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A geometric description of the Reidemeister-Turaev torsion of 3-manifolds

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Abstract

We give a relation between the Turaev-Reidemeister torsion and an invariant related to the Chern-Simons theory. As an application, we give a geometric description of the Turaev-Reidemeister torsion of closed oriented 3-manifolds from the point of view of a variant of a linking number.

1 Introduction

The Chern-Simons perturbation theory, established by M. Kontsevich in [12], S. Axelrod and I. M. Singer in [1], gives invariants of a 3-manifold with an acyclic representation of the fundamental group. Here a representation is said to be acyclic if the local system given by the representation is acyclic.

In the construction of the Chern-Simons perturbation theory, a 4-chain called a propagator plays an important role. A propagator is a 4-chain in the two point configuration space of given 3-manifold satisfying several homological conditions. There are some variations and related invariants of the Chern-Simons perturbation theory. Thus there are some variations of the homological conditions.

R. Bott and A. S. Cattaneo gave a refinement of the Chern-Simons perturbation theoretical invariants via a purely topological construction in [2] and [3]. Due to the homological conditions used in [2] and [3], the existence of propagators is not guaranteed in some cases. In [4] and [16], Cattaneo and the author refined the homological conditions and then showed that there exist refined propagators in any case.

In [16], the author defined an invariant d of an acyclic representation as a defect of the homological conditions for Bott-Cattaneo's propagators and that of refined propagators. The defect d was first introduced by C. Lescop in [14] for 3-manifold with $b_1 = 1$ when she defined an invariant of 3-manifolds with $b_1 = 1$. Her invariant can be considered as a generalization of the Chern-Simons perturbation theory for an abelian representation

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$H = \langle t \rangle \to \mathbb{Q}(t)^\times$ of the 1-dimensional homology group H of the manifold. She showed that the defect d can be computed from the Alexander polynomial of the 3-manifold.

In this article, we define an invariant $d'(M, e)$ of a closed oriented 3-manifold M with an Euler structure e. An Euler structure is an equivalence class of non-vanishing vector fields on the manifold. The invariant d' is a refinement of d in the following sense: There is a special non-vanishing vector filed v_0 such that $(-v_0)$ is equivalent to v_0. Then d' of the Euler structure represented by v_0 coincides with d.

We show that $d'(M, e)$ can be computed from the Reidemeister-Turaev torsion $\mathrm{Tor}(M, e)$ of M and e via an operator D defined by using logarithm derivatives (Theorem 1). Here the Reidemeister-Turaev torsion is a refinement of the Reidemeister torsion. The Alexander polynomial can be computed from the Reidemeister torsion. Theorem 1 is a generalization of Lescop's formula on the defect d to any closed oriented 3-manifold. The operator D holds information of $\mathrm{Tor}(M, e)$ up to constant multiplication. Namely, the Reidemeister-Turaev torsion $\mathrm{Tor}(M, e)$ can be computed from $d'(M, e)$ up to constant.

Theorem 1 (Section 6)**.**

$$D(\mathrm{Tor}(M, e)) = d'(M, e).$$

The invariant $d'(M, e)$ have the following topological description. Let Δ be the diagonal of $M \times M$. Take a non-vanishing vector field v_e of the normal bundle of Δ in $M \times M$ corresponding to the Euler structure e. Then we have a framed 3-manifold (Δ, v_e) in $M \times M$. We introduce a kind of a self-linking number $\mathrm{self.lk}(\Delta, v_e)$ of (Δ, v_e) with a twisted coefficient. Then we show that $d'(M, e) = \mathrm{self.lk}(\Delta, v_e)$. Thus Theorem 1 gives a geometric description of the Reidemeister-Turaev torsion from the point of view of a self-linking number:

Theorem 2 (Section 8.2)**.**

$$D(\mathrm{Tor}(M, e)) = \mathrm{self.lk}(\Delta, v_e).$$

Remark 1.1. There are several works on geometric approaches to the Reidemeister torsion. M. Hutchings and Y. J. Lee gave descriptions of the Reidemeister-Turaev torsion by using circle-valued Morse functions in [8],[9] and [10]. H. Goda, A. V. Pajitnov in [7] and T. Kitayama in [11] gave Morse theoretical description for the Reidemeister torsion for non-commutative representations in some cases. F. Deloup and G. Massuyeau in [5] showed that a quadratic function given by the Redemeister-Turaev torsion coincides with a quadratic form given by the intersection pairing of a cobordism.

Remark 1.2. The invariant d' can be defined for any acyclic representation on any oriented closed 3-manifold. The Reidemeister torsion is also defined for any acyclic representation.

The organization of this paper is as follows. In Section 2 we prepare some notations on manifolds and Euler structures. In Section 3 we review a definition of the Reidemeister-Turaev torsion. In Section 4 we define the invariant d'. In Section 5 we introduce an operator D to state Theorem 1. In Section 6 we state Theorem 1. In Section 7 we give a proof of Theorem 1. In Section 8 we state Theorem 2. In Section 9 we explain the relation of the invariant d' and Lescop's invariant.

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2 Preliminaries

Let M be a closed oriented 3-manifold. Let us denote $H = H_1(M;\mathbb{Z})/\mathrm{Tor}$. We fix a basis

$$\{t_1, \dots, t_k\} \subset H.$$

We interpret H as a multiplicative group. In this article, however, the operations of the all homology groups including $H_1(M;\mathbb{Z})$ are summation [2] with the exception of H:

$$H_1(M;\mathbb{Z})/\mathrm{Tor} = \{n_1 t_1 + \cdots + n_k t_k \mid n_i \in \mathbb{Z}\},$$

$$H = \{t_1^{n_1} \cdots t_k^{n_k} \mid n_i \in \mathbb{Z}\}.$$

For $h \in H$, we denote by

$$[h]_+ \in H_1(M;\mathbb{Z})/\mathrm{Tor}$$

the element of $H_1(M;\mathbb{Z})/\mathrm{Tor}$ corresponding to h. For example, $[x_1^2 x_2]_+ = 2x_1 + x_2$.

Let

$$\mathbb{R}H = \{\sum_{n_1,\dots,n_k} a_{n_1,\dots,n_k} t_1^{n_1} \cdots t_k^{n_k}\}$$

be the group ring of H over $\mathbb{R}$ and let

$$Q(H) = \{f/g \mid f, g \in \mathbb{R}H, g \neq 0\}$$

[2] We will use H as the torsion free part of the abelianization of the fundamental group. Usually, the operations of fundamental groups are multiplications and that of homology groups are summations.

be the quotient field of $\mathbb{R}H$. By using the basis, $Q(H)$ can be identified with

$$Q(H) = \mathbb{R}(t_1, \ldots, t_k).$$

The natural representation

$$\rho_0 : H \to Q(H)^\times$$

given by $x \mapsto (y \mapsto yx)$ induces a homomorphism

$$\rho_0 : \mathbb{R}H \to Q(H)^\times.$$

(We use the same symbol ρ_0 for the induced homomorphism).

Let $\Pi(M)$ be the fundamental groupoid of M; namely $\Pi(M)$ is a category such that the objects are the points of M and a morphism from $x \in M$ to $y \in M$ is a path from x to $y \in M$ up to homotopy relative to x, y. In particular, a morphism from a point $x \in M$ to x is an element of the fundamental group $\pi_1(M, x)$. A local system over M is a covariant functor from $\Pi(M)$ to a category of vector spaces. We denote by γ_* the isomorphism corresponding to a path γ via the local system. Let E be a vector space. For a homomorphism $\rho : \pi_1(M, x) \to \mathrm{Aut}E$, there is a local system on M such that each object is isomorphic to E and $\rho_* = \rho(\gamma)$ for any $\gamma \in \pi_1(M, x)$. We call such a local system *the local system corresponding to* ρ and we use the same symbol ρ for the local system.

The homomorphism $\rho_0 : H \to Q(H)^\times$ gives a representation of $\pi_1(M)$. Then there is a local system corresponding to ρ_0. We denote by $Q(H)_x$ the object corresponding to $x \in M$.

We assume that ρ_0 is acyclic. Namely, the corresponding local system is acyclic:

$$H_*(M; \rho_0) = 0.$$

2.1 Chains with local coefficients

Let X be a manifold. Let E be a vector space and let ρ be a local system corresponding to a representation $\rho : \pi_1(X) \to \mathrm{Aut}E$. Let $c : (A, a) \to (X, c(a))$ be a continuous map, where A is a compact oriented manifold with a base point $a \in A$. We denote by $E_{c(a)}^{c_*\pi_1(A,a)}$ the invariant part of $E_{c(a)}$ under the action of $c_*\pi_1(A, a) < \pi_1(X, c(a))$ via the representation ρ. Take an element $e \in E_{c(a)}^{c_*\pi_1(A,a)}$. Then c and $e \in E_{c(a)}$ give a $(\dim A)$-dimensional chain. We denote this chain by

$$\langle c : A \to X, a; e\rangle \in C_{\dim A}(X; \rho)$$

or shortly,

$$\langle A, a; e\rangle = \langle c : A \to X, a; e\rangle.$$

Let F be a compact oriented manifold. Let $\pi : \widetilde{X} \to X$ be a F-bundle over X. A homomorphism $\rho \circ \pi_* : \pi_1(\widetilde{X}) \stackrel{\pi_*}{\to} \pi_1(X) \to \mathrm{Aut}E$ gives a local system on $\widetilde{X}$. Let $c = \langle A \to X, a; e\rangle \in C_k(X; E)$ be a k-chain. We denote by $c^*\widetilde{X} \to A$ the pull back of $\pi : \widetilde{X} \to X$ along c. Then we have a bundle map $\widetilde{c} : c^*\widetilde{X} \to \widetilde{X}$ induced by $c : A \to X$. By using $\widetilde{c}$, the $(k + \dim F)$-chain $\pi^! c \in C_{k+\dim F}(\widetilde{X}; \pi^*E)$ can be described as follows:

$$\pi^! c = \langle \widetilde{c} : c^*\widetilde{X} \to \widetilde{X}, v_a; e\rangle \in C_{k+\dim F}(\widetilde{X}; \rho \circ \pi_*).$$

Here $v_a \in \pi^{-1}(a)$ is any point.

$$\begin{array}{ccc} c^*\widetilde{X} & \xrightarrow{\widetilde{c}} & \widetilde{X} \\ \downarrow & \circlearrowleft & \downarrow \pi \\ A & \xrightarrow{c} & X \end{array}$$

2.2 An Euler structure

Let v, v' be non-vanishing vector fields on M. We say that v and v' are *homologous* if $v|_{M\setminus B^3}$ is homotopic to $v'|_{M\setminus B^3}$ for a small ball B^3. An *Euler structure* on M is a homologous class of non-vanishing vectors field on M.

