RIMS-1944

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March 2021

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Abstract. Anabelian geometry has been developed over a much wider class of fields than Grothendieck, who is the originator of anabelian geometry, conjectured. So, it is natural to ask the following question: What kinds of fields are suitable for the base fields of anabelian geometry?

In the present paper, we consider this problem for higher local fields. First, to consider "anabelianness" of higher local fields themselves, we give mono-anabelian reconstruction algorithms of various invariants of higher local fields from their absolute Galois groups. As a result, the isomorphism classes of certain types of higher local fields are completely determined by their absolute Galois groups. Next, we prove that mixed-characteristic higher local fields are Kummer-faithful. This result affirms the above question for these higher local fields to a certain extent.

Contents

Introduction

Grothendieck gave the following conjecture in *Esquisse d'un Programme* and *Brief an G. Faltings* (cf. [SL]):

Conjecture
Let K be a field finitely generated over the prime field and G_K the absolute Galois group of K. Then an "anabelian variety" V over K may be "reconstructed" from the étale fundamental group (with the projection to G_K) $\pi_1(V) \twoheadrightarrow G_K$ of V (for some choice of basepoint).

This conjecture (which is referred to as the Grothendieck conjecture) is considered as a main issue of *anabelian geometry*.

2020 *Mathematics Subject Classification.* Primary 11S20; Secondary 14G20, 14H30.

Key words and phrases. anabelian geometry, complete discrete valuation field, Grothendieck conjecture, higher local field, hyperbolic curve, Kummer-faithful field, mono-anabelian reconstruction.

As in the conjecture, Grothendieck considered that anabelian geometry should be developed over fields finitely generated over prime fields. However, as we see below, anabelian geometry is developed over a much wider class of fields than he conjectured. So, it is natural to ask the following question:

What kinds of fields are suitable for the base fields of anabelian geometry?

This question is a main theme of the present paper and [Mu2]. In the present paper, we consider this problem for higher local fields. (In [Mu2], we consider this problem for complete discrete valuation fields with perfect residue fields (from the view point of "ramifications of fields").)

As he conjectured, fields finitely generated over prime fields are "suitable" in the sense as in the above question. Indeed, the Grothendieck conjecture for hyperbolic curves over fields finitely generated over $\mathbb{Q}$ was proved by Nakamura (the case where $g = 0$, cf. [Nak1, Theorem C], [Nak2, (1.1)]), Tamagawa (the case where X is affine, cf. [T, Theorem 0.3]) and Mochizuki (the general case, cf. [Mo1, Theorem A]), where X is a hyperbolic curve over such a field and g is the genus of the smooth compactification of X. On the other hand, the Grothendieck conjecture for hyperbolic curves over finite fields (resp. fields finitely generated over finite fields) was proved by Tamagawa (the case where X is affine, cf. [T, Theorem 0.6]) and Mochizuki (the general case, cf. [Mo6, Theorem 3.2]) (resp. by Stix (cf. [St1, Theorem 3.2], [St2, Theorem 5.1.1])). Moreover, the Grothendieck conjecture for infinite fields finitely generated over prime fields themselves was proved by Pop (cf. [P, Theorem 2]). (This result is a generalization of the theorem of Neukirch-Uchida, which is stated only for number fields.) Furthermore, by using the theorem of Neukirch-Uchida, in [Ho3, Theorem A], Hoshi established mono-anabelian reconstruction algorithms of number fields from the absolute Galois groups. (For mono-anabelian geometry, see also [Mo8, Introduction].)

Beyond the original version of Grothendieck's conjecture, p-adic local fields are also considered to be "suitable" (in the sense as in the above question). The Grothendieck conjecture for hyperbolic curves over p-adic local fields (in fact, more generally, over generalized sub-p-adic fields (i.e., fields isomorphic to subfields of fields finitely generated over the quotient field of the Witt ring with coefficients in an algebraic closure of $\mathbb{F}_p$)) was proved by Mochizuki (cf. [Mo3, Theorem A], [Mo4, Theorem 4.12]). Although the analogue of the theorem of Neukirch-Uchida for p-adic local fields fails to hold as it is, Mochizuki proved a certain analogue of the theorem for the absolute Galois groups with ramification filtrations (cf. [Mo2, Theorem 4.2]). Moreover, various invariants of p-adic local fields are recovered from their absolute Galois groups in the sense of mono-anabelian reconstruction (cf. [Ho2] and [Ho4]).

More generally, Abrashkin proved an analogue of the theorem of Neukirch-Uchida (for the absolute Galois groups with ramification filtrations) for higher local fields, (which we shall abbreviate to *HLF's* (cf. Definitions 1.1 and 1.4 (i)),) with (first) residue characteristics (cf. Definition 1.1 (i)) at least 3 (cf. [A2, Theorems 5, 6]) (and for complete discrete valuation fields of any positive characteristics with finite residue fields (cf. [A1, Theorem A], [A3, Theorem A])). So, HLF's are candidates for "suitable" fields in the sense as in the above question. Considering the above examples, it is natural to ask the following question:

(A) Which invariants of HLF's are reconstructed from the absolute Galois groups in the sense of mono-anabelian reconstruction?

(B) Does the Grothendieck conjecture hold for hyperbolic curves over HLF's?

In the present paper, we consider these problems.

For (A), we prove the following theorem:

Theorem A (cf. Proposition 2.1, Theorems 2.15, 2.16, 2.18)
Let K be an HLF, $d \in \mathbb{Z}_{\geq 0}$ the dimension of K (as an HLF), $K = K_d, K_{d-1}, \cdots, K_1, K_0$ a chain of residue fields of K (cf. Definition 1.1 (i), Remark 1.2) and G_K the absolute Galois group of K. Then we may determine group-theoretically from G_K which of the following statements holds:

(i) *K is an FF (cf. Definition 1.4 (i)).*

(ii) *K is an RPHLF (cf. Definition 1.4 (iii)) and is not an FF.*

(iii) *K is a PHLF (cf. Definition 1.4 (ii)) and is not an FF.*

(iv) *K is a ZRIHLF (cf. Definition 1.4 (iv)).*

In the case where one of (ii)$\sim$(iv) holds, there exist mono-anabelian reconstruction algorithms of the following invariants from G_K:

- *the characteristic p of K_0 (cf. Definition 1.1 (i));*
- *the cardinality of K_0;*
- *the dimension of K (as an HLF) (cf. Definition 1.1 (ii));*
- $\mathrm{Ker}(G_K \twoheadrightarrow G_{K_0})$ *(where G_{K_0} is the absolute Galois group of K_0).*

Moreover, in the case where (ii) holds, there exist mono-anabelian reconstruction algorithms of the following invariants from G_K:

- $[K_1 : \mathbb{Q}_p]$ *(cf. Definition 1.1 (i));*
- $\mathrm{Ker}(G_K \twoheadrightarrow G_{K_1})$ *(where G_{K_1} is the absolute Galois group of K_1).*

In particular, the isomorphism class of K is completely determined by G_K in the case where one of the following holds:

- *K is a PHLF and is not an FF;*
- *K is an RPHLF and is not an FF, and the profinite group $G_K/\mathrm{Ker}(G_K \twoheadrightarrow G_{K_1})$ (which is automatically a profinite group of MLF-type (cf. Section 0, Profinite groups)) is of GSMLF-type (cf.* [Ho4, Definition 6.8]*).*

This theorem shows that, the absolute Galois groups of HLF's (without the ramification filtrations) also have weak "anabelianness".

