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On endomorphisms of projective toric varieties

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ABSTRACT. For any projective toric variety X defined over a field of characteristic zero, there exist non-isomorphic surjective endomorphisms $f: X \to X$ in which the characteristic completely invariant divisor is a proper subset of the boundary divisor, the complement of the open torus. One can construct also an equivariant version of such endomorphisms with respect to involutions of X preserving the boundary divisor.

1. INTRODUCTION

We fix a field $\Bbbk$ and consider toric varieties as algebraic $\Bbbk$-schemes. A toric variety admits many non-isomorphic surjective endomorphisms (as $\Bbbk$-morphisms). As a typical example, we have an endomorphism induced by the k-th power map $u \mapsto u^k$ of the open torus for $k > 1$. In this article, we first prove:

Theorem 1.1. *If a toric variety X is* complete*, then any surjective endomorphism $f: X \to X$ is a finite morphism.*

Here, X is said to be *complete* if the structure morphism $X \to \operatorname{Spec} \Bbbk$ is proper. Note that the finiteness of surjective endomorphism is well known in the case of projective varieties (cf. Lemma 3.2 below).

The complement D of the open torus in X is called the *boundary divisor*. In many known examples of surjective endomorphisms f of X, D is f*-completely invariant*, i.e., $f^{-1}D = D$ (cf. Definition 3.5 below). The main purpose of this article is to construct explicitly non-isomorphic surjective endomorphisms f of a complete toric variety X under which D is not completely invariant. This is done in Sections 4 below, and we can prove:

Theorem 1.2. *Let X be a complete toric variety with the boundary divisor D. Let B be a union of prime components Γ of D such that some multiple of Γ is linearly equivalent to an effective divisor not containing Γ. Then there exist a positive integer $k > 1$ and a non-isomorphic surjective endomorphism $f: X \to X$ satisfying the following conditions*:

(1) *For any divisor F on X, the inverse image f^*F is $\mathbb{Q}$-linearly equivalent to kF, i.e., $mf^*F \sim mkF$ for some $m > 0$.*
(2) *If Γ is a prime component of $D - B$, then $f^*\Gamma = k\Gamma$.*
(3) *If Γ is a prime component of B, then $f^{-1}\Gamma \neq \Gamma$.*

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(4) *The degree of* f *equals* k^r *for* $r = \dim X$.

Moreover, when $\operatorname{char} \Bbbk = 0$ *and* X *is projective, one can impose the following additional conditions*:

(5) *The ramification index of* f *along a prime divisor* C *is less than* k *if* $C \not\subset D - B$.
(6) *The* characteristic completely invariant divisor S_f *equals* $D - B$.

Note that the pullback f^*F of a divisor F is well defined, since f is finite by Theorem 1.1. The characteristic completely invariant divisor S_f is defined as the union of prime divisors Γ on X such that $(f^k)^*\Gamma = b\Gamma$ for some $k \geq 1$ and $b \geq 2$ (cf. [12, §2.4] and Definition 3.5 below). Note also that (4) is a consequence of (1) when X is projective. For equivariant versions of Theorem 1.2, we can pose:

Problem. Let X be a projective toric variety of characteristic zero admitting an action of a finite group G which preserves the boundary divisor D. Under what conditions, one can find a G-equivariant non-isomorphic surjective endomorphism f of X such that the characteristic completely invariant divisor S_f is a proper subset of D?

As a partial answer to the problem, we have:

Theorem 1.3. *Assume that* $\Bbbk$ *is algebraically closed. Let* X *be a complete toric variety with the boundary divisor* D *and let* $\iota\colon X \to X$ *be an involution such that* $\iota(D) = D$. *For a prime component* Γ *of* D *and for the divisor* $B := \Gamma \cup \iota(\Gamma)$, *assume that some multiple of* Γ *is linearly equivalent to an effective divisor supported on* $D - B$. *Then there exist a positive integer* $k > 1$ *and a non-isomorphic surjective endomorphism* $f\colon X \to X$ *such that* $\iota \circ f = f \circ \iota$ *and that conditions* (1), (2), (3), *and* (4) *of Theorem* 1.2 *are all satisfied. Moreover, when* $\operatorname{char} \Bbbk = 0$ *and* X *is projective, one can impose conditions* (5) *and* (6) *of Theorem* 1.2.

Theorems 1.2 and 1.3 above are applied to proving Theorems 6.1 and 6.2 below on endomorphisms of projective toric surfaces and *half-toric surfaces* (cf. [10, §7.1]) over $\Bbbk = \mathbb{C}$. These theorems supply examples of a normal projective surface X over $\mathbb{C}$ admitting a non-isomorphic surjective endomorphism f such that $K_X + S_f$ is not pseudo-effective (cf. [12], [13]).

Construction of this article. In Section 2, we fix and explain basic terminologies and notions for toric varieties. We discuss the finiteness of endomorphisms in Section 3, where Theorem 1.1 is proved in Section 3.1. The notions of ramification divisor and characteristic completely invariant divisor are explained in Section 3.2 on remarks on finite surjective (endo-)morphisms.

Our method of constructing endomorphisms of complete toric varieties is given in Section 4. We introduce in Section 4.1 the notion of *trigger for endomorphisms* (cf. Definition 4.1) generalizing the notion of *root* in the study of automorphisms of toric varieties in [2]. Some basic properties of triggers are obtained also in Section 4.1. We define special endomorphisms $\Xi = \Xi_{\alpha_1,\alpha_2,k}(P_1, P_2)$ of complete toric varieties in Section 4.2 for triggers α_1, α_2, polynomials $P_1(\mathtt{x})$, $P_2(\mathtt{x})$ in $\Bbbk[\mathtt{x}]$, and a positive integer k satisfying certain conditions (cf. Definition 4.10 and Proposition 4.13). The definition is analogous to that of the automorphism $x_\alpha(\lambda)$ in [2, §4, n°5, Thm. 3, p. 573] defined by a root α and a constant $\lambda \in \Bbbk$ (cf. [14, Prop. 3.14]). Some basic properties of the endomorphism Ξ are obtained in Section 4.3, and we shall prove Theorem 1.2 in Section 4.4.

We shall study automorphisms of a complete toric variety X preserving the boundary divisor D in Section 5.1. When the automorphism is an involution, in Section 5.2, we shall construct some non-isomorphic surjective endomorphisms equivariant under the involution by using the endomorphism defined by a trigger in Section 5.2. The proof of Theorem 1.3 is included in Section 5.2.

Applications to the study of non-isomorphic surjective endomorphisms of projective toric surfaces and half-toric surfaces over $\mathbb{C}$ are given in Section 6, where Theorems 6.1 and 6.2 are proved.

Notation and conventions. We work in the category of algebraic $\Bbbk$-schemes for a fixed ground field $\Bbbk$. A *variety* means an integral separated $\Bbbk$-scheme. A variety X is said to be *complete* if the structure morphism $X \to \operatorname{Spec} \Bbbk$ is proper. Note that a proper subvariety of a complete variety is not necessarily complete. We use the same notation and conventions as in [10], [11], [12], and [13]. For example:

- Two divisors D_1 and D_2 are is said to be $\mathbb{Q}$-*linearly equivalent* if mD_1 is linearly equivalent to mD_2 for some $m > 0$. We write $D_1 \sim_{\mathbb{Q}} D_2$ for the $\mathbb{Q}$-linear equivalence.
- The number of prime components of a reduced divisor D is denoted by $\boldsymbol{n}(D)$ (cf. [13, Def. 4.1]).
- For a commutative algebra R, the group of invertible elements of R is denoted by $R^\star$.

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2. Some basic terminologies on toric varieties

For details of toric varieties, we refer the readers to [2], [6], [1], [14], and [3]. We fix a non-zero free abelian group N of finite rank, the dual abelian group $\mathsf{M} = \operatorname{Hom}(\mathsf{N}, \mathbb{Z})$, and the canonical bilinear map $\langle\ ,\ \rangle \colon \mathsf{M} \times \mathsf{N} \to \mathbb{Z}$. We write $\mathsf{N}_\mathbb{R}$ (resp. $\mathsf{M}_\mathbb{R}$) for the finite-dimensional real vector space $\mathsf{N} \otimes \mathbb{R}$ (resp. $\mathsf{M} \otimes \mathbb{R}$), and use the same symbol $\langle\ ,\ \rangle$ for the induced bilinear map $\mathsf{M}_\mathbb{R} \times \mathsf{N}_\mathbb{R} \to \mathbb{R}$. Our specific notation on toric varieties is listed in Table 1.

A *rational polyhedral cone* of N is a closed convex cone in $\mathsf{N}_\mathbb{R}$ generated by finitely many elements of N. A closed convex cone $\boldsymbol{\sigma}$ of $\mathsf{N}_\mathbb{R}$ is said to be *strictly convex* if $\boldsymbol{\sigma} \cap (-\boldsymbol{\sigma}) = \{0\}$. A *face* of a closed convex cone $\boldsymbol{\sigma}$ is the cone expressed as $\{m\}^\perp \cap \boldsymbol{\sigma}$ for some $m \in \boldsymbol{\sigma}^\vee$. A *fan* $\triangle$ of N is a *finite* collection of strictly convex rational polyhedral cones of N such that

- if $\boldsymbol{\sigma} \in \triangle$, then any face of $\boldsymbol{\sigma}$ belongs to $\triangle$;
- for any cones $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}' \in \triangle$, $\boldsymbol{\sigma} \cap \boldsymbol{\sigma}'$ is a face of $\boldsymbol{\sigma}$.

Table 1. List of notations on toric varieties

$\mathcal{S}^\vee$	dual cone $\{x \in \mathsf{M}_\mathbb{R} \mid \langle x, \mathcal{S}\rangle \subset \mathbb{R}_{\geq 0}\}$ (resp. $\{y \in \mathsf{N}_\mathbb{R} \mid \langle \mathcal{S}, y\rangle \subset \mathbb{R}_{\geq 0}\}$) for a subset $\mathcal{S}$ of $\mathsf{N}_\mathbb{R}$ (resp. $\mathsf{M}_\mathbb{R}$).
$\mathcal{S}^\perp$	linear subspace $\{x \in \mathsf{M}_\mathbb{R} \mid \langle x, \mathcal{S}\rangle = 0\}$ (resp. $\{y \in \mathsf{N}_\mathbb{R} \mid \langle \mathcal{S}, y\rangle = 0\}$) for a subset $\mathcal{S}$ of $\mathsf{N}_\mathbb{R}$ (resp. $\mathsf{M}_\mathbb{R}$).
$\mathbb{T}_\mathsf{N}$	algebraic torus $\operatorname{Spec} \Bbbk[\mathsf{M}]$ associated with N.
$\mathbb{T}_\mathsf{N}(\boldsymbol{\sigma})$	affine toric variety $\operatorname{Spec} \Bbbk[\boldsymbol{\sigma}^\vee \cap \mathsf{M}]$ associated with a strictly convex rational polyhedral cone $\boldsymbol{\sigma}$ in $\mathsf{N}_\mathbb{R}$.
$\mathbb{T}_\mathsf{N}(\triangle)$	toric variety $\bigcup_{\boldsymbol{\sigma}\in\triangle} \mathbb{T}_\mathsf{N}(\boldsymbol{\sigma})$ associated with a fan $\triangle$ of N.
$\mathbb{B}_\mathsf{N}(\triangle)$	boundary divisor, the complement of the open torus $\mathbb{T}_\mathsf{N}(\{0\})$ in $\mathbb{T}_\mathsf{N}(\triangle)$.
$\mathbb{T}_\phi$	morphism $\mathbb{T}_\mathsf{N}(\triangle) \to \mathbb{T}_{\mathsf{N}'}(\triangle')$ of toric varieties associated with a morphism $\phi\colon (\mathsf{N}, \triangle) \to (\mathsf{N}', \triangle')$ of fans (cf. Definition 2.1).
$\boldsymbol{e}(m)$	element $m \in \mathsf{M}$ in the group ring $\Bbbk[\mathsf{M}]$.
$\mathcal{P}_\mathsf{N}(\triangle)$	set of primary vectors of $(\mathsf{N}, \triangle)$ (cf. Definition 2.2).
$\mathsf{R}(v)$	1-dimensional cone $\mathbb{R}_{\geq 0}v$ for a primary vector v (cf. Definition 2.2).
$\boldsymbol{\Gamma}(v)$	prime component of $\mathbb{B}_\mathsf{N}(\triangle)$ defined by a primary vector v (cf. Definition 2.2).
$\boldsymbol{\gamma}_v$	1-parameter subgroup of $\mathbb{G}_\mathtt{m} \to \mathbb{T}_\mathsf{N}$ corresponding to an element $v \in \mathsf{N}$ or its extension (cf. Definition 2.9).

The fan is not necessary finite in [14], but we assume the finiteness for simplicity. A fan $\triangle$ is said to be *complete* if $\bigcup_{\boldsymbol{\sigma}\in\triangle} \boldsymbol{\sigma} = \mathsf{N}_\mathbb{R}$. Note that any strictly convex rational polyhedral cone $\boldsymbol{\sigma}$ is identified with the fan consisting of all the faces of $\boldsymbol{\sigma}$.

Definition. For a strictly convex rational polyhedral cone $\boldsymbol{\sigma}$ in $\mathsf{N}_\mathbb{R}$, the *affine toric variety* $\mathbb{T}_\mathsf{N}(\boldsymbol{\sigma})$ over the field $\Bbbk$ is defined as $\operatorname{Spec} \Bbbk[\boldsymbol{\sigma}^\vee \cap \mathsf{M}]$ for the semi-group ring $\Bbbk[\boldsymbol{\sigma}^\vee \cap \mathsf{M}]$. If $\boldsymbol{\tau}$ is a face of $\boldsymbol{\sigma}$, then $\mathbb{T}_\mathsf{N}(\boldsymbol{\tau})$ is canonically an open subset of $\mathbb{T}_\mathsf{N}(\boldsymbol{\sigma})$. Moreover, for any fan $\triangle$ of N, affine toric varieties $\mathbb{T}_\mathsf{N}(\boldsymbol{\sigma})$ for all $\boldsymbol{\sigma} \in \triangle$ are glued to an algebraic scheme $\mathbb{T}_\mathsf{N}(\triangle)$ over $\Bbbk$, which is called the *toric variety* associated with $\triangle$. The common open subset $\mathbb{T}_\mathsf{N}(\{0\})$ of $\mathbb{T}_\mathsf{N}(\boldsymbol{\sigma})$ for all $\boldsymbol{\sigma} \in \triangle$ is isomorphic to the algebraic torus $\mathbb{T}_\mathsf{N} = \operatorname{Spec} \Bbbk[\mathsf{M}]$ and is called the *open torus*. The complement of the open torus in $\mathbb{T}_\mathsf{N}(\triangle)$ is denoted by $\mathbb{B}_\mathsf{N}(\triangle)$ and is called the *boundary divisor*. For an element $m \in \mathsf{M}$, we set $\boldsymbol{e}(m)$ to be m in the group ring $\Bbbk[\mathsf{M}]$. This is regarded as a rational function on $\mathbb{T}_\mathsf{N}(\triangle)$.

Remark. The toric variety $\mathbb{T}_\mathsf{N}(\triangle)$ is a normal integral separated $\Bbbk$-scheme of finite type, i.e., a normal algebraic variety over $\Bbbk$. The fan $\triangle$ is complete if and only if $\mathbb{T}_\mathsf{N}(\triangle)$ is complete (cf. [2, §4, n$^\text{o}$2, Prop. 4, p. 561], [6, I, Thm. 8], [14, Thm. 1.11]). In [14], $\mathbb{T}_\mathsf{N}(\triangle)$ is denoted by $T_\mathsf{N}\operatorname{emb}(\triangle)$.

Remark. The group law $\mathbb{T}_\mathsf{N} \times \mathbb{T}_\mathsf{N} \to \mathbb{T}_\mathsf{N}$ of the algebraic torus $\mathbb{T}_\mathsf{N}$ corresponds to the $\Bbbk$-algebra homomorphism $\mu^*\colon \Bbbk[\mathsf{M}] \to \Bbbk[\mathsf{M}] \otimes_\Bbbk \Bbbk[\mathsf{M}]$ given by $\mu^*\boldsymbol{e}(m) = \boldsymbol{e}(m) \otimes \boldsymbol{e}(m)$ for $m \in \mathsf{M}$. The group law extends to an action of $\mathbb{T}_\mathsf{N}$ on $\mathbb{T}_\mathsf{N}(\triangle)$ so that the open immersion $\mathbb{T}_\mathsf{N} = \mathbb{T}_\mathsf{N}(\{0\}) \hookrightarrow \mathbb{T}_\mathsf{N}(\triangle)$ is equivariant under the action. In particular, the open torus is a unique open dense orbit of $\mathbb{T}_\mathsf{N}$ in $\mathbb{T}_\mathsf{N}(\triangle)$.

Definition 2.1. Let N' be another free abelian group of finite rank and let $\phi\colon \mathsf{N} \to \mathsf{N}'$ be a homomorphism of abelian groups. The dual of ϕ is denoted by $\phi^\vee$, which is a homomorphism $\mathsf{M}' = \mathrm{Hom}(\mathsf{N}', \mathbb{Z}) \to \mathsf{M} = \mathrm{Hom}(\mathsf{N}, \mathbb{Z})$ satisfying $\langle \phi^\vee(m'), n\rangle = \langle m', \phi(n)\rangle$ for any $m' \in \mathsf{M}'$ and $n \in \mathsf{N}$. The morphism $\mathbb{T}_\phi\colon \mathbb{T}_{\mathsf{N}} \to \mathbb{T}_{\mathsf{N}'}$ of algebraic tori associated with ϕ is defined by the $\Bbbk$-algebra homomorphism $\Bbbk[\mathsf{M}'] \to \Bbbk[\mathsf{M}]$ sending $\boldsymbol{e}(m')$ to $\boldsymbol{e}(\phi^\vee(m'))$. Let $\triangle'$ be a fan of N' and assume that, for any $\boldsymbol{\sigma} \in \triangle$, the image $\phi_{\mathbb{R}}(\boldsymbol{\sigma})$ under $\phi_{\mathbb{R}} = \phi \otimes \mathbb{R}\colon \mathsf{N}_{\mathbb{R}} \to \mathsf{N}'_{\mathbb{R}}$ is contained in some cone $\boldsymbol{\sigma}' \in \triangle'$. In this case, $\phi\colon (\mathsf{N}, \triangle) \to (\mathsf{N}', \triangle')$ is called a *morphism of fans*, and $\mathbb{T}_\phi$ extends to a morphism $\mathbb{T}_{\mathsf{N}}(\triangle) \to \mathbb{T}_{\mathsf{N}'}(\triangle')$ equivariant under the actions of $\mathbb{T}_{\mathsf{N}}$ and $\mathbb{T}_{\mathsf{N}'}$ with respect to $\mathbb{T}_\phi$ (cf. [14, Thm. 1.13]). This extended morphism is also denoted by $\mathbb{T}_\phi$.

Definition 2.2. Let $\triangle$ be a fan of N. A *primary vector* of $\triangle$ with respect to N, or a primary vector of $(\mathsf{N}, \triangle)$, is defined as a primitive element of N generating a 1-dimensional cone belonging to $\triangle$. The set of primary vectors is denoted by $\mathcal{P}_{\mathsf{N}}(\triangle)$. Here, $\mathcal{P}_{\mathsf{N}}(\boldsymbol{\sigma}) = \mathcal{P}_{\mathsf{N}}(\triangle) \cap \boldsymbol{\sigma}$ for any $\boldsymbol{\sigma} \in \triangle$. For a primary vector v, $\mathsf{R}(v)$ denotes the 1-dimensional cone $\mathbb{R}_{\geq 0}v$, and the prime divisor $\boldsymbol{\Gamma}(v)$ on $\mathbb{T}_{\mathsf{N}}(\triangle)$ is defined as the complement of

$$\bigcup\nolimits_{\boldsymbol{\sigma} \in \triangle,\, v \notin \boldsymbol{\sigma}} \mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma}).$$

In particular, $\boldsymbol{\Gamma}(v)$ is the closure of $\mathbb{T}_{\mathsf{N}}(\mathsf{R}(v)) \setminus \mathbb{T}_{\mathsf{N}}(\{0\})$ in $\mathbb{T}_{\mathsf{N}}(\triangle)$.

Remark. A complete fan $\triangle$ of N is determined by $\mathcal{P}_{\mathsf{N}}(\triangle)$ when $\operatorname{rank} \mathsf{N} = 2$. In fact, for $l := \#\mathcal{P}_{\mathsf{N}}(\triangle)$, there is a bijection $\mathbb{Z}/l\mathbb{Z} \ni i \mapsto v_i \in \mathcal{P}_{\mathsf{N}}(\triangle)$ such that the set of 2-dimensional cones belonging to $\triangle$ is $\{\mathbb{R}_{\geq 0}v_i + \mathbb{R}_{\geq 0}v_{i+1} \mid i \in \mathbb{Z}/l\mathbb{Z}\}$ (cf. [10, Exam. 3.4]).

