}_X)$$

for the *topological fundamental group* of $\mathbb{G}_X$ [relative to some choice of basepoint], and

$$\pi_1^{\mathrm{top}}(\mathbb{G}_X)^\wedge$$

for the profinite completion of $\pi_1^{\mathrm{top}}(\mathbb{G}_X)$. Thus, we have a natural $\pi_1(X)$-conjugacy class of continuous isomorphisms

$$\Pi_{\mathcal{G}_X} \xrightarrow{\ \sim\ } \pi_1(X \times_{\mathbb{K}} \overline{\mathbb{K}}),$$

by means of which let us identify $\Pi_{\mathcal{G}_X}$ with $\pi_1(X \times_{\mathbb{K}} \overline{\mathbb{K}})$, and a natural outer surjective continuous homomorphism

$$\Pi_{\mathcal{G}_X} \longrightarrow\!\!\!\!\rightarrow \pi_1^{\mathrm{top}}(\mathbb{G}_X)^\wedge.$$

LEMMA 6.4. — *In the situation of Definition 6.3, let*

$$G \subseteq \mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K})$$

*be a closed subgroup of $\mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K})$ such that the composite $G \hookrightarrow \mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K}) \twoheadrightarrow \mathrm{Gal}(\overline{k}/k)$ is a continuous **isomorphism**. Write*

$$\Pi_G \overset{\mathrm{def}}{=} \pi_1(X) \times_{\mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K})} G$$

for the fiber product of $\pi_1(X) \twoheadrightarrow \mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K})$ and $G \hookrightarrow \mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K})$. Thus, we have an exact sequence of profinite groups

$$1 \longrightarrow \Pi_{\mathcal{G}_X} \longrightarrow \Pi_G \longrightarrow G \longrightarrow 1.$$

Then the following assertions hold:

(i) *Let v be a vertex of $\mathbb{G}_X$ and $\Pi_v \subseteq \Pi_{\mathcal{G}_X}$ a verticial subgroup of $\Pi_{\mathcal{G}_X}$ associated to v [cf. [12, Definition 1.1, (ii)]]. Write $G_v \subseteq G$ for the [necessarily open] subgroup of G obtained by forming the image of the composite $C_{\Pi_G}(\Pi_v) \hookrightarrow \Pi_G \twoheadrightarrow G$. Then there exist a finite extension $k_v \subseteq \overline{k}$ of k, a **hyperbolic curve** X_v over k_v, and a commutative diagram of profinite groups*

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \Pi_v & \longrightarrow & C_{\Pi_G}(\Pi_v) & \longrightarrow & G_v & \longrightarrow & 1 \\
 & & \downarrow{\wr} & & \downarrow{\wr} & & \downarrow{\wr} & & \\
1 & \longrightarrow & \pi_1(X_v \times_{k_v} \overline{k}_v) & \longrightarrow & \pi_1(X_v) & \longrightarrow & \mathrm{Gal}(\overline{k}/k_v) & \longrightarrow & 1
\end{array}
$$

— *where we apply the notational conventions introduced in Definition 6.1 in the case where we take the "$(k, \overline{k}, X)$" of Definition 6.1 to be $(k_v, \overline{k}, X_v)$, the horizontal sequences are **exact**, and the vertical arrows are continuous **isomorphisms**.*

(ii) *Let v be a vertex of $\mathbb{G}_X$ and $\Pi_v \subseteq \Pi_{\mathcal{G}_X}$ a verticial subgroup of $\Pi_{\mathcal{G}_X}$ associated to v. Consider the composite*

$$\Pi_v^{\mathrm{ab}} \xrightarrow{\ \sim\ } \pi_1(X_v \times_{k_v} \overline{k})^{\mathrm{ab}} \longrightarrow \left(\pi_1^{\mathrm{top}}(\mathbb{G}_{X_v})^\wedge\right)^{\mathrm{ab}}$$

of the continuous isomorphism $\Pi_v^{\mathrm{ab}} \xrightarrow{\sim} \pi_1(X_v \times_{k_v} \overline{k})^{\mathrm{ab}}$ induced by the left-hand vertical continuous isomorphism of the diagram of (i) and the surjective continuous homomorphism

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$\pi_1(X_v \times_{k_v} \overline{k})^{\mathrm{ab}} \twoheadrightarrow (\pi_1^{\mathrm{top}}(\mathbb{G}_{X_v})^{\wedge})^{\mathrm{ab}}$ *induced by the outer surjective continuous homomorphism of the final display of Definition* 6.1, (ii), *in the case where we take the "$(k, \overline{k}, X)$" of Definition* 6.1 *to be* $(k_v, \overline{k}, X_v)$. *Then this composite* **factors** *through the quotient* $\Pi_v^{\mathrm{ab}} \twoheadrightarrow \Pi_v^{\mathrm{ab/edge}}$ *defined in* [6, *Definition* 1.3, (iii)].