The following question gives an approach to the question (B):

(B′) Are HLF's Kummer-faithful (cf. Definition 3.1 (ii))?

Known results in anabelian geometry introduced above are mainly the relative versions of Grothendieck conjecture (for schemes over fields finitely generated over $\mathbb{Q}$ or p-adic local fields). On the other hand, Mochizuki proved the absolute version of the Grothendieck conjecture for hyperbolic curves over number fields (cf. [Mo5, Corollary 1.3.5]). However, the absolute version of the Grothendieck conjecture for hyperbolic curves over p-adic local fields is still open in general (for studies approaching this problem, see, e.g., [Mo6], [Mo7, §3] and [Mu]). In absolute situations, there are essential difficulties which are not found in relative situations. As another (semi-)absolute result, Hoshi proved that a point-theoretic and Galois-preserving isomorphism (cf. [Ho1, Definition 3.1]) between the étale fundamental groups of affine hyperbolic curves over Kummer-faithful fields arises from an isomorphism of schemes (cf. [Ho1, Theorem A]). Although (B′) does not necessarily give an affirmative answer to (B), Hoshi's theorem ensures that, to some extent, Kummer-faithful fields are suitable for the base fields of anabelian geometry.

The following is an answer to (B′):

Theorem B (cf. Theorem 3.8)
MHLF's (cf. Definition 1.4 (ii)) are Kummer-faithful.

We shall review the contents of the present paper. In Section 1, we discuss some generalities on higher local fields. In Section 2, we treat the problem (A) by using the theories in Section 1. In Section 3, we prove Theorem B by considering Néron models and formal groups of abelian varieties.

Acknowledgments

The author would like to express deep gratitude to Professor Akio Tamagawa for his helpful advices and encouragement. The author was supported by JSPS KAKENHI Grant Number 19J10214. This research was supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.

0. Notations and conventions

Numbers:

We shall write

- $\mathbb{Z}$ for the set of integers;
- $\mathbb{Q}$ for the set of rational numbers;
- $\mathbb{R}$ for the set of real numbers;
- $\mathfrak{Primes}$ for the set of prime numbers.

For $a \in \mathbb{R}$ and $\mathbb{X} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$, we shall write $\mathbb{X}_{\geq a}$ (resp. $\mathbb{X}_{>a}$, resp. $\mathbb{X}_{\leq a}$, resp. $\mathbb{X}_{<a}$) for

$$\{b \in \mathbb{X} \,|\, b \geq a \text{ (resp. } b > a, \text{ resp. } b \leq a, \text{ resp. } b < a)\}.$$

For $n \in \mathbb{Z}$ and a subset $A \subset \mathbb{Z}$, we shall write nA for $\{na \in \mathbb{Z} \,|\, a \in A\}$.

Fields:

For $p \in \mathfrak{Primes}$ and $n \in \mathbb{Z}_{>0}$, we shall write

- $\mathbb{Z}_p$ for the p-adic completion of $\mathbb{Z}$;
- $\mathbb{Q}_p$ for the quotient field of $\mathbb{Z}_p$;
- $\mathbb{F}_{p^n}$ for the finite field of cardinality p^n.

Let K be a field. We shall say that K is an *MLF* (resp. a *PLF*) if K is isomorphic to a finite extension of $\mathbb{Q}_p$ for some $p \in \mathfrak{Primes}$ (resp. a complete discrete valuation field of positive characteristic whose residue field is finite) (where "MLF" (resp. "PLF") is understood as an abbreviation for "Mixed-characteristic Local Field" (resp. "Positive-characteristic Local Field")).

Modules:

Let M be a $\mathbb{Z}$-module. We shall write M_{div} for the submodule which consists of divisible elements of M, i.e.,

$$M_{\mathrm{div}} = \bigcap_{n \in \mathbb{Z}_{>0}} nM.$$

For $n \in \mathbb{Z}_{>0}$, we shall write $M[n]$ for the kernel of multiplication by n.

Profinite groups:

Let G be a profinite group and $p \in \mathfrak{Primes}$. Then we shall write G^{ab} for the abelianization of G (i.e., the quotient of G by the closure of the commutator subgroup of G), and $G(p)$ for the maximal pro-p quotient of G.

We denote the cohomological p-dimension of G by $\mathrm{cd}_p G$, and set:

$$\mathrm{cd} G := \sup_{p \in \mathfrak{Primes}} \mathrm{cd}_p G.$$

We shall say that a profinite group G is of *MLF-type* (resp. *PLF-type*) if G is isomorphic, as a profinite group, to the absolute Galois group of an MLF (resp. a PLF).

1. Generalities on higher local fields

In this section, we discuss some generalities on higher local fields.

For a field K, we shall write

- $\overline{K}$ for an algebraic closure of K;
- K^{sep} for the separable closure of K in $\overline{K}$;
- G_K for the Galois group $\mathrm{Gal}(K^{\mathrm{sep}}/K)$;
- $\zeta_l \in K^{\mathrm{sep}}$ for a primitive l-th root of unity (for a prime number l different from the characteristic of K).

Definition 1.1 (cf. [FK, §1.1])

(i) Let d be a non-negative integer. Then a field K is a *d-dimensional local field* if there is a chain of fields $K = K_d, K_{d-1}, \cdots, K_1, K_0$ where K_{i+1} is a complete discrete valuation field with residue field K_i for $0 \leq i \leq d-1$ and K_0 is a finite field. The field K_{d-1} (resp. K_0) is said to be the *first* (resp. the *last*) *residue field of* K.

(ii) A field K is a *higher local field* if K is a d-dimensional local field for some non-negative integer d.

Remark 1.2
Let K be a higher local field of dimension $d \in \mathbb{Z}_{\geq 0}$. Since a complete discrete valuation field admits only one (normalized) discrete valuation, a chain of residue fields $K = K_d, K_{d-1}, \cdots, K_1, K_0$ in Definition 1.1 (i) is unique up to isomorphism. In particular, the dimension d of a higher local field K is well-defined.