3 The Reidemeister-Turaev torsion $\mathrm{Tor}(M, e)$

Let M be a closed oriented 3-manifold and let e be an Euler structure on M. The Reidemeister-Turaev torsion

$$\mathrm{Tor}(M, e) \in Q(H)^\times$$

is an invariant of (M, e) defined by V. Turaev as a refinement of the Reidemeister torsion for abelian representations. In this section, we review the definition of $\mathrm{Tor}(M, e)$. For example, see [17] or [18] for more details.

Let $C_* = (C_i(M; \rho_0), \partial_i : C_i(M, \rho_0) \to C_{i-1}(M, \rho_0))_i$ be a chain complex of M with the local system associated to ρ_0. Since ρ_0 is acyclic, there exist homomorphisms

$$g_i : C_{i-1}(M; \rho) \to C_i(M; \rho) \quad (i = 0, 1, 2, 3)$$

satisfying

$$\partial_{i+1} \circ g_{i+1} + g_i \circ \partial_i = \mathrm{id}_{C_i}.$$

Then $\partial + g$ gives an isomorphism

$$\partial + g = g_3 + \partial_2 + g_1 : C_{even} \stackrel{\cong}{\to} C_{odd}.$$

Here $C_{even} = C_2(M;\rho_0) \oplus C_0(M;\rho_0)$ and $C_{odd} = C_3(M;\rho) \oplus C_1(M;\rho)$. It is known that an Euler structure gives a basis of both C_{even} and C_{odd} (in Section 7.2, we construct such a basis when C_* is given as a Morse-Smale complex). Therefore we can compute the determinant of the matrix $\partial + g$ with respect to the basis given by the Euler structure e.

Definition 3.1.

$$\mathrm{Tor}(M,e) = \det(\partial + g) \in Q(H)^\times.$$

4 The invariant $d'(M,e)$

We define an invariant

$$d'(M,e) \in H_1(M;\mathbb{Z})/\mathrm{Tor} \otimes_{\mathbb{Z}} Q(H)$$

of (M,e). The invariant $d'(M,e)$ is a refinement of an invariant d defined in [16]. See Remark 4.3 for more details.

4.1 Preparation for local systems

Let $Q(H)^* = \mathrm{Hom}_{Q(H)}(Q(H), Q(H))$ be the dual space of $Q(H)$. We denote by $\rho_0^* : H \to \mathrm{Aut}_{Q(H)}(Q(H)^*) = (Q(H)^*)^\times$ the dual representation of ρ_0, namely ρ_0^* is given by

$$\rho_0^*(h)(f) = f \circ \rho_0(h^{-1})$$

for $h \in H$ and $f \in Q(H)^*$. Thanks to the Künneth theorem, $\rho_0 \boxtimes_{Q(H)} \rho_0^* : \pi_1(M \times M) = \pi_1(M) \times \pi_1(M) \to H_1(M;\mathbb{Z}) \times H_1(M;\mathbb{Z}) \to \mathrm{Aut}_{Q(H)}(Q(H) \otimes Q(H)^*)$ gives an acyclic local system on a 6-dimensional manifold $M \times M$. Here $\rho_0 \boxtimes \rho_0^*$ is the exterior tensor product of ρ_0 and ρ_0^*.

Let $\mathrm{ev} : Q(H) \otimes Q(H)^* \to Q(H)$ be the evaluation map defined by $\mathrm{ev}(x \otimes f) = f(x)$ for $x \in Q(H)$ and $f \in Q(H)^*$. For $x \in Q(H)$, $f \in Q(H)^*$ and $h \in H$, we have

$$\mathrm{ev}((\rho_0 \otimes \rho_0^*)(h)(x \otimes f)) = \mathrm{ev}(\rho_0(h)(x) \otimes \rho_0^*(h)(x)) = f \circ \rho_0(h^{-1})(\rho_0(h)x) = f(x).$$

Namely, the action $H \curvearrowright Q(H) \otimes_{Q(H)} Q(H)^*$ is compatible with the evaluation map. This implies that $\rho_0 \otimes \rho_0^*$ is trivial:

$$\left(\rho_0 \otimes \rho_0^*\right)(h) = \mathrm{id} : Q(H) \otimes Q(H)^* \to Q(H) \otimes Q(H)^*$$

for any $h \in H$. Therefore the restriction $\rho_0 \otimes \rho_0^* = (\rho_0 \boxtimes \rho_0^*)|_\Delta$ of $\rho_0 \boxtimes \rho_0^*$ to the diagonal $\Delta = \{(x,x) \mid x \in M\} \subset M \times M$ gives the trivial local system. Thus we have

$$\begin{aligned} H_*(\Delta : \rho_0 \boxtimes \rho_0^*|_\Delta) &= H_*(\Delta : \rho_0 \otimes \rho_0^*) = H_1(\Delta;\mathbb{Z}) \otimes_{\mathbb{Z}} Q(H) \\ &= H_1(M;\mathbb{Z}) \otimes_{\mathbb{Z}} Q(H) = H_1(M;\mathbb{Z})/\mathrm{Tor} \otimes_{\mathbb{Z}} Q(H). \end{aligned}$$

4.2 Preparation for manifolds

Let ν_Δ be the normal bundle of Δ in M^2. Set $D(\nu_\Delta) = \{(x,v) \mid \|v\| \leq 1\}$. $D(\nu_\Delta)$ forms a D^3-bundle over Δ. Let $S\nu_\Delta = \partial D(\nu_\Delta)$ be (the total space of) the unit sphere bundle of Δ. We denote by $B\ell(M^2, \Delta)$ the real blowing up of M^2 along Δ, namely

$$\begin{aligned} B\ell(M^2,\Delta) &= (M^2 \setminus \Delta) \cup \{((x,tu),u) \in \nu_\Delta \times S\nu_\Delta \mid x \in \Delta, u \in (S\nu_\Delta)_x, t \in [0,\infty)\} \\ &\cong (M^2 \setminus \Delta) \cup S\nu_\Delta. \end{aligned}$$

We denote by $\pi : B\ell(M^2,\Delta) \to M^2$ the blow down map. Namely, $\pi|_{M^2\setminus\Delta} = \mathrm{id} : M^2 \setminus \Delta \to M^2 \setminus \Delta \subset M^2$ and $\pi|_{S\nu_\Delta} : S\nu_\Delta \to \Delta \subset M^2$ are the projections. By the construction,

$$\mathrm{int} B\ell(M^2,\Delta) = M^2 \setminus \Delta,$$

$$\partial B\ell(M^2,\Delta) = \pi^{-1}(\Delta) = S\nu_\Delta.$$

We identified ν_Δ with the tangent bundle TM by the bundle isomorphism given by

$$\nu_\Delta \ni ((x,x),(v,-v)) \mapsto (x,v) \in TM,$$

where $x \in M, v \in T_xM$. Under the identification, we have $D(\nu_\Delta) = D(TM)$ and $S\nu_\Delta = STM$.

Take an open tubular neighborhood $N(\Delta)$ of Δ in M^2 and then we identify $N(\Delta)$ with ν_Δ. We can take a diffeomorphism between $B\ell(M^2,\Delta)$ and $M^2 \setminus \mathrm{int} D(\nu_\Delta)$ by "shrinking" a collar of $\partial B\ell(M^2,\Delta)$. More precisely, we take the inclusion map

$$\iota : B\ell(M^2,\Delta) \hookrightarrow M^2$$

as follows: Let $D_2(\nu_\Delta) = \{(x,v) \in \nu_\Delta \mid \|v\| \leq 2\} \subset M^2$. Take a C^∞ function $h : [0,\infty) \to [1,\infty)$ satisfying

$$\frac{dh}{dt}(t) \geq 0 \ \ \forall t \ \text{ and } \ h(t) = \begin{cases} t & (t \geq 2) \\ 1 & (t = 0) \end{cases}.$$

The inclusion $\iota : B\ell(M^2,\Delta) \hookrightarrow M^2$ is given as the canonical extension of the map $\iota_0 : (M^2 \setminus \Delta =)\mathrm{int} B\ell(M^2,\Delta) \to B\ell(M^2,\Delta)$ defined by

$$\begin{aligned} \iota_0|_{B\ell(M^2)\setminus D_2(\nu_\Delta)} &= \mathrm{id}, \\ \iota_0|_{D_2(\nu_\Delta)\setminus\Delta}(x,v) &= \left(x, \frac{h(\|v\|)}{\|v\|}v\right) \quad \text{for } (x,v) \in D_2(\nu_\Delta) \setminus \Delta. \end{aligned}$$

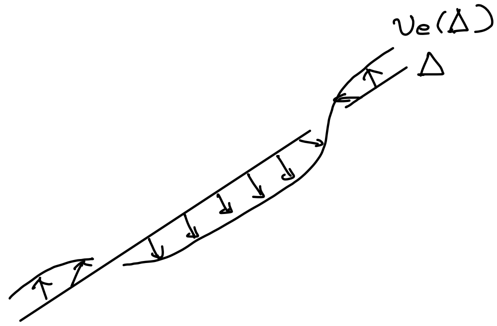
Obviously, $\iota(B\ell(M^2,\Delta)) = M^2 \setminus N(\Delta)$. Thus we can identify $B\ell(M^2,\Delta)$ with $M^2 \setminus N(\Delta)$ via ι. We will consider $B\ell(M^2,\Delta) \subset M^2$ via ι, if need be.

4.3 The definition of $d'(M,e)$

Take a non-vanishing vector filed v_e on M representing the Euler structure e. Set

$$v_e(\Delta) = \left\{ \left(x, \frac{v_e(x)}{\|v_e(x)\|} \right) \right\} \subset STM \cong S\nu_\Delta = \partial B\ell(M^2, \Delta).$$

When we consider $B\ell(M^2,\Delta) \subset M^2$, $v_e(\Delta)$ is a 3-manifold given by pushing Δ along v_e. Namely, $v_e(\Delta)$ is a parallel copy of Δ).



Via a triangulation, the closed 3-manifold Δ gives a 3-cycle of $C_3(M;\mathbb{Z})$ and then Δ together with $1 \in Q(H)$ gives a 3-cycle of $C_3(\Delta, \rho_0 \boxtimes \rho_0^*|_\Delta)$:

$$\Delta \otimes 1 \in C_3(\Delta;\mathbb{Z}) \otimes_{\mathbb{Z}} Q(H) = C_3(\Delta, \rho_0 \boxtimes \rho_0^*|_\Delta).$$

$\Delta \otimes 1$ also gives a 3-cycle of $C_3(M \times M; \rho_0 \boxtimes \rho_0^*)$ via the morphism

$$C_3(\Delta, \rho_0 \boxtimes \rho_0^*|_\Delta) \to C_3(M \times M; \rho_0 \boxtimes \rho^*).$$

Since $\pi^*(\rho_0 \boxtimes \rho_0^*)|_{\partial B\ell(M^2,\Delta)} = \pi^*(\rho_0 \otimes \rho_0^*)$, $v_e(\Delta) \otimes 1$ is a 3-cycle in $C_3(\partial B\ell(M^2,\Delta); \pi^*(\rho_0 \boxtimes \rho_0^*))$, and also gives a 3-cycle in $C_3(B\ell(M^2,\Delta); \pi^*(\rho_0 \boxtimes \rho_0^*))$ as in the case of $\Delta \otimes 1$.