(iii) *For each vertex v of $\mathbb{G}_X$, take a verticial subgroup $\Pi_v \subseteq \Pi_{\mathcal{G}_X}$ of $\Pi_{\mathcal{G}_X}$ associated to v. Then the various natural inclusions $\Pi_v \hookrightarrow \Pi_{\mathcal{G}_X}$ — where v ranges over the vertices of $\mathbb{G}_X$ — and the natural outer surjective continuous homomorphism $\Pi_{\mathcal{G}_X} \twoheadrightarrow \pi_1^{\mathrm{top}}(\mathbb{G}_X)^{\wedge}$ of the final display of Definition* 6.3, (ii), *determine an* **exact** *sequence of abelian profinite groups*

$$0 \longrightarrow \bigoplus_v \Pi_v^{\mathrm{ab/edge}} \longrightarrow \Pi_{\mathcal{G}_X}^{\mathrm{ab/edge}} \longrightarrow \left(\pi_1^{\mathrm{top}}(\mathbb{G}_X)^{\wedge}\right)^{\mathrm{ab}} \longrightarrow 0$$

— where we write "$(-)^{\mathrm{ab/edge}}$" for the quotient of "$(-)$" defined in [6, *Definition* 1.3, (i), (iii)].

(iv) *Write*

$$\Pi_{\mathcal{G}_X}^{\mathrm{ab/edge}} \longrightarrow\!\!\!\!\!\rightarrow M$$

for the quotient of the abelian profinite group $\Pi_{\mathcal{G}_X}^{\mathrm{ab/edge}}$ by the [necessarily normal closed] subgroup generated by the kernels of the resulting [cf. (ii)] surjective continuous homomorphisms $\Pi_v^{\mathrm{ab/edge}} \twoheadrightarrow (\pi_1^{\mathrm{top}}(\mathbb{G}_{X_v})^{\wedge})^{\mathrm{ab}}$ [cf. (iii)] — where v ranges over the vertices of $\mathbb{G}_X$. [Thus, we have a commutative diagram of abelian profinite groups

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus_v \Pi_v^{\mathrm{ab/edge}} & \longrightarrow & \Pi_{\mathcal{G}_X}^{\mathrm{ab/edge}} & \longrightarrow & (\pi_1^{\mathrm{top}}(\mathbb{G}_X)^{\wedge})^{\mathrm{ab}} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \bigoplus_v (\pi_1^{\mathrm{top}}(\mathbb{G}_{X_v})^{\wedge})^{\mathrm{ab}} & \longrightarrow & M & \longrightarrow & (\pi_1^{\mathrm{top}}(\mathbb{G}_X)^{\wedge})^{\mathrm{ab}} & \longrightarrow & 0
\end{array}
$$

— where the horizontal sequences are **exact**, *and the vertical arrows are* **surjective**.] *Then there exist a* **finitely generated free** $\mathbb{Z}$-*module M_0 equipped with a continuous action of* $\mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K})$ *and a* **$\mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K})$-equivariant** *continuous isomorphism $M_0 \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}} \xrightarrow{\sim} M$.*

(v) *Write*

$$\Pi_{\mathcal{G}_X}^{\mathrm{ab}} \longrightarrow\!\!\!\!\!\rightarrow Q$$

for the **maximal G-stable torsion-free quotient** *of the abelian profinite group $\Pi_{\mathcal{G}_X}^{\mathrm{ab}}$ on which the resulting action of G is* **trivial**. *Then this surjective continuous homomorphism $\Pi_{\mathcal{G}_X}^{\mathrm{ab}} \twoheadrightarrow Q$* **factors** *through the surjective continuous homomorphism $\Pi_{\mathcal{G}_X}^{\mathrm{ab}} \twoheadrightarrow M$ of (iv).*

(vi) *The quotient Q of $\Pi_{\mathcal{G}_X}^{\mathrm{ab}}$ of (v) has a natural structure of* **free $\widehat{\mathbb{Z}}$-module**.