We have a classification theorem for higher local fields (cf. [FK, §1.1]):

Theorem 1.3
Let K be a d-dimensional local field whose last residue field is of characteristic p. Then one (and only one) of the following holds:

(i) *K is isomorphic to $\mathbb{F}_q((T_1))\cdots((T_d))$ where q is the cardinality of the last residue field of K;*
(ii) *K is isomorphic to a finite extension of $k\{\{T_1\}\}\cdots\{\{T_{d-1}\}\}$ where k is a finite extension of $\mathbb{Q}_p$. (For the definition of $k\{\{T\}\}$, see [FK, §1.1]);*
(iii) *$d \geq 3$ and K is isomorphic to a finite extension of $k\{\{T_1\}\}\cdots\{\{T_m\}\}((T_{m+1}))\cdots((T_{d-1}))$ where $1 \leq m \leq d-2$ and k is a finite extension of $\mathbb{Q}_p$;*
(iv) *$d \geq 2$ and K is isomorphic to $k((T_1))\cdots((T_{d-1}))$ where k is a finite extension of $\mathbb{Q}_p$.*

Definition 1.4
Let K be a field and G a profinite group.

(i) We shall say that K is an *FF* (resp. *HLF*) if K is a finite field (resp. higher local field) (where "FF" (resp. "HLF") is understood as an abbreviation for "Finite Field" (resp. "Higher Local Field")).
(ii) Suppose that K is an HLF. If Theorem 1.3 (i) (resp. (ii), resp. (i) or (ii), resp. (iii) or (iv)) holds for K and for some non-negative integer d and some prime number p, we shall say that K is a(n) *PHLF* (resp. *MHLF*, resp. *PFHLF*, resp. *ZFHLF*) (where "PHLF" (resp. "MHLF", resp. "PFHLF", resp. "ZFHLF") is understood as an abbreviation for "Positive-characteristic Higher Local Field" (resp. "Mixed-characteristic Higher Local Field", resp. "Positive-First-residue-characteristic Higher Local Field", resp. "Zero-First-residue-characteristic Higher Local Field")).
(iii) Suppose that K is an HLF, and one (and only one) of the following conditions holds:
   - K is of dimension 0 (i.e., K is an FF).
   - K is of dimension 1 and Theorem 1.3 (ii) holds for K and for some prime number p (i.e., K is an MLF).
   - Theorem 1.3 (iv) holds for K and for some non-negative integer d and some prime number p. (In this case, automatically we have $d \geq 2$.)

   Then we shall say that K is an *RPHLF* (where "RPHLF" is understood as an abbreviation for "Residually-Perfect Higher Local Field"). (Here, we also regard K itself as a "residue field".)
(iv) Suppose that K is an HLF, and one (and only one) of the following conditions holds:

- K is of dimension $d \geq 2$ and Theorem 1.3 (ii) holds for K and for some prime number p.
- Theorem 1.3 (iii) holds for K and for some non-negative integer d and some prime number p. (In this case, automatically we have $d \geq 3$.)

Then we shall say that K is a *ZRIHLF* (where "ZRIHLF" is understood as an abbreviation for "Zero-characteristic Residually-Imperfect Higher Local Field").

(v) We shall say that G is of *FF-type* (resp. *HLF-type*, resp. *PHLF-type*, resp. *MHLF-type*, resp. *PFHLF-type*, resp. *RPHLF-type*, *ZRIHLF-type*) if G is isomorphic, as a profinite group, to the absolute Galois group of a(n) FF (resp. HLF, resp. PHLF, resp. MHLF, resp. PFHLF, resp. RPHLF, resp. ZRIHLF).

Remark 1.5
Let K be an HLF. Then any finite extension L of K is an HLF and has the same type and the same dimension as K.

We give a group-theoretic characterization of profinite groups of FF-type and RPHLF-type:

Proposition 1.6
Let G be a profinite group of HLF-type. Then G is topologically finitely generated if and only if G is of RPHLF-type. Moreover, G is isomorphic to $\hat{\mathbb{Z}}$ if and only if G is of FF-type.

Proof.
Suppose that G is of RPHLF-type. Then, since profinite groups of MLF-type are topologically finitely generated and the inertia subgroups of the absolute Galois groups of RPHLF's of dimensions greater than or equal to 2 are isomorphic to $\hat{\mathbb{Z}}(1)$, it follows immediately that G is topologically finitely generated.

Let K be an HLF of dimension $d\,(\in \mathbb{Z}_{\geq 0})$ such that G_K is isomorphic to G as a profinite group, and $K = K_d, K_{d-1}, \cdots, K_1, K_0$ a chain of residue fields of K (cf. Definition 1.1 (i), Remark 1.2). Suppose that K is not an RPHLF. Then clearly $d > 0$. Moreover, K_1 is a PLF, hence G_{K_1} is not topologically finitely generated (cf. [NSW, Proposition 6.1.7]). By considering the surjection $G_K \twoheadrightarrow G_{K_1}$, $G\,(\simeq G_K)$ is not topologically finitely generated.

The latter equivalence is immediate (note that for G not of FF-type, there exists a surjection from G to a profinite group of MLF or PLF-type, which is not cyclic). □

In the remainder of this section, let K be an HLF of dimension $d \in \mathbb{Z}_{>0}$. Moreover, we shall write:

- $K = K_d, K_{d-1}, \cdots, K_1, K_0$ for a chain of residue fields of K (cf. Definition 1.1 (i), Remark 1.2);
- p_K for the characteristic of K;
- p_0 for the characteristic of K_0;
- q_0 for the cardinality of K_0.

Lemma 1.7
Suppose that K is an RPHLF and $d > 1$. Let I be the kernel of the surjection $G_K \twoheadrightarrow G_{K_{d-1}}$. Then, for any prime number l, the following two conditions are equivalent:

(i) *The natural homomorphism* $\mathrm{Hom}(G_K, \mathbb{Z}/l\mathbb{Z}) \to \mathrm{Hom}(I, \mathbb{Z}/l\mathbb{Z})$ *is non-trivial.*
(ii) *K contains a primitive l-th root of unity.*

Proof.
By considering an extension of K generated by a l-th root of a uniformizer of K, the implication (ii)$\Longrightarrow$(i) holds. The implication (i)$\Longrightarrow$(ii) is immediate from the fact that $I \simeq \hat{\mathbb{Z}}(1)$. $\square$

Proposition 1.8
Suppose that K is an RPHLF. Set $p := p_0$ and $N := [K_1 : \mathbb{Q}_p]$. Then, for any prime number l,

$$\dim_{\mathbb{Z}/l\mathbb{Z}}(\mathrm{Hom}(G_K, \mathbb{Z}/l\mathbb{Z})) = \begin{cases} N\delta_{pl} + d + 1, & (\zeta_l \in K); \\ N\delta_{pl} + 1, & (\zeta_l \notin K), \end{cases}$$

where

$$\delta_{pl} = \begin{cases} 0, & (l \neq p); \\ 1, & (l = p). \end{cases}$$

Proof.
We prove this proposition by induction on d. The case where $d = 1$ follows immediately from local class field theory. Suppose that $d > 1$. Let I be the kernel of the surjection $G_K \to G_{K_{d-1}}$ (note that I is isomorphic to $\hat{\mathbb{Z}}$ as a profinite group). Then we have the following exact sequence:

$$0 \longrightarrow \mathrm{Hom}(G_{K_{d-1}}, \mathbb{Z}/l\mathbb{Z}) \longrightarrow \mathrm{Hom}(G_K, \mathbb{Z}/l\mathbb{Z}) \longrightarrow \mathrm{Hom}(I, \mathbb{Z}/l\mathbb{Z})(\simeq \mathbb{Z}/l\mathbb{Z}).$$