Let

$$\partial_* : H_4(M^2, B\ell(M^2,\Delta); \rho_0 \boxtimes \rho_0^*) \to H_3(B\ell(M^2,\Delta); \rho_0 \boxtimes \rho_0^*)$$

be the connecting homomorphism in the long exact sequence of homologies of the pair $(M^2, B\ell(M^2,\Delta))$. Since $H_4(M^2; \rho_0 \boxtimes \rho_0^*) = 0$ and $H_3(M^2; \rho_0 \boxtimes \rho_0^*) = 0$, ∂_* is an isomorphism. Let

$$E : H_4(M^2, B\ell(M^2,\Delta)), \rho_0 \boxtimes \rho_0^*) \overset{\cong}{\to} H_4(N(\Delta)), \partial N(\Delta); \rho_0 \boxtimes \rho_0^*)$$

be an excision isomorphism. Let

$$\tau : H_4((N(\Delta)), \partial N(\Delta); \rho_0 \boxtimes \rho_0^*) \overset{\cong}{\to} H_1(\Delta; \rho_0 \otimes \rho_0^*)$$

be the Thom isomorphism. Therefore we have an isomorphism

$$\Phi = \tau \circ E \circ (\partial_*)^{-1} : H_3(B\ell(M^2, \Delta); \rho_0 \boxtimes \rho_0^*) \overset{\cong}{\to} H_1(\Delta; \rho_0 \otimes \rho_0^*).$$

Definition 4.1.

$$d'(M,e) = \Phi([v_e(\Delta) \otimes 1]) \in H_1(\Delta; \rho_0 \otimes \rho_0^*) = H_1(M;\mathbb{Z})/\mathrm{Tor} \otimes_{\mathbb{Z}} Q(H).$$

Here $[v_e(\Delta) \otimes 1] \in H_3(B\ell(M^2, \Delta); \rho_0 \boxtimes \rho_0^*)$ is the homology class represented by the 3-cycle $v_3(\Delta) \otimes 1$.

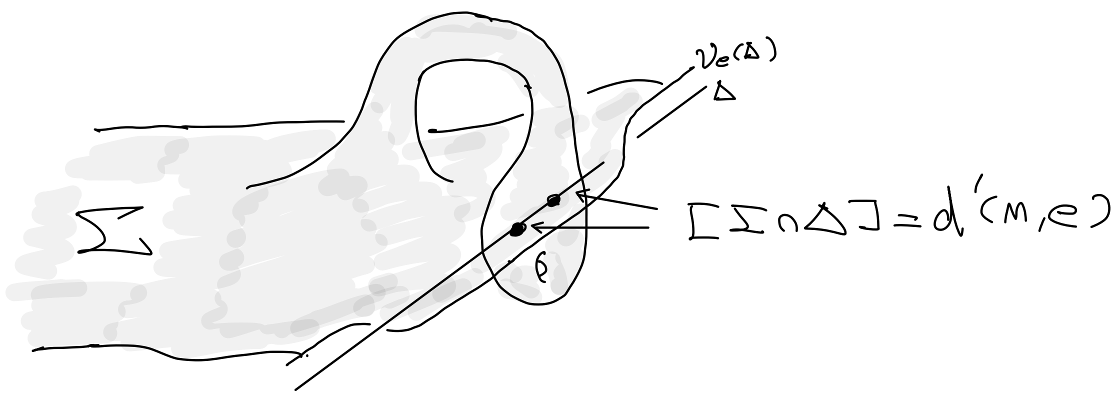
4.4 A geometric description of $d'(M,e)$

Since $H_3(M^2, \rho_0 \boxtimes \rho_0^*) = 0$, there exists a 4-chain $\Sigma \in C_4(M^2; \rho_0 \boxtimes \rho_0^*)$ bounded by $v_e(\Delta) \otimes 1$. Σ gives a 4-cycle of $C_4(M^2, B\ell(M^2, \Delta); \rho_0 \boxtimes \rho_0^*)$. The 3-manifold $\Delta \subset M^2$ gives a 3-cycle of $C_3(M^2; \mathbb{Z})$.

Proposition 4.2. *The intersection $\Sigma \cap \Delta \in C_1(\Delta; \rho_0 \otimes \rho_0^*)$ represents $d'(M,e)$:*

$$d'(M,e) = [\Sigma \cap \Delta] \in H_1(\Delta; \rho_0 \otimes \rho_0^*) = H_1(M;\mathbb{Z})/\mathrm{Tor} \otimes_{\mathbb{Z}} Q(H).$$

Proof. The 4-chain Σ represents $\partial_*^{-1}([v_e(\Delta) \otimes 1]) \in H_4(M^2, B\ell(M^2, \Delta); \rho_0 \boxtimes \rho_0^*)$. The composition of the Thom isomorphism τ and the excision isomorphism E is realized as $\tau \circ E([\Sigma]) = [\Sigma \cap \Delta]$. Then $d'(M,e) = \Phi(v_e(\Delta) \otimes 1) = [\Sigma \cap \Delta]$. □



Remark 4.3. An invariant $d(\rho_0)$ defined in [16] can be computed from $d'(M,e)$ as follows:

$$d(\rho_0) = d'(M, e_0).$$

Here e_0 is an Euler structure represented by a non-vanishing vector field v_0, which is homologous to $-v_0$.

5 An operator D

In this section, we introduce an operator D from $Q(H)$ to $(H_1(M;\mathbb{Z})/\mathrm{Tor}) \otimes_{\mathbb{Z}} Q(H)$. Recall that for $h \in H$, $[h]_+ \in H_1(M;\mathbb{Z})/\mathrm{tor}$ is the element corresponding to h.

Definition 5.1. (1) For $g, h \in \mathbb{R}H$, we have $g/h \in Q(H)$. We set $D(g/f) = D(g) - D(f)$.

(2) For $f = \sum_{h\in H} a_h h \in \mathbb{R}H, a_h \in \mathbb{R}$, we set $D(f) = D\left(\sum_{h\in H} a_h h\right) = \sum_h [h]_+ \otimes \dfrac{a_h h}{f}$.

Example 5.2. Let $H = \langle t_1, t_2 \rangle$.

$$\begin{aligned}
& D(1 + t_1^3 + t_1^2 t_2^6) \\
= \ & [1]_+ \otimes \frac{1}{1 + t_1^3 + t_1^2 t_2^6} + [t_1^3]_+ \otimes \frac{t_1^3}{1 + t_1^3 + t_1^2 t_2^6} + [t_1^2 t_2^6]_+ \otimes \frac{t_1^2 t_2^6}{1 + t_1^3 + t_1^2 t_2^6} \\
= \ & 0 \otimes \frac{1}{1 + t_1^3 + t_1^2 t_2^6} + 3[t_1]_+ \otimes \frac{t_1^3}{1 + t_1^3 + t_1^2 t_2^6} + (2[t_1]_+ + 6[t_2]_+) \otimes \frac{t_1^2 t_2^6}{1 + t_1^3 + t_1^2 t_2^6} \\
= \ & [t_1]_+ \otimes \frac{3t_1^3 + 2t_1^2 t_2^6}{1 + t_1^3 + t_1^2 t_2^6} + [t_2]_+ \otimes \frac{6 t_1^2 t_2^6}{1 + t_1^3 + t_1^2 t_2^6}.
\end{aligned}$$

The following lemma and the corollary guarantee that the definition is well defined. Namely, $D(f)$ is independent of the representation of $f \in Q(H)$.

Lemma 5.3. *For any $f, g \in \mathbb{R}H$,*

(1) $D(f+g) = (1 \otimes \frac{f}{f+g})D(f) + (1 \otimes \frac{g}{f+g})D(g)$,

(2) $D(fg) = D(f) + D(g)$.

Proof. (1) Let $f = \sum_h a_h h, g = \sum_h b_h h$. We have

$$\begin{aligned}
\left(1 \otimes \frac{f}{f+g}\right) D(f) + \left(1 \otimes \frac{g}{f+g}\right) D(g) &= \sum_h [h]_+ \otimes \left(\frac{f}{f+g} \cdot \frac{a_h h}{f} + \frac{g}{f+g} \cdot \frac{b_h h}{g}\right) \\
&= \sum_h [h]_+ \otimes \left(\frac{a_h h}{f+g} + \frac{b_h h}{f+g}\right) \\
&= D(f+g).
\end{aligned}$$

(2) Let $f = \sum_h a_h h, g = \sum_{h'} b_{h'} h'$. Then $fg = \sum_{h,h'} a_h b_{h'} hh'$. We have

$$\begin{aligned}
D(fg) &= \sum_{h,h'} [hh']_+ \otimes \frac{a_h b_{h'} hh'}{fg} \\
&= \sum_{h,h'} ([h]_+ + [h']_+) \otimes \frac{a_h b_{h'} hh'}{fg} \\
&= \sum_{h,h'} [h]_+ \otimes \frac{a_h b_{h'} hh'}{fg} + \sum_{h,h'} [h']_+ \otimes \frac{a_h b_{h'} hh'}{fg} \\
&= \sum_h \left([h]_+ \otimes \sum_{h'} \frac{a_h b_{h'} hh'}{fg} \right) + \sum_{h'} \left([h']_+ \otimes \sum_h \frac{a_h b_{h'} hh'}{fg} \right) \\
&= \sum_h \left([h]_+ \otimes \frac{a_h h}{f} \frac{\sum_{h'} b_{h'} h'}{g} \right) + \sum_{h'} \left([h']_+ \otimes \frac{b_{h'} h'}{g} \frac{\sum_h a_h h}{f} \right) \\
&= \sum_h \left([h]_+ \otimes \frac{a_h h}{f} \right) + \sum_{h'} \left([h']_+ \otimes \frac{b_{h'} h'}{g} \right) \\
&= D(f) + D(g).
\end{aligned}$$

□

As a direct consequence of the above lemma, we have the following corollary.

Corollary 5.4. *Let $f_1, g_1, f_2, g_2 \in \mathbb{R}H$. If $g_1/f_1 = g_2/f_2 \in Q(H)$ then*

$$D(g_1/f_1) = D(g_2/f_2).$$

Proof. For any $h \in \mathbb{R}H$, we have $D\left(\frac{g_1 h}{f_1 h}\right) = D\left(\frac{g_1}{h_1}\right)$. Actually,

$$\begin{aligned}
D\left(\frac{g_1 h}{f_1 h}\right) &= D(g_1 h) - D(f_1 h) \\
&= D(g_1) + D(h) - (D(f_1) + D(h)) \\
&= D(g_1) - D(f_1) = D\left(\frac{g_1}{f_1}\right).
\end{aligned}$$

This implies that $D\left(\frac{g_1}{f_1}\right) = D\left(\frac{g_2}{f_2}\right)$ if $\frac{g_1}{f_1} = \frac{g_2}{f_2} \in Q(H)$. □

The properties in Lemma 5.3 characterizes the operator D:

Lemma 5.5. *If a morphism $D' : Q(H) \to H_1(M;\mathbb{Z})/\mathrm{Tor} \otimes Q(H)$ satisfies the following properties (1), (2) and (3), then $D' = D$.*

(1) $D'(f+g) = (1 \otimes \frac{f}{f+g})D'(f) + (1 \otimes \frac{g}{f+g})D'(g)$, for any $f, g \in \mathbb{R}H$.