PROOF. — Assertion (i) follows immediately from the various definitions involved, together with the *commensurable terminality* of Π_v in $\Pi_{\mathcal{G}_X}$ [cf. [12, Proposition 1.2, (ii)]]. Assertion (ii) is immediate. Assertion (iii) follows from [6, Lemma 1.4].

Next, we verify assertion (iv). Let us first observe that, by associating,

• to a vertex v of $\mathbb{G}_X$, the connected temperoid [cf. [11, Definition 3.1, (ii)]] determined by the category of topological coverings with countably many connected components of the topological space associated to the [necessarily finite connected] semi-graph $\mathbb{G}_{X_v}$ and,

• to an edge of $\mathbb{G}_X$, the connected temperoid $\mathcal{B}^{\text{temp}}(\{1\})$ [cf. the discussion at the beginning of [11, §3]],

we obtain a connected semi-graph $\underline{\mathcal{G}}_X$ of temperoids [cf. [8, Definition 2.1, (i)]] whose underlying semi-graph is given by $\mathbb{G}_X$. Then one verifies immediately from the various definitions involved that the category $\mathcal{B}^{\text{tp}}(\underline{\mathcal{G}}_X)$ defined in [8, Definition 2.1, (vi)] is a *connected temperoid*, and, moreover, the abelian profinite group M may be naturally identified with the profinite completion of the abelianization of the tempered fundamental group [cf. [11, Remark 3.2.1]] of this connected temperoid $\mathcal{B}^{\text{tp}}(\underline{\mathcal{G}}_X)$ [cf. also the proof of [6, Lemma 1.4]]. In particular, we conclude that the abelianization M_0 of the tempered fundamental group of the connected temperoid $\mathcal{B}^{\text{tp}}(\underline{\mathcal{G}}_X)$, equipped with the natural action of $\text{Gal}(\overline{\mathbb{K}}/\mathbb{K})$ [i.e., induced by the natural actions of $\text{Gal}(\overline{\mathbb{K}}/\mathbb{K})$ on the semi-graphs $\mathbb{G}_X$ and $\mathbb{G}_{X_v}$'s — where v ranges over the vertices of $\mathbb{G}_X$], satisfies the desired condition. This completes the proof of assertion (iv).

Next, we verify assertion (v). Let us first observe that since [one verifies easily that] there is *no nontrivial G-stable torsion-free quotient* of the abelian profinite group $\widehat{\mathbb{Z}}(1)$ — where the "(1)" denotes a "Tate twist" — on which the resulting action of G is *trivial*, the natural surjective continuous homomorphism $\Pi_{\mathcal{G}_X}^{\text{ab}} \twoheadrightarrow Q$ *factors* through the natural surjective continuous homomorphism $\Pi_{\mathcal{G}_X}^{\text{ab}} \twoheadrightarrow \Pi_{\mathcal{G}_X}^{\text{ab/edge}}$. Thus, it follows from assertion (i) and Lemma 6.2 that the natural surjective continuous homomorphism $\Pi_{\mathcal{G}_X}^{\text{ab}} \twoheadrightarrow Q$ *factors* through the natural surjective continuous homomorphism $\Pi_{\mathcal{G}_X}^{\text{ab}} \twoheadrightarrow (\Pi_{\mathcal{G}_X}^{\text{ab/edge}} \twoheadrightarrow) M$, as desired. This completes the proof of assertion (v). Assertion (vi) follows from assertions (iv), (v), together with the [well-known] *flatness* of $\widehat{\mathbb{Z}}$ over $\mathbb{Z}$. This completes the proof of Lemma 6.4. □

THEOREM 6.5. — *In the situation of Lemma* 6.4, *the following assertions hold:*

(i) *The profinite group Π_G is a* **hyperbolic MLF-curvoid** *whose* **geometric subgroup** *is given by $\Pi_{\mathcal{G}_X} \subseteq \Pi_G$. Moreover, the normal closed subgroup $\Pi_{\mathcal{G}_X} \subseteq \Pi_G$ of Π_G is* **pseudo-MLF-geometric**.