Therefore, by Lemma 1.7,

$$\dim_{\mathbb{Z}/l\mathbb{Z}}(\mathrm{Hom}(G_K, \mathbb{Z}/l\mathbb{Z})) = \begin{cases} \dim_{\mathbb{Z}/l\mathbb{Z}}(\mathrm{Hom}_{\mathbb{Z}/l\mathbb{Z}}(G_{K_{d-1}}, \mathbb{Z}/l\mathbb{Z})) + 1, & (\zeta_l \in K); \\ \dim_{\mathbb{Z}/l\mathbb{Z}}(\mathrm{Hom}_{\mathbb{Z}/l\mathbb{Z}}(G_{K_{d-1}}, \mathbb{Z}/l\mathbb{Z})), & (\zeta_l \notin K). \end{cases}$$

By induction hypothesis, this completes the proof of Proposition 1.8 (note that $\zeta_l \in K$ if and only if $\zeta_l \in K_{d-1}$). $\square$

The following proposition gives group-theoretic characterizations of p_0 and $N := [K_1 : \mathbb{Q}_{p_0}]$ of an RPHLF:

Proposition 1.9
Let K, p and N be as in Proposition 1.8, and l a prime number. Then $l = p$ if and only if there exists an open subgroup H of G_K of index l such that

$$\dim_{\mathbb{Z}/l\mathbb{Z}}(\mathrm{Hom}(H, \mathbb{Z}/l\mathbb{Z})) > \dim_{\mathbb{Z}/l\mathbb{Z}}(\mathrm{Hom}(G_K, \mathbb{Z}/l\mathbb{Z})).$$

Moreover, if such an open subgroup H of G_K exists (hence $l = p$), we have

$$\dim_{\mathbb{Z}/p\mathbb{Z}}(\mathrm{Hom}(H, \mathbb{Z}/p\mathbb{Z})) - \dim_{\mathbb{Z}/p\mathbb{Z}}(\mathrm{Hom}(G_K, \mathbb{Z}/p\mathbb{Z})) = (p-1)N.$$

Proof.
Suppose that $l = p$. Let L_1 be a finite extension of K_1 of degree l. Set $L := K \cdot L_1$, and let H be the open subgroup of G_K corresponding to L. Then, by Proposition 1.8, we have

$$\dim_{\mathbb{Z}/l\mathbb{Z}}(\mathrm{Hom}(H, \mathbb{Z}/l\mathbb{Z})) > \dim_{\mathbb{Z}/l\mathbb{Z}}(\mathrm{Hom}(G_K, \mathbb{Z}/l\mathbb{Z})).$$

Conversely, suppose that $l \neq p$. Then, by Proposition 1.8, for any open subgroup of H of index l, we have

$$\dim_{\mathbb{Z}/l\mathbb{Z}}(\mathrm{Hom}(H, \mathbb{Z}/l\mathbb{Z})) = \dim_{\mathbb{Z}/l\mathbb{Z}}(\mathrm{Hom}(G_K, \mathbb{Z}/l\mathbb{Z})).$$

(Note that, by the choice of H, $\zeta_l \in K$ if and only if $\zeta_l \in L$, where L is the finite extension of K corresponding to H.)

The latter portion of the statement follows immediately from Proposition 1.8. $\square$

The following proposition determines group-theoretically whether or not an RPHLF contains a primitive p_0-th root of unity:

Proposition 1.10
Let K, p and N as in Proposition 1.8. Then $\zeta_p \notin K$ if and only if there exists an open normal subgroup H of G_K of index n such that n divides $p-1$ and

$$\dim_{\mathbb{Z}/p\mathbb{Z}}(\mathrm{Hom}(H, \mathbb{Z}/p\mathbb{Z})) - \dim_{\mathbb{Z}/p\mathbb{Z}}(\mathrm{Hom}(G_K, \mathbb{Z}/p\mathbb{Z})) > (n-1)N.$$

Proof.
Immediate from Proposition 1.8 (note that $[K(\zeta_p) : K]$ divides $p-1$). $\square$

We may describe explicitly the structure of the abelianization of the maximal pro-l quotient of the absolute Galois group of an HLF of positive dimension (for $l \in \mathfrak{Primes}$ different from p_0):

Proposition 1.11
Let K be an HLF (of dimension $d \in \mathbb{Z}_{>0}$). Then, for any prime number l different from p_0, we have

$$G_K(l)^{\mathrm{ab}} \simeq \mathbb{Z}_l \oplus (\mathbb{Z}/l^{v_l(q_0-1)}\mathbb{Z})^{\oplus d},$$

where v_l is the valuation of $\mathbb{Q}_l$ such that $v_l(\mathbb{Q}_l^\times) = \mathbb{Z}$.

Proof.
Immediate from [EF, Remark 5.4]. $\square$

The following proposition gives a group-theoretic characterization of p_0 for a PHLF or ZRIHLF:

Proposition 1.12
Let K be a PHLF or ZRIHLF (of dimension $d \in \mathbb{Z}_{>0}$), l a prime number. Set $p := p_0$. Then $l = p$ if and only if $G_K(l)$ is not topologically finitely generated.

Proof.
Suppose that $l \neq p$. Then $G_K(l)$ is topologically finitely generated by [EF, Remark 5.4]. On the other hand, there exists a surjection $G_K \twoheadrightarrow G_{K_1}$. Since K_1 is a PLF (of

characteristic p), $G_{K_1}(p)$ (hence also $G_K(p)$) is not topologically finitely generated (cf. [NSW, Proposition 6.1.7]). $\square$

The following proposition, together with Proposition 1.6, gives a group-theoretic characterization of profinite groups of PHLF-type and ZRIHLF-type.

Proposition 1.13
Let K be a PHLF or ZRIHLF (of dimension $d \in \mathbb{Z}_{>0}$). Set $p := p_0$.

(i) *Suppose that K is a PHLF. Then*

$$\mathrm{cd}_p G_K = 1.$$

(ii) *Suppose that K is a ZRIHLF. Then*

$$\mathrm{cd}_p G_K = d + 1.$$

Proof.
Immediate from [K, §0, Corollary] and [NSW, Propositions 6.1.7, 6.5.10]. $\square$

Remark 1.14
Note that, by definition, a ZRIHLF is of dimension at least 2. Therefore, for a ZRIHLF K, we have

$$\mathrm{cd}_p G_K \geq 3,$$

where $p := p_0$.

Remark 1.15
Let G be a profinite group of HLF-type. Then, by the above arguments, one and only one of the following condition holds:

(i) G is of FF-type.
(ii) G is of RPHLF-type and is not of FF-type.
(iii) G is of PHLF-type and is not of FF-type.
(iv) G is of ZRIHLF-type.

2. Reconstruction of various invariants of higher local fields

In this section, by applying theories in Section 1, we establish mono-anabelian reconstruction algorithms of various invariants of HLF's.