(2) $D'(fg) = D'(f) + D'(g)$, for any $f, g \in \mathbb{R}H$.

(3) For any $h \in H$ and for any $0 \neq a \in \mathbb{R}$, $D'(ah) = D(ah) = [h]_+ \otimes 1$.

Proof. Any element of $Q(H)$ can be written as a quotient of elements in $\mathbb{R}H$, and any element of $\mathbb{R}H$ can be written as a linear combination of elements of H. Thus we can compute D' completely by the rule (1), (2) and (3). $\square$

We next give an alternative description of the operator D by using the basis $t_1, \ldots, t_k$ of H for the convenience of computation.

Take a representation of the group H:

$$H = \langle x_1, \ldots, x_m \mid R \rangle.$$

By using generators $x_1, \ldots, x_m$ of H, the operator D can be written as follows:

Proposition 5.6. *For any polynomial $f(x_1, \ldots, x_m) \in \mathbb{R}H$ of $x_1, \ldots, x_m$,*

$$D(f) = \sum_{i=1}^{m} \left([x_i]_+ \otimes x_i \frac{\partial}{\partial x_i} \log |f(x_1, \ldots, x_m)| \right).$$

In particular, for any monomial $ax_1^{n_1} \cdots x_m^{n_m} \in \mathbb{R}H$, where $0 \neq a \in \mathbb{R}$ and $n_1, \ldots, n_m \in \mathbb{Z}$, we have

$$D(ax_1^{n_1} \cdots x_m^{n_m}) = (n_1[x_1]_+ + \cdots + n_m[x_m]_+) \otimes 1.$$

Proof. Set $D'(f) = \sum_{i=1}^{m} \left([x_i]_+ \otimes x_i \frac{\partial}{\partial x_i} \log |f(x_1, \ldots, x_m) \right) |$ for $f \in \mathbb{R}H$. Clearly, D' satisfies the conditions in Lemma 5.5. Thus $D' = D$. $\square$

Example 5.7. Let $H = \langle t_1, t_2 \rangle$ and $f = 1 + t_1^3 + t_1^2 t_2^6$.

$$\begin{aligned} D(f) &= [t_1]_+ \otimes t_1 \frac{\partial}{\partial t_1} \log |1 + t_1^3 + t_1^2 t_2^6| + [t_2]_+ \otimes t_2 \frac{\partial}{\partial t_2} \log |1 + t_1^3 + t_1^2 t_2^6| \\ &= [t_1]_+ \otimes \frac{3t_1^3 + 2t_1^2 t_2^6}{f} + [t_2]_+ \otimes \frac{6t_1^2 t_2^6}{f}. \end{aligned}$$

We can use other representation of H_1, for example

$$H_1 = \langle t_1, t_2, T | T = t_1^2 \rangle \text{ and } f = 1 + t_1 T + t_2^6 T = 1 + t_1^3 + t_1^2 t_2^6.$$

$$
\begin{aligned}
& D(1+t_1T+t_2^6T) \\
= \ & [t_1]_+ \otimes t_1 \frac{\partial}{\partial t_1} \log|1+t_1T+t_2^6T| + [t_2]_+ \otimes t_2 \frac{\partial}{\partial t_2} \log|1+t_1T+t_2^6| \\
& +[T]_+ \otimes T \frac{\partial}{\partial T} \log|1+t_1T+t_2^6T| \\
= \ & [t_1]_+ \otimes \frac{t_1T}{f} + [t_2]_+ \otimes \frac{6t_2^6T}{f} + [T]_+ \otimes \frac{t_1T+t_2^6T}{f} \\
= \ & [t_1]_+ \otimes \frac{t_1^3}{f} + [t_2]_+ \otimes \frac{6t_1^3t_2^6}{f} + 2[t_1]_+ \otimes \frac{t_1^3+t_1^2t_2^6}{f} \\
= \ & [t_1]_+ \otimes \frac{3t_1^3+2t_1^2t_2^6}{f} + [t_2]_+ \otimes \frac{6t_1^3t_2^6}{f}.
\end{aligned}
$$

6 Main theorem

The following theorem is the main theorem of this article.

Theorem 1.

$$D(\mathrm{Tor}(M,e)) = -d'(\rho,e).$$

Example 6.1. Let $M = S^1 \times S^2$. In this case $H = H_1(M;\mathbb{Z}) = \langle t \rangle$. Here $t \in H$ is represented by $S^1 \times \{\mathrm{pt}\}$ for a point $\mathrm{pt} \in S^2$. It is known that, for a suitable Euler structure e, the Reidemeister-Turaev torsion is given as

$$\mathrm{Tor}(S^1 \times S^2, e) = \frac{1}{(t-1)^2} \in Q(H).$$

Thus

$$d'(M,e) = -D\left(\frac{1}{(t-1)^2}\right) = -[t]_+ \otimes \frac{-2t}{t-1}.$$

7 Proof of Theorem 1

In this Section we give the proof of Theorem 1. In Section 7.1 and 7.2 we prepare a Morse function and some notations. In Ssection 7.3 we give a description of $d'(M,e)$ by using the Morse function. In Section 7.4 we give explicit descriptions of the boundary homomorphisms and related homomorphisms associated with the Morse function. By using these descriptions, in Section 7.5 and 7.6 we compute $d'(M,e)$ and $\mathrm{Tor}(M,e)$. And then in Section 7.7 we compare these computations to get a proof of the theorem.

7.1 A Morse function and a combinatorial propagator

We give a Morse theoretic description of Σ which is a 4-chain of $C_4(M^2;\rho_0 \boxtimes \rho_0^*)$ bounded by $v_e(\Delta)\otimes 1$ (see Section 4.4). The description is related to the construction in [15], [16] and it is inspired by [6] and [19].

Take a Morse function $f : M \to \mathbb{R}$ and a Riemannian metric on M satisfying the Morse-Smale condition. We denote by $\mathrm{Crit}_i(f)$ the set of critical points of index i. Let $\mathrm{Crit}(f) = \sum_i \mathrm{Crit}_i(f)$. We denote by $\mathrm{ind}(p)$ the Morse index of a critical point p.

Let $(\Phi_t^f : M \overset{\cong}{\to} M)_{t\in\mathbb{R}}$ be the one-parameter family of diffeomorphisms associated to the gradient vector field $\mathrm{grad} f$. For $p \in \mathrm{Crit}(f)$, let $\mathcal{A}_p$ and $\mathcal{D}_p$ be the ascending manifold of p and the descending manifold of p respectively:

$$\mathcal{A}_p = \{x \in M \mid \lim_{t\to-\infty} \Phi_t^f(x) = p\},$$

$$\mathcal{D}_p = \{x \in M \mid \lim_{t\to\infty} \Phi_t^f(x) = p\}.$$

We orient $\mathcal{A}_p$ and $\mathcal{D}_p$ by imposing the condition $T_p\mathcal{A}_p \oplus T_p\mathcal{D}_p = T_pM$. Let $\mathcal{M}(p,q)$ be the set of all trajectories from $p \in \mathrm{Crit}_i(f)$ to $q \in \mathrm{Crit}_{i-1}(f)$:

$$\mathcal{M}(p,q) = \{\gamma : \mathbb{R} \to M \mid d\gamma/dt = \mathrm{grad} f, \lim_{t\to-\infty} \gamma(t) = q, \lim_{t\to\infty} \gamma(t) = p\}/\mathbb{R}.$$

Here $s \in \mathbb{R}$ acts as $s \cdot \gamma(t) = \gamma(t+s)$. We consider γ as both a path and a compact oriented 1-manifold (or 1-chain). When we consider γ as a path, γ is a path from p to q (the opposite direction induced by the parameter $\mathbb{R}$). Thus γ determines an isomorphism $\gamma_* : Q(H)_p \overset{\cong}{\to} Q(H)_q$. When we consider γ as an oriented manifold, the orientation is given as a closure of a part of the intersection $\mathcal{D}_p \cap \mathcal{A}_q$. By using the orientation of γ, we assign a signature $\varepsilon(\gamma) \in \{+1,-1\}$ for each $\gamma \in \mathcal{M}(p,q)$ as follows: If the orientation is from p to q, we set $\varepsilon(\gamma) = (-1)^{\mathrm{ind}(p)}$. If the orientation is from q to p, we set $\varepsilon(\gamma) = (-1)^{\mathrm{ind}(p)+1}$.

We denote by $C_*^f = (C_i^f(M;\rho_0), \partial_i^f : C_i^f(M;\rho_0) \to C_{i-1}^f(M;\rho_0))_i$ the Morse-Smale complex of f, namely $C_i^f(M;\rho_0) = \oplus_{p\in\mathrm{Crit}_i(f)} Q(H)_p$ and

$$\partial_i(x) = \sum_{q\in\mathrm{Crit}_{i-1}(f)} \sum_{\gamma\in\mathcal{M}(p,q)} \varepsilon(\gamma)\gamma_*(x)$$

for any $x \in Q(H)_p$ and $p \in \mathrm{Crit}_i(f)$.

Take homomorphisms $g^f = \{g_i^f : C_{i-1}^f(M;\rho_0) \to C_i^f(M;\rho_0)\}_{i=1,2,3,4}$ satisfying

$$\partial_{i+1}^f \circ g_{i+1}^f + g_i^f \circ \partial_i^f = \mathrm{id}_{C_i^f(M;\rho_0)}$$

as in the definition of $\mathrm{Tor}(M;\rho_0)$ in Section 3. This g^f is called a *combinatorial propagator* in [6] and [19].

A combinatorial propagator g^f gives a homomorphism

$$g_{q,p}=\pi_{Q(H)_p}\circ g_i^f|_{Q(H)_q}:Q(H)_q\to Q(H)_p$$

for each $q\in \mathrm{Crit}_{i-1}(f)$ and $p\in \mathrm{Crit}_i(f)$, where $\pi_{Q(H)_p}:\oplus_{r\in \mathrm{Crit}_i(f)}Q(H)_r\to Q(H)_p$ is the projection.