(ii) *For a closed subgroup of $\Pi_{\mathcal{G}_X}$, the following three conditions are equivalent:*

(1) *The closed subgroup is an* **inertia subgroup** *of $\pi_1(X \times_{\mathbb{K}} \overline{\mathbb{K}}) = \Pi_{\mathcal{G}_X}$ associated to a cusp of the hyperbolic curve X over $\mathbb{K}$.*

(2) *The closed subgroup is a* **cuspidal subgroup** *of $\Pi_{\mathcal{G}_X}$ associated to a cusp of $\mathcal{G}_X$.*

(3) *The closed subgroup is a* **cuspidal inertia subgroup** *in $\Delta(\Pi_G) = \Pi_{\mathcal{G}_X}$ of the hyperbolic MLF-curvoid Π_G [cf. (i)].*

(iii) *Write $\Pi_{(\mathcal{G}_X)_{\leadsto \text{Node}(\mathcal{G}_X)}}$ for the PSC-fundamental group of the semi-graph $(\mathcal{G}_X)_{\leadsto \text{Node}(\mathcal{G}_X)}$ of anabelioids of pro-$\mathfrak{Primes}$ PSC-type defined in [7, Definition 2.8] [cf. also [6, Definition 1.1, (i)]] and $\Pi_{\mathcal{G}(\Pi_G)}$ for the PSC-fundamental group of the semi-graph $\mathcal{G}(\Pi_G)$ of anabelioids of pro-$\mathfrak{Primes}$ PSC-type of (1) of Definition 2.10, (iii) [i.e., in the case where we take the "Π" of Definition 2.10 to be Π_G — cf. (i)]. Then the composite of the natural outer continuous isomorphism*

$$\Pi_{(\mathcal{G}_X)_{\leadsto \text{Node}(\mathcal{G}_X)}} \overset{\sim}{\longrightarrow} \Pi_{\mathcal{G}_X}$$

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of [7, Proposition 2.9, (i)] *and the outer continuous isomorphism*

$$\Pi_{\mathcal{G}_X} = \Delta(\Pi_G) \overset{\sim}{\longrightarrow} \Pi_{\mathcal{G}(\Pi_G)}$$

[*cf.* (i)] *of* (2) *of Definition* 2.10, (iii) [*i.e., in the case where we take the "*Π*" of Definition* 2.10 *to be* Π_G — *cf.* (i)], *is* **graphic** [*cf.* [12, Definition 1.4, (i)]], *i.e., arises from an isomorphism of semi-graphs of anabelioids*

$$(\mathcal{G}_X)_{\rightsquigarrow \mathrm{Node}(\mathcal{G}_X)} \overset{\sim}{\longrightarrow} \mathcal{G}(\Pi_G).$$

(iv) *Suppose that the hyperbolic curve* X *over* $\mathbb{K}$ *is of type* (g, r). *Then the* **equality** $(g(\Pi_G), r(\Pi_G)) = (g, r)$ [*cf.* (i)] *holds.*

PROOF. — First, we verify assertion (i). Let us first observe that it follows from the exact sequence preceding assertion (i) of Lemma 6.4 that the normal closed subgroup $\Pi_{\mathcal{G}_X} \subseteq \Pi_G$ of Π_G is *of co-MLF-type*. Next, let us observe that it follows from [12, Remark 1.1.3] that the profinite group $\Pi_{\mathcal{G}_X}$ is *topologically finitely generated*. Moreover, it follows immediately from Lemma 6.4, (vi), that the normal closed subgroup $\Pi_{\mathcal{G}_X} \subseteq \Pi_G$ of Π_G satisfies condition (2) of Definition 1.5, (ii). In particular, one concludes that

(a) the normal closed subgroup $\Pi_{\mathcal{G}_X} \subseteq \Pi_G$ of Π_G is *pseudo-MLF-geometric*, hence also satisfies conditions (1), (2) of Definition 2.1.