Let K be an HLF. We shall write

- K^{sep} for a separable closure of K;
- G_K for the Galois group $\mathrm{Gal}(K^{\mathrm{sep}}/K)$;
- $K = K_d, K_{d-1}, \cdots, K_1, K_0$ for a chain of residue fields of K (cf. Definition 1.1 (i), Remark 1.2);
- $d = d(K)$ for the dimension of an HLF K;
- $p(K)$ for the characteristic of K_0;
- $q(K)$ for the cardinality of K_0.

If K is an RPHLF of positive dimension, set $N(K) := [K_1 : \mathbb{Q}_{p(K)}]$ (which clearly does not depend on the choice of a chain $K = K_d, \cdots, K_0$ (cf. Remark 1.2)).

Let G be a profinite group of HLF-type.

We may determine the type of G group-theoretically:

Proposition 2.1

(i) *G is of FF-type if and only if G is isomorphic to $\hat{\mathbb{Z}}$ as a profinite group.*
(ii) *G is of RPHLF-type and is not of FF-type if and only if G is topologically finitely generated and is not isomorphic to $\hat{\mathbb{Z}}$.*
(iii) *G is of PHLF-type and is not of FF-type if and only if there exists a unique prime number p such that $G(p)$ is not topologically finitely generated and $\mathrm{cd}_p G = 1$.*
(iv) *G is of ZRIHLF-type if and only if there exists a unique prime number p such that $G(p)$ is not topologically finitely generated and $\mathrm{cd}_p G \geq 3$.*

Proof.
Immediate from Propositions 1.6, 1.12, 1.13 and Remark 1.14. □

Definition 2.2
Suppose that G is of PHLF-type or ZRIHLF-type and is not of FF-type. We shall write $p(G)$ for the uniquely determined prime number in Proposition 2.1.

Lemma 2.3
Suppose that G is of RPHLF-type and is not of FF-type. Then there exists a uniquely determined prime number p satisfying the following condition:

There exists an open subgroup H of G of index p such that

$$\dim_{\mathbb{Z}/p\mathbb{Z}}(\mathrm{Hom}(H, \mathbb{Z}/p\mathbb{Z})) > \dim_{\mathbb{Z}/p\mathbb{Z}}(\mathrm{Hom}(G, \mathbb{Z}/p\mathbb{Z})).$$

Proof.
Immediate from Proposition 1.9. □

Definition 2.4
Suppose that G is of RPHLF-type and is not of FF-type. We shall write $p(G)$ for the uniquely determined prime number in Lemma 2.3.

Lemma 2.5
Suppose that G is of HLF-type and is not of FF-type. For a prime number l different from $p(G)$, let $v_l(G) \in \mathbb{Z}_{\geq 0}$ be the integer such that

$$l^{v_l(G)} = \max\{\mathrm{ord}(\sigma) \,|\, \sigma \in G(l)^{\mathrm{ab}} \textit{ is a torsion element}\},$$

where $\mathrm{ord}(\sigma)$ is the order of σ. Then $v_l(G) = 0$ for all but finitely many prime numbers l different from $p(G)$.

Proof.
Immediate from Proposition 1.11. □

Definition 2.6
Suppose that G is of HLF-type and is not of FF-type. For a prime number l different from $p(G)$, let $v_l(G)$ be as in Lemma 2.5. Set:

$$q(G) := 1 + \prod_{l \in \mathfrak{Primes} \setminus \{p(G)\}} l^{v_l(G)} \in \mathbb{Z}.$$

(cf. Lemma 2.5.)

Definition 2.7
Suppose that G is of HLF-type and is not of FF-type. Let $\mathcal{J}_0$ be the set of open subgroups H of G satisfying the following condition:

$$q(H) = q(G)^{[G:H]}.$$

Define a closed subgroup $J_0(G)$ of G as follows:

$$J_0(G) := \bigcap_{H \in \mathcal{J}_0} H.$$

Definition 2.8
Suppose that G is of HLF-type and is not of FF-type. Set:

$$d(G) := \max_{\substack{l \in \mathfrak{Primes} \setminus \{p(G)\} \\ H \subset G}} \Big(\dim_{\mathbb{Z}/l\mathbb{Z}}(H(l)^{\mathrm{ab}} \otimes_{\mathbb{Z}_l} \mathbb{Z}/l\mathbb{Z})\Big) - 1,$$

where H runs through the set of open subgroups of G.

Lemma 2.9
Suppose that G is of RPHLF-type and is not of FF-type. Then let $\mathcal{H}$ be the set of open subgroups H of G of index $p(G)$ satisfying

$$\dim_{\mathbb{Z}/p(G)\mathbb{Z}}(\mathrm{Hom}(H,\, \mathbb{Z}/p(G)\mathbb{Z})) > \dim_{\mathbb{Z}/p(G)\mathbb{Z}}(\mathrm{Hom}(G,\, \mathbb{Z}/p(G)\mathbb{Z})).$$

(Note that $\mathcal{H}$ is non-empty by the definition of $p(G)$.) Then, for any $H \in \mathcal{H}$,

$$\dim_{\mathbb{Z}/p(G)\mathbb{Z}}(\mathrm{Hom}(H,\, \mathbb{Z}/p(G)\mathbb{Z})) - \dim_{\mathbb{Z}/p(G)\mathbb{Z}}(\mathrm{Hom}(G,\, \mathbb{Z}/p(G)\mathbb{Z}))$$

takes the same value. Moreover, $p(G) - 1$ divides this integer .

Proof.
Immediate from Proposition 1.9. □

Definition 2.10
Suppose that G is of RPHLF-type and is not of FF-type. Let $\mathcal{H}$ be as in Lemma 2.9. Then, for an element $H \in \mathcal{H}$, set:

$$N(G) := \frac{\dim_{\mathbb{Z}/p(G)\mathbb{Z}}(\mathrm{Hom}(H,\, \mathbb{Z}/p(G)\mathbb{Z})) - \dim_{\mathbb{Z}/p(G)\mathbb{Z}}(\mathrm{Hom}(G,\, \mathbb{Z}/p(G)\mathbb{Z}))}{p(G) - 1}.$$

(Note that, by Lemma 2.9, this does not depend on the choice of $H \in \mathcal{H}$.)

Definition 2.11
Suppose that G is of RPHLF-type and is not of FF-type. We shall say that G *satisfies* (†) if the following condition holds:

There exists an open normal subgroup H of G of index n such that n divides $p(G)-1$ and

$$\dim_{\mathbb{Z}/p(G)\mathbb{Z}}(\mathrm{Hom}(H,\,\mathbb{Z}/p(G)\mathbb{Z}))-\dim_{\mathbb{Z}/p(G)\mathbb{Z}}(\mathrm{Hom}(G,\,\mathbb{Z}/p(G)\mathbb{Z}))>(n-1)N(G).$$

Lemma 2.12

Suppose that G is of RPHLF-type and is not of FF-type. Then there exists a uniquely determined maximal open normal subgroup of G which does not satisfy (†).