7.2 A Morse theoretical description of Euler structures

We assign an orientation $(-1)^{\mathrm{ind}(p)}$ for each critical point $p\in \mathrm{Crit}(f)$. Then $\mathrm{Crit}(f)=\sum_p p$ becomes a 0-cycle of $C_0(M,\mathbb{Z})$. Let $c\in C_1(M;\mathbb{Z})$ be a 1-chain consists of ± 1-wighted $\sharp\mathrm{Crit}(f)/2$-trajectories satisfying the following conditions:

$$\partial c=\mathrm{Crit}(f).$$

We deform $\mathrm{grad}f$ to a non-vanishing vector field by canceling the critical points pairwise along each component of c. We denote by $\mathrm{grad}f/c$ such a non-vanishing vector field.

We take a 1-chain e_f such that the non-vanishing vector field $\mathrm{grad}f/e_f$ represents the given Euler structure e.

Notations on $\mathrm{grad}f/e_f$

For the subsequent sections, we introduce some notations related to $\mathrm{grad}f/e_f$. We deform $\mathrm{grad}f$ by moving critical points along each component of e_f and finally canceling. Then we have a homotopy from $\mathrm{grad}f$ to a non-vanishing vector field. This homotopy gives a vector filed $\widetilde{s}\in\Gamma(p_2^*TM)$, namely a section of the vector bundle p_2^*TM. Here $p_2:[0,1]\times M\to M$ is the projection. We take the deformation of $\mathrm{grad}f$ to satisfy the following conditions:

- There is an embedding $i:(e_f,\partial e_f)\hookrightarrow(M\times[0,1],M\times\{0\})$ satisfying $i|_{\partial e_f}=\mathrm{id}$ and $\widetilde{s}^{-1}(0)=i(e_f)$.
- There is a regular neighborhood $N(e_f)\subset M$ of e_f satisfying $\widetilde{s}|_{[0,1]\times(M\setminus N(e_f))}=p_2^*\widetilde{s}|_{M\setminus N(e_f)}$, namely we do not change $\mathrm{grad}f$ on $M\setminus N(e_f)$.
- $\widetilde{s}|_{\{0\}\times M}=\mathrm{grad}f$,
- $\widetilde{s}$ is transverse to the zero section 0.

Set

$$\mathrm{grad}f/e_f=\widetilde{s}|_{\{1\}\times M}.$$

7.3 A Morse theoretic description of $d'(M, e)$

By using the Morse function f and the metric, we first construct a 4-chain

$$\Sigma_f^0 \in C_4(B\ell(M^2;\Delta);\rho_0\boxtimes\rho_0^*)$$

bounded by $v_e(\Delta)\otimes 1$ (Definition 7.5 and Proposition 7.6).

Let $M_\to(f)$ be the set of all pairs of two points in M connected by a trajectory:

$$M_\to(f)=\{(x,\Phi_t^f(x))\mid x\in M\setminus \mathrm{Crit}(f), t>0\}\subset M^2\setminus\Delta.$$

There is a compactification $\overline{M}_\to(f)\subset B\ell(M^2,\Delta)(\subset M^2)$ which consists of all broken trajectories (see [15, Lemma 5.13] or [13, Lemma 4.3], [19, Proposition 3.14]). $\overline{M}_\to(f)$ with the local coefficient given by $1\in Q(H)$ near Δ gives a 4-chain in $C_4(B\ell(M^2,\Delta);\rho_0\boxtimes\rho_0^*)$ and in $C_4(M^2;\rho_0\boxtimes\rho_0)$:

$$\overline{M}_\to(f)=\langle\overline{M}_\to(f),\mathrm{pt};1\rangle,$$

where $\mathrm{pt}\in\partial B\ell(M^2;\Delta)$ is any point.

Similarly, we have compactifications $\overline{\mathcal{D}_p}$ of $\mathcal{D}_p$ and $\overline{\mathcal{A}_q}$ of $\mathcal{A}_q$ by adding all the broken trajectories (see [15, Lemma 5.14] or [19, Proposition 3.14]). For $p\in\mathrm{Crit}_i(f)$ and $q\in\mathrm{Crit}_{i-1}(f)$, the combinatorial propagator g^f gives a homomorphism $g^f_{q,p}\otimes 1: Q(H)_q\otimes Q(H)_q^*\to Q(H)_p\otimes Q(H)_q^*$. Thus we have a 4-chain

$$\langle\overline{\mathcal{D}_p}\times\overline{\mathcal{A}_q},(p,q);(g^f_{q,p}\otimes 1)1\rangle\in C_4(M^2;\rho_0\boxtimes\rho_0^*).$$

Let

$$\langle(\overline{\mathcal{D}_p}\times\overline{\mathcal{A}_q})^{B\ell},(p,q);(g^f_{q,p}\otimes 1)1\rangle\in C_4(B\ell(M^2;\Delta);\rho_0\boxtimes\rho_0^*).$$

be the 4-chain given from the closure of $\overline{\mathcal{D}_p}\times\overline{\mathcal{A}_q}\setminus\Delta$ in $B\ell(M^2,\Delta)$. See [19, Proposition 3.14] for more details.

Set

$$\mathcal{M}_0(f)=\langle\overline{M}_\to(f),\mathrm{pt};1\rangle+\sum_{p,q;\mathrm{ind}(p)=\mathrm{ind}(q)+1}\langle(\overline{\mathcal{D}_p}\times\overline{\mathcal{A}_q})^{B\ell},(p,q);(g^f_{q,p}\otimes 1)1\rangle \in C_4(B\ell(M^2;\Delta);\rho_0\boxtimes\rho_0^*)$$

The 4-chain $\overline{M}_\to(f)$ has

$$\left\{\frac{\mathrm{grad}_x f}{\|\mathrm{grad}_x f\|}\mid x\in M\setminus\mathrm{Crit}(f)\right\}\otimes 1 \qquad \begin{aligned}&\in C_3(\partial B\ell(M^2,\Delta);\mathbb{Z})\otimes_{\mathbb{Z}}Q(H)\\ &=C_3(STM;\mathbb{Z})\otimes_{\mathbb{Z}}Q(H)\end{aligned}$$

as a part of the boundary $\partial\overline{M}_\to(f)$.

The intersection of manifolds $(\overline{\mathcal{D}_p} \times \overline{\mathcal{A}_q}) \cap \Delta = \overline{\mathcal{D}_p} \cap \overline{\mathcal{A}_q}$ forms a sum of trajectories connecting p and q:

$$\overline{\mathcal{D}_p} \cap \overline{\mathcal{A}_q} = \sum_{\gamma \in \mathcal{M}(p,q)} \gamma.$$

Thus, $\langle (\overline{\mathcal{D}_p} \times \overline{\mathcal{A}_q})^{B\ell}, (p,q); (g^f_{q,p} \otimes 1)1 \rangle$ has

$$\begin{aligned} & \sum_{\gamma \in \mathcal{M}(p,q)} \pi^! \langle \gamma, q; ((\gamma_* \circ g^f_{q,p}) \otimes 1)1 \rangle \in C_3(S\nu_\Delta; \rho_0 \boxtimes \rho_0^*) = C_3(S\nu_\Delta; \pi^*(\rho_0 \otimes \rho_0^*)) \\ = & \sum_{\gamma \in \mathcal{M}(p,q)} \pi^{-1}(\gamma) \otimes \gamma_* g^f_{q,p}(1) \in C_3(S\nu_\Delta; \mathbb{Z}) \otimes Q(H). \end{aligned}$$

as a part of the boundary. Here $\pi : S\nu_\Delta = STM \to M$ is the projection.

By a similar argument as in [15, Lemma 5.16] (or [16, Proposition 6.2]), we can check that the other boundary strata of $\partial \overline{M}_{\to}(f)$ and $\partial(\mathcal{A}_p \times \mathcal{D}_q)^{B\ell}$ with local coefficients are cancelled:

Lemma 7.1.

$$\begin{aligned} & \partial \mathcal{M}_0(f) \\ = & \left\langle \overline{\left\{ \left(x, \frac{\mathrm{grad}_x f}{\|\mathrm{grad}_x(f)\|} \right) \mid x \in M \setminus \mathrm{Crit}(f) \right\}}, \mathrm{pt}; 1 \right\rangle \\ & + \sum_{p,q;\mathrm{ind}(p)=\mathrm{ind}(q)+1} \sum_{\gamma \in \mathcal{M}(p,q)} \pi^! \langle \gamma, q; ((\gamma_* \circ g^f_{q,p}) \otimes 1)1 \rangle \quad (\in C_3(\partial B\ell(M^2; \Delta); \rho_0 \otimes \rho_0^*)) \\ = & \overline{\left\{ \left(x, \frac{\mathrm{grad}_x f}{\|\mathrm{grad}_x(f)\|} \right) \mid x \in M \setminus \mathrm{Crit}(f) \right\}} \otimes 1 + \sum_{p,q;\mathrm{ind}(p)=\mathrm{ind}(q)+1} \sum_{\gamma \in \mathcal{M}(p,q)} \pi^{-1}(\gamma) \otimes \gamma_* g^f_{q,p}(1) \\ & \quad (\in C_3(\partial B\ell(M^2; \Delta); \mathbb{Z}) \otimes Q(H)). \end{aligned}$$

Here $\pi : \partial D(v_\Delta) \to \Delta$ is the projection.

Set

$$\begin{aligned} \Gamma &= \sum_{p,q;\mathrm{ind}(p)=\mathrm{ind}(q)+1} \sum_{\gamma \in \mathcal{M}(p,q)} \langle \gamma, q; 1 \otimes ((\gamma_* \circ g^f_{q,p}) \otimes 1)1 \rangle \in C_1(M; \rho_0 \otimes \rho_0^*) \\ &= \sum_{p,q;\mathrm{ind}(p)=\mathrm{ind}(q)+1} \sum_{\gamma \in \mathcal{M}(p,q)} \gamma \otimes \gamma_* g^f_{q,p}(1) \in C_1(M; \mathbb{Z}) \otimes Q(H). \end{aligned}$$

The 1-chain Γ has the following property:

Lemma 7.2.

$$\partial\Gamma = -\mathrm{Crit(f)} \otimes 1 \in C_0(M;\mathbb{Z}) \otimes Q(H).$$