Remark. The boundary divisor $\mathbb{B}_{\mathsf{N}}(\triangle)$ is just the union of $\mathbb{T}_{\mathsf{N}}$-invariant prime divisors, and hence,

$$\mathbb{B}_{\mathsf{N}}(\triangle) = \sum\nolimits_{v \in \mathcal{P}_{\mathsf{N}}(\triangle)} \boldsymbol{\Gamma}(v).$$

Remark 2.3. For any $m \in \mathsf{M}$, the principal divisor associated with the rational function $\boldsymbol{e}(m)$ on $\mathbb{T}_{\mathsf{N}}(\triangle)$ is written as

$$\mathrm{div}(\boldsymbol{e}(m)) = \sum\nolimits_{v \in \mathcal{P}_{\mathsf{N}}(\triangle)} \langle m, v\rangle \boldsymbol{\Gamma}(v) \tag{II-1}$$

by [3, §3.3, Lemma], which is a special case of the equality in [14, Prop. 2.1(ii)]. Let $\mathcal{I}$ be the ideal sheaf $\mathcal{O}_{\mathbb{T}_{\mathsf{N}}(\triangle)}(-B)$ of an effective divisor

$$B = \sum\nolimits_{v \in \mathcal{P}_{\mathsf{N}}(\triangle)} b_v \boldsymbol{\Gamma}(v)$$

defined by integers $b_v \geq 0$. By (II-1), we see that, for any cone $\boldsymbol{\sigma} \in \triangle$, $\mathcal{I}|_{\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})}$ is generated by $\boldsymbol{e}(m)$ for all $m \in \mathsf{M}$ such that $\langle m, v\rangle \geq b_v$ for any $v \in \mathcal{P}_{\mathsf{N}}(\boldsymbol{\sigma})$.

Remark 2.4. Similarly to $\boldsymbol{e}(m)$, every element $Q \in \Bbbk[\mathsf{M}]$ is regarded as a rational function on $\mathbb{T}_{\mathsf{N}}(\triangle)$ which is regular on the open torus. Assume that $Q \neq 0$. Then we can consider the associated principal divisor $\mathrm{div}(Q)$ on $\mathbb{T}_{\mathsf{N}}(\triangle)$. For a cone $\boldsymbol{\sigma} \in \triangle$, if $\mathrm{mult}_{\boldsymbol{\Gamma}(v)} \mathrm{div}(Q) \geq 0$ for any $v \in \mathcal{P}_{\mathsf{N}}(\boldsymbol{\sigma})$, then $Q \in \Bbbk[\boldsymbol{\sigma}^\vee \cap \mathsf{M}]$. In fact, in this case, $\mathrm{mult}_\Gamma \mathrm{div}(Q) \geq 0$ for any prime divisor Γ on the affine toric variety

$\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$, and hence, Q belongs to the affine coordinate ring $\Bbbk[\boldsymbol{\sigma}^{\vee} \cap \mathsf{M}]$, since it is normal.

Lemma 2.5. *Let $P(\mathtt{x})$ be a polynomial in $\Bbbk[\mathtt{x}]$ for a variable $\mathtt{x}$ such that $P(0) \neq 0$. Then, for any $v \in \mathcal{P}_{\mathsf{N}}(\triangle)$ and $\alpha \in \mathsf{M}$, one has*

$$\text{(II-2)} \qquad \operatorname{mult}_{\boldsymbol{\Gamma}(v)} \operatorname{div}(P(\boldsymbol{e}(\alpha))) = \min\{0, (\deg P)\langle \alpha, v\rangle\},$$

where we regard $P(\boldsymbol{e}(\alpha))$ as a rational function on $\mathbb{T}_{\mathsf{N}}(\triangle)$.

Proof. The assertion holds if $\deg P = 0$, since P is a constant and $\operatorname{div}(P(\boldsymbol{e}(\alpha))) = 0$. Thus, we may assume that $d := \deg P > 0$. For the expansion $P(\mathtt{x}) = \sum_{s=0}^{d} p_s \mathtt{x}^s$ with $p_s \in \Bbbk$, we have $p_0 \neq 0$ and $p_d \neq 0$. If $\langle \alpha, v\rangle > 0$, then $\boldsymbol{e}(\alpha)$ is regular on $\mathbb{T}_{\mathsf{N}}(\mathsf{R}(v))$ and vanishing along $\boldsymbol{\Gamma}(v)$ by (II-1) in Remark 2.3; hence, $P(\boldsymbol{e}(\alpha))$ is regular on $\mathbb{T}_{\mathsf{N}}(\mathsf{R}(v))$ and not vanishing along $\boldsymbol{\Gamma}(v)$ by $P(0) \neq 0$. Therefore, (II-2) holds when $\langle \alpha, v\rangle > 0$.

Assume that $\langle \alpha, v\rangle < 0$. Let us consider another polynomial

$$P^{\circ}(\mathtt{x}) := \mathtt{x}^d P(\mathtt{x}^{-1}) = \sum_{s=0}^{d} p_{d-s}\mathtt{x}^s \in \Bbbk[\mathtt{x}].$$

Note that $P^{\circ}(0) \neq 0$ and $P(\boldsymbol{e}(\alpha)) = \boldsymbol{e}(\alpha)^d P^{\circ}(\boldsymbol{e}(-\alpha))$. Since $\langle -\alpha, v\rangle > 0$, we have

$$\operatorname{mult}_{\boldsymbol{\Gamma}(v)} \operatorname{div}(P(\boldsymbol{e}(\alpha))) = \operatorname{mult}_{\boldsymbol{\Gamma}(v)} \operatorname{div}(\boldsymbol{e}(\alpha)^d) + \operatorname{mult}_{\boldsymbol{\Gamma}(v)} \operatorname{div}(P^{\circ}(\boldsymbol{e}(-\alpha))) = d\langle \alpha, v\rangle$$

by (II-1) in Remark 2.3 and by (II-2) in the case where $\langle \alpha, v\rangle > 0$. Thus, (II-2) holds when $\langle \alpha, v\rangle < 0$.

Finally, assume that $\langle \alpha, v\rangle = 0$. We can regard α as a homomorphism $\mathsf{N} \to \mathbb{Z}$, and now, it descends to a non-zero homomorphism $\bar{\alpha}\colon \mathsf{N}(v) := \mathsf{N}/\mathbb{Z}v \to \mathbb{Z}$. Let $\mathbb{T}_{\bar{\alpha}}\colon \mathbb{T}_{\mathsf{N}(v)} \to \mathbb{T}_{\mathbb{Z}} = \mathbb{G}_{\mathtt{m}}$ be the homomorphism of algebraic torus induced by $\bar{\alpha}$, where $\mathbb{G}_{\mathtt{m}}$ stands for the 1-dimensional torus $\operatorname{Spec} \Bbbk[\mathtt{t}, \mathtt{t}^{-1}]$. Then $\mathbb{T}_{\bar{\alpha}}$ is dominant, and the induced morphism

$$\boldsymbol{\Gamma}(v) \cap \mathbb{T}_{\mathsf{N}}(\mathsf{R}(v)) \simeq \mathbb{T}_{\mathsf{N}(v)} \xrightarrow{\mathbb{T}_{\bar{\alpha}}} \mathbb{G}_{\mathtt{m}}$$

is just the restriction of the rational function $\boldsymbol{e}(\alpha)\colon \mathbb{T}_{\mathsf{N}}(\triangle) \dashrightarrow \mathbb{P}^1$. Since $\deg P > 0$, $\boldsymbol{\Gamma}(v) \cap \mathbb{T}_{\mathsf{N}}(\mathsf{R}(v))$ dominates $\mathbb{P}^1$ by the rational function $P(\boldsymbol{e}(\alpha))\colon \mathbb{T}_{\mathsf{N}}(\triangle) \dashrightarrow \mathbb{P}^1$. Hence, $\operatorname{mult}_{\boldsymbol{\Gamma}(v)} \operatorname{div}(P(\boldsymbol{e}(\alpha))) = 0$, and (II-2) holds. Thus, we are done. □

Corollary 2.6. *Let r be a positive integer. For variables $\mathtt{t}_1, \dots, \mathtt{t}_r$, we set*

$$A = \Bbbk[\mathtt{t}_1^{\pm 1}, \mathtt{t}_2^{\pm 1}, \dots, \mathtt{t}_r^{\pm 1}] \quad \textit{and} \quad A_i = \Bbbk[\mathtt{t}_1^{\pm 1}, \dots, \mathtt{t}_{i-1}^{\pm 1}, \mathtt{t}_i, \mathtt{t}_{i+1}^{\pm 1}, \dots, \mathtt{t}_r^{\pm 1}]$$

as $\Bbbk$-algebras, where $1 \leq i \leq r$. Let $\alpha = \mathtt{t}_1^{a_1}\mathtt{t}_2^{a_2}\cdots\mathtt{t}_r^{a_r}$ be a monomial in A defined by integers $a_1, \dots, a_r$, and let $P(\mathtt{x})$ be a polynomial in $\Bbbk[\mathtt{x}]$ of degree d such that $P(0) \neq 0$. Then

$$\max\{q \in \mathbb{Z} \mid P(\alpha) \in \mathtt{t}_i^q A_i\} = \min\{0, da_i\}$$

for any $1 \leq i \leq r$.

Proof. The polynomial ring $\widetilde{A} = \Bbbk[\mathtt{t}_1, \mathtt{t}_2, \dots, \mathtt{t}_r]$ is identified with the semi-group ring $\Bbbk[\boldsymbol{\sigma}^{\vee} \cap \mathsf{M}]$ for a free abelian group N of rank r with a free basis $(n_1, \dots, n_r)$, the cone $\boldsymbol{\sigma} = \sum_{i=1}^{r} \mathbb{R}_{\geq 0} n_i$, and $\mathsf{M} = \operatorname{Hom}(\mathsf{N}, \mathbb{Z})$. For the dual basis $(m_1, \dots, m_r)$ of $(n_1, \dots, n_r)$, we have $\mathtt{t}_i = \boldsymbol{e}(m_i)$ for any $1 \leq i \leq r$ under the identification. We apply Lemma 2.5 to the toric variety $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma}) = \operatorname{Spec} \widetilde{A}$. Here, $\mathcal{P}_{\mathsf{N}}(\boldsymbol{\sigma}) = \{n_1, \ldots, n_r\}$. Then $\operatorname{div}(\mathtt{t}_i) = \boldsymbol{\Gamma}(n_i)$, and A (resp. A_i) is identified with the affine coordinate ring of $\mathbb{T}_{\mathsf{N}}(\{0\})$ (resp. $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma}) \setminus \boldsymbol{\Gamma}(n_i)$). Moreover, $\alpha = \boldsymbol{e}(m_\alpha)$ for $m_\alpha = \sum a_i m_i$, where $\langle m_\alpha, n_i \rangle = a_i$ for any i. Hence,

$$\max\{q \in \mathbb{Z} \mid P(\alpha) \in \mathtt{t}_i^q A_i\} = \operatorname{mult}_{\boldsymbol{\Gamma}(n_i)} \operatorname{div}(P(\alpha)) = \min\{0, da_i\}$$

for any i by Lemma 2.5. □

Convention. For $\Bbbk$-schemes Y and T, the set $\operatorname{Hom}_{\Bbbk}(T, Y)$ of morphisms of $\Bbbk$-schemes is denoted by $Y\langle T\rangle$ and an element of it is called a *T-valued point* of Y (cf. [4, I, §3.4]). Note that the correspondence $T \mapsto Y\langle T\rangle$ is a presheaf on the category of $\Bbbk$-schemes. When $T = \operatorname{Spec} R$ for a $\Bbbk$-algebra R, T-valued points of Y are called R-valued points, and $Y\langle T\rangle$ is written as $Y\langle R\rangle$, for simplicity.

Definition 2.7. Let u be a T-valued point of $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$ for a $\Bbbk$-scheme T and for a strictly convex rational polyhedral cone $\boldsymbol{\sigma}$ in $\mathsf{N}_{\mathbb{R}}$.

(1) For any $m \in \boldsymbol{\sigma}^\vee \cap \mathsf{M}$, we define $u(m)$ to be the image of $\boldsymbol{e}(m)$ under the associated $\Bbbk$-algebra homomorphism $u^*\colon \Bbbk[\boldsymbol{\sigma}^\vee \cap \mathsf{M}] \to H^0(T, \mathcal{O}_T)$.

(2) Let $\boldsymbol{\sigma}'$ be a strictly convex rational polyhedral cone in $\mathsf{N}'_{\mathbb{R}}$ for a free abelian group N', and let $\phi\colon \mathsf{N} \to \mathsf{N}'$ be a homomorphism of abelian groups such that $\phi_{\mathbb{R}}(\boldsymbol{\sigma}) \subset \boldsymbol{\sigma}'$. The image of u under $\mathbb{T}_\phi\langle T\rangle\colon \mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})\langle T\rangle \to \mathbb{T}_{\mathsf{N}'}(\boldsymbol{\sigma}')\langle T'\rangle$ is denoted by $\phi(u)$.

Remark. The set $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})\langle T\rangle$ of T-valued points is in one-to-one correspondence with the set of homomorphisms $\boldsymbol{\sigma}^\vee \cap \mathsf{M} \to R = (R, \times)$ of semi-groups for $R = H^0(T, \mathcal{O}_T)$. The correspondence is given by $u \mapsto (m \mapsto u(m))$. In particular, $\mathbb{T}_{\mathsf{N}}\langle T\rangle \simeq \mathbb{T}_{\mathsf{N}}\langle R\rangle \simeq \operatorname{Hom}(\mathsf{M}, R^\star)$ for the group $R^\star$ of invertible elements of R. For the homomorphism ϕ in (2) and its dual $\phi^\vee\colon \mathsf{M}' = \operatorname{Hom}(\mathsf{N}', \mathbb{Z}) \to \mathsf{M} = \operatorname{Hom}(\mathsf{N}, \mathbb{Z})$, the image $\phi(u)$ corresponds to the homomorphism $(\boldsymbol{\sigma}')^\vee \cap \mathsf{M}' \to R$ given by $m' \mapsto u(\phi^\vee m')$ for any $m' \in \mathsf{M}'$.

Remark 2.8. Let v be a primary vector of $(\mathsf{N}, \triangle)$. For a cone $\boldsymbol{\sigma} \in \triangle$ containing v, a T-valued point $u \in \mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})\langle T\rangle$ is contained in $\boldsymbol{\Gamma}(v)\langle T\rangle$ if and only if $u(m) = 0$ for any $m \in \boldsymbol{\sigma}^\vee \cap \mathsf{M}$ such that $\langle m, v\rangle > 0$. This follows from the description of the ideal sheaf of $\boldsymbol{\Gamma}(v)$ in Remark 2.3.

Definition 2.9. For $v \in \mathsf{N}$, let $j_v\colon \mathbb{Z} \to \mathsf{N}$ be the homomorphism sending 1 to v. The 1-*parameter subgroup* associated with v (cf. [6, I, §1]) is defined as the associated morphism $\boldsymbol{\gamma}_v := \mathbb{T}_{j_v}\colon \mathbb{G}_{\mathtt{m}} = \mathbb{T}_{\mathbb{Z}} \to \mathbb{T}_{\mathsf{N}}$ of algebraic tori (cf. Definition 2.1). Assume that $v \in \boldsymbol{\sigma}$ for a cone $\boldsymbol{\sigma}$ belonging to a fan $\triangle$ of N. Then j_v is a morphism $(\mathbb{Z}, \mathbb{R}_{\geq 0}) \to (\mathsf{M}, \triangle)$ of fans, and the associated morphism

$$\mathbb{T}_{j_v}\colon \mathbb{T}_{\mathbb{Z}}(\mathbb{R}_{\geq 0}) = \operatorname{Spec} \Bbbk[\mathtt{t}] \to \mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma}) \subset \mathbb{T}_{\mathsf{N}}(\triangle)$$

of toric varieties is an extension of $\boldsymbol{\gamma}_v$ above. This extended morphism is also denoted by $\boldsymbol{\gamma}_v$.

Remark. The toric variety $\mathbb{T}_{\mathbb{Z}}(\mathbb{R}_{\geq 0})$ is identified with the affine line $\operatorname{Spec} \Bbbk[\mathtt{t}]$ under which the open torus $\mathbb{T}_{\mathbb{Z}}(\{0\})$ is identified with $\operatorname{Spec} \Bbbk[\mathtt{t}, \mathtt{t}^{-1}]$, where $\mathtt{t} = \boldsymbol{e}(\mathtt{m})$ for the element $\mathtt{m}$ of $\mathbb{Z}^\vee = \mathrm{Hom}(\mathbb{Z},\mathbb{Z})$ corresponding to the identity homomorphism $\mathrm{id}_\mathbb{Z}$. For $v \in \mathsf{N}$, the 1-parameter subgroup $\boldsymbol{\gamma}_v \colon \mathbb{G}_\mathtt{m} = \mathrm{Spec}\,\Bbbk[\mathtt{t},\mathtt{t}^{-1}] \to \mathbb{T}_\mathsf{N}$ is defined by $\boldsymbol{\gamma}_v^* \boldsymbol{e}(m) = \mathtt{t}^{\langle m,v\rangle}$ for $m \in \mathsf{M}$. Assume that v is a primary vector of $(\mathsf{N},\triangle)$. Then

$$\mathbb{T}_\mathsf{N}(\mathsf{R}(v)) \simeq \mathbb{T}_{\mathbb{Z}v}(\mathsf{R}(v)) \times \mathbb{T}_{\mathsf{N}(v)}$$

for $\mathsf{N}(v) := \mathsf{N}/\mathbb{Z}v$, and $\mathbb{T}_\mathbb{Z}(\mathbb{R}_{\geq 0}) \simeq \mathbb{T}_{\mathbb{Z}v}(\mathsf{R}(v))$ by the isomorphism $j_v \colon \mathbb{Z} \to \mathbb{Z}v$. For the $\Bbbk$-valued point $\mathsf{0}$ of $\mathrm{Spec}\,\Bbbk[\mathtt{t}]$ defined by $\mathtt{t} = 0$, its image $\boldsymbol{\gamma}_v(\mathsf{0})$ under

$$\boldsymbol{\gamma}_v \colon \mathrm{Spec}\,\Bbbk[\mathtt{t}] \simeq \mathbb{T}_{\mathbb{Z}v}(\mathsf{R}(v)) \to \mathbb{T}_\mathsf{N}(\mathsf{R}(v)) \subset \mathbb{T}_\mathsf{N}(\triangle)$$

is considered as the "limit point" of the 1-parameter subgroup $\boldsymbol{\gamma}_v \colon \mathbb{G}_\mathtt{m} \to \mathbb{T}_\mathsf{N}$ (cf. [6, I, Thm. 1′]), and it corresponds to the $\Bbbk$-valued point $(\mathsf{0},\mathsf{e})$ of $\mathbb{T}_{\mathbb{Z}v}(\mathsf{R}(v)) \times \mathbb{T}_{\mathsf{N}(v)}$ for the unit element e of the group $\mathbb{T}_{\mathsf{N}(v)}\langle\Bbbk\rangle = \mathsf{N}(v) \otimes_\mathbb{Z} \Bbbk^\star$.

3. Finiteness of surjective endomorphisms

In Section 3.1, we shall prove Theorem 1.1 in the introduction, which states that every surjective endomorphism of a complete toric variety is finite. In Section 3.2, we shall give some remarks on finite morphisms and finite endomorphisms.