Proof.

Immediate from Proposition 1.10. □

Definition 2.13

Suppose that G is of RPHLF-type and is not of FF-type. We shall write $H_{p(G)}(G)$ for the uniquely determined maximal open subgroup of G which does not satisfy (†).

Definition 2.14

Suppose that G is of RPHLF-type and is not of FF-type. Let $\mathcal{J}_1$ be the set of open subgroups H of $H_{p(G)}(G)$ satisfying the following condition:

$$\begin{aligned}\dim_{\mathbb{Z}/p(G)\mathbb{Z}}(\mathrm{Hom}(H,\,\mathbb{Z}/p(G)\mathbb{Z}))-\dim_{\mathbb{Z}/p(G)\mathbb{Z}}(\mathrm{Hom}(H_{p(G)}(G),\,\mathbb{Z}/p(G)\mathbb{Z}))\\ =([H_{p(G)}(G):H]-1)N(H_{p(G)}(G)).\end{aligned}$$

Define a closed subgroup $J_1(G)$ of G as follows:

$$J_1(G):=\bigcap_{H\in\mathcal{J}_1}H.$$

Theorem 2.15

Let K be an HLF which is not an FF.

(i) *It holds that*
- $p(K)=p(G_K)$;
- $q(K)=q(G_K)$;
- $d(K)=d(G_K)$;
- $\mathrm{Ker}(G_K\twoheadrightarrow G_{K_0})$ *does not depend on the choice of a chain* $K=K_d,\cdots,K_0$ *and coincides with* $J_0(G_K)$.

(ii) *Suppose that K is an RPHLF. Then it holds that*
- $N(K)=N(G_K)$;
- $\mathrm{Ker}(G_K\twoheadrightarrow G_{K_1})$ *does not depend on the choice of a chain* $K=K_d,\cdots,K_0$ *and coincides with* $J_1(G_K)$, *(hence* $G_{K_1}\simeq G_K/J_1(G_K)$*).*

Proof.

The assertion for $p(K)$ follows immediately from Propositions 1.9, 1.12 and the definition of $p(G)$. The assertion for $q(K)$ follows immediately from Proposition 1.11 and the definition of $q(G)$. The assertion for $d(K)$ follows immediately from Remark 1.5, Proposition 1.11 and the definition of $d(K)$. The assertion for $J_0(G_K)$ follows immediately from the definition of $J_0(G_K)$ and the assertion for $q(K)$. The assertion for $N(K)$ follows immediately from Proposition 1.9, the assertion for $q(K)$ and the definition of $N(G)$. The assertion for $J_1(G_K)$ follows immediately from the definition of $J_1(G_K)$ and the assertion for $N(K)$. □

We may determine group-theoretically the isomorphism class of a PHLF:

Theorem 2.16
For $i = 1, 2$, let K_i be a PHLF which is not an FF. Suppose that there exists an isomorphism of profinite groups $G_{K_1} \simeq G_{K_2}$. Then there exists an isomorphism $K_1 \simeq K_2$ of fields.

Proof.
Since the isomorphism class of K_i is determined by $q(K_i)$ and $d(K_i)$, this theorem follows immediately from Theorem 2.15 (i). $\square$

Remark 2.17
In [A2, Theorem 5], Abrashkin recovered a PHLF (with characteristic at least 3) group-theoretically from the absolute Galois group with ramification filtration in a functorial fashion. On the other hand, the reconstruction in Theorem 2.16 is not functorial (such reconstruction is sometimes referred to as a "weak (Isom-)version of Grothendieck conjecture"). However, Theorem 2.16 does not require the "preservation of ramification filtration".

We may also determine group-theoretically the isomorphism class of an RPHLF under a certain condition:

Theorem 2.18
For $i = 1, 2$, let K_i be an RPHLF which is not an FF. Suppose that there exists an isomorphism of profinite groups $G_{K_1} \simeq G_{K_2}$. Suppose, moreover, that $G_{K_i}/J_1(G_{K_i})$ (which is clearly a profinite group of MLF-type) is of GSMLF-type for some $i = 1, 2$ (for the definition of GSMLF-type, see [Ho4, Definition 6.8]*). Then there exists an isomorphism $K_1 \simeq K_2$ of fields.*

Proof.
By Theorem 2.15, an isomorphism $G_{K_1} \simeq G_{K_2}$ induces an isomorphism $G_{K_1}/J_1(G_{K_1}) \simeq G_{K_2}/J_1(G_{K_2})$. Therefore, by the definition of GSMLF-type, $G_{K_i}/J_1(G_{K_i})$ is of GSMLF-type for $i = 1, 2$. Since the isomorphism class of K_i is determined by the isomorphism class of $(K_i)_1$ and $d(K_i)$, this theorem follows immediately from [Ho4, Theorem 6.10] and Theorem 2.15. $\square$

3. Kummer-faithfulness of higher local fields of MHLF-type

In this section, we prove that MHLF's are Kummer-faithful.

Definition 3.1 (cf. [Mo8, Definition 1.5], [Ho1, Definition 1.2])
Let K be a field.

(i) We shall say that K is pre-Kummer-faithful if, for every finite extension L of K and every semi-abelian variety A over L, it holds that $A(L)_{\mathrm{div}} = \{0\}$.
(ii) We shall say that K is Kummer-faithful if K is pre-Kummer-faithful and perfect.

Remark 3.2
Let K be a field and K' a finite extension of K. Then it is immediate that K is pre-Kummer-faithful (resp. Kummer-faithful) if and only if K' is pre-Kummer faithful (resp. Kummer-faithful).

Lemma 3.3
A field K is pre-Kummer-faithful if and only if, for any finite extension L of K, $\mathbb{G}_m(L)_{\mathrm{div}} = \{0\}$ and $A(L)_{\mathrm{div}} = \{0\}$ for any abelian variety A over L.

Proof.
Immediate from the definition of pre-Kummer-faithfulness. (See also [OT, Proposition 2.3].) □

Lemma 3.4
Let M be a $\mathbb{Z}$-module.

(i) *M is a divisible group if and only if $M = M_{\mathrm{div}}$.*

(ii) *Suppose that M satisfies the following condition:*

The inverse system $(M[n!], \varphi_{m,n})$ satisfies the Mittag-Leffler condition (i.e., for any $n \in \mathbb{Z}_{>0}$, there exists an $N \in \mathbb{Z}_{>0}$ such that for all $m, m' \geq N$, $\varphi_{m,n}(M[m!]) = \varphi_{m',n}(M[(m')!])$), where $\varphi_{m,n} : M[m!] \to M[n!]$ is given by multiplication by $\dfrac{m!}{n!}$ $(m \geq n)$.

Then M_{div} is a divisible group. In particular, if $M[n]$ is a finite group for any $n \in \mathbb{Z}_{>0}$, M_{div} is a divisible group.