Next, let us observe that one verifies immediately that the composite

$$\mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K}) = \pi_1(X)/\pi_1(X \times_{\mathbb{K}} \overline{\mathbb{K}}) \longrightarrow \mathrm{Out}\big(\pi_1(X \times_{\mathbb{K}} \overline{\mathbb{K}})\big) \overset{\sim}{\longrightarrow} \mathrm{Out}(\Pi_{\mathcal{G}_X})$$

— where the "=" is the equality that arises from the exact sequence of the first line of the final display of Definition 6.3, (i); the first arrow is the outer continuous action by conjugation; the second arrow is the isomorphism obtained by conjugation by a continuous isomorphism $\Pi_{\mathcal{G}_X} \overset{\sim}{\to} \pi_1(X \times_{\mathbb{K}} \overline{\mathbb{K}})$ that lifts the $\pi_1(X)$-conjugacy class of continuous isomorphisms of the sixth display of Definition 6.3, (ii) — *factors* through the closed subgroup $\mathrm{Aut}(\mathcal{G}_X) \subseteq \mathrm{Out}(\Pi_{\mathcal{G}_X})$ [*cf.* condition (3) of Definition 2.1]; moreover, one also verifies immediately from [7, Corollary 3.9, (ii), (iii)] that, for each prime number l, the continuous character $\mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K}) \to \mathbb{Z}_l^\times$ obtained by forming the composite of the resulting homomorphism $\mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K}) \to \mathrm{Aut}(\mathcal{G}_X)$ and the pro-l cyclotomic character $\mathrm{Aut}(\mathcal{G}_X) \to \mathbb{Z}_l^\times$ of [12, Lemma 2.1] *coincides* with the l-adic cyclotomic character of $\mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K})$. Thus, one concludes from [4, Theorem 1.4, (iv)] [*cf.* also (a)] that assertion (i) holds. This completes the proof of assertion (i).

Assertion (ii) follows immediately from the argument of the second paragraph of the proof of assertion (i), together with Lemma 2.7, (iv). Assertion (iii) follows immediately, in light of [12, Proposition 1.5, (ii)], from assertion (ii), condition (c) of Definition 2.10, (iii), and condition (1) of [7, Proposition 2.9, (i)].

Finally, we verify assertion (iv). The equality $r(\Pi_G) = r$ follows from assertion (ii) and [12, Proposition 1.2, (i)]. Thus, the equality $g(\Pi_G) = g$ follows from the well-known structure of the étale fundamental groups of hyperbolic curves over algebraically closed fields of characteristic zero. This completes the proof of assertion (iv), hence also of Theorem 6.5. □

THEOREM 6.6. — *Let* F *be an NF,* $\overline{F}$ *an algebraic closure of* F, V *a normal variety over* F [*cf.* [2, Definition 1.4]], *and* X *a* **hyperbolic curve** *over* V. *Write* $\pi_1(X)$, $\pi_1(V)$,

$\pi_1(V \times_F \overline{F})$ for the respective étale fundamental groups of X, V, $V \times_F \overline{F}$ [relative to some choices of basepoints]. Thus, we have an exact sequence of profinite groups

$$1 \longrightarrow \pi_1(V \times_F \overline{F}) \longrightarrow \pi_1(V) \longrightarrow \mathrm{Gal}(\overline{F}/F) \longrightarrow 1.$$

Moreover, the structure morphism $X \to V$ induces an outer **surjective** continuous homomorphism

$$\pi_1(X) \longrightarrow \pi_1(V)$$

[cf. [2, Proposition 2.4, (i)]], whose kernel we denote by

$$\Delta_{X/V}.$$

Let $s \colon \mathrm{Gal}(\overline{F}/F) \to \pi_1(V)$ be a **pro-$\mathfrak{Primes}$ Galois section** of V [cf. [3, Definition 5.2]]. Write

$$\Pi_s \overset{\text{def}}{=} \pi_1(X) \times_{\pi_1(V)} \mathrm{Gal}(\overline{F}/F)$$

for the fiber product of $\pi_1(X) \twoheadrightarrow \pi_1(V)$ and $s \colon \mathrm{Gal}(\overline{F}/F) \to \pi_1(V)$. Thus, we have an exact sequence of profinite groups

$$1 \longrightarrow \Delta_{X/V} \longrightarrow \Pi_s \longrightarrow \mathrm{Gal}(\overline{F}/F) \longrightarrow 1.$$

Then the following assertions hold:

(i) The profinite group Π_s is a **hyperbolic NF-curvoid** whose **geometric subgroup** is given by $\Delta_{X/V} \subseteq \Pi_s$. Moreover, the normal closed subgroup $\Delta_{X/V} \subseteq \Pi_s$ of Π_s is **pseudo-NF-geometric**.