3.1. **Proof of Theorem 1.1.**

For the proof of Theorem 1.1, we need the following lemma, which seems to be well known:

Lemma 3.1. *Assume that* $\Bbbk$ *is algebraically closed. Let* $f \colon X \to Y$ *be a proper morphism of varieties such that* $\mathcal{O}_Y \simeq f_*\mathcal{O}_X$*. Let* G *be a connected algebraic group acting on* X*. Then the action of* G *descends to* Y *so that* f *is* G*-equivariant.*

Proof. Let C be a complete curve contained in a fiber of f and let us consider the composite

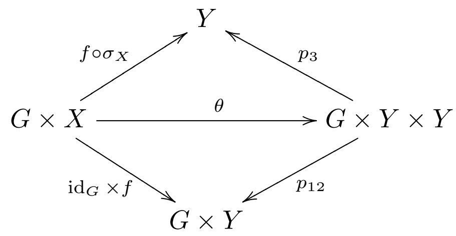
$$\eta \colon G \times C \xrightarrow{\mathrm{id}_G \times \iota} G \times X \xrightarrow{\sigma_X} X \xrightarrow{f} Y,$$

where $\iota \colon C \hookrightarrow X$ is the closed immersion and σ_X is the morphism of action of G. For the unit element e of G, the image of $\{e\} \times C$ under η is a point. We have a commutative digram

$$\begin{array}{ccc} G \times C & \xrightarrow{(p_G,\eta)} & G \times Y \\ & {\scriptstyle p_G}\searrow \quad \swarrow{\scriptstyle q_G} & \\ & G & \end{array}$$

for first projections $p_G \colon G \times C \to G$ and $q_G \colon G \times Y \to G$. By rigidity lemma [9, Prop. 6.1] applied to the diagram, there is a morphism $\zeta \colon G \to Y$ such that $\eta = \zeta \circ p_G$. As a consequence, $f(\sigma_X(\{g\} \times C)) = \eta(\{g\} \times C)$ is a point for any $g \in G$. Moreover, $f(\sigma_X(\{g\} \times f^{-1}(y)))$ is a point for any $y \in Y$ and $g \in G$, since $f^{-1}(y)$ is connected. Equivalently, every fiber of $\mathrm{id}_G \times f \colon G \times X \to G \times Y$ is mapped to a point by the proper morphism $\theta \colon G \times X \to G \times Y \times Y$ defined as

$(\mathrm{id}_G \times f, f \circ \sigma_X)$: there is a commutative diagram



for the projection p_{12} to the first and second factor and the projection p_3 to the third factor. For the Stein factorization $G \times X \to Z \to G \times Y \times Y$ of θ, the induced morphism $\phi\colon Z \to G \times Y \times Y \xrightarrow{p_{12}} G \times Y$ is finite and

$$\mathcal{O}_{G\times Y} \simeq (\mathrm{id}_G \times f)_* \mathcal{O}_{G\times X} \simeq \phi_* \mathcal{O}_Z.$$

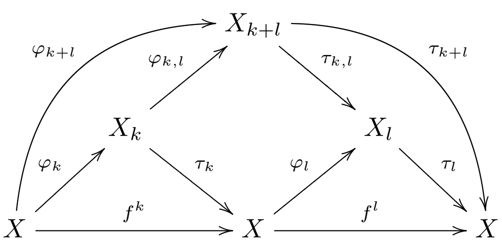
Hence, ϕ is an isomorphism. By ϕ^{-1} and by composing with p_3, we have a morphism $\sigma_Y : G \times Y \to Y$ such that $\sigma_Y \circ (\mathrm{id}_G \times f) = f \circ \sigma_X$. Therefore, σ_Y is a morphism of an action of G on Y, and f is G-equivariant. □

Proof of Theorem 1.1. By base change, we may assume that $\Bbbk$ is algebraically closed (cf. [5, Prop. (2.7.1)]). We may write $X = \mathbb{T}_{\mathsf{N}}(\triangle)$ for a complete fan $\triangle$ of a free abelian group N of finite rank. For the given surjective endomorphism $f\colon X \to X$, let $X \xrightarrow{\varphi} X' \xrightarrow{\tau} X$ be the Stein factorization of f. By Lemma 3.1, the action of $\mathbb{T}_{\mathsf{N}}$ on X descends to X' and φ is $\mathbb{T}_{\mathsf{N}}$-equivariant. Then the exceptional locus of φ is a union of orbits of $\mathbb{T}_{\mathsf{N}}$, and hence, it is contained in the boundary divisor $\mathbb{B}_{\mathsf{N}}(\triangle)$ of X. Thus, X' contains an open orbit of $\mathbb{T}_{\mathsf{N}}$, which is isomorphic to $\mathbb{T}_{\mathsf{N}}$. Therefore, X' is also a toric variety. By [6, I, Thms. 6 and 7], there is a complete fan $\triangle'$ of N such that

- $X' \simeq \mathbb{T}_{\mathsf{N}}(\triangle')$,
- each $\boldsymbol{\sigma} \in \triangle$ is contained in some $\boldsymbol{\sigma}' \in \triangle'$,
- φ is associated with the morphism $\mathrm{id}_{\mathsf{N}} : (\mathsf{N}, \triangle) \to (\mathsf{N}, \triangle')$ of fans.

In particular, we have an inequality $\#\triangle \geq \#\triangle'$ of cardinalities, where the equality holds if and only if φ is an isomorphism.

We shall derive a contradiction assuming that f is not finite. Then φ is not an isomorphism. For any integer $k \geq 1$, let $X \xrightarrow{\varphi_k} X_k \xrightarrow{\tau_k} X$ be the Stein factorization of f^k. Then $X_1 = X'$, $\varphi_1 = \varphi$, $\tau_1 = \tau$, and we have a commutative diagram



for any k, $l \geq 1$, where morphisms $\varphi_{k,l}$ and $\tau_{k,l}$, respectively, are birational and finite. By induction, φ_k and $\varphi_{k,l}$ are not isomorphisms for any k, $l \geq 1$. Thus, we have an infinite sequence $X \to X_1 \to \cdots \to X_k \to X_{k+1} \to \cdots$ of birational morphisms by φ_1 and $\varphi_{k,1}$. By the previous argument, $X_k \simeq \mathbb{T}_{\mathsf{N}}(\triangle_k)$ for a fan $\triangle_k$ of N, and we have an infinite sequence $\#\triangle > \#\triangle_1 > \#\triangle_2 > \cdots$, which contradicts the finiteness of the set $\triangle$. Thus, we are done. □

3.2. **Remarks on finite surjective (endo-)morphisms.** We shall explain the pullback of (Weil) divisors by a finite surjective morphism. In characteristic zero, we shall also explain the ramification formula for a finite surjective morphism and the characteristic completely invariant divisor for a finite endomorphism introduced in [12, §2.4].

Lemma 3.2. *Let X be a normal variety with a surjective endomorphism $f\colon X \to X$. If X is projective, then f is finite.*

Proof. By base change, we may assume that $\Bbbk$ is algebraically closed. Let $\mathrm{Num}(X)$ denote the group of Cartier divisors on X modulo the numerical equivalence relation. Then $\mathrm{Num}(X)$ is a free abelian group of finite rank (cf. [7, IV, §1, Prop. 4]): the rank is just the Picard number of X. The pullback homomorphism $E \mapsto f^*E$ for Cartier divisors E on X induces an injection $f^\star\colon \mathrm{Num}(X) \to \mathrm{Num}(X)$ by [7, IV, §1, Prop. 2]. Thus, $f^\star \otimes \mathbb{Q}\colon \mathrm{Num}(X) \otimes \mathbb{Q} \to \mathrm{Num}(X) \otimes \mathbb{Q}$ is bijective. If an irreducible curve C is contained in a fiber of f, then $DC = 0$ for any Cartier divisor D on X by $(f^*E)C = E(f_*C)$. This is a contradiction. Therefore, f is finite. □

Definition (Pullback of a divisor)**.** Let $f\colon X \to Y$ be a finite surjective morphism of normal varieties. For a divisor E on Y, the pullback f^*E is defined as a divisor on X as follows: For the non-singular part Y_{reg} of Y, the complement of $f^{-1}Y_{\mathrm{reg}}$ in X has codimension ≥ 2. Thus, we can define f^*E by

$$f^*E|_{f^{-1}Y_{\mathrm{reg}}} = f'^*(E|_{Y_{\mathrm{reg}}})$$

for the restriction $f' = f|_{f^{-1}Y_{\mathrm{reg}}}\colon f^{-1}Y_{\mathrm{reg}} \to Y_{\mathrm{reg}}$ of f, where f'^* indicates the pullback of a Cartier divisor. The correspondence $E \mapsto f^*E$ gives rise to a homomorphism $\mathrm{Div}(Y) \to \mathrm{Div}(X)$ of divisor groups, which is also denoted by f^*. When E is reduced, we write

$$f^{-1}E = (f^*E)_{\mathrm{red}}$$

by abuse of notation. Here, $\mathrm{Supp}\, f^*E = f^{-1}\,\mathrm{Supp}\, E$, since f is finite and surjective.

Remark. If E is Cartier on Y, then f^*E coincides with the pullback as a Cartier divisor. Recall that the push-forward homomorphism $f_*\colon \mathrm{Div}(X) \to \mathrm{Div}(Y)$ is defined by $f_*\Gamma = d_\Gamma f(\Gamma)$ for a prime divisor Γ, where d_Γ is the degree of the finite morphism $f|_\Gamma\colon \Gamma \to f(\Gamma)$. Then $(\deg f)E = f_*(f^*E)$ for any divisor E on Y.

Definition 3.3 (Ramification divisor)**.** Assume that $\mathrm{char}\,\Bbbk = 0$ and let $f\colon X \to Y$ be a finite surjective morphism of normal varieties. For a prime divisor Γ on X, the *ramification index* of f along Γ is defined as the multiplicity $\mathrm{mult}_\Gamma\, f^*(f(\Gamma))$ of the divisor $f^*(f(\Gamma))$ along Γ. The *ramification divisor* R_f of f is an effective divisor on X defined by

$$\mathrm{mult}_\Gamma\, R_f = \mathrm{mult}_\Gamma\, f^*(f(\Gamma)) - 1$$

for any prime divisor Γ.

Remark. Since $\operatorname{char} \Bbbk = 0$, f is étale on a non-empty open subset of X, and moreover, f is étale at the generic point of Γ if and only if the ramification index of f along Γ equals 1. In particular, the number of prime components of R_f is finite.

Remark 3.4 (Ramification formula). Let η be a rational n-form on Y_{reg}, where $n = \dim Y$. The pullback $f^*\eta$ is defined as a rational n-form on X_{reg}, since the complement of $X_{\mathrm{reg}} \cap f^{-1}Y_{\mathrm{reg}}$ in X_{reg} has codimension ≥ 2. We can consider the canonical divisor $K_{Y_{\mathrm{reg}}}$ (resp. $K_{X_{\mathrm{reg}}}$) as the divisor of zeros (and the minus of poles) of η (resp. $f^*\eta$). We define the canonical divisor K_Y (resp. K_X) by $K_Y|_{Y_{\mathrm{reg}}} = K_{Y_{\mathrm{reg}}}$ (resp. $K_X|_{X_{\mathrm{reg}}} = K_{X_{\mathrm{reg}}}$). Then

$$K_X = f^*K_Y + R_f$$

as a divisor on X. This equality is called the *ramification formula*. For a reduced divisor E on Y, there is a divisor F on X such that F and f^*E have no common prime components and $R_f = F + f^*E - f^{-1}E$, i.e.,

$$K_X + f^{-1}E = f^*(K_Y + E) + F$$

(cf. [11, Lem. 1.39]). This is also called the ramification formula.

Definition 3.5. Assume that $\operatorname{char} \Bbbk = 0$ and let $f \colon X \to X$ be a non-isomorphic finite surjective endomorphism of a complete normal variety. A reduced divisor D on X (including 0) is said to be *completely invariant* under f, or *f-completely invariant* if $f^{-1}D = D$ (cf. [12, Def. 2.12]). Let $\mathcal{S}(X, f)$ be the set of prime divisors Γ on X such that $(f^k)^*\Gamma = b\Gamma$ for some $k > 0$ and $b > 1$. Then $\mathcal{S}(X, f)$ is finite by the same argument as in the proof of [12, Prop. 2.15]. We define

$$S_f := \sum\nolimits_{\Gamma \in \mathcal{S}(X,f)} \Gamma \quad \text{and} \quad \Delta_f := \sum\nolimits_{\Gamma \notin \mathcal{S}(X,f)} (\operatorname{mult}_\Gamma R_f)\Gamma$$

as in [12, Def. 2.16], where S_f is called the *characteristic completely invariant divisor* and Δ_f is called the *refined ramification divisor*.

Remark. The divisor S_f is f-completely invariant and we have $K_X + S_f = f^*(K_X + S_f) + \Delta_f$ as a ramification formula for f (cf. [12, Lem. 2.17]).

Lemma 3.6. *Let X be a normal complete variety with a reduced divisor D and let $f \colon X \to X$ be a finite endomorphism such that $f^{-1}D = D$. Assume that* $\operatorname{char} \Bbbk = 0$. *Then $S_f \subset D$ if there exist positive integers k and a satisfying the following conditions for any prime divisor Γ not contained in D*:

(i) *The pullback $(f^k)^*\Gamma$ is $\mathbb{Q}$-linearly equivalent to $a^k\Gamma$.*
(ii) *The ramification index of f along Γ is less than a.*

Proof. Let Γ be a prime component of S_f. Then $(f^l)^*\Gamma = b\Gamma$ for some $l > 0$ and $b > 1$. Assume that $\Gamma \not\subset D$. Then $(f^{kl})^*\Gamma = b^k\Gamma \sim_{\mathbb{Q}} a^{kl}\Gamma$ by (i), and we have $b = a^l$. We set $\Gamma_j = f^j(\Gamma)$ for $1 \leq j \leq l$. Then $\Gamma_0 = \Gamma_l = \Gamma$, and if $j > 0$, then $f^*\Gamma_j = b_j\Gamma_{j-1}$ for some $b_j \geq 1$. Here, $\prod_{j=1}^{l} b_j = b = a^l$. Furthermore, $\Gamma_j \not\subset D$ and $b_j < a$ for any $1 \leq j \leq l$ by $f^{-1}D = D$ and by (ii), since the ramification index of f along Γ_{j-1} equals b_j; this contradicts $\prod_{j=1}^{l} b_j = a^l$. Therefore, $S_f \subset D$. $\square$

4. Surjective endomorphisms of complete toric varieties

We shall construct some special endomorphisms of complete toric varieties and study their properties. We fix a free abelian group N of finite rank $r > 0$ and a *complete* fan $\triangle$ of N, and consider the toric variety $X = \mathbb{T}_{\mathsf{N}}(\triangle)$ with the boundary divisor $D = \mathbb{B}_{\mathsf{N}}(\triangle)$. We keep the notation in Section 2.

4.1. **Triggers for endomorphisms.**

Definition 4.1. An element α of $\mathsf{M} = \mathrm{Hom}_{\mathbb{Z}}(\mathsf{N}, \mathbb{Z})$ is called a *trigger for endomorphisms* on $(\mathsf{N}, \triangle)$ if there is a unique primary vector $v_\alpha \in \mathcal{P}_{\mathsf{N}}(\triangle)$ such that $\langle \alpha, v_\alpha \rangle > 0$. A trigger α is called a *root* of $(\mathsf{N}, \triangle)$ if $\langle \alpha, v_\alpha \rangle = 1$.

The following result is mentioned in [2, §4, n°5, Déf. 4, Rem. 3, p. 572], when α is a root:

Lemma 4.2. *For a trigger α for endomorphisms on $(\mathsf{N}, \triangle)$ and for a cone $\boldsymbol{\sigma} \in \triangle$, if $\boldsymbol{\sigma} \subset \{\alpha\}^\perp$, then $\boldsymbol{\sigma} + \mathsf{R}(v_\alpha) \in \triangle$.*

Proof. There is a cone $\tilde{\boldsymbol{\sigma}} \in \triangle$ of dimension r such that $\boldsymbol{\sigma}$ is a face $\tilde{\boldsymbol{\sigma}}$. If $v_\alpha \notin \tilde{\boldsymbol{\sigma}}$, then $\langle \alpha, v \rangle \leq 0$ for any $v \in \mathcal{P}_{\mathsf{N}}(\tilde{\boldsymbol{\sigma}})$, and hence, $\tilde{\boldsymbol{\sigma}}$ is contained in the half space $\{-\alpha\}^\vee$. Thus, we may assume that $v_\alpha \in \tilde{\boldsymbol{\sigma}}$, since $\triangle$ is complete. We can write $\boldsymbol{\sigma} = \{\beta\}^\perp \cap \tilde{\boldsymbol{\sigma}}$ for some $\beta \in \tilde{\boldsymbol{\sigma}}^\vee \cap \mathsf{M}$. Then $\langle \beta, v_\alpha \rangle > 0$ by $\langle \alpha, v_\alpha \rangle > 0$. We set $\gamma := \langle \alpha, v_\alpha \rangle \beta - \langle \beta, v_\alpha \rangle \alpha$. Then $\langle \gamma, v_\alpha \rangle = 0$ and $\langle \gamma, v \rangle \geq \langle \alpha, v_\alpha \rangle \langle \beta, v \rangle \geq 0$ for any $v \in \mathcal{P}_{\mathsf{N}}(\tilde{\boldsymbol{\sigma}}) \setminus \{v_\alpha\}$. Thus, $\gamma \in \tilde{\boldsymbol{\sigma}}^\vee \cap \mathsf{M}$, and $\{\gamma\}^\perp \cap \tilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma} + \mathsf{R}(v_\alpha)$ by $\boldsymbol{\sigma} \subset \{\alpha\}^\perp$. Therefore, $\boldsymbol{\sigma} + \mathsf{R}(v_\alpha) \in \triangle$. □

Remark 4.3. For two triggers α and β for endomorphisms on $(\mathsf{N}, \triangle)$, if $v_\alpha = v_\beta$, then $\alpha + \beta$ is a trigger with $v_{\alpha+\beta} = v_\alpha$. In fact, $\langle \alpha, v_\alpha \rangle > 0$, $\langle \beta, v_\alpha \rangle > 0$, and $\langle \alpha, v \rangle \leq 0$ and $\langle \beta, v \rangle \leq 0$ for any $v \in \mathcal{P}_{\mathsf{N}}(\triangle) \setminus \{v_\alpha\}$.

Lemma 4.4. *For the toric variety $X = \mathbb{T}_{\mathsf{N}}(\triangle)$, let E be a divisor on X supported on the boundary divisor $D = \mathbb{B}_{\mathsf{N}}(\triangle)$. Then $H^0(X, \mathcal{O}_X(E))$ has a canonical graded M-module structure*

$$\bigoplus\nolimits_{m \in \mathsf{M}} H^0(X, \mathcal{O}_X(E))_m$$

in which $\dim H^0(X, \mathcal{O}_X(E))_m \leq 1$ *for any* $m \in \mathsf{M}$.

Proof. Let Y be the non-singular locus X_{reg} of X. Then $E|_Y$ is Cartier and the restriction homomorphism $H^0(X, \mathcal{O}_X(E)) \to H^0(Y, \mathcal{O}_Y(E|_Y))$ is an isomorphism. Here, $\mathcal{O}_Y(E|_Y)$ is a $\mathbb{T}_{\mathsf{N}}$-*linearized invertible sheaf* in the sense of [9, Ch. 1, §3, Def. 1.6], since E is $\mathbb{T}_{\mathsf{N}}$-invariant. In particular, $H^0(X, \mathcal{O}_X(E)) \simeq H^0(Y, \mathcal{O}_Y(E|_Y))$ has a structure of graded M-modules by the dual action of $\mathbb{T}_{\mathsf{N}}$. Let $j \colon U \hookrightarrow Y$ be the open immersion from the open torus $U = \mathbb{T}_{\mathsf{N}}(\{0\}) = X \setminus D$. Then we have an isomorphism $\mathcal{O}_Y(E|_Y)|_U \simeq \mathcal{O}_U$ of $\mathbb{T}_{\mathsf{N}}$-linearized invertible sheaves by $\operatorname{Supp} E \subset D$. By the canonical injection $\mathcal{O}_Y(E|_Y) \hookrightarrow j_*(\mathcal{O}_Y(E|_Y)|_U)$, and we have an injection of graded M-modules

$$H^0(X, \mathcal{O}_X(E)) \simeq H^0(Y, \mathcal{O}_Y(E|_Y)) \hookrightarrow H^0(U, \mathcal{O}_U) = \Bbbk[\mathsf{M}] = \bigoplus\nolimits_{m \in \mathsf{M}} \Bbbk\boldsymbol{e}(m).$$

Therefore, $H^0(X, \mathcal{O}_X(E))_m = H^0(X, \mathcal{O}_X(E)) \cap \Bbbk \boldsymbol{e}(m)$ for any $m \in \mathsf{M}$, and it is at most 1-dimensional. Thus, we are done. □

Remark 4.5. By the proof, we see that $m \in \mathsf{M}$ satisfies $H^0(X, \mathcal{O}_X(E))_m \neq 0$ if and only if $\operatorname{div}(\boldsymbol{e}(m)) + E \geq 0$ for the principal divisor $\operatorname{div}(\boldsymbol{e}(m))$, i.e., $\langle m, v \rangle + \operatorname{mult}_{\boldsymbol{\Gamma}(v)} E \geq 0$ for any $v \in \mathcal{P}_{\mathsf{N}}(\triangle)$ (cf. (II-1) in Remark 2.3).

Remark. Lemma 4.4 and its higher cohomology version are well known at least in the case where E is Cartier (cf. [6, I, §3, Thm. 12], [1, Thm. 7.2], [14, Thm. 2.6]).