Proof. Let c_p be the coefficient of $p \in \mathrm{Crit}(f)$ in $\partial\Gamma$. We show that $c_p = (-1)^{\mathrm{ind}(p)+1}$. If a trajectory γ is from a critical point $r \in \mathrm{Crit}_{\mathrm{ind}(p)+1}(f)$ to p and γ is oriented from p to r, then the coefficient of p in $\partial\gamma$ is -1 and $\varepsilon(\gamma) = (-1)^{\mathrm{ind}(r)+1} = (-1)^{\mathrm{ind}(p)}$. If the orientation of γ is opposite, namely from r to p, then the coefficient is $+1$ and $\varepsilon(\gamma) = (-1)^{\mathrm{ind}(p)+1}$. If a trajectory γ connecting p and $q \in \mathrm{Crit}_{\mathrm{ind}(p)-1}(f)$ is oriented from p to q, then the coefficient of p in $\partial\gamma$ is -1 and $\varepsilon(\gamma) = (-1)^{\mathrm{ind}(p)}$. If the orientation of γ is opposite, namely from q to p, then the coefficient is $+1$ and $\varepsilon(\gamma) = (-1)^{\mathrm{ind}(p)+1}$. Therefore c_p is computed as follows:

$$\begin{aligned}
c_p &= \sum_{r \in \mathrm{Crit}_{\mathrm{ind}(p)+1}(f)} \sum_{\gamma \in \mathcal{M}(r,p)} (-1)^{\mathrm{ind}(p)+1} \varepsilon(\gamma)\gamma_* g^f_{p,r}(1) \\
&\quad + \sum_{q \in \mathrm{Crit}_{\mathrm{ind}(p)-1}(f)} \sum_{\gamma \in \mathcal{M}(p,q)} (-1)^{\mathrm{ind}(p)+1} \varepsilon(\gamma)\gamma_* g^f_{q,p}(1) \\
&= \sum_{r \in \mathrm{Crit}_{\mathrm{ind}(p)+1}(f)} (-1)^{\mathrm{ind}(p)} \partial^f_{r,p} g^f_{p,r} + \sum_{q \in \mathrm{Crit}_{\mathrm{ind}(p)+1}(f)} (-1)^{\mathrm{ind}(p)-1} \partial^f_{p,q} g^f_{q,p} \\
&= (-1)^{\mathrm{ind}(p)+1} \sum_{r \in \mathrm{Crit}_{\mathrm{ind}(p)+1}(f)} \sum_{q \in \mathrm{Crit}_{\mathrm{ind}(p)-1}(f)} \pi_{Q(H)_p} \circ \left(g^f \circ \partial^f + \partial^f \circ g^f \right) |_{Q(H)_p}(1) \\
&= (-1)^{\mathrm{ind}(p)+1}.
\end{aligned}$$

Here $\pi_{Q(H)_p} : \sum_{p' \in \mathrm{Crit}_{\mathrm{ind}(p)}} Q(H)_{p'} \to Q(H)_p$ is the projection.

□

We prepare two more chains.

We recall that $i(e_f)$ is a submanifold of $[0,1] \times M$ and $\widetilde{s}$ is a section of the $\mathbb{R}^3$ bundle $p_2^* TM \to [0,1] \times M$ introduced in Section 7.2.

The image of the 4-manifold

$$\widetilde{S} = \overline{\left\{ \frac{\widetilde{s}(x)}{\|\widetilde{s}(x)\|} \mid x \in [0,1] \times M \setminus \widetilde{s}^{-1}(0) \right\}} \subset [0,1] \times STM$$

under the projection $\widetilde{p_2} : [0,1] \times STM \to STM$ gives a 4-chain in $STM = \partial B\ell(M^2, \Delta)$:

$$\left((p_2)_* \widetilde{S} \right) \otimes 1 \in C_4(\partial B\ell(M^2, \Delta); \mathbb{Z}) \otimes Q(H).$$

Since $\widetilde{s}|_{\{0\} \times M} = \mathrm{grad} f$ and $\widetilde{s}|_{\{1\} \times M} = \mathrm{grad} f / e_f$, we have the following lemma.

Lemma 7.3.

$$\begin{aligned}&\partial\Big(((p_2)_*\widetilde{S})\otimes 1\Big)\\ =\;& \pi^{-1}(e_f)\otimes 1-\overline{\left\{\frac{\mathrm{grad}_x f}{\|\mathrm{grad}_x f\|}\mid x\in M\setminus \mathrm{Crit}(f)\right\}}\otimes 1+\overline{\left\{\frac{(\mathrm{grad} f/e_f)_x}{\|(\mathrm{grad} f/e_f)_x\|}\mid x\in M\right\}}\otimes 1.\end{aligned}$$

$D(\nu_\Delta)=D(TM)$ forms a D^3-bundle over M. Let $\widetilde{\pi}:D(\nu_\Delta)=D(TM)\to M$ be the projection. Thanks to Lemma 7.2,

$$e_f\otimes 1+\Gamma$$

is a 1-cycle of $C_1(M;\mathbb{Z})\otimes_{\mathbb{Z}} Q(H)$. Thus we have a 4-chain

$$\widetilde{\pi}^{-1}(e_f\otimes 1+\Gamma)\in C_4(D(\nu_\Delta);\mathbb{Z})\otimes_{\mathbb{Z}} Q(H).$$

By the construction, the boundary of this chain is written as follows:

Lemma 7.4.

$$\begin{aligned}\partial(\widetilde{\pi}^!(e_f\otimes 1+\Gamma)) &= \pi^!(e_f\otimes 1+\Gamma)\\ &= \pi^{-1}(e_f)\otimes 1+\pi^!\Gamma.\end{aligned}$$

Here $\pi: STM(=D(\nu_\Delta)\to M$ is the projection.

Definition 7.5.

$$\Sigma_0^f=\mathcal{M}_0(f)+((p_2)_*\widetilde{S})\otimes 1-\widetilde{\pi}^{-1}(e_f\otimes 1+\Gamma)\in C_4(M^2;\Delta).$$

The following proposition is a direct consequence of Lemma 7.1, Lemma 7.3 and Lemma 7.4.

Proposition 7.6.

$$\partial\Sigma_0^f=\overline{\left\{\frac{(\mathrm{grad} f/e)_x}{\|(\mathrm{grad} f/e)_x\|}\mid x\in M\right\}}\otimes 1=v_e(\Delta)\otimes 1.$$

The above proposition implies that we can take Σ_0^f as Σ in the description of $d'(M,e)$ given in Section 4.4. Thus we have the following formula.

Proposition 7.7.

$$\begin{aligned}d'(M,e) &= -[e_f\otimes 1+\Gamma]\\ &= -[e_f\otimes 1+\sum_{p,q;\mathrm{ind}(p)=\mathrm{ind}(q)+1}\ \sum_{\gamma\in\mathcal{M}(p,q)}\gamma\otimes\gamma_* g^f_{p,q}(1)]\\ &\qquad\in H_1(M;\mathbb{Z})\otimes Q(H).\end{aligned}$$

Proof. Thanks to Proposition 4.2,

$$\begin{aligned} d'(M,e) &= [\Sigma_0^f \cap \Delta] \\ &= [\Big(\mathcal{M}_0(f) + ((p_2)_*\widetilde{S}) \otimes 1 - \widetilde{\pi}^{-1}(e_f \otimes 1 + \Gamma)\Big) \cap \Delta]. \end{aligned}$$

Since $\mathcal{M}_0(f)$ and $(p_2)_*\widetilde{S}$ are in $B\ell(M^2;\Delta)$, these are far from Δ. Thus we have

$$\begin{aligned} d'(M,e) &= [\Sigma_0^f \cap \Delta] \\ &= [-\widetilde{\pi}^{-1}(e_f \otimes 1 + \Gamma) \cap \Delta] \\ &= -[e_f \otimes 1 + \Gamma] \\ &= -\left[e_f \otimes 1 + \sum_{p,q;\mathrm{ind}(p)=\mathrm{ind}(q)+1} \sum_{\gamma \in \mathcal{M}(p,q)} \gamma \otimes \gamma_* g^f_{q,p}(1) \right]. \end{aligned}$$

□

By collapsing e_f to a point, each trajectory γ becomes a 1-cycle. We denote by γ/e_f such a 1-cycle. Thus we have the following slightly different description of $d'(M,e)$:

Proposition 7.8.

$$d'(M,e) = - \sum_{p,q;\mathrm{ind}(p)=\mathrm{ind}(q)+1} \sum_{\gamma \in \mathcal{M}(p,q)} [\gamma/e_f] \otimes \gamma_* g^f_{q,p}(1) \in H_1(M;\mathbb{Z}) \otimes Q(H).$$

Here $[\gamma/e_f]$ is a 1-cycle of $H_1(M;\mathbb{Z})$ represented by γ/e_f.

7.4 An explicit descriptions of the boundary operators ∂_*^f and a propagator g^f

To compute $d'(M,e)$ and $\mathrm{Tor}(M,e)$ explicitly, we first introduce two bases of $C_*^f(M,\rho_0)$.

Tow bases $\mathbf{b}_e$ and $\mathbf{b}_f$ of $C_*^f(M;\rho_0)$

To simplify the computations, we assume that $\sharp\mathrm{Crit}_3(f) = \sharp\mathrm{Crit}_0(f) = 1$. Let

$$\begin{aligned} \mathrm{Crit}_3(f) &= \{\mathrm{NP}\}, \\ \mathrm{Crit}_2(f) &= \{p_1, \ldots, p_n\}, \\ \mathrm{Crit}_1(f) &= \{q_1, \ldots, q_n\}, \\ \mathrm{Crit}_0(f) &= \{\mathrm{SP}\}. \end{aligned}$$

We denote by

$$
\begin{aligned}
C_{\mathrm{NP}} &= C_3^f(M;\rho_0) = Q(H)_{NP},\\
C_{\mathbf{p}} &= C_2^f(M;\rho_0) = \oplus_{i=1}^n Q(H)_{p_i},\\
C_{\mathbf{q}} &= C_1^f(M;\rho_0) = \oplus_{i=1}^n Q(H)_{q_i},\\
C_{\mathrm{SP}} &= C_0^f(M;\rho_0) = Q(H)_{SP}.
\end{aligned}
$$

We take isomorphisms

$$
Q(H)_{\mathrm{NP}} \cong Q(H), Q(H)_{p_1} \cong Q(H), \ldots, Q(H)_{p_n} \cong Q(H),
$$

$$
Q(H)_{q_1} \cong Q(H), \ldots, Q(H)_{q_n} \cong Q(H), Q(H)_{\mathrm{SP}} \cong Q(H)
$$

such that these are compatible with the Euler structure e_f. The compatibility means that, for example, if a component γ of e_f connects p_i and q_j, thus the following diagram should commutes:

$$
\begin{array}{ccc}
Q(H)_{p_i} & \xrightarrow{\cong} & Q(H) \\
\downarrow{\scriptstyle \gamma_*} & \circlearrowleft & \nearrow{\scriptstyle \cong} \\
Q(H)_{q_j} & &
\end{array}
$$

For each $p \in \mathrm{Crit}(f)$, we denote by $p \in Q(H)_p$ the element corresponding to the generator $1 \in Q(H)$ under the isomorphism $Q(H)_p \cong Q(H)$. Therefore, we now have a basis $\mathbf{b}_e$ of $C_*^f(M, \rho_0)$:

$$
\mathbf{b}_e = \{\mathrm{NP}, p_1, \ldots, p_n, q_1, \ldots, q_n, \mathrm{SP}\}.
$$