(ii) For a closed subgroup of $\Delta_{X/V}$, it holds that the closed subgroup is an **inertia subgroup** associated to a cusp of a geometric fiber of the hyperbolic curve X over V [cf. the final portion of [2, Proposition 2.4, (i)]] if and only if the closed subgroup is a **cuspidal inertia subgroup** of the hyperbolic NF-curvoid Π_s [cf. (i)].

(iii) Suppose that the hyperbolic curve X over V is of type (g, r). Then the **equality** $(g(\Pi_s), r(\Pi_s)) = (g, r)$ [cf. (i)] holds.

(iv) Let D be an element of $\widetilde{\mathcal{V}}(G(\Pi_s))$ [cf. (i); Proposition 2.5, (ii); [4, Proposition 3.5, (1)]]. Suppose that V is a **smooth curve** over F, and that the pro-$\mathfrak{Primes}$ Galois section s of V is **geometric** at the nonarchimedean prime of F [cf. [3, Definition 5.11]] determined by $D \in \widetilde{\mathcal{V}}(G(\Pi_s))$ [cf. (i); [4, Proposition 3.5, (i)]]. Then the element $D \in \widetilde{\mathcal{V}}(G(\Pi_s))$ is **curvoidal** [cf. (i)].

PROOF. — First, we verify assertion (i). Let us first observe that it follows from the exact sequence preceding assertion (i) that the normal closed subgroup $\Delta_{X/V} \subseteq \Pi_s$ of Π_s is *of co-NF-type*. Next, let us observe that it follows from [2, Proposition 2.4, (iii)] that the profinite group $\Delta_{X/V}$ is *topologically finitely generated*. In particular, one concludes that

(a) the normal closed subgroup $\Delta_{X/V} \subseteq \Pi_s$ of Π_s is *pseudo-NF-geometric*, hence also satisfies conditions (1), (2) of Definition 2.1.

Next, write $\mathcal{G}_X$ for the semi-graph of anabelioids of pro-$\mathfrak{Primes}$ PSC-type determined by a fixed geometric fiber of the hyperbolic curve X over V [cf. [12, Definition 1.1, (i)]]

and $\Pi_{\mathcal{G}_X}$ for the PSC-fundamental group of $\mathcal{G}_X$. Thus, it follows from [2, Proposition 2.4, (i)] that there exists a natural $\pi_1(X)$-conjugacy class of continuous isomorphisms

$$\Pi_{\mathcal{G}_X} \overset{\sim}{\longrightarrow} \Delta_{X/V}.$$

Next, let us observe that one verifies immediately that the composite

$$\pi_1(V) = \pi_1(X)/\Delta_{X/V} \longrightarrow \mathrm{Out}(\Delta_{X/V}) \overset{\sim}{\longrightarrow} \mathrm{Out}(\Pi_{\mathcal{G}_X})$$

— where the "=" is the equality that arises from the definition of $\Delta_{X/V}$, the first arrow is the outer continuous action by conjugation, the second arrow is the isomorphism obtained by conjugation by a continuous isomorphism $\Pi_{\mathcal{G}_X} \overset{\sim}{\to} \Delta_{X/V}$ that lifts the above $\pi_1(X)$-conjugacy class of continuous isomorphisms — *factors* through the closed subgroup $\mathrm{Aut}(\mathcal{G}_X) \subseteq \mathrm{Out}(\Pi_{\mathcal{G}_X})$ [cf. condition (3) of Definition 2.1; moreover, one also verifies immediately from [7, Corollary 3.9, (ii), (iii)] that, for each prime number l, the continuous character $\pi_1(V) \to \mathbb{Z}_l^\times$ obtained by forming the composite of the resulting homomorphism $\pi_1(V) \to \mathrm{Aut}(\mathcal{G}_X)$ and the pro-l cyclotomic character $\mathrm{Aut}(\mathcal{G}_X) \to \mathbb{Z}_l^\times$ of [12, Lemma 2.1] *factors* through the quotient $\pi_1(V) \twoheadrightarrow \mathrm{Gal}(\overline{F}/F)$, and, moreover, the resulting character $\mathrm{Gal}(\overline{F}/F) \to \mathbb{Z}_l^\times$ *coincides* with the l-adic cyclotomic character of $\mathrm{Gal}(\overline{F}/F)$. Thus, one concludes from [4, Proposition 3.7, (iii)] [cf. also (a)] that assertion (i) holds. This completes the proof of assertion (i).