Corollary 4.6. *The following hold for any* $\alpha \in \mathsf{M} \setminus \{0\}$ *and* $v \in \mathcal{P}_{\mathsf{N}}(\triangle)$:

(1) α *is a trigger for endomorphisms on* $(\mathsf{N}, \triangle)$ *with* $v = v_\alpha$ *if and only if* $H^0(X, \mathcal{O}_X(k\boldsymbol{\Gamma}(v)))_{-\alpha} \neq 0$ *for some* $k > 0$;
(2) α *is a root of* $(\mathsf{N}, \triangle)$ *with* $v = v_\alpha$ *if and only if* $H^0(X, \mathcal{O}_X(\boldsymbol{\Gamma}(v)))_{-\alpha} \neq 0$.

Proof. For α and v above, α is a trigger for endomorphisms on (resp. a root of) $(\mathsf{N}, \triangle)$ with $v = v_\alpha$ if and only if $\langle \alpha, v \rangle = k$ for some $k > 1$ (resp. for $k = 1$) and $\langle \alpha, v' \rangle \leq 0$ for any $v' \in \mathcal{P}_{\mathsf{N}}(\triangle) \setminus \{v\}$: By Remark 4.5, this is also equivalent to $H^0(X, \mathcal{O}_X(k\boldsymbol{\Gamma}(v)))_{-\alpha} \neq 0$. □

Corollary 4.7. *Let* Γ *be a prime component of the boundary divisor* $D = \mathbb{B}_{\mathsf{N}}(\triangle)$.

(1) *If some multiple of* Γ *is linearly equivalent to a non-zero effective divisor not containing* Γ, *then* $\Gamma = \boldsymbol{\Gamma}(v_\alpha)$ *for a trigger* α *for endomorphisms on* $(\mathsf{N}, \triangle)$.
(2) *If some multiple of* Γ *is linearly equivalent to a non-zero effective divisor* E *supported on* $D - \Gamma$, *then there is a trigger* α *for endomorphisms on* $(\mathsf{N}, \triangle)$ *such that* $\operatorname{div}(\boldsymbol{e}(\alpha)) = k\Gamma - E$ *for some* $k > 0$.

Proof. (1): By assumption, there is a non-zero element $\zeta \in H^0(X, \mathcal{O}_X(k\Gamma))$ for a positive integer k such that the divisor $\operatorname{div}(\zeta)$ of zeros does not contain Γ as a prime component. By Lemma 4.4, we can write $\zeta = \sum_\alpha \zeta_\alpha$ for elements $\zeta_\alpha \in H^0(X, \mathcal{O}_X(k\Gamma))_\alpha$ for $\alpha \in \mathsf{M}$. Since X is complete, $k\Gamma \not\sim 0$. Thus, there is an element $\alpha \in \mathsf{M} \setminus \{0\}$ such that $\zeta_{-\alpha} \neq 0$ and that the divisor $\operatorname{div}(\zeta_{-\alpha})$ of zeros does not contain Γ as a prime component. This α is a trigger for endomorphisms on $(\mathsf{N}, \triangle)$, and $\Gamma = \boldsymbol{\Gamma}(v_\alpha)$ by Corollary 4.6(1).

(2): There is a rational function f on X such that the principal divisor $\operatorname{div}(f)$ is expressed as $k\Gamma - E$ for a positive integer k. Then $f|_U$ is nowhere vanishing on the open torus $U = X \setminus D$. The group $H^0(U, \mathcal{O}_U)^\star \simeq \Bbbk[\mathsf{M}]^\star$ of invertible functions on U consists of $\lambda \boldsymbol{e}(m)$ for $\lambda \in \Bbbk^\star$ and $m \in \mathsf{M}$. Thus, we may write $f = \boldsymbol{e}(\alpha)$ for some $\alpha \in \mathsf{M}$. Then α is a trigger for endomorphisms on $(\mathsf{N}, \triangle)$ by $\operatorname{div}(\boldsymbol{e}(\alpha)) = k\Gamma - E$, where $\Gamma = \boldsymbol{\Gamma}(v_\alpha)$ and $k = \langle \alpha, v_\alpha \rangle$. □

Lemma 4.8. *Let* α *and* β *be two triggers for endomorphisms on* $(\mathsf{N}, \triangle)$. *If* $\langle \alpha, v_\beta \rangle \leq 0$ *and* $\langle \beta, v_\alpha \rangle \leq 0$, *then one of the following holds*:

(1) *There exist triggers* α' *and* β' *such that* $v_\alpha = v_{\alpha'}$, $v_\beta = v_{\beta'}$, *and* $\langle \alpha', v_{\beta'} \rangle = \langle \beta', v_{\alpha'} \rangle = 0$.
(2) *There exist positive integers* a *and* b *such that* $a\alpha + b\beta = 0$.

Proof. By assumption, we have $v_\alpha \neq v_\beta$. For prime components $\Gamma_\alpha := \boldsymbol{\Gamma}(v_\alpha)$ and $\Gamma_\beta := \boldsymbol{\Gamma}(v_\beta)$ of D and for non-negative integers

$$h_\alpha := \langle \alpha, v_\alpha \rangle, \quad h_\beta := \langle \beta, v_\beta \rangle, \quad e_{\alpha,\beta} := -\langle \alpha, v_\beta \rangle, \quad e_{\beta,\alpha} := -\langle \beta, v_\alpha \rangle,$$

we consider effective divisors

$$G_\alpha := h_\alpha \Gamma_\alpha - e_{\alpha,\beta}\Gamma_\beta - \mathrm{div}(\boldsymbol{e}(\alpha)) \quad \text{and} \quad G_\beta := -e_{\beta,\alpha}\Gamma_\alpha + h_\beta \Gamma_\beta - \mathrm{div}(\boldsymbol{e}(\beta))$$

on X supported on $D - (\Gamma_\alpha + \Gamma_\beta)$. Then

$$\text{(IV-1)} \qquad \begin{aligned} h_\beta G_\alpha + e_{\alpha,\beta} G_\beta &= (h_\alpha h_\beta - e_{\alpha,\beta} e_{\beta,\alpha})\Gamma_\alpha - \mathrm{div}(\boldsymbol{e}(h_\beta \alpha + e_{\alpha,\beta}\beta)), \\ e_{\beta,\alpha} G_\alpha + h_\alpha G_\beta &= (h_\alpha h_\beta - e_{\alpha,\beta} e_{\beta,\alpha})\Gamma_\beta - \mathrm{div}(\boldsymbol{e}(e_{\beta,\alpha} \alpha + h_\alpha \beta)). \end{aligned}$$

Hence, $h_\alpha h_\beta - e_{\alpha,\beta} e_{\beta,\alpha} \geq 0$. If $h_\alpha h_\beta - e_{\alpha,\beta} e_{\beta,\alpha} > 0$, then $\alpha' := h_\beta \alpha + e_{\alpha,\beta}\beta$ and $\beta' := e_{\beta,\alpha}\alpha + h_\alpha \beta$ are triggers such that $v_{\alpha'} = v_\alpha$, $v_{\beta'} = v_\beta$, and $\langle \alpha', v_\beta \rangle = \langle \beta', v_\alpha \rangle = 0$ by (IV-1). Thus, (1) holds in this case. Assume that $h_\alpha h_\beta = e_{\alpha,\beta} e_{\beta,\alpha}$. Then $e_{\alpha,\beta} > 0$ and $e_{\beta,\alpha} > 0$ by $h_\alpha h_\beta > 0$, and hence, $G_\alpha = G_\beta = 0$ by (IV-1). Consequently, $h_\alpha \Gamma_\alpha \sim e_{\alpha,\beta}\Gamma_\beta$, $e_{\beta,\alpha}\Gamma_\alpha \sim h_\beta \Gamma_\beta$, and

$$e_{\beta,\alpha}\,\mathrm{div}(\boldsymbol{e}(\alpha)) + h_\alpha\,\mathrm{div}(\boldsymbol{e}(\beta)) = h_\beta\,\mathrm{div}(\boldsymbol{e}(\alpha)) + e_{\alpha,\beta}\,\mathrm{div}(\boldsymbol{e}(\beta)) = 0.$$

Therefore, $e_{\beta,\alpha}\alpha + h_\alpha \beta = h_\beta \alpha + e_{\alpha,\beta}\beta = 0$, and (2) holds. Thus, we are done. □

Lemma 4.9. *Let $P = P(\mathtt{x})$ be a polynomial in $\Bbbk[\mathtt{x}]$ for one variable $\mathtt{x}$ such that $P(0) \neq 0$. For a trigger α for endomorphisms on $(\mathsf{N}, \triangle)$, there is an effective divisor $E_\alpha(P)$ on X such that any prime component of $E_\alpha(P)$ is not contained in D and*

$$\mathrm{div}(P(\boldsymbol{e}(-\alpha))) = E_\alpha(P) - (\deg P)\langle \alpha, v_\alpha \rangle \boldsymbol{\Gamma}(v_\alpha),$$

where $P(\boldsymbol{e}(-\alpha))$ is regarded as a rational function on X.

Proof. Since $\boldsymbol{e}(-\alpha) \in \Bbbk[\mathsf{M}]$, $P(\boldsymbol{e}(-\alpha))$ is regular on the open torus $X \setminus D$. Thus, the assertion follows from Lemma 2.5, since $\langle \alpha, v \rangle \leq 0$ for any $v \in \mathcal{P}_{\mathsf{N}}(\triangle) \setminus \{v_\alpha\}$. □

4.2. **Endomorphisms of complete toric varieties defined by triggers.**

Definition 4.10. Let α_1 and α_2 be triggers for endomorphisms on $(\mathsf{N}, \triangle)$ and let $P_1(\mathtt{x})$ and $P_2(\mathtt{x})$ be polynomials in $\Bbbk[\mathtt{x}]$ for one variable $\mathtt{x}$ such that $P_1(0)P_2(0) \neq 0$. Let $Z_{1,2} \subset \mathbb{T}_{\mathsf{N}}$ be the zero locus of the function $P_1(\boldsymbol{e}(-\alpha_1))P_2(\boldsymbol{e}(-\alpha_2))$. For a positive integer k, a morphism $\Xi = \Xi_{\alpha_1,\alpha_2,k}(P_1, P_2) \colon \mathbb{T}_{\mathsf{N}} \setminus Z_{1,2} \to \mathbb{T}_{\mathsf{N}}$ is defined by

$$\Xi(u) = u^k \cdot \boldsymbol{\gamma}_{v_{\alpha_1}}(P_1(u(-\alpha_1)) \cdot \boldsymbol{\gamma}_{v_{\alpha_2}}(P_2(u(-\alpha_2)))$$

for T-valued points u of $\mathbb{T}_{\mathsf{N}} \setminus Z_{1,2}$ for any $\Bbbk$-scheme T, where $\cdot$ stands for the multiplication in $\mathbb{T}_{\mathsf{N}}\langle T \rangle$. For a trigger α and a polynomial $P(\mathtt{x})$ with $P(0) \neq 0$ and for the zero locus $Z \subset \mathbb{T}_{\mathsf{N}}$ of the function $P(\boldsymbol{e}(-\alpha))$, we set $\Xi_{\alpha,k}(P) := \Xi_{\alpha,\alpha,k}(P, 1)$ as a morphism $\mathbb{T}_{\mathsf{N}} \setminus Z \to \mathbb{T}_{\mathsf{N}}$, in which

$$\Xi_{\alpha,k}(P)u = u^k \cdot \boldsymbol{\gamma}_{v_\alpha}(P(u(-\alpha)))$$

for any T-valued point u of $\mathbb{T}_{\mathsf{N}} \setminus Z$ for any $\Bbbk$-scheme T.

Remark 4.11. By definition,

$$\Xi_{\alpha_1,\alpha_2,k}(P_1,P_2) = \Xi_{\alpha_2,\alpha_1,k}(P_2,P_1) \quad \text{and} \quad \Xi_{\alpha,\alpha,k}(P_1,P_2) = \Xi_{\alpha,k}(P_1P_2).$$

If $P(\mathtt{x}) = 1 + \lambda\mathtt{x}$ for a constant $\lambda \in \Bbbk$ and if α is a root, then $\Xi_{\alpha,1}(P)$ is just the morphism $x_\alpha(\lambda)$ defined in [2, §4, n°5, Thm. 3, p. 573] (cf. [14, Prop. 3.14]).

Remark 4.12. For $\Xi = \Xi_{\alpha_1,\alpha_2,k}(P_1,P_2)$, we have

$$\Xi(u)(m) = u(km)P_1(u(-\alpha_1))^{\langle m,v_{\alpha_1}\rangle} P_2(u(-\alpha_2))^{\langle m,v_{\alpha_2}\rangle} \tag{IV-2}$$

for any $m \in \mathsf{M}$ and for any T-valued point u of $\mathbb{T}_\mathsf{N} \setminus Z_{1,2}$ for any $\Bbbk$-scheme T. This implies that

$$\Xi^*\boldsymbol{e}(m) = \boldsymbol{e}(km)P_1(\boldsymbol{e}(-\alpha_1))^{\langle m,v_{\alpha_1}\rangle} P_2(\boldsymbol{e}(-\alpha_2))^{\langle m,v_{\alpha_2}\rangle} \tag{IV-3}$$

for any $m \in \mathsf{M}$. Hence, Ξ is regarded as a rational map $X \dashrightarrow X$ by identifying $\mathbb{T}_\mathsf{N}$ with $X \setminus D$, since $P_1(\boldsymbol{e}(-\alpha_1))$ and $P_2(\boldsymbol{e}(-\alpha_2))$ are rational functions on X.

Proposition 4.13. *In Definition* 4.10, *assume that one of the following holds*:

(i) $v_{\alpha_1} = v_{\alpha_2}$, *and the integer k satisfies*

$$k \geq \langle \alpha_1, v_{\alpha_1}\rangle \deg P_1 + \langle \alpha_2, v_{\alpha_2}\rangle \deg P_2;$$

(ii) *either* $\langle \alpha_1, v_{\alpha_2}\rangle = 0$ *or* $\langle \alpha_2, v_{\alpha_1}\rangle = 0$, *and the integer k satisfies*

$$k \geq \max\{\langle \alpha_1, v_{\alpha_1}\rangle \deg P_1,\ \langle \alpha_2, v_{\alpha_2}\rangle \deg P_2\}.$$

Then $\Xi_{\alpha_1,\alpha_2,k}(P_1,P_2)$ *is regarded as an endomorphism of* $X = \mathbb{T}_\mathsf{N}(\triangle)$ *and it induces an endomorphism of the affine toric variety* $\mathbb{T}_\mathsf{N}(\boldsymbol{\sigma})$ *for any cone* $\boldsymbol{\sigma} \in \triangle$ *containing* v_{α_1} *and* v_{α_2}.

Proof. For simplicity, we set $v_i = v_{\alpha_i}$ and $d_i = \deg P_i$ for $i = 1, 2$, and set $\Xi := \Xi_{\alpha_1,\alpha_2,k}(P_1,P_2)$. In (ii), we may assume that $\langle \alpha_2, v_1\rangle = 0$ by Remark 4.11. Note that $v_1 \neq v_2$ in (ii). We shall prove the assertion by the following three steps modifying arguments in the proof of [14, Prop. 3.14].

Step 1. We shall prove the last assertion on $\mathbb{T}_\mathsf{N}(\boldsymbol{\sigma})$ for any $\boldsymbol{\sigma} \in \triangle$ such that $\{v_1, v_2\} \subset \boldsymbol{\sigma}$. In this case, $\langle m, v_1\rangle \geq 0$ and $\langle m, v_2\rangle \geq 0$ for any $m \in \boldsymbol{\sigma}^\vee \cap \mathsf{M}$. Hence, $\Xi^*\boldsymbol{e}(m) \in \Bbbk[\mathsf{M}]$ for the same m by (IV-3) in Remark 4.12. We set

$$\xi_v(m) := \mathrm{mult}_{\boldsymbol{\Gamma}(v)} \mathrm{div}(\Xi^*\boldsymbol{e}(m))$$

for $m \in \mathsf{M}$ and $v \in \mathcal{P}_\mathsf{N}(\triangle)$. Then

$$\xi_v(m) = \langle km, v\rangle + d_1\langle m, v_1\rangle \min\{0, -\langle \alpha_1, v\rangle\} + d_2\langle m, v_2\rangle \min\{0, -\langle \alpha_2, v\rangle\}$$

by Lemma 2.5 and by equalities (II-1) in Remark 2.3 and (IV-3) in Remark 4.12. Thus, the following hold for any $m \in \boldsymbol{\sigma}^\vee \cap \mathsf{M}$ and $v \in \mathcal{P}_\mathsf{N}(\boldsymbol{\sigma})$ by (i) and (ii):

- If $v \notin \{v_1, v_2\}$, then $\xi_v(m) = k\langle m, v\rangle \geq 0$.
- If $v = v_1 = v_2$, then $\xi_v(m) = (k - d_1\langle \alpha_1, v_1\rangle - d_2\langle \alpha_2, v_2\rangle)\langle m, v\rangle \geq 0$.
- If $v_1 \neq v_2$ and if $v = v_i$ for $i = 1, 2$, then $\xi_v(m) = (k - d_i\langle \alpha_i, v_i\rangle)\langle m, v\rangle \geq 0$.

Therefore, $\Xi^*\boldsymbol{e}(m) \in \Bbbk[\boldsymbol{\sigma}^\vee \cap \mathsf{M}]$ for any $m \in \boldsymbol{\sigma}^\vee \cap \mathsf{M}$ by Remark 2.4. This implies that Ξ induces an endomorphism $\mathbb{T}_\mathsf{N}(\boldsymbol{\sigma}) \to \mathbb{T}_\mathsf{N}(\boldsymbol{\sigma})$.

Step 2. We shall show that Ξ induces a morphism $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma}) \to \mathbb{T}_{\mathsf{N}}(\triangle)$ when $v_1 \in \boldsymbol{\sigma}$ but $v_2 \notin \boldsymbol{\sigma}$. Note that $\langle \alpha_2, v_1 \rangle = 0$ by our assumption. Now, $-\alpha_2 \in \boldsymbol{\sigma}^\vee \cap \mathsf{M}$, and $\boldsymbol{e}(-\alpha_2)$ is regular on $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$. Let U_2 and V_2 be Zariski-open subsets of $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$ defined as the complements of zero loci of regular functions $P_2(\boldsymbol{e}(-\alpha_2))$ and $\boldsymbol{e}(-\alpha_2)$, respectively. Then $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma}) = U_2 \cup V_2$ by $P_2(0) \neq 0$, and $V_2 = \mathbb{T}_{\mathsf{N}}(\{\alpha_2\}^\perp \cap \boldsymbol{\sigma})$ by [14, Prop. 1.3]. Let $\boldsymbol{\tau}_2$ be the cone $\mathsf{R}(v_2) + (\{\alpha_2\}^\perp \cap \boldsymbol{\sigma})$. Then $\boldsymbol{\tau}_2 \in \triangle$ by Lemma 4.2, and $V_2 \subset \mathbb{T}_{\mathsf{N}}(\boldsymbol{\tau}_2)$. Moreover, $\{v_1, v_2\} \subset \boldsymbol{\tau}_2$, since $\langle \alpha_2, v_1 \rangle = 0$. Hence, Ξ induces a morphism $V_2 \subset \mathbb{T}_{\mathsf{N}}(\boldsymbol{\tau}_2) \to \mathbb{T}_{\mathsf{N}}(\boldsymbol{\tau}_2) \subset \mathbb{T}_{\mathsf{N}}(\triangle)$ by Step 1 applied to $\boldsymbol{\tau}_2$. It suffices to prove that Ξ induces a morphism $U_2 \to \mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$. Let ψ_2 be the nowhere vanishing function $P_2(\boldsymbol{e}(-\alpha_2))$ on U_2. Then

$$\Xi^* \boldsymbol{e}(m)|_{U_2} = \psi_2 \boldsymbol{e}(km) P_1(\boldsymbol{e}(-\alpha_1))^{\langle m, v_1 \rangle}|_{U_2}$$

for $m \in \mathsf{M}$ by (IV-3) in Remark 4.12. Hence, it is enough to show that

$$\xi_v^{(1)}(v) := \operatorname{mult}_{\boldsymbol{\Gamma}(v)} \operatorname{div}(\boldsymbol{e}(km) P_1(\boldsymbol{e}(-\alpha_1))^{\langle m, v_1 \rangle}) \geq 0$$

for any $m \in \boldsymbol{\sigma}^\vee \cap \mathsf{M}$ and any $v \in \mathcal{P}_{\mathsf{N}}(\boldsymbol{\sigma})$, by Remark 2.4. Now,

$$\xi_v^{(1)}(m) = \langle km, v \rangle + d_1 \langle m, v_1 \rangle \min\{0, -\langle \alpha_1, v \rangle\}$$

by Lemma 2.5, and we have $\langle m, v \rangle \geq 0$ and $\langle m, v_1 \rangle \geq 0$. If $v \neq v_1$, then $\xi_v^{(1)}(m) = k\langle m, v \rangle \geq 0$ by $\langle \alpha_1, v \rangle \leq 0$. If $v = v_1$, then

$$\xi_v^{(1)}(m) = (k - d_1 \langle \alpha_1, v_1 \rangle) \langle m, v_1 \rangle \geq 0$$

by (ii). Therefore, $\xi_v^{(1)}(m) \geq 0$ for any such v and m. Consequently, Ξ induces a morphism $U_2 \to \mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$ and a morphism $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma}) \to \mathbb{T}_{\mathsf{N}}(\boldsymbol{\tau}_2) \cup \mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma}) \subset \mathbb{T}_{\mathsf{N}}(\triangle)$.