To simplify the computations, we introduce an alternative basis. Let $\pi_{Q(H)_{p_1}} : C_{\mathbf{p}} \to Q(H)_{p_1}$ be the projection. Without loss of generality, we assume that $\pi_{Q(H)_{p_1}} \circ \partial_3^f(\mathrm{NP}) \neq 0$. We take a basis of $C_{\mathbf{p}}$ as

$$
\partial_3(\mathrm{NP}), p_2, \ldots, p_n.
$$

At least one of $q_1, \ldots, q_n$ does not belong to $\mathrm{Im}\partial_2 \subset C_{\mathrm{q}}$. We may assume that $q_1 \notin \mathrm{Im}\partial_2$. We take a basis of $C_{\mathbf{q}}$ as

$$
q_1, \partial_2(p_2), \ldots, \partial_2(p_n).
$$

Since $H_0(M;\rho_0) = 0$, then $\partial_1(q_1) \neq 0$. We take $\partial_1(q_1)$ as a basis of C_{SP}. Now we have a new basis $\mathbf{b}_f$:

$$
\mathbf{b}_f = \{\mathrm{NP}, \partial_3(\mathrm{NP}), p_2, \ldots, p_n, q_1, \partial_2(p_2), \ldots, \partial_2(p_n), \partial_1(q_1)\}.
$$

The transformation matrix from $\mathbf{b}_e$ to $\mathbf{b}_f$

Let $(\partial_1^{ij})_{ij}, (\partial_2^{ij})_{ij}$ and $(\partial_3^{ij})_{ij}$ be representation matrices of $\partial_1^f, \partial_2^f$ and ∂_3^f respectively under the basis $\mathbf{b}_f$. Namely,

$$\begin{aligned}
\partial_3^f(\mathrm{NP}) &= (p_1, \ldots, p_n) \begin{pmatrix} \partial_3^{11} \\ \vdots \\ \partial_3^{1n} \end{pmatrix}, \\
\partial_2^f(p_1, \ldots, p_n) &= (q_1, \ldots, q_n) \begin{pmatrix} \partial_2^{11} & \cdots & \partial_2^{1n} \\ \vdots & \ddots & \vdots \\ \partial_2^{n1} & \cdots & \partial_2^{nn} \end{pmatrix}, \\
\partial_1^f(q_1, \ldots, q_n) &= (\mathrm{SP})(\partial_1^{11} \ \cdots \ \partial_1^{1n}).
\end{aligned}$$

Then the transformation matrices are written as the following:

Lemma 7.9. *(1)*

$$(\partial_3(\mathrm{NP}), p_2, \ldots, p_n) = (p_1, p_2, \ldots, p_n) \begin{pmatrix} \partial_3^{11} & 0 & \cdots & 0 \\ \partial_3^{12} & 1 & & 0 \\ \vdots & & \ddots & \\ \partial_3^{1n} & 0 & & 1 \end{pmatrix}.$$

(2)

$$(q_1, \partial_2(p_2), \ldots, \partial_2(p_n)) = (q_1, q_2, \ldots, q_n) \begin{pmatrix} 1 & \partial_2^{21} & \cdots & \partial_2^{n1} \\ 0 & \partial_2^{22} & \cdots & \partial_2^{n2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \partial_2^{2n} & \cdots & \partial_2^{nn} \end{pmatrix}.$$

(3)

$$\partial_1^f(q_1) = \partial_1^{11}\mathrm{SP}.$$

A propagator g^f

We give a combinatorial propagator g^f explicitly written by using the basis $\mathbf{b}_f$ as follows:

- $g_3^f(\partial_3^f(\mathrm{NP})) = \mathrm{NP}$, $g_3^f(p_2) = 0, \ldots, g_3^f(p_n) = 0$,
- $g_2^f(q_1) = 0$, $g_2^f(\partial_2^f(p_2)) = p_2, \ldots, g_2^f(\partial_2^f(p_n)) = p_n$,
- $g_1^f(\partial_1^f(q_1)) = q_1$.

Obviously, these homomorphisms $g^f = \{g_i\}_i$ satisfies $\partial^f_{i+1} \circ g^f_{i+1} + g^f_i \circ \partial^f_i = \mathrm{id}$ for any i, namely g^f is a combinatorial propagator.

An explicit description of g^f under the basis b_e.

A description of g^f_3

Since

$$g^f_3(\partial_3(\mathrm{NP}), p_2, \ldots, p_n) = (\mathrm{NP})\,(1\ 0\ \cdots\ 0) \quad \text{and}$$

$$(\partial_3(\mathrm{NP}), p_2, \ldots, p_n) = (p_1, p_2, \ldots, p_n) \begin{pmatrix} \partial_3^{11} & 0 & \cdots & 0 \\ \partial_3^{12} & 1 & & 0 \\ \vdots & & \ddots & \\ \partial_3^{1n} & 0 & & 1 \end{pmatrix},$$

we have

$$\begin{aligned} g^f_3(p_1, p_2, \ldots, p_n) &= (NP)(1\ 0\ \ldots\ 0) \begin{pmatrix} \partial_3^{11} & 0 & \cdots & 0 \\ \partial_3^{12} & 1 & & 0 \\ \vdots & & \ddots & \\ \partial_3^{1n} & 0 & & 1 \end{pmatrix}^{-1} \\ &= (NP)(1\ 0\ \ldots\ 0) \begin{pmatrix} (\partial_3^{11})^{-1} & 0 & \cdots & 0 \\ -\partial_3^{12}/\partial_3^{11} & 1 & & 0 \\ \vdots & & \ddots & \\ -\partial_3^{1n}/\partial_3^{11} & 0 & & 1 \end{pmatrix} \\ &= (\mathrm{NP})((\partial_3^{11})^{-1}\ 0\ \cdots\ 0). \end{aligned}$$

Thus we have

$$g^f_{p_1,\mathrm{NP}} = (\partial_3^{11})^{-1}, g^f_{p_2,\mathrm{NP}} = 0, \ldots, g^f_{p_n,\mathrm{NP}} = 0.$$

A description of g^f_2

Since

$$g^f_2(q_1, \partial_2(p_2), \ldots, \partial_2(p_n)) = (\partial_3(\mathrm{NP}), p_2, \ldots, p_n) \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \end{pmatrix},$$

$$(\partial_3(\mathrm{NP}), p_2, \ldots, p_n) = (p_1, p_2, \ldots, p_n) \begin{pmatrix} \partial_3^{11} & 0 & \cdots & 0 \\ \partial_3^{12} & 1 & & 0 \\ \vdots & & \ddots & \\ \partial_3^{1n} & 0 & & 1 \end{pmatrix} \quad \text{and}$$

$$(q_1, \partial_2(p_2), \ldots, \partial_2(p_n)) = (q_1, q_2, \ldots, q_n) \begin{pmatrix} 1 & \partial_2^{21} & \cdots & \partial_2^{n1} \\ 0 & \partial_2^{22} & \cdots & \partial_2^{n2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \partial_2^{2n} & \cdots & \partial_2^{nn} \end{pmatrix},$$

We have

$$\begin{aligned} & g_2^f(q_1, q_2, \ldots, q_n) \\ = \quad & (p_1, p_2, \ldots, p_n) \begin{pmatrix} \partial_3^{11} & 0 & \cdots & 0 \\ \partial_3^{12} & 1 & & 0 \\ \vdots & & \ddots & \\ \partial_3^{1n} & 0 & & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \end{pmatrix} \begin{pmatrix} 1 & \partial_2^{21} & \cdots & \partial_2^{n1} \\ 0 & \partial_2^{22} & \cdots & \partial_2^{n2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \partial_2^{2n} & \cdots & \partial_2^{nn} \end{pmatrix}^{-1} \\ = \quad & (p_1, p_2, \ldots, p_n) \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \end{pmatrix} \begin{pmatrix} 1 & * & \cdots & * \\ 0 & \frac{1}{\det A} A_2^{22} & \cdots & \frac{1}{\det A} A_2^{n2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{1}{\det A} A_2^{2n} & \cdots & \frac{1}{\det A} A_2^{nn} \end{pmatrix} \\ = \quad & (p_1, p_2, \ldots, p_n) \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{\det A} A_2^{22} & \cdots & \frac{1}{\det A} A_2^{n2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{1}{\det A} A_2^{2n} & \cdots & \frac{1}{\det A} A_2^{nn} \end{pmatrix}. \end{aligned}$$

Here the matrix A is defined by

$$A_2 = \begin{pmatrix} \partial_2^{22} & \cdots & \partial_2^{n2} \\ \vdots & \ddots & \vdots \\ \partial_2^{2n} & \cdots & \partial_2^{nn} \end{pmatrix}$$

and (A_2^{ij}) is the (i, j)-cofactor of A_2, namely

$$\left(\frac{1}{\det A_2} A_2^{ji} \right)_{i,j} = A_2^{-1}.$$

Thus we have

$$g^f_{q_i,p_j} = \begin{cases} \frac{1}{\det A_2} A_2^{ij} & (i, j \geq 2), \\ g^f_{q_i,p_j} = 0 & i = 1 \text{ or} j = 1. \end{cases}$$

A description of g_1^f

We have

$$\begin{aligned} g_1^f(\partial_1(q_1)) &= (q_1, \partial_2(p_2), \ldots, \partial_2(p_n)) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\ &= (q_1, q_2, \ldots, q_n) \begin{pmatrix} 1 & \partial_2^{21} & \cdots & \partial_2^{n1} \\ 0 & \partial_2^{22} & \cdots & \partial_2^{n2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \partial_2^{2n} & \cdots & \partial_2^{nn} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\ &= (q_1, q_2, \ldots, q_n) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \end{aligned}$$

and

$$\partial_1^f(q_1) = \partial_1^{11}\mathrm{SP}.$$

Then we have

$$g_1^f(\mathrm{SP}) = (q_1, q_2, \ldots, q_n) \begin{pmatrix} (\partial_1^{11})^{-1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Therefore we have

$$g^f_{\mathrm{SP},q_1} = (\partial_1^{11})^{-1}, g^f_{\mathrm{SP},q_2} = 0, \ldots, g^f_{\mathrm{SP},q_n} = 0.$$

7.5 The computation of $D(\mathrm{Tor}(M,e))$

Recall that $\mathrm{Tor}(M,e)$ is defined to be $\mathrm{Top}(M,e) = \det(\partial^f + g^f)$.