Proof.
(i) follows immediately. For (ii), consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & M[m!] \arrow[r] \arrow[d, "\times \frac{m!}{n!}"] & M \arrow[r, "\times m!"] \arrow[d, "\times \frac{m!}{n!}"] & m!M \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & M[n!] \arrow[r] & M \arrow[r, "\times n!"] & n!M \arrow[r] & 0,
\end{tikzcd}$$

where $m \geq n$ and the horizontal sequences are exact. Since the inverse system $(M[n!], \varphi_{m,n})$ satisfies the Mittag-Leffler condition, by taking the inverse limits, we have the following exact sequence:

$$0 \longrightarrow \varprojlim_{n\to\infty} M[n!] \longrightarrow \varprojlim M \longrightarrow M_{\mathrm{div}} \longrightarrow 0.$$

Therefore, for any $x \in M_{\mathrm{div}}$, there exists an element $(y_n)_{n\in\mathbb{Z}_{>0}} \in \varprojlim M$ such that $n!y_n = x$ for all $n \in \mathbb{Z}_{>0}$. Since $y_n = \dfrac{m!}{n!}y_m$ for all $m, n \in \mathbb{Z}_{>0}$ satisfying $m \geq n$, we have that $y_n \in M_{\mathrm{div}}$ for all $n \in \mathbb{Z}_{>0}$. This implies that M_{div} is a divisible group, as desired. □

In the remainder of this section, we assume that K is a complete discrete valuation field with positive residue characteristic.

We shall write

- $\mathcal{O}_K$ for the ring of integers of K;
- $\mathfrak{M}_K$ for the maximal ideal of $\mathcal{O}_K$;
- k for the residue field of K;
- p for the characteristic of k;
- v_K for the valuation of K such that $v_K(K^\times) = \mathbb{Z}$.

For $a \in \mathcal{O}_K$, we denote the image of a in k by $\overline{a}$.

For an abelian variety A over K of dimension $g \in \mathbb{Z}_{>0}$, we shall write $\mathbf{F}_A = (F_{A,1}, \cdots, F_{A,g})$ for the formal group associated to A ($F_{A,1}, \cdots, F_{A,g} \in K[\![X_1, \cdots, X_g, Y_1, \cdots, Y_g]\!]$). (For the definition of the formal groups associated to abelian varieties, see [HS, §C.2].)

Remark 3.5
Let A be an abelian variety over K, and $\mathfrak{A}$ the Néron model of A (cf. [BLR, §1.3, Corollary 2]). Then we may define the formal group associated to $\mathfrak{A}$ (for a fixed choice of local parameters at the identity element) similarly to that of A. Therefore, by choosing local parameters of $\mathfrak{A}$ at the identity element which also determine local parameters of A at the identity element, we obtain the formal group $\mathbf{F}_A = (F_{A,1}, \cdots, F_{A,g})$ associated to A defined over $\mathcal{O}_K$ (i.e., $F_{A,1}, \cdots, F_{A,g} \in \mathcal{O}_K[\![X_1, \cdots, X_g, Y_1, \cdots, Y_g]\!]$).

Suppose that $\mathbf{F}_A$ is defined over $\mathcal{O}_K$. Then we shall write $\mathbf{F}_A(\mathfrak{M}_K)$ for the group associated to $\mathbf{F}_A/\mathcal{O}_K$ (cf. [HS, §C.2]).

Remark 3.6
Let $\mathbf{F}_A$ and $\mathbf{F}_A(\mathfrak{M}_K)$ be as above. For $i \in \mathbb{Z}_{>0}$, denote the subset $(\mathfrak{M}_K^i)^{\oplus g} \subset \mathfrak{M}_K^{\oplus g} = \mathbf{F}_A(\mathfrak{M}_K)$ by $\mathbf{F}_A(\mathfrak{M}_K^i)$. Then $\mathbf{F}_A(\mathfrak{M}_K^i)$ is clearly a subgroup of $\mathbf{F}_A(\mathfrak{M}_K)$, and we have

$$\bigcap_{i \in \mathbb{Z}_{>0}} \mathbf{F}_A(\mathfrak{M}_K^i) = \{\mathbf{0}\}.$$

Moreover, for any $i \in \mathbb{Z}_{>0}$, we have an isomorphism of groups $\mathbf{F}_A(\mathfrak{M}_K^i)/\mathbf{F}_A(\mathfrak{M}_K^{i+1}) \xrightarrow{\sim} (\mathfrak{M}_K^i/\mathfrak{M}_K^{i+1})^{\oplus g}$, where the group law of the right hand side is the usual one. In particular, we have

$$\bigcap_{i \in \mathbb{Z}_{>0}} p^i \mathbf{F}_A(\mathfrak{M}_K) = \{\mathbf{0}\}.$$

Proposition 3.7
Suppose that k is pre-Kummer-faithful. Then K is pre-Kummer-faithful.

Proof.
By considering finite extensions of K, it suffices to show that $(K^\times)_{\mathrm{div}} = \{1\}$ and $A(K)_{\mathrm{div}} = \{0\}$ for any abelian variety A over K (cf. Remark 3.2 and Lemma 3.3). First, we prove that $(K^\times)_{\mathrm{div}} = \{1\}$. Let us take any element $a \in (K^\times)_{\mathrm{div}}$. Then $v_K(a) = 0$ and $\overline{a}$ belongs to $(k^\times)_{\mathrm{div}}$. By the pre-Kummer-faithfulness of k, it follows that $\overline{a} = 1$, i.e., $a \in 1 + \mathfrak{M}_K$. Since $a \in (K^\times)_{\mathrm{div}}$, for any $n \geq 1$, there exists $b \in \mathcal{O}_K^\times$ such that $b^{p^n} = a$. As $\mathrm{char}(k) = p$, we have $\overline{b} = 1$ in $k^\times$ and hence $a = b^{p^n} \in (1 + \mathfrak{M}_K)^{p^n} \subset 1 + \mathfrak{M}_K^n$. Therefore, we have $(K^\times)_{\mathrm{div}} = \{1\}$.

Next, we prove that $A(K)_{\mathrm{div}} = \{0\}$ for any abelian variety A over K. Let g be the dimension of A and $\mathfrak{A}$ the Néron model of A. By taking a finite Galois extension of K if necessary, we may assume without loss of generality that A has semi-abelian reduction (cf. [BLR, §7.4, Theorem 1]). Thus, $\mathfrak{A}_k^0$ is a semi-abelian variety, where $\mathfrak{A}_k := \mathfrak{A} \times_{\mathrm{Spec}\mathcal{O}_K} \mathrm{Spec}\, k$. Moreover, we have the following exact sequence:

$$0 \longrightarrow \mathfrak{A}_k^0(k) \longrightarrow \mathfrak{A}_k(k) \longrightarrow \pi_0(\mathfrak{A}_k)(k),$$

where $\pi_0(\mathfrak{A}_k)$ is the group of connected components of $\mathfrak{A}_k$. Note that $\pi_0(\mathfrak{A}_k)(k)$ is a finite group. Therefore, it holds that $\mathfrak{A}_k(k)_{\mathrm{div}} \subset \mathfrak{A}_k^0(k)$. On the other hand, since k is pre-Kummer-faithful, we have $\mathfrak{A}_k^0(k)_{\mathrm{div}} = \{0\}$. So, it holds that $\mathfrak{A}_k(k)_{\mathrm{div}} = \{0\}$. Indeed, $\mathfrak{A}_k(k)_{\mathrm{div}} \subset \mathfrak{A}_k^0(k)$ implies that $(\mathfrak{A}_k(k)_{\mathrm{div}})_{\mathrm{div}} \subset \mathfrak{A}_k^0(k)_{\mathrm{div}} = \{0\}$. Since $\mathfrak{A}_k(k)[n]$ (hence also $\mathfrak{A}_k(k)_{\mathrm{div}}[n]$) is finite for any $n \in \mathbb{Z}_{>0}$, it holds that $\mathfrak{A}_k(k)_{\mathrm{div}} = (\mathfrak{A}_k(k)_{\mathrm{div}})_{\mathrm{div}}$ by Lemma 3.4.