Step 3. The final step. By Steps 1 and 2, it suffices to prove that Ξ induces a morphism $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma}) \to \mathbb{T}_{\mathsf{N}}(\triangle)$ for any cone $\boldsymbol{\sigma} \in \triangle$ not containing v_1. In this case, $-\alpha_1 \in \boldsymbol{\sigma}^\vee \cap \mathsf{M}$, and $\boldsymbol{e}(-\alpha_1)$ is regular on $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$. Let U_1 and V_1 be Zariski-open subsets of $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$ defined as the complements of zero loci of $P_1(\boldsymbol{e}(-\alpha_1))$ and $\boldsymbol{e}(-\alpha_1)$, respectively. Then $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma}) = U_1 \cup V_1$, $V_1 = \mathbb{T}_{\mathsf{N}}(\{\alpha_1\}^\perp \cap \boldsymbol{\sigma})$, and the cone $\boldsymbol{\tau}_1 := \mathsf{R}(v_1) + (\{\alpha_1\}^\perp \cap \boldsymbol{\sigma})$ belongs to $\triangle$ by the same argument as in Step 2. Since $V_1 \subset \mathbb{T}_{\mathsf{N}}(\boldsymbol{\tau}_1)$ and since $v_1 \in \boldsymbol{\tau}_1$, Ξ induces a morphism $V_1 \subset \mathbb{T}_{\mathsf{N}}(\boldsymbol{\tau}_1) \to \mathbb{T}_{\mathsf{N}}(\triangle)$ by Steps 1 and 2. It remains to prove that Ξ induces a morphism $U_1 \to \mathbb{T}_{\mathsf{N}}(\triangle)$. Let ψ_1 be the nowhere vanishing function $P_1(\boldsymbol{e}(-\alpha_1))$ on U_1 and set $\Xi' := \Xi_{\alpha_2}(P_2) = \Xi_{\alpha_2, \alpha_2}(1, P_2)$ (cf. Definition 4.10). Then

$$\Xi^* \boldsymbol{e}(m)|_{U_1} = \psi_1 \boldsymbol{e}(km) P_2(\boldsymbol{e}(-\alpha_2))^{\langle m, v_2 \rangle}|_{U_1} = \psi_1 (\Xi'^* \boldsymbol{e}(m))|_{U_1} \tag{IV-4}$$

for any $m \in \mathsf{M}$ by (IV-3) in Remark 4.12.

Assume that $v_2 \in \boldsymbol{\sigma}$. Then $\Xi'^* \boldsymbol{e}(m) \in \boldsymbol{\sigma}^\vee \cap \mathsf{M}$ for any $m \in \boldsymbol{\sigma}^\vee \cap \mathsf{M}$ by Step 1, and $\Xi^* \boldsymbol{e}(m)|_{U_1}$ is regular for the same m by (IV-4). Thus, Ξ induces a morphism $U_1 \to \mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$.

Assume that $v_2 \notin \boldsymbol{\sigma}$. Then $-\alpha_2 \in \boldsymbol{\sigma}^\vee \cap \mathsf{M}$. Let U_2 and V_2 be Zariski-open subsets of $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$ defined as the zero loci of regular functions $P_2(\boldsymbol{e}(-\alpha_2))$ and $\boldsymbol{e}(-\alpha_2)$ on $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$, respectively. Then $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma}) = U_2 \cup V_2$, $V_2 = \mathbb{T}_{\mathsf{N}}(\{\alpha_2\}^\perp \cap \boldsymbol{\sigma})$, and $\boldsymbol{\tau}_2 := \mathsf{R}(v_2) + (\{\alpha_2\}^\perp \cap \boldsymbol{\sigma}) \in \triangle$ as in Step 2. Since $v_2 \in \boldsymbol{\tau}_2$, Ξ' induces a morphism $V_2 \subset \mathbb{T}_{\mathsf{N}}(\boldsymbol{\tau}_2) \to \mathbb{T}_{\mathsf{N}}(\boldsymbol{\tau}_2) \subset \mathbb{T}_{\mathsf{N}}(\triangle)$ by Step 1. Hence, $\Xi'^* \boldsymbol{e}(m)$ is regular on V_2 for any $m \in \boldsymbol{\tau}_2^\vee \cap \mathsf{M}$, and $\Xi^*\boldsymbol{e}(m)$ is also regular on $U_1 \cap V_2$ for the same m by (IV-4). Therefore, Ξ induces a morphism $U_1 \cap V_2 \to \mathbb{T}_{\mathsf{N}}(\boldsymbol{\tau}_2)$. On the other hand, by (IV-3) in Remark 4.12, we have

$$\Xi^*\boldsymbol{e}(m)|_{U_1\cap U_2} = \boldsymbol{e}(km)\psi_1^{\langle m,v_1\rangle}\psi_2^{\langle m,v_2\rangle}|_{U_1\cap U_2}$$

for the nowhere vanishing function $\psi_2 = P_2(-\boldsymbol{e}(\alpha_2))$ on U_2. Thus, $\Xi^*\boldsymbol{e}(m)$ is regular on $U_1 \cap U_2$ for any $m \in \boldsymbol{\sigma}^\vee \cap \mathsf{M}$, and hence, Ξ induces a morphism $U_1 \cap U_2 \to \mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma})$. Consequently, Ξ induces a morphism $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma}) \to \mathbb{T}_{\mathsf{N}}(\triangle)$. Thus, we are done. □

Example 4.14. We consider the case where $X = \mathbb{P}^2$. Here, N is of rank 2 with a free basis (e_1, e_2), i.e., $\mathsf{N} = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$, and the complete fan $\triangle$ of N consists of $\{0\}$, $\mathsf{R}_1 = \mathbb{R}_{\geq 0}e_1$, $\mathsf{R}_2 = \mathbb{R}_{\geq 0}e_2$, $\mathsf{R}_3 = \mathbb{R}_{\geq 0}(-e_1 - e_2)$, $\boldsymbol{\sigma}_1 = \mathsf{R}_2 + \mathsf{R}_3$, $\boldsymbol{\sigma}_2 = \mathsf{R}_3 + \mathsf{R}_1$, and $\boldsymbol{\sigma}_3 = \mathsf{R}_1 + \mathsf{R}_2$. In particular, $\mathcal{P}_{\mathsf{N}}(\triangle) = \{e_1, e_2, -(e_1 + e_2)\}$. For a dual basis (f_1, f_2) of M, i.e., $\langle f_i, e_j\rangle = \delta_{i,j}$, we set $\mathtt{t}_1 := \boldsymbol{e}(f_1)$ and $\mathtt{t}_2 := \boldsymbol{e}(f_2)$, which are inhomogeneous coordinate functions of $X = \mathbb{P}^2$. Let α_1 and α_2 be triggers for endomorphisms on $(\mathsf{N}, \triangle)$ such that $v_{\alpha_1} = e_1$, $v_{\alpha_2} = e_2$, and $\langle \alpha_1, v_{\alpha_2}\rangle = 0$. Then

$$\alpha_1 = a_1 f_1 \quad \text{and} \quad \alpha_2 = -b_2 f_1 + a_2 f_2$$

for positive integers $a_1 = \langle \alpha_1, v_{\alpha_1}\rangle$, $a_2 = \langle \alpha_2, v_{\alpha_2}\rangle$, and an integer $0 \leq b_2 \leq a_2$. In particular, $\boldsymbol{e}(-\alpha_1) = \mathtt{t}_1^{-a_1}$ and $\boldsymbol{e}(-\alpha_2) = \mathtt{t}_1^{b_2}\mathtt{t}_2^{-a_2}$. For $i = 1$, 2, let $P_i(\mathtt{x})$ be a polynomial in $\Bbbk[\mathtt{x}]$ such that $P_i(0) \neq 0$. We set $d_i := \deg P_i$. For a positive integer $k \geq \max\{d_1 a_1, d_2 a_2\}$, we have an endomorphism $\Xi = \Xi_{\alpha_1,\alpha_2,k}(P_1, P_2)$ of X by Proposition 4.13. Here,

$$\Xi^*\mathtt{t}_1 = \mathtt{t}_1^k P_1(\mathtt{t}_1^{-a_1}) \quad \text{and} \quad \Xi^*\mathtt{t}_2 = \mathtt{t}_2^k P_2(\mathtt{t}_2^{-a_2}\mathtt{t}_1^{b_2})$$

by (IV-3) in Remark 4.12, and Ξ is determined by these equalities. We can describe Ξ by a homogeneous coordinate $(\mathtt{Z}_0 : \mathtt{Z}_1 : \mathtt{Z}_2)$ of $\mathbb{P}^2$ such that $\mathtt{Z}_1/\mathtt{Z}_0 = \mathtt{t}_1$ and $\mathtt{Z}_2/\mathtt{Z}_0 = \mathtt{t}_2$: For $i = 1$, 2, the homogeneous polynomial

$$F_i(\mathtt{U}, \mathtt{V}) := \mathtt{V}^{d_i} P_i(\mathtt{U}/\mathtt{V}) \in \Bbbk[\mathtt{U}, \mathtt{V}]$$

is of degree d_i, and $F_i(0, 1) \neq 0$ and $F_i(1, 0) \neq 0$. Then Ξ is determined by

$$\Xi^*\mathtt{Z}_0 = \mathtt{Z}_0^k, \quad \Xi^*\mathtt{Z}_1 = \mathtt{Z}_1^{k-d_1a_1} F_1(\mathtt{Z}_0^{a_1}, \mathtt{Z}_1^{a_1}), \quad \Xi^*\mathtt{Z}_2 = \mathtt{Z}_2^{k-d_2a_2} F_2(\mathtt{Z}_0^{a_2-b_2}\mathtt{Z}_1^{b_2}, \mathtt{Z}_2^{a_2}).$$

4.3. **Properties of endomorphisms defined by triggers.** We shall study the endomorphism $\Xi_{\alpha_1,\alpha_2,k}(P_1, P_2)$ of $X = \mathbb{T}_{\mathsf{N}}(\triangle)$ defined in Proposition 4.13. For simplicity, we set

$$\Xi := \Xi_{\alpha_1,\alpha_2,k}(P_1, P_2), \quad v_i := v_{\alpha_i} \quad \text{and} \quad d_i := \deg P_i$$

for $i = 1$, 2 as in the proof of Proposition 4.13.

Lemma 4.15. *The degree of* Ξ *equals* k^r.

Proof. Let $(m_1, m_2, \ldots, m_r)$ be a free basis of M, and set $t_j := \boldsymbol{e}(m_j)$ for $1 \leq j \leq r$. Then the function field of X is equal to $\Bbbk(t_1, t_2, \ldots, t_r)$, which is pure transcendental over $\Bbbk$. The pullback $\Xi^* t_j$ is a rational function for any $1 \le j \le r$, and $\deg \Xi$ equals the degree of the field extension

$$\Bbbk(t_1, t_2, \dots, t_r)/\Bbbk(\Xi^* t_1, \dots, \Xi^* t_t).$$

We shall verify $k^r = \deg \Xi$ by a suitable choice of $(m_1, \dots, m_r)$. For a fixed $(m_1, \dots, m_r)$, we define integers $a_{i,j}$ for $1 \le i \le 2$ and $1 \le j \le r$ by

$$\alpha_1 = \sum\nolimits_{j=1}^{r} a_{1,j} m_j \quad \text{and} \quad \alpha_2 = \sum\nolimits_{j=1}^{r} a_{2,j} m_j. \tag{IV-5}$$

Case 1. First, we consider the case where $v_1 = v_2$. We set $v^\circ := v_1 = v_2$. Since v° is a primitive element of N, we can take a free basis $(m_1, m_2, \dots, m_r)$ such that $\langle m_1, v^\circ \rangle = 1$ and $\langle m_j, v^\circ \rangle = 0$ for any $j > 1$. Then

$$\langle \alpha_1, v^\circ \rangle = a_{1,1} > 0 \quad \text{and} \quad \langle \alpha_2, v^\circ \rangle = a_{2,1} > 0$$

by (IV-5). From (IV-3) in Remark 4.12, we have $\Xi^* t_i = t_i^k$ for any $i \ge 2$, and

$$\Xi^* t_1 = t_1^k P_1(\boldsymbol{e}(-\alpha_1)) P_2(\boldsymbol{e}(-\alpha_2)).$$

By Corollary 2.6, $\Xi^* t_1 \in \Bbbk[t_1, t_2^{\pm 1}, \dots, t_r^{\pm 1}]$, since

$$k + \min\{0, -d_1 a_{1,1}\} + \min\{0, -d_2 a_{2,1}\} = k - d_1 a_{1,1} - d_2 a_{2,1} \ge 0$$

by Proposition 4.13(i). Hence, there is a polynomial $Q(\mathtt{x}) \in K[\mathtt{x}]$ of degree k for the field $K = \Bbbk(t_2, \dots, t_r)$ such that $Q(t_1) - \Xi^* t_1 = 0$. Thus,

$$\deg K(t_1)/K(\Xi^* t_1) = k$$

by Lemma 4.16 below. On the other hand, $\deg K(\Xi^* t_1)/K^\dagger(\Xi^* t_1) = \deg K/K^\dagger = k^{r-1}$ for the subfield $K^\dagger = \Bbbk(\Xi^* t_2, \dots, \Xi^* t_r)$ of K. Thus,

$$\deg \Xi = \deg \Bbbk(t_1, t_2, \dots, t_r)/\Bbbk(\Xi^* t_1, \Xi^* t_2, \dots, \Xi^* t_r) = \deg K(t_1)/K^\dagger(\Xi^* t_1) = k^r.$$

Case 2. Second, we consider the case where $v_1 \ne v_2$. We may assume that $\langle \alpha_2, v_1 \rangle = 0$ in Proposition 4.13(ii). Then primitive elements v_1 and v_2 of N are linearly independent. Hence, we can find a free basis $(m_1, m_2, \dots, m_r)$ of M and integers p and q such that $0 \le p < q$, $\gcd(p, q) = 1$,

$$\langle m_1, v_1 \rangle = 1, \quad \langle m_1, v_2 \rangle = p, \quad \langle m_2, v_1 \rangle = 0, \quad \langle m_2, v_2 \rangle = q, \tag{IV-6}$$

and $(m_3, m_4, \dots, m_r)$ is a free basis of $\{v_1, v_2\}^\perp \cap \mathsf{M}$. Then

$$\begin{aligned} \langle \alpha_1, v_1 \rangle &= a_{1,1} > 0, \quad \langle \alpha_1, v_2 \rangle = a_{1,1}\, p + a_{1,2}\, q \le 0, \\ \langle \alpha_2, v_1 \rangle &= a_{2,1} = 0, \quad \langle \alpha_2, v_2 \rangle = a_{2,1}\, p + a_{2,2}\, q = a_{2,2}\, q > 0, \end{aligned} \tag{IV-7}$$

by (IV-5). In particular, $a_{1,2} \le -a_{1,1} p/q \le 0$, and $a_{2,2} > 0$. From (IV-3) in Remark 4.12, we have $\Xi^* t_j = t_j^k$ for any $j \ge 3$,

$$\Xi^* t_2 = t_2^k P_2(\boldsymbol{e}(-\alpha_2))^q, \quad \text{and} \quad \Xi^* t_1 = t_1^k P_1(\boldsymbol{e}(-\alpha_1)) P_2(\boldsymbol{e}(-\alpha_2))^p.$$

By Corollary 2.6, $\Xi^* t_2 \in \Bbbk[t_2, t_3^{\pm 1}, \dots, t_r^{\pm 1}]$ and $\Xi^* t_1 \in \Bbbk[t_1, t_2^{\pm 1}, \dots, t_r^{\pm 1}]$, since

$$\begin{aligned} &k + d_2 q \min\{0, -a_{2,2}\} = k - d_2 q a_{2,2} \ge 0 \quad \text{and} \\ &k + d_1 \min\{0, -a_{1,1}\} + p d_2 \min\{0, -a_{2,1}\} = k - d_1 a_{1,1} \ge 0 \end{aligned}$$

by Proposition 4.13(ii). We set $L_2 := \Bbbk(t_3, \dots, t_r)$ and $L_1 := \Bbbk(t_2, \dots, t_r) = L_2(t_2)$ as subfields of $\Bbbk(t_1, \dots, t_r)$. Then there exist polynomials $Q_2(\mathtt{x}) \in L_2[\mathtt{x}]$ and $Q_1(\mathtt{x}) \in L_1[\mathtt{x}]$ of degree k such that $Q_2(t_2) - \Xi^* t_2 = 0$ and $Q_1(t_1) - \Xi^* t_1 = 0$. By Lemma 4.16 below, we have

$$\deg L_2(t_2)/L_2(\Xi^* t_2) = \deg L_2(\Xi^* t_1, t_2)/L_2(\Xi^* t_1, \Xi^* t_2) = \deg L_1(t_1)/L_1(\Xi^* t_1) = k,$$

where $L_2(\Xi^* t_1, t_2) = L_1(\Xi^* t_1)$. On the other hand,

$$\deg L_2(\Xi^* t_1, \Xi^* t_2)/L_2^{\dagger}(\Xi^* t_1, \Xi^* t_2) = \deg L_2/L_2^{\dagger} = k^{r-2}$$

for the subfield $L_2^{\dagger} = \Bbbk(\Xi^* t_3, \dots, \Xi^* t_r)$ of L_2. Hence,

$$\deg \Xi = \deg \Bbbk(t_1, \dots, t_r)/\Bbbk(\Xi^* t_1, \dots, \Xi^* t_r) = \deg L_1(t_1)/L_2^{\dagger}(\Xi^* t_1, \Xi^* t_2) = k^r.$$

Thus, we are done. □

In the proof of Lemma 4.15 above, we use the following:

Lemma 4.16. *For a field K and two variables $\mathtt{x}$ and $\mathtt{y}$, let $L = K(\mathtt{y})$ be the field pure transcendental over K and let $Q(\mathtt{x})$ be a polynomial in $K[\mathtt{x}]$ of degree $k > 0$. Then $\mathtt{y} - Q(\mathtt{x})$ is irreducible in $L[\mathtt{x}]$ and $\deg M/L = k$ for the field $M = L[\mathtt{x}]/(\mathtt{y} - Q(\mathtt{x}))$.*

Proof. This follows from the irreducibility of $\mathtt{y} - Q(\mathtt{x})$ in $K[\mathtt{x}, \mathtt{y}]$. □

The endomorphism $\Xi \colon X \to X$ is finite by Theorem 1.1. Thus, we can consider the pullback of a divisor by Ξ (cf. Section 3.2).

Proposition 4.17. *For any divisor F on X, the pullback $\Xi^* F$ is $\mathbb{Q}$-linearly equivalent to kF. Moreover, the following hold, where E_i is the effective divisor $E_{\alpha_i}(P_i)$ defined in Lemma 4.9, for $i = 1$, 2:*

1. *If $v \in \mathcal{P}_{\mathsf{N}}(\triangle) \setminus \{v_1, v_2\}$, then $\Xi^* \boldsymbol{\Gamma}(v) = k \boldsymbol{\Gamma}(v)$.*
2. *If $v_1 = v_2$, then*
$$\Xi^* \boldsymbol{\Gamma}(v^\circ) = k^\circ \boldsymbol{\Gamma}(v^\circ) + E_1 + E_2$$
for $v^\circ := v_1 = v_2$, where $k^\circ := k - d_1 \langle \alpha_1, v^\circ \rangle - d_2 \langle \alpha_2, v^\circ \rangle$.
3. *If $v_1 \neq v_2$, then*
$$\Xi^* \boldsymbol{\Gamma}(v_i) = k_i \boldsymbol{\Gamma}(v_i) + E_i$$
for any $i = 1$, 2, where $k_i := k - d_i \langle \alpha_i, v_i \rangle$.