Assertion (ii) follows immediately from the argument of the second paragraph of the proof of assertion (i), together with Lemma 2.7, (iv). Next, we verify assertion (iii). The equality $r(\Pi_s) = r$ follows from assertion (ii) and [12, Proposition 1.2, (i)]. Thus, the equality $g(\Pi_s) = g$ follows from [2, Proposition 2.4, (i)], together with the well-known structure of the étale fundamental groups of hyperbolic curves over algebraically closed fields of characteristic zero. This completes the proof of assertion (iii). Assertion (iv) follows from assertion (i) and Theorem 6.5, (i). This completes the proof of Theorem 6.6. $\qquad\square$

COROLLARY 6.7. — *Let K be an MLF (respectively, NF), $\overline{K}$ an algebraic closure of K, and X a **hyperbolic orbicurve** over K. Write $\pi_1(X)$, $\pi_1(X \times_K \overline{K})$ for the respective étale fundamental groups of X, $X \times_K \overline{K}$ [relative to some choices of basepoints]. Thus, we have an exact sequence of profinite groups*

$$1 \longrightarrow \pi_1(X \times_K \overline{K}) \longrightarrow \pi_1(X) \longrightarrow \mathrm{Gal}(\overline{K}/K) \longrightarrow 1.$$

Then the following assertions hold:

(i) *The profinite group $\pi_1(X)$ is a **hyperbolic MLF-orbicurvoid** (respectively, **hyperbolic NF-orbicurvoid**) whose **geometric subgroup** is given by $\pi_1(X \times_K \overline{K}) \subseteq \pi_1(X)$.*

(ii) *Suppose that the hyperbolic orbicurve X over K is a **hyperbolic curve** over K. Then the hyperbolic MLF-orbicurvoid (respectively, hyperbolic NF-orbicurvoid) $\pi_1(X)$ [cf. (i)] is a **hyperbolic MLF-curvoid** (respectively, **hyperbolic NF-curvoid**). If, moreover, the hyperbolic curve X is of type (g, r), then the **equality** $(g(\pi_1(X)), r(\pi_1(X))) = (g, r)$ holds.*

(iii) *For a closed subgroup of $\pi_1(X \times_K \overline{K})$, it holds that the closed subgroup is an **inertia subgroup** associated to a cusp of the hyperbolic orbicurve X over K if and only*

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if the closed subgroup is a **cuspidal inertia subgroup** *of the hyperbolic MLF-orbicurvoid* (*respectively, hyperbolic NF-orbicurvoid*) $\pi_1(X)$ [*cf.* (i)].

(iv)   *It holds that the hyperbolic orbicurve X over K is a* **K-core** [*cf.* [10, Remark 2.1.1]] *if and only if the hyperbolic MLF-orbicurvoid* (*respectively, hyperbolic NF-orbicurvoid*) $\pi_1(X)$ [*cf.* (i)] *is* **relatively core-like**.

PROOF. — First, we verify assertion (i). Let us first observe that it follows from the exact sequence in the display in the statement of Corollary 6.7 that the normal closed subgroup $\pi_1(X \times_K \overline{K}) \subseteq \pi_1(X)$ of $\pi_1(X)$ is *of co-MLF-type* (respectively, *of co-NF-type*). Next, let us observe that it follows from [13, Proposition 2.3, (i)] that the profinite group $\pi_1(X \times_K \overline{K})$ is *slim*. In particular, one concludes that the normal closed subgroup $\pi_1(X \times_K \overline{K}) \subseteq \pi_1(X)$ of $\pi_1(X)$ satisfies condition (1) of Definition 3.1. Thus, one verifies immediately from Theorem 6.5, (i) (respectively, Theorem 6.6, (i)), that assertion (i) holds. This completes the proof of assertion (i).

The first portion of assertion (ii) follows from Theorem 6.5, (i) (respectively, Theorem 6.6, (i)). The final portion of assertion (ii) follows from Theorem 6.5, (iv) (respectively, Theorem 6.6, (iii)). Assertion (iii) follows from Theorem 6.5, (ii) (respectively, Theorem 6.6, (ii)), together with Proposition 3.8, (iii). Assertion (iv) follows immediately from [9, Theorem A] and [2, Proposition 3.2, (i)], together with the various definitions involved. This completes the proof of Corollary 6.7. $\square$

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