On the other hand, we have a bijection $\mathfrak{A}(\mathcal{O}_K) \xrightarrow{\sim} A(K)$ and a reduction map $\rho : \mathfrak{A}(\mathcal{O}_K) \to \mathfrak{A}_k(k)$. Let A_1 be the kernel of ρ. Thus, we have the following exact sequence:

$$0 \longrightarrow A_1 \longrightarrow \mathfrak{A}(\mathcal{O}_K) \longrightarrow \mathfrak{A}_k(k).$$

Since $\mathfrak{A}_k(k)_{\mathrm{div}} = \{0\}$, we have $A_1 \supset \mathfrak{A}(\mathcal{O}_K)_{\mathrm{div}} (\simeq A(K)_{\mathrm{div}})$. Therefore, it suffices to show that $(A_1)_{\mathrm{div}} = \{0\}$. (Note that $A(K)_{\mathrm{div}}$ is divisible by Lemma 3.4 (ii).)

Let e be the identity element of $\mathfrak{A}(\mathcal{O}_K)$ and $\hat{\mathcal{O}}_{\mathfrak{A},e}$ be the completion of the local ring $\mathcal{O}_{\mathfrak{A},e}$ of $\mathfrak{A}$ at e with respect to the maximal ideal $\mathfrak{M}_{\mathfrak{A},e}$. Then we can choose local coordinates $x_1, \cdots, x_g$ so that $\hat{\mathcal{O}}_{\mathfrak{A},e} \simeq \mathcal{O}_K[[x_1, \cdots, x_g]]$. Let $\mathbf{F}_A$ be the formal group associated to A defined over $\mathcal{O}_K$ (cf. Remark 3.5). By an argument similar to the proof of [HS, Theorem C.2.6], we have $A_1 \simeq \mathbf{F}_A(\mathfrak{M}_K)$ where $\mathbf{F}_A(\mathfrak{M}_K)$ is the group associated to $\mathbf{F}_A/\mathcal{O}_K$. By Remark 3.6, we have

$$\bigcap_{n \in \mathbb{Z}_{>0}} p^n \mathbf{F}_A(\mathfrak{M}_K) = \{\mathbf{0}\}.$$

This implies that $(A_1)_{\mathrm{div}} = \{0\}$, as desired. □

Theorem 3.8
Let K be a PFHLF. Then K is pre-Kummer-faithful. In particular, if K is an MHLF, K is Kummer-faithful.

Proof.

This theorem follows immediately from the fact that finite fields are Kummer-faithful (cf. [Ho1, Remark 1.2.3 (i)]) and Proposition 3.7. □

Remark 3.9
If K is a ZFHLF, K is not (pre-)Kummer-faithful. Indeed, let $\mathfrak{M}_K$ be the maximal ideal of the ring of integers of K. Then any element of $1 + \mathfrak{M}_K$ is contained in $(K^\times)_{\mathrm{div}}$.

In the remainder of this section, for $i = 1, 2$, we shall write

- K_i for a Kummer-faithful field;
- $\overline{K_i}$ for an algebraic closure of K_i;
- G_{K_i} for the Galois group $\mathrm{Gal}(\overline{K_i}/K_i)$;
- X_i for a hyperbolic curve over K_i;
- X_i^{cpt} for the smooth compactification of X_i;
- $\pi_1(X_i)$ for the étale fundamental group of X_i (for some choice of basepoint);

- $\Delta_{X_i} = \pi_1(X_i \times_{\mathrm{Spec}\, K_i} \mathrm{Spec}\, \overline{K_i}) \subset \pi_1(X_i)$ for the geometric fundamental group of X_i (for some choice of basepoint).

Proposition 3.10
Suppose that K_i is an MHLF for $i = 1, 2$. Then any isomorphism of profinite groups $\phi : \pi_1(X_1) \xrightarrow{\sim} \pi_1(X_2)$ is Galois-preserving. (For the definition of Galois-preserving isomorphisms, see [Ho1, Definition 3.1 (ii)]*.)*

Proof.
Immediate from [MT, Corollary D]. □

Corollary 3.11
Suppose that K_i is an MHLF for $i = 1, 2$, and that either X_1 or X_2 is affine. Write $\mathrm{Isom}(\pi_1(X_1), \pi_1(X_2))$ *for the set of isomorphisms of profinite groups $\pi_1(X_1) \xrightarrow{\sim} \pi_1(X_2)$,* $\mathrm{Isom}^{\mathrm{PT}}(\pi_1(X_1), \pi_1(X_2)) \subset \mathrm{Isom}(\pi_1(X_1), \pi_1(X_2))$ *for the subset of point-theoretic isomorphisms of profinite groups $\pi_1(X_1) \xrightarrow{\sim} \pi_1(X_2)$,* $\mathrm{Isom}(X_1, X_2)$ *for the set of isomorphisms of schemes $X_1 \xrightarrow{\sim} X_2$, and* $\mathrm{Inn}(\pi_1(X_2))$ *for the group of inner automorphisms of $\pi_1(X_2)$. (For the definition of point-theoretic isomorphisms, see* [Ho1, Definition 3.1 (i)]*.) Then the natural map*

$$\mathrm{Isom}(X_1, X_2) \to \mathrm{Isom}(\pi_1(X_1), \pi_1(X_2))/\mathrm{Inn}(\pi_1(X_2))$$

determines a bijection

$$\mathrm{Isom}(X_1, X_2) \xrightarrow{\sim} \mathrm{Isom}^{\mathrm{PT}}(\pi_1(X_1), \pi_1(X_2))/\mathrm{Inn}(\pi_1(X_2)).$$

Proof.
Immediate from [Ho1, Theorem 3.4], Theorem 3.8 and Proposition 3.10. □

Remark 3.12
Minamide and Tsujimura proved (a certain weak version of) the Grothendieck conjecture for hyperbolic curves of genus 0 over MHLF (cf. [MT, Theorem A]). (In fact, they proved the conjecture for hyperbolic curves of genus 0 over a wider class of fields.)

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