Proof. We may assume that $\langle \alpha_1, v_2 \rangle = 0$ in Proposition 4.13(ii). Note that the convex cone $\boldsymbol{\sigma}_\alpha := \mathsf{R}(v_1) + \mathsf{R}(v_2)$ belongs to $\triangle$. In fact, this is trivial in case $v_1 = v_2$, and if $v_1 \neq v_2$, then $\{\alpha_1\}^\perp \cap \mathsf{R}(v_2) = \mathsf{R}(v_2)$ by Proposition 4.13(ii), and $\boldsymbol{\sigma}_\alpha \in \triangle$ by Lemma 4.2. We set

$$D_\alpha := \sum\nolimits_{v \in \mathcal{P}_{\mathsf{N}}(\triangle) \setminus \{v_1, v_2\}} \boldsymbol{\Gamma}(v).$$

First, we shall show (1). Since $\langle \alpha_i, v \rangle \leq 0$, $\boldsymbol{e}(-\alpha_i)$ is regular on $\mathbb{T}_{\mathsf{N}}(\mathsf{R}(v))$. Note that $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma}_\alpha) \cap \boldsymbol{\Gamma}(v) = \emptyset$. Since Ξ induces an endomorphism of $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma}_\alpha)$ (cf. Proposition 4.13), we have $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma}_\alpha) \cap \Xi^{-1} \boldsymbol{\Gamma}(v) = \emptyset$. In particular, $\Xi^{-1} \boldsymbol{\Gamma}(v) \subset D_\alpha$.

By Lemma 4.9, $P_i(\boldsymbol{e}(-\alpha_i))$ is a local defining equation of E_i in $\mathbb{T}_{\mathsf{N}}(\mathsf{R}(v))$ for $i = 1$, 2. We set $\mathcal{U} := \mathbb{T}_{\mathsf{N}}(\mathsf{R}(v)) \setminus (E_1 \cup E_2)$ and $\psi_i := P_i(\boldsymbol{e}(-\alpha_i))|_{\mathcal{U}}$ for $i = 1$, 2. Then

$$\Xi^*\boldsymbol{e}(m)|_{\mathcal{U}} = \boldsymbol{e}(m)^k \psi_1^{\langle m, v_1\rangle} \psi_2^{\langle m, v_2\rangle} \tag{IV-8}$$

for any $m \in \mathsf{M}$ by (IV-3) in Remark 4.12. We can take an element $m_1 \in \mathsf{M}$ such that $\langle m_1, v\rangle = 1$, since v is primitive. Then $m_1 \in \mathsf{R}(v)^\vee \cap \mathsf{M}$, and $\boldsymbol{e}(m_1)$ is a local defining equation of $\boldsymbol{\Gamma}(v)$ in $\mathbb{T}_{\mathsf{N}}(\mathsf{R}(v))$. Hence, $\Xi^*\boldsymbol{\Gamma}(v)|_{\mathcal{U}} = k\boldsymbol{\Gamma}(v)|_{\mathcal{U}}$ by (IV-8) for m_1. Therefore,

$$\Xi^*\boldsymbol{\Gamma}(v) = k\boldsymbol{\Gamma}(v) + G$$

for an effective divisor G supported on $D_\alpha - \boldsymbol{\Gamma}(v)$. If $\boldsymbol{\Gamma}(v') \subset \operatorname{Supp} G$ for some $v' \in \mathcal{P}_{\mathsf{N}}(\triangle)$, then $\boldsymbol{\Gamma}(v') \subset \Xi^{-1}\boldsymbol{\Gamma}(v')$ by the same argument as above for v', and hence, $\Xi(\boldsymbol{\Gamma}(v')) \subset \boldsymbol{\Gamma}(v) \cap \boldsymbol{\Gamma}(v')$. This contradicts the finiteness of Ξ, since $\dim \boldsymbol{\Gamma}(v) \cap \boldsymbol{\Gamma}(v') < \dim \boldsymbol{\Gamma}(v')$. Therefore, $G = 0$, and we have proved (1).

Next, we shall prove the first assertion of Proposition 4.17. Every divisor on X is linearly equivalent to a divisor supported on the boundary divisor D, since the divisor class group of the open torus $X \setminus D$ is zero. By (1), it is enough to show that, for $i = 1$, 2, some multiple of $\boldsymbol{\Gamma}(v_i)$ is linearly equivalent to a divisor supported on D_α. Now, we have

$$\operatorname{div}(\boldsymbol{e}(\alpha_i)) = \langle \alpha_i, v_1\rangle \boldsymbol{\Gamma}(v_1) + \langle \alpha_i, v_2\rangle \boldsymbol{\Gamma}(v_2) + \sum\nolimits_{v \in \mathcal{P}_{\mathsf{N}}(\triangle)\setminus\{v_1, v_2\}} \langle \alpha_i, v\rangle \boldsymbol{\Gamma}(v).$$

Thus, the assertion holds if $v_1 = v_2$. Even in case $v_1 \neq v_2$, since we have assumed $\langle \alpha_1, v_2\rangle = 0$, the assertion holds by equalities above for $\operatorname{div}(\boldsymbol{e}(\alpha_1))$ and $\operatorname{div}(\boldsymbol{e}(\alpha_2))$. The remaining assertions (2) and (3) are shown as follows:

(2): For $v^\circ = v_1 = v_2$, we have $\boldsymbol{\sigma}_\alpha = \mathsf{R}(v^\circ)$. Let $(m_1, m_2, \ldots, m_r)$ be the free basis of M in the proof of Lemma 4.15 taken for the case: $v_1 = v_2$. Then $(m_2, m_3, \ldots, m_r)$ is a free basis of $\{v^\circ\}^\perp \cap \mathsf{M}$ and $\langle m_1, v^\circ\rangle = 1$. As in the proof of (1), $\boldsymbol{e}(m_1)$ is a defining equation of $\boldsymbol{\Gamma}(v^\circ)$ in $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma}_\alpha)$. Hence,

$$\Xi^*\boldsymbol{\Gamma}(v^\circ)|_{\mathbb{T}_{\mathsf{N}}(v^\circ)} = \operatorname{div}(\Xi^*\boldsymbol{e}(m_1))|_{\mathbb{T}_{\mathsf{N}}(v^\circ)}.$$

On the other hand, we have

$$\Xi^*\boldsymbol{e}(m_1) = \boldsymbol{e}(km_1) P_1(\boldsymbol{e}(-\alpha_1)) P_2(\boldsymbol{e}(-\alpha_2))$$

by (IV-3) in Remark 4.12. Hence,

$$\Xi^*\boldsymbol{\Gamma}(v^\circ)|_{\mathbb{T}_{\mathsf{N}}(v^\circ)} = (k^\circ \boldsymbol{\Gamma}(v^\circ) + E_1 + E_2)|_{\mathbb{T}_{\mathsf{N}}(v^\circ)}$$

by Lemma 4.9. For any $v \in \mathcal{P}_{\mathsf{N}}(\triangle) \setminus \{v^\circ\}$, the prime divisor $\boldsymbol{\Gamma}(v)$ is not contained in $\Xi^{-1}\boldsymbol{\Gamma}(v^\circ)$ by (1) and by the finiteness of Ξ. Thus, we have the equality in (2).

(3): Let $(m_1, m_2, \ldots, m_r)$ be the free basis of M in the proof of Lemma 4.15 taken for the case: $v_1 \neq v_2$. Then $(m_3, m_4, \ldots, m_r)$ is a free basis of $\boldsymbol{\sigma}_\alpha^\perp \cap \mathsf{M}$, and values of $\langle m_i, v_j\rangle$ for $1 \leq i, j \leq 2$ are given as in (IV-6) in the proof of Lemma 4.15 for mutually coprime integers p and q such that $0 \leq p < q$. We set $\widetilde{m}_1 := qm_1 - pm_2 \in \boldsymbol{\sigma}_\alpha^\vee \cap \mathsf{M}$. Then $\langle \widetilde{m}_1, v_1\rangle = q$, $\langle \widetilde{m}_1, v_2\rangle = 0$, $\langle m_2, v_1\rangle = 0$,

$\langle m_2, v_2\rangle = q$, and moreover,

$$\{m \in \boldsymbol{\sigma}_\alpha^\vee \cap \mathsf{M} \mid \langle m, v_1\rangle \geq q\} = \widetilde{m}_1 + \boldsymbol{\sigma}_\alpha^\vee \cap \mathsf{M},$$
$$\{m \in \boldsymbol{\sigma}_\alpha^\vee \cap \mathsf{M} \mid \langle m, v_2\rangle \geq q\} = m_2 + \boldsymbol{\sigma}_\alpha^\vee \cap \mathsf{M}.$$

Hence, $q\boldsymbol{\Gamma}(v_1)$ and $q\Gamma(v_2)$ are Cartier divisors on $\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma}_\alpha)$ with local defining equations $\boldsymbol{e}(\widetilde{m}_1)$ and $\boldsymbol{e}(m_2)$, respectively (cf. Remark 2.3). Thus,

$$\Xi^*(q\boldsymbol{\Gamma}(v_1))|_{\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma}_\alpha)} = \mathrm{div}(\Xi^*\boldsymbol{e}(\widetilde{m}_1))|_{\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma}_\alpha)} \quad \text{and}$$
$$\Xi^*(q\boldsymbol{\Gamma}(v_2))|_{\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma}_\alpha)} = \mathrm{div}(\Xi^*\boldsymbol{e}(m_2))|_{\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma}_\alpha)}.$$

On the other hand, we have

$$\Xi^*\boldsymbol{e}(\widetilde{m}_1) = \boldsymbol{e}(k\widetilde{m}_1)P_1(\boldsymbol{e}(-\alpha_1))^q \quad \text{and} \quad \Xi^*\boldsymbol{e}(m_2) = \boldsymbol{e}(km_2)P_2(\boldsymbol{e}(-\alpha_2))^q$$

by (IV-3) in Remark 4.12, and it implies that

$$\Xi^*(q\boldsymbol{\Gamma}(v_i))|_{\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma}_\alpha)} = (qk_i\boldsymbol{\Gamma}(v_1) + qE_1)\,|_{\mathbb{T}_{\mathsf{N}}(\boldsymbol{\sigma}_\alpha)}$$

for $i = 1, 2$, by Lemma 4.9. Now, $\Xi^{-1}\boldsymbol{\Gamma}(v_1)$ and $\Xi^{-1}\boldsymbol{\Gamma}(v_2)$ do not contain $\boldsymbol{\Gamma}(v)$ for any $v \in \mathcal{P}_{\mathsf{N}}(\triangle) \setminus \{v_1, v_2\}$ by (1) and by finiteness of Ξ. Thus, (3) holds, and we are done. □

Remark. When X is projective, we have another proof of Lemma 4.15 by applying Proposition 4.17, since $\Xi^*A \sim_{\mathbb{Q}} kA$ for an ample divisor A on X and since $(\Xi^*A)^r = (\deg \Xi)A^r$ for $r = \dim X$.

When $\mathrm{char}\,\Bbbk = 0$, we can consider the ramification divisor R_Ξ and the *characteristic completely invariant divisor* S_Ξ of the endomorphism Ξ (cf. Definition 3.5). In some special cases, we have a simple description of R_Ξ and S_Ξ.

Example 4.18. Assume that $\mathrm{char}\,\Bbbk = 0$. For a positive integer k, the multiplication map $\mathsf{N} \to \mathsf{N}$ by k induces a morphism $(\mathsf{N}, \triangle) \to (\mathsf{N}, \triangle)$ of fans, and we have an associated endomorphism μ_k of $X = \mathbb{T}_{\mathsf{N}}(\triangle)$. Then μ_k is a k-th power map, i.e., it induces $\mathbb{T}_{\mathsf{N}}\langle R\rangle \simeq \mathsf{N} \otimes_{\mathbb{Z}} R^\star \ni u \mapsto u^k$ for any $\Bbbk$-algebra R. In particular, μ_k induces a finite étale endomorphism of the open torus $X \setminus D$. Moreover, $\mu_k = \Xi_{\alpha,k}(1)$ for any trigger α in the sense of Definition 4.10.

For a root α of $(\mathsf{N}, \triangle)$ and for $\lambda \in \Bbbk \setminus \{0\}$, let $x_\alpha(\lambda)$ be the automorphism $\Xi_{\alpha,1}(1 + \lambda\mathtt{x})$ as in Remark 4.11. For integers k and l greater than 1, we set

$$\Xi := \mu_l \circ x_\alpha \circ \mu_k$$

as an endomorphism of X and consider its ramification divisor. Note that $\deg \Xi = (kl)^r$ and that $\Xi = \Xi_{k\alpha,kl}(P)$ for the polynomial $P(\mathtt{x}) = (1 + \lambda\mathtt{x})^l$ by equalities

$$\mu_k^*\boldsymbol{e}(m) = \boldsymbol{e}(km) \quad \text{and} \quad x_\alpha(\lambda)^*\boldsymbol{e}(m) = \boldsymbol{e}(m)(1 + \lambda\boldsymbol{e}(-\alpha))^{\langle m, v_\alpha\rangle}$$

for any $m \in \mathsf{M}$ (cf. (IV-3) in Remark 4.12). Now, $\mu_k^*\boldsymbol{\Gamma}(v) = k\boldsymbol{\Gamma}(v)$ for any $v \in \mathcal{P}_{\mathsf{N}}(\triangle)$: This is a special case of Proposition 4.17(1). Moreover, $x_\alpha(\lambda)^*\boldsymbol{\Gamma}(v) = \boldsymbol{\Gamma}(v)$ for any $v \in \mathcal{P}_{\mathsf{N}}(\triangle) \setminus \{v_\alpha\}$. For $v = v_\alpha$, note that $x_\alpha(\lambda)^*\boldsymbol{\Gamma}(v_\alpha)$ is a prime divisor not contained in the boundary divisor D, since we have assumed $\lambda \neq 0$. Hence,

$$F_\alpha := \mu_k^*(x_\alpha(\lambda)^*\boldsymbol{\Gamma}(v_\alpha))$$

is a reduced divisor having no common prime components with D. Since the ramification divisor R_{μ_k} of μ_k equals $(k-1)D$, the ramification divisor R_Ξ of Ξ is expressed as

$$\begin{aligned} R_\Xi &= R_{\mu_k} + \mu_k^*(x_\alpha(\lambda)^* R_{\mu_l}) = (k-1)D + (l-1)\mu_k^*(x_\alpha(\lambda)^* D)) \\ &= (k-1)D + (l-1)k(D - \boldsymbol{\Gamma}(v_\alpha)) + (l-1)F_\alpha \\ &= (kl-1)(D - \boldsymbol{\Gamma}(v_\alpha)) + (k-1)\boldsymbol{\Gamma}(v_\alpha) + (l-1)F_\alpha. \end{aligned}$$

In particular, for a prime divisor Θ not contained in $D - \boldsymbol{\Gamma}(v_\alpha)$, the ramification index of Ξ along Θ is less than kl, since we have assumed $k,\ l > 1$. By Proposition 4.17, we know that $\Xi^* F \sim_{\mathbb{Q}} klF$ for any divisor F on X. Hence, the characteristic completely invariant divisor S_Ξ equals $D - \boldsymbol{\Gamma}(v_\alpha)$ by Lemma 3.6.

For a general $\Xi = \Xi_{\alpha_1,\alpha_2}(P_1, P_2)$, we have only the following weaker result on R_Ξ and S_Ξ.

Lemma 4.19. *Assume that* $\Bbbk$ *is of characteristic zero,* X *is projective, and*

$$k - 2 \geq (d_1\langle \alpha_1, v_1\rangle \boldsymbol{\Gamma}(v_1) + d_2\langle \alpha_2, v_2\rangle \boldsymbol{\Gamma}(v_2))H^{r-1} \tag{IV-9}$$

for an ample divisor H *on* X*. Then the following hold for the divisor*

$$D_\alpha := \sum\nolimits_{v \in \mathcal{P}_{\mathsf{N}}(\triangle) \setminus \{v_{\alpha_1}, v_{\alpha_2}\}} \boldsymbol{\Gamma}(v) :$$

(1) *The ramification index of* Ξ *along any prime divisor* Θ *on* X *not contained in* D_α *is less than* k.
(2) *Let* g *be an automorphism of* X *such that* $g^{-1}D = D$ *and* $g^{-1}D_\alpha = D_\alpha$. *Then* D_α *equals the characteristic completely invariant divisor* $S_{g\circ\Xi}$ *of the endomorphism* $g \circ \Xi$.

Proof. For a prime divisor Θ, let r_Θ be the ramification index of Ξ along Θ, i.e., $r_\Theta = \operatorname{mult}_\Theta \Xi^*(\Xi(\Theta))$. For the automorphism g in (2), the ramification index of $g \circ \Xi$ along Θ is also equal to r_Θ. Hence, (2) is a consequence of (1) by Lemma 3.6 and Proposition 4.17.

Assertion (1) is shown as follows: If $\Theta = \boldsymbol{\Gamma}(v_i)$ for some $i = 1, 2$, then $r_\Theta = k_i$ (resp. $= k^\circ$) in case $v_1 \neq v_2$ (resp. $v_1 = v_2$) by (3) (resp. (2)) of Proposition 4.17. Here, $k > k_i$ (resp. $k > k^\circ$) by (IV-9), and (1) holds for this Θ.

Assume that Θ is a prime component of E_i for some $i = 1, 2$, where E_1 and E_2 are as in Proposition 4.17 (cf. Lemma 4.9). Then $\Theta|_{X \setminus D}$ is defined by an irreducible element $Q \in \Bbbk[\mathsf{M}]$ which is a factor of $P_i(\boldsymbol{e}(-\alpha_i))$. Here, $d_i = \deg P_i > 0$ by the existence of Θ. Let

$$P_i(\mathtt{x}) = c \prod\nolimits_{t=1}^{l} (\mathtt{x} - \lambda_t)^{n_t}$$

be the polynomial factorization in $\overline{\Bbbk}[\mathtt{x}]$ for an algebraic closure $\overline{\Bbbk}$ of $\Bbbk$, where $c \in \Bbbk \setminus \{0\}$, $\lambda_t \in \overline{\Bbbk} \setminus \{0\}$, and $d_i = \sum_{t=1}^{l} n_t$. If $\alpha \in \mathsf{M}$ is written as $a\beta$ for a primitive element $\beta \in \mathsf{M}$ and an integer $a > 0$, then

$$\boldsymbol{e}(\alpha) - \lambda^a = \prod\nolimits_{s=1}^{a-1} (\boldsymbol{e}(\beta) - \zeta^s \lambda)$$

for any $\lambda \in \overline{\Bbbk}$, where ζ is a primitive a-th root of unity, and each factor $\boldsymbol{e}(\beta) - \zeta^s\lambda$ is irreducible in $\overline{\Bbbk}[\mathsf{M}]$. This implies that the multiplicity of Q in $P_i(\boldsymbol{e}(-\alpha_i))$ is at most $d_i = \deg P_i$, or equivalently, $\operatorname{mult}_\Theta E_i \le d_i$. On the other hand, $r_\Theta = \operatorname{mult}_\Theta E_i$ by (2) and (3) of Proposition 4.17. Therefore, $r_\Theta \le k - 1$ by (IV-9). Thus, (1) holds for this Θ.

For the rest of the proof, it suffices to prove $r_\Theta \le k - 1$ for any prime divisor Θ not contained in $D + E_1 + E_2$. By Proposition 4.17, we have $\Xi^{-1}(D) = D + E_1 + E_2$. As a ramification formula for Ξ, we have

$$K_{\mathbb{T}_{\mathsf{N}}(\triangle)} + D + (E_1 + E_2)_{\mathrm{red}} = \Xi^*(K_{\mathbb{T}_{\mathsf{N}}(\triangle)} + D) + \Delta$$

for the effective divisor

$$\Delta := \sum\nolimits_{\Theta \not\subset D + (E_1+E_2)_{\mathrm{red}}} (r_\Theta - 1)\Theta$$

(cf. Remark 3.4). Hence, $(E_1 + E_2)_{\mathrm{red}} \sim \Delta$ by $K_{\mathbb{T}_{\mathsf{N}}(\triangle)} + D \sim 0$. If $v^\circ = v_1 = v_2$, then

$$E_1 + E_2 \sim \Xi^*\Gamma(v^\circ) - k^\circ \boldsymbol{\Gamma}(v^\circ) \sim (k - k^\circ)\boldsymbol{\Gamma}(v^\circ) = \sum\nolimits_{i=1}^{2} d_i \langle \alpha_i, v_i \rangle \boldsymbol{\Gamma}(v_i)$$

by Proposition 4.17. If $v_1 \ne v_2$, then

$$E_1 + E_2 \sim \sum\nolimits_{i=1}^{2} (\Xi^*\boldsymbol{\Gamma}(v_i) - k_i \boldsymbol{\Gamma}(v_i)) \sim \sum\nolimits_{i=1}^{2} d_i \langle \alpha_i, v_i \rangle \boldsymbol{\Gamma}(v_i)$$

by Proposition 4.17. Hence, in both cases, by (IV-9), we have

$$\begin{aligned} k - 2 \ge (E_1 + E_2)H^{r-1} &\ge (E_1 + E_2)_{\mathrm{red}} H^{r-1} \\ &= \Delta H^{r-1} \ge (r_\Theta - 1)\Theta H^{r-1} \ge r_\Theta - 1 \end{aligned}$$

for any $\Theta \subset \operatorname{Supp} \Delta$. Thus, we are done. □

4.4. **Proof of Theorem 1.2.**

Proof of Theorem 1.2. We write $X = \mathbb{T}_{\mathsf{N}}(\triangle)$ for a free abelian group N of rank r and for a complete fan $\triangle$ of N. For the boundary divisor $D = \mathbb{B}_{\mathsf{N}}(\triangle)$, we are given a reduced divisor B contained in D such that, for any prime component Γ of B, some multiple of Γ is linearly equivalent to an effective divisor not containing Γ. We shall construct an endomorphism of X satisfying conditions (1)–(5) of Theorem 1.2. Note that Theorem 1.2(6) follows from (1), (2), and (5) of Theorem 1.2 by Lemma 3.6.

First, we consider the case where B is a prime divisor. Then there is a trigger α for endomorphisms on $(\mathsf{N}, \triangle)$ such that $B = \boldsymbol{\Gamma}(v_\alpha)$, by Corollary 4.7(1). Let Ξ be the endomorphism of X defined as $\Xi_{\alpha,k}(P)$ for an integer $k > 1$ and a polynomial $P = P(\mathtt{x})$ in $\Bbbk[\mathtt{x}]$ with $P(0) \ne 0$ such that $\deg P > 0$ and $k \ge \langle \alpha, v_\alpha \rangle \deg P$. Then Ξ satisfies (1), (2), (3) and (4) of Theorem 1.2 by Proposition 4.17 and Lemma 4.15. Suppose that $\operatorname{char} \Bbbk = 0$ and X is projective. In this situation, we take k so that

$$k - 2 \ge \langle \alpha, v_\alpha \rangle (\deg P) B H^{r-1}$$

for an ample divisor H on X. Then Ξ satisfies Theorem 1.2(5) by Lemma 4.19(1).

Second, we shall prove Theorem 1.2 by induction on the number $\boldsymbol{n}(B)$ of prime components of B. Let C be a prime component of B. By induction and by the argument above in the case of prime divisor, we may assume the existence of endomorphisms $f_1\colon X \to X$ and $f_2\colon X \to X$ with positive integers k_1 and k_2 satisfying the following conditions

(i) $f_1^*F \sim_{\mathbb{Q}} k_1 F$ and $f_2^*F \sim_{\mathbb{Q}} k_2 F$ for any divisor F on X;
(ii) $f_1^*\Gamma = k_1\Gamma$ (resp. $f_2^*\Gamma = k_2\Gamma$) for any prime component Γ of $D - C$ (resp. $D - (B - C)$);
(iii) $f_1^{-1}C \neq C$ and $f_2^{-1}\Theta \neq \Theta$ for any prime component Θ of $B - C$;
(iv) $\deg f_1 = k_1^r$ and $\deg f_2 = k_2^r$.

Then the composite $f = f_2 \circ f_1 \colon X \to X$ and the integer $k = k_1 k_2$ satisfy (1), (2), and (4) of Theorem 1.2 by (i), (ii), and (iv), respectively. We have $f^{-1}C \neq C$ by (ii) and (iii), since $f_2^{-1}C = C$. Similarly, $f^{-1}\Theta \neq \Theta$ for any prime component Θ of $B - C$. In fact, if $f^{-1}\Theta = f_1^{-1}(f_2^{-1}\Theta) = \Theta$, then we have $f_2^{-1}\Theta = \Theta$ by $f_1^{-1}\Theta = \Theta$: This is a contradiction. Thus, Theorem 1.2(3) is also satisfied. In the case where $\operatorname{char}\Bbbk = 0$ and X is projective, we may assume the following additional condition by induction:

(v) the ramification index of f_1 (resp. f_2) along a prime divisor not contained in $D - C$ (resp. $D - (B - C)$) is less than k_1 (resp. k_2).

Then Theorem 1.2(5) holds for $f = f_2 \circ f_1$ and for $k = k_1 k_2$. Thus, we are done. □

5. Endomorphisms commuting with an involution

In Section 5.1, we shall study automorphisms of a complete toric variety preserving the boundary divisor, and compare endomorphisms defined by triggers. When the automorphism is an involution, in Section 5.2, we shall prove Theorem 1.3 on the existence of certain non-isomorphic surjective endomorphisms equivariant under the involution. In this section, we fix a complete fan $\triangle$ of a non-zero free abelian group N of finite rank, and set X to be the toric variety $\mathbb{T}_{\mathsf{N}}(\triangle)$ and D to be the boundary divisor $\mathbb{B}_{\mathsf{N}}(\triangle)$.

5.1. Automorphisms of a toric variety preserving the boundary divisor.

Definition 5.1. We define $\operatorname{Aut}(X, D)$ to be the group of automorphisms $g\colon X \to X$ such that $g(D) = D$, and define $\operatorname{Aut}(\mathsf{N}, \triangle)$ to be the group of automorphisms $\phi\colon \mathsf{N} \to \mathsf{N}$ such that $\phi_{\mathbb{R}}(\boldsymbol{\sigma}) \in \triangle$ for any $\boldsymbol{\sigma} \in \triangle$. For a $\Bbbk$-valued point u of $\mathbb{T}_{\mathsf{N}}$, we define L_u to be the action of u as an automorphism in $\operatorname{Aut}(X, D)$.

Remark. The automorphism L_u is the composite of the morphism $\mathbb{T}_{\mathsf{N}} \times X \to X$ of action of $\mathbb{T}_{\mathsf{N}}$ on X and the morphism $u \times \mathrm{id}_X \colon \operatorname{Spec}\Bbbk \times X \simeq X \to \mathbb{T}_{\mathsf{N}} \times X$ defined by $u \in \mathbb{T}_{\mathsf{N}}\langle\Bbbk\rangle = \operatorname{Hom}_{\Bbbk}(\operatorname{Spec}\Bbbk, \mathbb{T}_{\mathsf{N}})$. The correspondence $u \mapsto L_u$ gives rise to a group homomorphism $\mathbb{T}_{\mathsf{N}}\langle\Bbbk\rangle = \mathsf{N} \otimes_{\mathbb{Z}} \Bbbk^\star \to \operatorname{Aut}(X, D)$.

Remark. For any $\phi \in \operatorname{Aut}(\mathsf{N}, \triangle)$, the associated morphism $\mathbb{T}_\phi$ in Definition 2.1 is considered as an automorphism in $\operatorname{Aut}(X, D)$, since ϕ is a morphism $(\mathsf{N}, \triangle) \to (\mathsf{N}, \triangle)$ of fans. Moreover, $\phi \mapsto \mathbb{T}_\phi$ gives rise to a group homomorphism $\operatorname{Aut}(\mathsf{N}, \triangle) \to \operatorname{Aut}(X, D)$.

Lemma 5.2. *The group* $\mathrm{Aut}(X, D)$ *is isomorphic to the semi-direct product* $\mathbb{T}_{\mathsf{N}}\langle\Bbbk\rangle \rtimes \mathrm{Aut}(\mathsf{N}, \triangle)$ *with respect to the homomorphism* $\mathrm{Aut}(\mathsf{N}, \triangle) \to \mathrm{Aut}(\mathbb{T}_{\mathsf{N}}\langle\Bbbk\rangle)$ *given by* $\phi \mapsto (u \mapsto \phi(u))$ *(cf. Definition* 2.7(2)*).*

Proof. The restriction homomorphism $\mathrm{Aut}(X, D) \to \mathrm{Aut}(X \setminus D)$ is injective, and $\mathrm{Aut}(X \setminus D)$ is anti-isomorphic to the group $\mathrm{Aut}(\Bbbk[\mathsf{M}]/\Bbbk)$ of $\Bbbk$-algebra automorphisms of $\Bbbk[\mathsf{M}]$. Since an invertible element of $\Bbbk[\mathsf{M}]$ is expressed as $c\boldsymbol{e}(m)$ for some $c \in \Bbbk^\star$ and $m \in \mathsf{M}$, we have an isomorphism

$$\mathrm{Aut}(\Bbbk[\mathsf{M}]/\Bbbk) \simeq \mathrm{Hom}(\mathsf{M}, \Bbbk^\star) \rtimes \mathrm{Aut}(\mathsf{M}),$$

where $\mathrm{Hom}(\mathsf{M}, \Bbbk^\star)$ is a right $\mathrm{Aut}(\mathsf{M})$-module by $\chi^\varphi(m) = \chi(\varphi(m))$ for $m \in \mathsf{M}$, $\varphi \in \mathrm{Aut}(\mathsf{M})$, and $\chi \in \mathrm{Hom}(\mathsf{M}, \Bbbk^\star)$. Therefore, $\mathrm{Aut}(X \setminus D)$ is isomorphic to $\mathbb{T}_{\mathsf{N}}\langle\Bbbk\rangle \rtimes \mathrm{Aut}(\mathsf{N})$. In particular, $\mathrm{Aut}(X \setminus D)$ is generated by actions L_u on the open torus $X \setminus D$ for all $u \in \mathbb{T}_{\mathsf{N}}\langle\Bbbk\rangle$ and by automorphisms $\mathbb{T}_\phi$ of $\mathbb{T}_{\mathsf{N}} = \mathbb{T}_{\mathsf{N}}(\{0\}) = X \setminus D$ for all $\phi \in \mathrm{Aut}(\mathsf{N})$. The automorphism $\mathbb{T}_\phi$ extends to an automorphism of X if and only if $\phi \in \mathrm{Aut}(\mathsf{N}, \triangle)$, since $\mathbb{T}_\phi$ is equivariant under the action of $\mathbb{T}_{\mathsf{N}}$ and $\triangle$ is in one-to-one correspondence with the set of orbits of $\mathbb{T}_{\mathsf{N}}$ in X (cf. [14, Prop. 1.6]). Hence, $\mathrm{Aut}(X, D)$ is generated by L_u for $u \in \mathbb{T}_{\mathsf{N}}\langle\Bbbk\rangle$ and $\mathbb{T}_\phi$ for $\phi \in \mathrm{Aut}(\mathsf{N}, \triangle)$. Therefore, $\mathrm{Aut}(X, D) \simeq \mathbb{T}_{\mathsf{N}}\langle\Bbbk\rangle \rtimes \mathrm{Aut}(\mathsf{N}, \triangle)$. □

Remark 5.3. For any $u \in \mathbb{T}_{\mathsf{N}}\langle\Bbbk\rangle$ and $\phi \in \mathrm{Aut}(\mathsf{N}, \triangle)$, we have

$$\text{(V-1)} \qquad \mathbb{T}_\phi \circ L_u \circ \mathbb{T}_\phi^{-1} = L_{\phi(u)}$$

by Lemma 5.2. This is shown directly by equalities

$$\text{(V-2)} \qquad L_u^*\boldsymbol{e}(m) = u(m)\boldsymbol{e}(m) \quad \text{and} \quad \mathbb{T}_\phi^*\boldsymbol{e}(m) = \boldsymbol{e}(\phi^\vee m)$$

for any $m \in \mathsf{M}$, where $\phi^\vee \colon \mathsf{M} \to \mathsf{M}$ stands for the dual of ϕ (cf. Definitions 2.1 and 2.7).

Definition 5.4. For an automorphism $g \in \mathrm{Aut}(X, D)$, we define $g_* \in \mathrm{Aut}(\mathsf{N}, \triangle)$ to be the image of g under the projection

$$\mathrm{Aut}(X, D) \simeq \mathbb{T}_{\mathsf{N}}\langle\Bbbk\rangle \rtimes \mathrm{Aut}(\mathsf{N}, \triangle) \to \mathrm{Aut}(\mathsf{N}, \triangle).$$

We define g^* to be the dual $(g_*)^\vee$ as an automorphism of M.

Remark 5.5. If $g = L_u \circ \mathbb{T}_\phi$ for some $u \in \mathbb{T}_{\mathsf{N}}\langle\Bbbk\rangle$ and $\phi \in \mathrm{Aut}(\mathsf{N}, \triangle)$, then $g_* = \phi$. When $\Bbbk = \mathbb{C}$, g_* is just the induced automorphism of $H_1((X \setminus D)^{\mathrm{an}}, \mathbb{Z}) \simeq H_1((\mathbb{T}_{\mathsf{N}})^{\mathrm{an}}, \mathbb{Z}) \simeq \mathsf{N}$, where ${}^{\mathrm{an}}$ indicates the associated complex analytic space.

Remark 5.6. For $u \in \mathbb{T}_{\mathsf{N}}\langle\Bbbk\rangle$ and $\phi \in \mathrm{Aut}(\mathsf{N}, \triangle)$, the automorphism $L_u \circ \mathbb{T}_\phi$ in $\mathrm{Aut}(X, D)$ is an involution if and only if $\phi^2 = \mathrm{id}_{\mathsf{N}}$ and $\phi(u) = u^{-1}$. This is shown by (V-1) in Remark 5.3.

Lemma 5.7. *For any* $u \in \mathbb{T}_{\mathsf{N}}\langle\Bbbk\rangle$ *and* $\phi \in \mathrm{Aut}(\mathsf{N}, \triangle)$ *and for the endomorphism* $\Xi_{\alpha_1,\alpha_2,k}(P_1, P_2)$ *of* $X = \mathbb{T}_{\mathsf{N}}(\triangle)$ *in Proposition* 4.13*, the equality*

$$\text{(V-3)} \qquad L_u \circ \mathbb{T}_\phi \circ \Xi_{\alpha_1,\alpha_2,k}(P_1, P_2) \circ (L_u \circ \mathbb{T}_\phi)^{-1} = L_{u^{1-k}} \circ \Xi_{\alpha_1^\dagger,\alpha_2^\dagger}(P_1^\dagger, P_2^\dagger)$$

holds as an endomorphism of X*, where* $\alpha_i^\dagger := (\phi^{-1})^\vee\alpha_i$ *and* $P_i^\dagger(\mathtt{x}) := P_i(u(\alpha_i^\dagger)\mathtt{x})$ *for* $i = 1$, 2*.*

Proof. The automorphism ϕ induces a permutation of $\mathcal{P}_{\mathsf{N}}(\triangle)$, and $\langle(\phi^{-1})^{\vee}m, v\rangle = \langle m, \phi^{-1}v\rangle$ for any $v \in \mathcal{P}_{\mathsf{N}}(\triangle)$ and $m \in \mathsf{M}$. Hence, $\alpha_i^{\dagger}$ is also a trigger and $v_{\alpha_i^{\dagger}} = \phi(v_{\alpha_i})$ for $i = 1, 2$. The equality (V-3) is deduced from two equalities

$$L_u \circ \Xi_{\alpha_1,\alpha_2,k}(P_1, P_2) \circ L_u^{-1} = L_{u^{1-k}} \circ \Xi_{\alpha_1,\alpha_2,k}(P_1^{\ddagger}, P_2^{\ddagger}), \tag{V-4}$$

$$\mathbb{T}_{\phi} \circ \Xi_{\alpha_1,\alpha_2,k}(P_1, P_2) \circ \mathbb{T}_{\phi}^{-1} = \Xi_{\alpha_1^{\dagger},\alpha_2^{\dagger},k}(P_1, P_2), \tag{V-5}$$

where $P_i^{\ddagger}(\mathtt{x}) := P_i(u(\alpha_i)\mathtt{x})$ for $i = 1, 2$ in (V-4). We can verify them using (IV-3) in Remark 4.12 and (V-2) in Remark 5.3 for any $m \in \mathsf{M}$: We set

$$\Xi := \Xi_{\alpha_1,\alpha_2,k}, \quad \Xi^{\ddagger} := \Xi_{\alpha_1,\alpha_2,k}(P_1^{\ddagger}, P_2^{\ddagger}), \quad \text{and} \quad \Xi^{\dagger} := \Xi_{\alpha_1^{\dagger},\alpha_2^{\dagger},k}(P_1, P_2).$$

Then (V-4) is equivalent to $L_{u^k} \circ \Xi = \Xi^{\ddagger} \circ L_u$, and this is shown by

$$\begin{aligned}
&L_u^*(\Xi^{\ddagger *}\boldsymbol{e}(m)) = L_u^*\left(\boldsymbol{e}(km)P_1^{\ddagger}(\boldsymbol{e}(-\alpha_1))^{\langle m,v_{\alpha_1}\rangle}P_2^{\ddagger}(\boldsymbol{e}(-\alpha_2))^{\langle m,v_{\alpha_2}\rangle}\right)\\
&= L_u^*\left(\boldsymbol{e}(km)P_1(u(\alpha_1)\boldsymbol{e}(-\alpha_1))^{\langle m,v_{\alpha_1}\rangle}P_2(u(\alpha_2)\boldsymbol{e}(-\alpha_2))^{\langle m,v_{\alpha_2}\rangle}\right)\\
&= u(km)\boldsymbol{e}(km)P_1(\boldsymbol{e}(-\alpha_1))^{\langle m,v_{\alpha_1}\rangle}P_2(\boldsymbol{e}(-\alpha_2))^{\langle m,v_{\alpha_2}\rangle}\\
&= u(km)\Xi^*\boldsymbol{e}(m) = \Xi^*(u^k(m)\boldsymbol{e}(m)) = \Xi^*(L_{u^k}^*\boldsymbol{e}(m))
\end{aligned}$$

for any $m \in \mathsf{M}$. The other equality (V-5) is equivalent to $\mathbb{T}_{\phi} \circ \Xi = \Xi^{\dagger} \circ \mathbb{T}_{\phi}$, and this is shown by

$$\begin{aligned}
&\mathbb{T}_{\phi}^*(\Xi^{\dagger *}\boldsymbol{e}(m)) = \mathbb{T}_{\phi}^*\left(\boldsymbol{e}(km)P_1(\boldsymbol{e}(-\alpha_1^{\dagger}))^{\langle m,v_{\alpha_1^{\dagger}}\rangle}P_2(\boldsymbol{e}(-\alpha_2^{\dagger}))^{\langle m,v_{\alpha_2^{\dagger}}\rangle}\right)\\
&= \mathbb{T}_{\phi}^*\left(\boldsymbol{e}(km)P_1(\boldsymbol{e}(-(\phi^{-1})^{\vee}\alpha_1))^{\langle m,\phi(v_{\alpha_1})\rangle}P_2(\boldsymbol{e}(-(\phi^{-1})^{\vee}\alpha_2))^{\langle m,\phi(v_{\alpha_2})\rangle}\right)\\
&= \boldsymbol{e}(\phi^{\vee}(km))P_1(\boldsymbol{e}(-\alpha_1))^{\langle m,\phi(v_{\alpha_1})\rangle}P_2(\boldsymbol{e}(-\alpha_2))^{\langle m,\phi(v_{\alpha_2})\rangle}\\
&= \boldsymbol{e}(k\phi^{\vee}m)P_1(\boldsymbol{e}(-\alpha_1))^{\langle \phi^{\vee}m,v_{\alpha_1}\rangle}P_2(\boldsymbol{e}(-\alpha_2))^{\langle \phi^{\vee}m,v_{\alpha_2}\rangle}\\
&= \Xi^*\boldsymbol{e}(\phi^{\vee}m) = \Xi^*(\mathbb{T}_{\phi}^*\boldsymbol{e}(m))
\end{aligned}$$

for any $m \in \mathsf{M}$. Thus, we are done. □

5.2. **Equivariant endomorphisms under involutions.** By an endomorphism defined by triggers studied in Section 4 and by results on automorphisms in Section 5.1, we shall prove Theorem 1.3 in the introduction.

Lemma 5.8. *Let $\phi\colon \mathsf{N} \to \mathsf{N}$ be an involution and let u be a $\Bbbk$-valued point of $\mathbb{T}_{\mathsf{N}}$ such that $\phi(u) = u^{-1}$. If the square map $\Bbbk \ni \lambda \to \lambda^2 \in \Bbbk$ is surjective, then there is a $\Bbbk$-valued point u_o of $\mathbb{T}_{\mathsf{N}}$ such that $u = \phi(u_o)^{-1}u_o$.*

Proof. Let N_1 be the kernel of the endomorphism $\mathrm{id}_{\mathsf{N}} + \phi\colon \mathsf{N} \to \mathsf{N}$ and let N_2 be the image of $\mathrm{id}_{\mathsf{N}} - \phi\colon \mathsf{N} \to \mathsf{N}$. Then $2\mathsf{N}_1 \subset \mathsf{N}_2 \subset \mathsf{N}_1$, since $\phi^2 = \mathrm{id}_{\mathsf{N}}$ and since $2n = n - \phi(n)$ for any $n \in \mathsf{N}_1$. Hence, by the assumption of the square map, the homomorphism

$$(\mathrm{id}_{\mathsf{N}} - \phi) \otimes \mathrm{id}_{\Bbbk^\star} : \mathsf{N} \otimes_{\mathbb{Z}} \Bbbk^\star \to \mathsf{N}_1 \otimes_{\mathbb{Z}} \Bbbk^\star$$

is surjective, which maps $u' \in \mathbb{T}_{\mathsf{N}}\langle\Bbbk\rangle = \mathsf{N} \otimes \Bbbk^\star$ to $u'\phi(u')^{-1}$. Since $u \in \mathsf{N}_1 \otimes \Bbbk^\star$, we can find an expected u_o. □

Proposition 5.9. *Let* α_1 *and* α_2 *be triggers for endomorphisms on* $(\mathsf{N}, \triangle)$ *and let* g *be an involution of* X *in* $\mathrm{Aut}(X, D)$ *such that* $g^*(\alpha_1) = \alpha_2$ *(cf. Definition* 5.4*). Suppose that either* $\langle \alpha_1, v_{\alpha_2} \rangle = 0$ *or* $\langle \alpha_2, v_{\alpha_1} \rangle = 0$*. Then*

$$\text{(V-6)} \quad v_{\alpha_1} \neq v_{\alpha_2}, \quad \langle \alpha_1, v_{\alpha_1} \rangle = \langle \alpha_2, v_{\alpha_2} \rangle > 0, \quad \text{and} \quad \langle \alpha_1, v_{\alpha_2} \rangle = \langle \alpha_2, v_{\alpha_1} \rangle = 0.$$

If $\mathbb{k}$ *is algebraically closed, then there exist polynomials* $P_1(\mathtt{x})$*,* $P_2(\mathtt{x})$ *in* $\mathbb{k}[\mathtt{x}]$ *and an element* $u_o \in \mathbb{T}_{\mathsf{N}}\langle \mathbb{k} \rangle$ *such that* $P_1(0) \neq 0$*,* $P_2(0) \neq 0$*,* $\deg P_1 = \deg P_2 > 0$*, and*

$$\text{(V-7)} \qquad g \circ L_{u_o} \circ \Xi_{\alpha_1, \alpha_2, k}(P_1, P_2) = L_{u_o} \circ \Xi_{\alpha_1, \alpha_2, k}(P_1, P_2) \circ g$$

for the endomorphism $\Xi_{\alpha_1, \alpha_2, k}(P_1, P_2)$ *of* X *in Proposition* 4.13 *defined for any integer* $k \geq \langle \alpha_1, v_{\alpha_1} \rangle \deg P_1 = \langle \alpha_2, v_{\alpha_2} \rangle \deg P_2$*.*

Proof. By Lemma 5.2, we can write $g = L_u \circ \mathbb{T}_\phi$ for some $u \in \mathbb{T}_{\mathsf{N}}\langle \mathbb{k} \rangle$ and $\phi \in \mathrm{Aut}(\mathsf{N}, \triangle)$, where $\phi^2 = \mathrm{id}_{\mathsf{N}}$ and $\phi(u) = u^{-1}$ by Remark 5.6. We have $g^*\alpha_1 = \phi^\vee \alpha_1 = \alpha_2$ and $g^*\alpha_2 = \alpha_1$ (cf. Definition 5.4). Moreover, $v_{\alpha_1} = \phi(v_{\alpha_2})$, $v_{\alpha_2} = \phi(v_{\alpha_1})$, and we have (V-6) by $\langle \phi^\vee m, n \rangle = \langle m, \phi(n) \rangle$ for any $m \in \mathsf{M}$ and $n \in \mathsf{N}$.

For the second assertion, we set $c := u(\alpha_1)$. Then $u(\alpha_2) = c^{-1}$ by $u^{-1}(\alpha_2) = \phi(u)(\alpha_2) = u(\phi^\vee \alpha_2) = u(\alpha_1)$. We can take polynomials $P_1(\mathtt{x})$ and $P_2(\mathtt{x})$ in $\mathbb{k}[\mathtt{x}]$ such that $P_2(0) \neq 0$ and $P_1(\mathtt{x}) = P_2(c\mathtt{x})$. For any $u' \in \mathbb{T}_{\mathsf{N}}\langle \mathbb{k} \rangle$, we have

$$\text{(V-8)} \quad g \circ L_{u'} \circ g^{-1} = L_u \circ \mathbb{T}_\phi \circ L_{u'} \circ \mathbb{T}_\phi^{-1} \circ L_{u^{-1}} = L_u \circ L_{\phi(u')} \circ L_{u^{-1}} = L_{\phi(u')}$$

by (V-1) in Remark 5.3. Then

$$\begin{aligned} g \circ L_{u'} \circ \Xi_{\alpha_1, \alpha_2, k}(P_1, P_2) \circ g^{-1} &= (g \circ L_{u'} \circ g^{-1}) \circ g \circ \Xi_{\alpha_1, \alpha_2, k}(P_1, P_2) \circ g^{-1} \\ &= L_{\phi(u')u^{1-k}} \circ \Xi_{\alpha_1^\dagger, \alpha_2^\dagger, k}(P_1^\dagger, P_2^\dagger) \end{aligned}$$

by (V-3) in Lemma 5.7, where $\alpha_1^\dagger = (g^{-1})^*\alpha_1 = \alpha_2$, $\alpha_2^\dagger = (g^{-1})^*\alpha_2 = \alpha_1$, and

$$\begin{aligned} P_1^\dagger(\mathtt{x}) &= P_1(u(\alpha_1^\dagger)\mathtt{x}) = P_1(c^{-1}\mathtt{x}) = P_2(\mathtt{x}), \\ P_2^\dagger(\mathtt{x}) &= P_2(u(\alpha_2^\dagger)\mathtt{x}) = P_2(c\mathtt{x}) = P_1(\mathtt{x}) \end{aligned}$$

by the choices of P_1 and P_2. In particular,

$$\Xi_{\alpha_1^\dagger, \alpha_2^\dagger, k}(P_1^\dagger, P_2^\dagger) = \Xi_{\alpha_2, \alpha_1, k}(P_2, P_1) = \Xi_{\alpha_1, \alpha_2, k}(P_1, P_2).$$

On the other hand, by Lemma 5.8, we can find an element $u_o \in \mathbb{T}_{\mathsf{N}}\langle \mathbb{k} \rangle$ such that

$$u_o = \phi(u_o)u^{1-k},$$

since $\phi(u^{1-k}) = u^{k-1}$. Thus, (V-7) holds for this u_o, and we are done. □

Proposition 5.10. *Let* α *be a trigger for endomorphisms on* $(\mathsf{N}, \triangle)$ *and let* g *be an involution of* X *in* $\mathrm{Aut}(X, D)$ *such that* $g^*(\alpha) = \alpha$ *(cf. Definition* 5.4*). If* $\mathbb{k}$ *is algebraically closed, then there exist a polynomial* $P(\mathtt{x})$ *in* $\mathbb{k}[\mathtt{x}]$ *and an element* $u_o \in \mathbb{T}_{\mathsf{N}}\langle \mathbb{k} \rangle$ *such that* $P(0) \neq 0$*,* $\deg P > 0$*, and*

$$g \circ L_{u_o} \circ \Xi_{\alpha, k}(P) = L_{u_o} \circ \Xi_{\alpha, k}(P) \circ g$$

for the endomorphism $\Xi_{\alpha, k}(P) = \Xi_{\alpha, \alpha, k}(P, 1)$ *of* X *in Proposition* 4.13 *(cf. Definition* 4.10*) defined for any integer* $k \geq \langle \alpha, v_\alpha \rangle \deg P$*.*

Proof. By Lemma 5.2, we can write $g = L_u \circ \mathbb{T}_\phi$ for some $u \in \mathbb{T}_{\mathsf{N}}\langle \Bbbk \rangle$ and $\phi \in \mathrm{Aut}(\mathsf{N}, \triangle)$, where $\phi^2 = \mathrm{id}_{\mathsf{N}}$ and $\phi(u) = u^{-1}$ by Remark 5.6. Since $g^*\alpha = \phi^\vee\alpha = \alpha$, we have $u(\alpha) = \pm 1$ by $u(\alpha) = u(\phi^\vee\alpha) = \phi(u)(\alpha) = u(-\alpha)$. Let $P(\mathtt{x})$ be a polynomial in $\Bbbk[\mathtt{x}^2] \subset \Bbbk[\mathtt{x}]$ such that $P(0) \neq 0$. Then $P(-\mathtt{x}) = P(\mathtt{x})$. By (V-3) in Lemma 5.7 and by (V-8) in the proof of Proposition 5.9, we have

$$\begin{aligned} g \circ L_{u'} \circ \Xi_{\alpha,k}(P) \circ g^{-1} &= g \circ L_{u'} \circ g^{-1} \circ g \circ \Xi_{\alpha,\alpha,k}(P, 1) \circ g^{-1} \\ &= L_{\phi(u')u^{1-k}} \circ \Xi_{\alpha^\dagger,k}(P^\dagger) \end{aligned}$$

for any $u' \in \mathbb{T}_{\mathsf{N}}\langle \Bbbk \rangle$, where $\alpha^\dagger = (\phi^{-1})^\vee\alpha = (g^{-1})^*\alpha = \alpha$ and $P^\dagger(\mathtt{x}) = P(u(\alpha^\dagger)\mathtt{x}) = P(u(\alpha)\mathtt{x}) = P(\mathtt{x})$. Then $\Xi_{\alpha^\dagger,k}(P_1^\dagger) = \Xi_{\alpha,k}(P)$. On the other hand, by Lemma 5.8, we can find an element $u_o \in \mathbb{T}_{\mathsf{N}}\langle \Bbbk \rangle$ such that

$$u_o = \phi(u_o)u^{1-k},$$

since $\phi(u^{1-k}) = u^{k-1}$. Thus, we are done. □

Proof of Theorem 1.3. By the assumption on the prime component Γ of D and by Corollary 4.7(2), there is a trigger α for endomorphisms on $(\mathsf{N}, \triangle)$ such that $\Gamma = \boldsymbol{\Gamma}(v_\alpha)$ and $\mathrm{div}(\boldsymbol{e}(\alpha)) = \langle \alpha, v_\alpha \rangle \Gamma - E$ for an effective divisor E supported on $D - B$, where $B = \Gamma \cup \iota(\Gamma)$. Here, $\iota^*\alpha$ is another trigger such that $\iota^*(\Gamma) = \iota(\Gamma) = \boldsymbol{\Gamma}(\iota_*(v_\alpha)) = \boldsymbol{\Gamma}(v_{\iota^*\alpha})$.

Assume that B is irreducible, i.e., $B = \Gamma = \iota(\Gamma)$. Then $v_\alpha = v_{\iota^*\alpha}$. By replacing α with $\alpha + \iota^*\alpha$, we may assume that $\iota^*\alpha = \alpha$ (cf. Remark 4.3). By applying Proposition 5.10 to the involution ι, we have a non-isomorphic surjective endomorphism f of X such that $\iota \circ f = f \circ \iota$ and that

$$f = L_{u_o} \circ \Xi_{\alpha,k}(P)$$

for some $u_o \in \mathbb{T}_{\mathsf{N}}\langle \Bbbk \rangle$, $P(\mathtt{x}) \in \Bbbk[\mathtt{x}]$, and $k \geq \langle \alpha, v_\alpha \rangle \deg P$. Here, the following hold by properties of $\Xi_{\alpha,k}(P)$ shown in Propositions 4.15 and 4.17:

- $f^*F \sim_{\mathbb{Q}} kF$ for any divisor F on X;
- $f^*\Theta = k\Theta$ for any prime component Θ of $D - \Gamma$;
- $f^{-1}\Gamma \neq \Gamma$;
- $\deg f = k^r$ for $r = \dim X = \mathrm{rank}\, \mathsf{N}$.

Thus, f satisfies conditions (1), (2), (3), and (4) of Theorem 1.2. When $\mathrm{char}\, \Bbbk = 0$ and X is projective, we may assume that

$$k - 2 \geq \langle \alpha, v_\alpha \rangle (\deg P) \Gamma H^{r-1}$$

for an ample divisor H, and the following holds by Lemma 4.19:

- the ramification index of f along any prime divisor C not contained in $D - \Gamma$ is less than k, and $S_f = D - \Gamma$.

Thus, f satisfies (5) and (6) of Theorem 1.2. Therefore, we have proved Theorem 1.3 in the case where B is irreducible.

Next, assume that B is reducible, i.e., $B = \Gamma + \iota(\Gamma)$. Then $\langle \alpha, v_{\iota^*\alpha} \rangle = 0$ for the trigger α, since $\iota(\Gamma) \not\subset \mathrm{Supp}\, E \subset D - B$. By Proposition 5.9 applied to the involution ι and triggers α and $\iota^*\alpha$, we have a non-isomorphic surjective endomorphism

f of X such that $\iota \circ f = f \circ \iota$ and that

$$f = L_{u_o} \circ \Xi_{\alpha,\, \iota^*\alpha,\, k}(P_1, P_2)$$

for some $u_o \in \mathbb{T}_{\mathsf{N}}\langle \Bbbk \rangle$, polynomials $P_1(\mathtt{x})$, $P_2(\mathtt{x}) \in \Bbbk[\mathtt{x}]$ with $P_1(0) \neq 0$, $P_2(0) \neq 0$, $\deg P_1 = \deg P_2$, and a positive integer $k \geq \langle \alpha, v_\alpha \rangle \deg P_1$. Here, the following hold by properties of $\Xi_{\alpha,\, \iota^*\alpha,\, k}(P_1, P_2)$ shown in Propositions 4.15 and 4.17:

- $f^*F \sim_{\mathbb{Q}} kF$ for any divisor F on X;
- $f^*\Theta = k\Theta$ for any prime component Θ of $D - B$;
- $f^{-1}\Gamma \neq \Gamma$ and $f^{-1}\iota(\Gamma) \neq \iota(\Gamma)$;
- $\deg f = k^r$ for $r = \dim X = \operatorname{rank} \mathsf{N}$.

Thus, f satisfies conditions (1), (2), (3), and (4) of Theorem 1.2. When $\operatorname{char} \Bbbk = 0$ and X is projective, we may assume

$$k - 2 \geq \langle \alpha, v_\alpha \rangle (\deg P_1) B H^{r-1}$$

for an ample divisor H, and the following holds by Lemma 4.19:

- the ramification index of f along any prime divisor C not contained in $D - B$ is less than k, and $S_f = D - B$.

Thus, f satisfies (5) and (6) of Theorem 1.2. Therefore, we have proved Theorem 1.3 also in the case where B is reducible. Thus, we are done. □

6. Endomorphisms of toric and half-toric surfaces

We shall apply results in Sections 4 and 5 to the study of non-isomorphic surjective endomorphisms of projective toric surfaces surfaces and *half-toric surfaces* defined in [10, §7.1] over $\Bbbk = \mathbb{C}$. A prime divisor Γ on a normal projective surface is said to be *negative* if the self-intersection number Γ^2 is negative, where we note that the intersection number of two (Weil) divisors on a normal projective surface is well defined by Mumford's numerical pullback (cf. [8, II, (b), p. 17], [15, §1], [10, §2.1]). We set $\Bbbk = \mathbb{C}$ in this section.

Theorem 6.1. *Let (X, D) be a* projective toric surface*, i.e., X is a projective toric variety of dimension 2 and D is the boundary divisor. Let B be a reduced divisor contained in D such that any prime component of B is not a negative curve. Then there is a non-isomorphic surjective endomorphism f of X such that $S_f = D - B$.*

Proof. By assumption, any prime component Γ of B is semi-ample by [12, Thm. 1.5], since $-K_X \sim D$ is big. Thus, the assertion is a consequence of Theorem 1.2. □

Theorem 6.2. *Let (X, D) be a* half-toric *surface and let C be an end component of D such that a prime component of τ^*C is not a negative curve for the* characteristic double cover $\tau \colon \widetilde{X} \to X$ *(cf. Remark 6.3 below). Then X admits a non-isomorphic surjective endomorphism f such that $S_f = D - C$.*

Remark 6.3. The pair (X, D) of a normal projective surface X and a reduced divisor D is called a *half-toric surface* if $K_X + D \not\sim 0$, $2(K_X + D) \sim 0$, and if there is a double cover $\tau \colon \widetilde{X} \to X$ such that

- $(\widetilde{X}, \widetilde{D})$ is a toric surface for $\widetilde{D} = \tau^{-1}D$,

- τ is étale outside a finite set, i.e., τ is *étale in codimension* 1

(cf. [10, Def. 7.1]). We call τ the *characteristic double cover*. It is an *index* 1 *cover* with respect to $K_X + D \sim_{\mathbb{Q}} 0$ in the sense of [11, Def. 4.18]. The divisor D is a *linear chain of rational curves* (cf. [10, Def. 4.1]), and the number $\boldsymbol{n}(D)$ of prime components equals $\boldsymbol{\rho}(X) + 1$ (cf. [10, Thm. 1.7(1)]). Conversely, by [10, Thm. 1.3], a half-toric surface is characterized as a pair (X, D) of a normal projective surface X and a reduced divisor D such that (X, D) is log-canonical, $\boldsymbol{n}(D) = \boldsymbol{\rho}(X) + 1$, $K_X + D \not\sim 0$, and $K_X + D \sim_{\mathbb{Q}} 0$.

Proof of Theorem 6.2. We may write $\widetilde{X} = \mathbb{T}_{\mathsf{N}}(\triangle)$ for a complete fan $\triangle$ of a free abelian group N of rank 2, where $\widetilde{D} = \tau^{-1}D$ equals the boundary divisor $\mathbb{B}_{\mathsf{N}}(\triangle)$. For the Galois involution $\iota\colon \widetilde{X} \to \widetilde{X}$ for τ, we have $\iota(\widetilde{D}) = \widetilde{D}$. It suffices to construct a non-isomorphic surjective endomorphism $\tilde{f}$ of $\widetilde{X}$ such that $\iota \circ \tilde{f} = \tilde{f} \circ \iota$ and that $S_{\tilde{f}} = \widetilde{D} - \tau^*C = \tau^{-1}(D - C)$. In fact, $\tilde{f}$ induces a non-isomorphic surjective endomorphism f of X such that $\tau \circ \tilde{f} = f \circ \tau$, and we have $S_f = D - C$ by $S_{\tilde{f}} = \tau^{-1}S_f$ (cf. [12, Lem. 2.19(3)]).

Let Γ be a prime component of τ^*C, which is not a negative curve by assumption. Then Γ is semi-ample by [12, Thm. 1.5], since $-K_{\widetilde{X}} \sim \widetilde{D}$ is big. The semi-ampleness of Γ implies that $\Gamma = \boldsymbol{\Gamma}(v_\alpha)$ for a trigger α for endomorphisms on $(\mathsf{N}, \triangle)$ by Corollary 4.7(1). Thus, if τ^*C is irreducible, i.e., $\tau^*C = \Gamma$, then we have an expected endomorphism $\tilde{f}$ of $\widetilde{X}$ by Theorem 1.3.

Assume that τ^*C is reducible. Then $\tau^*C = \Gamma + \iota(\Gamma)$, and $\iota(\Gamma) = \boldsymbol{\Gamma}(v_{\iota^*\alpha})$ for the trigger $\iota^*\alpha$. If we can take α to satisfy $\langle \alpha, v_{\iota^*\alpha} \rangle = 0$, then $\langle \alpha, v_\alpha \rangle \Gamma$ is linearly equivalent to an effective divisor supported on $\widetilde{D} - \tau^*C$, and we have an expected endomorphism $\tilde{f}$ of $\widetilde{X}$ by Theorem 1.3. Thus, by Corollary 4.8, we may assume that $\iota^*\alpha$ and $-\alpha$ are proportional. In particular, some positive multiples of Γ and $\iota(\Gamma)$ are linearly equivalent. Let Θ be a prime component of $\widetilde{D}$ intersecting Γ. Then Θ intersects $\iota(\Gamma)$ also, and it implies that $\Gamma + \iota(\Gamma) + \Theta$ is a cyclic chain of rational curves, i.e., $\widetilde{D} = \Gamma + \iota(\Gamma) + \Theta$, by a well-known property of projective toric surfaces (cf. [10, Exam. 3.4]). Moreover, $\boldsymbol{\rho}(\widetilde{X}) = 1$ (cf. [10, Lem. 3.11]). Then some multiples of Γ and Θ are linearly equivalent, and we have an expected endomorphism $\tilde{f}$ of $\widetilde{X}$ by Theorem 1.3. Thus, we are done. □

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