The isomorphism

$$\partial^f + g^f = g_3^f + \partial_2^f + g_1^f : C^f_{even} \to C^f_{odd}$$

is represented by the following matrix under the basis $\mathbf{b}_e$:

$$\begin{aligned}
& (g_3^f + \partial_2^f + g_1^f)(p_1, p_2, \ldots, p_n, \mathrm{SP}) \\
= \quad & (\mathrm{NP}, q_1, q_2, \ldots, q_n) \begin{pmatrix} (\partial_3^{11})^{-1} & 0 & \cdots & 0 & 0 \\ \partial_2^{11} & \partial_2^{21} & \cdots & \partial_2^{n1} & (\partial_1^{11})^{-1} \\ \partial_2^{12} & \partial_2^{22} & \cdots & \partial_2^{n2} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \partial_2^{1n} & \partial_2^{2n} & \cdots & \partial_2^{nn} & 0 \end{pmatrix}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\mathrm{Tor}(M, e) \quad &= \quad \det \begin{pmatrix} (\partial_3^{11})^{-1} & 0 & \cdots & 0 & 0 \\ \partial_2^{11} & \partial_2^{21} & \cdots & \partial_2^{n1} & (\partial_1^{11})^{-1} \\ \partial_2^{12} & \partial_2^{22} & \cdots & \partial_2^{n2} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \partial_2^{1n} & \partial_2^{2n} & \cdots & \partial_2^{nn} & 0 \end{pmatrix} \\
&= \quad (-1)^{n-1} (\partial_3^{11})^{-1} \det \begin{pmatrix} \partial_2^{22} & \cdots & \partial_2^{n2} \\ \vdots & \ddots & \vdots \\ \partial_2^{2n} & \cdots & \partial_2^{nn} \end{pmatrix} (\partial_1^{11})^{-1} \\
&= \quad (-1)^{n-1} (\partial_3^{11})^{-1} (\det A_2) (\partial_1^{11})^{-1}.
\end{aligned}$$

We next compute $D(\mathrm{Tor}(M, e))$. Recall that we already have a representation

$$H = \langle t_1, \ldots, t_k \rangle.$$

To simplify the computations, we use the following alternative representation:

$$H = \langle t_1, \ldots, t_k, \{\gamma_*(1)\}_{\gamma:\text{trajectory}} \mid R \rangle,$$

where R is a family of appropriate relations. To simplify the notations, we denote by

$$\gamma' = \gamma_*(1).$$

Thus our representation of H is written as

$$H = \langle t_1, \ldots, t_k, \{\gamma'\}_{\gamma:\text{trajectory}} \mid R \rangle.$$

Thanks to Proposition 5.6, D is written as follows:

$$D = \sum_i \left([t_i]_+ \otimes t_i \frac{\partial}{\partial t_i} \log \right) + \sum_\gamma \left([\gamma']_+ \otimes \gamma' \frac{\partial}{\partial \gamma'} \log \right).$$

Lemma 7.10. *For any trajectory* $\gamma \in \mathcal{M}(p,q)$,

$$[\gamma']_+ = (-1)^{\mathrm{ind}(p)+1}\varepsilon(\gamma)[\gamma/e_f] \in H_1(M;\mathbb{Z})/\mathrm{Tor}.$$

Proof. $\gamma' = \gamma_*(1) \in Q(H) = \mathrm{Hom}(Q(H)_p, Q(H)_q)$. The identification $Q(H)_p = Q(H)_q = Q(H)$ is compatible with the Euler structure e_f. The homology class $[\gamma']_+ = [\gamma_*(1)]_+$ is represented by a cycle γ/e_f with the orientation directed from p to q along γ. Whereas, γ/e_f already has an orientation induced by that of γ as a part of $\mathcal{D}_p \cap \mathcal{A}_q$. Thus we have $[\gamma']_+ = (-1)^{\mathrm{ind}(p)+1}\varepsilon(\gamma)[\gamma/e_f]$. □

The terms $t_1, \ldots, t_k$ do not appear in our representation of $\mathrm{Tor}(M,e)$. Therefore,

$$D(\mathrm{Tor(M,e)}) \;=\; \sum_\gamma [\gamma/e_f] \otimes \gamma' \frac{\partial}{\partial \gamma'} \log\left|(-1)^{n-1}(\partial_3^{11})^{-1}(\det A_2)(\partial_1^{11})^{-1}\right|.$$

We compute $\gamma'\frac{\partial}{\partial\gamma'}\log|(-1)^{n-1}(\partial_3^{11})^{-1}(\det A_2)(\partial_1^{11})^{-1}|$ for each trajectory γ. We denote by $s(\gamma) = p, t(\gamma) = q$ for a trajectory γ from p to q. We break the computation into cases depending on $s(\gamma)$ and $t(\gamma)$.

Case 1:$s(\gamma) = \mathrm{NP}, t(\gamma) = p_i$

In this case the term γ' only appears in ∂_3^{1i} (We note that $\partial_3^{1i} = \sum_{\overline{\gamma}\in\mathcal{M}(\mathrm{NP},p_i)} \varepsilon(\overline{\gamma})\overline{\gamma}'$). Thus,

$$\begin{aligned}
&\gamma'\frac{\partial}{\partial\gamma'}\log\left|(-1)^{n-1}(\partial_3^{11})^{-1}(\det A_2)(\partial_1^{11})^{-1}\right| \\
=\;& \gamma'\frac{\partial}{\partial\gamma'}\log\left|(-1)^{n-1}(\partial_3^{11})^{-1}\det\begin{pmatrix}\partial_2^{22} & \cdots & \partial_2^{n2}\\ \vdots & \ddots & \vdots \\ \partial_2^{2n} & \cdots & \partial_2^{nn}\end{pmatrix}(\partial_1^{11})^{-1}\right| \\
=\;& -\gamma'\frac{\partial}{\partial\gamma'}\log|\partial_3^{11}| + \gamma'\frac{\partial}{\partial\gamma'}\log\left|\det\begin{pmatrix}\partial_2^{22} & \cdots & \partial_2^{n2}\\ \vdots & \ddots & \vdots \\ \partial_2^{2n} & \cdots & \partial_2^{nn}\end{pmatrix}\right| - \gamma'\frac{\partial}{\partial\gamma'}\log|\partial_1^{11}| \\
=\;& -\gamma'\frac{\partial}{\partial\gamma'}\log|\partial_3^{11}| \\
=\;& -\gamma'\frac{1}{\partial_3^{11}}\frac{\partial}{\partial\gamma'}(\partial_3^{11}) \\
=\;& \begin{cases} -\varepsilon(\gamma)\frac{\gamma'}{\partial_3^{11}} & (i=1), \\ 0 & (i \neq 1). \end{cases}
\end{aligned}$$

Case 2-1: $s(\gamma) = \mathrm{p}_i, t(\gamma) = q_j$, $i = 1$ or $j = 1$

In this case there are no γ' in $(\partial_3^{11})^{-1}(\det A_2)(\partial_1^{11})^{-1}$. Then we have

$$\gamma' \frac{\partial}{\partial \gamma'} \log \left| (-1)^{n-1} (\partial_3^{11})^{-1} (\det A_2)(\partial_1^{11})^{-1} \right| = 0.$$

Case 2-2: $s(\gamma) = \mathrm{p}_i, t(\gamma) = q_j$, $i, j \geq 2$

$$\begin{aligned}
& \gamma' \frac{\partial}{\partial \gamma'} \log \left| (-1)^{n-1} (\partial_3^{11})^{-1} (\det A_2)(\partial_1^{11})^{-1} \right| \\
= \quad & \gamma' \frac{\partial}{\partial \gamma'} \log \left| \det \begin{pmatrix} \partial_2^{22} & \cdots & \partial_2^{n2} \\ \vdots & \ddots & \vdots \\ \partial_2^{2n} & \cdots & \partial_2^{nn} \end{pmatrix} \right| \\
= \quad & \gamma' \frac{\partial}{\partial \gamma'} \log \left| \sum_{a=2}^{n} \partial_2^{aj} A_2^{aj} \right| \\
= \quad & \gamma' \frac{1}{\det A_2} \cdot \frac{\partial}{\partial \gamma'} \sum_{a=2}^{n} \partial_2^{aj} A_2^{aj} \\
= \quad & \gamma' \frac{1}{\det A_2} \varepsilon(\gamma) A_2^{ij}.
\end{aligned}$$

Case 3: $s(\gamma) = \mathrm{q}_j, t(\gamma) = \mathrm{SP}$

$$\begin{aligned}
& \gamma' \frac{\partial}{\partial \gamma'} \log \left| (-1)^{n-1} (\partial_3^{11})^{-1} (\det A_2)(\partial_1^{11})^{-1} \right| \\
= \quad & -\gamma' \frac{\partial}{\partial \gamma'} \log |\partial_1^{11}| \\
= \quad & -\gamma' \frac{1}{\partial_1^{11}} \frac{\partial}{\partial \gamma'} (\partial_1^{11}) \\
= \quad & \begin{cases} -\varepsilon(\gamma) \frac{\gamma'}{\partial_1^{11}} & (j = 1), \\ 0 & (j \neq 1). \end{cases}
\end{aligned}$$

Therefore, we can compute $D(\mathrm{Tor}(M,e))$ as follows:

$$\begin{aligned} & D(\mathrm{Tor}(M,e)) \\ = & \sum_{\gamma\in\mathcal{M}(\mathrm{NP},p_1)} -[\gamma']\otimes\frac{\varepsilon(\gamma)\gamma'}{\partial_3^{11}} + \sum_{i,j\geq 2}\sum_{\gamma\in\mathcal{M}(p_i,q_j)}[\gamma']\otimes\frac{\varepsilon(\gamma)\gamma'}{\det A_2}A_2^{ij} - \sum_{\gamma\in\mathcal{M}(q_1,\mathrm{SP})}[\gamma']\otimes\frac{\varepsilon(\gamma)\gamma'}{\partial_1^{11}} \\ = & \sum_{\gamma\in\mathcal{M}(\mathrm{NP},p_1)}[\gamma/e_f]_+\otimes\frac{\gamma'}{\partial_3^{11}} + \sum_{i,j\geq 2}\sum_{\gamma\in\mathcal{M}(p_i,q_j)}[\gamma/e_f]_+\otimes\frac{\gamma'}{\det A_2}A_2^{ij} + \sum_{\gamma\in\mathcal{M}(q_1,\mathrm{SP})}[\gamma/e_f]_+\otimes\frac{\gamma'}{\partial_1^{11}}. \end{aligned}$$

Lemma 7.11.

$$\begin{aligned} & D(\mathrm{Tor}(M,e)) \\ = & \sum_{\gamma\in\mathcal{M}(\mathrm{NP},p_1)}[\gamma/e_f]_+\otimes\frac{\gamma'}{\partial_3^{11}} + \sum_{i,j\geq 2}\sum_{\gamma\in\mathcal{M}(p_i,q_j)}[\gamma/e_f]_+\otimes\frac{\gamma'}{\det A_2}A_2^{ij} + \sum_{\gamma\in\mathcal{M}(q_1,\mathrm{SP})}[\gamma/e_f]_+\otimes\frac{\gamma'}{\partial_1^{11}}. \end{aligned}$$

7.6 The Computation of $\overline{d'}(M,e)$ by using g^f

By using the explicit description of g^f under the basis $\mathbf{b}_e$ and the formula given in Proposition 7.8, we have the following description of $\overline{d'}(M,e)$:

$$\begin{aligned} \overline{d}'(M,e) &= -\sum_{p,q;\mathrm{ind}(p)=\mathrm{ind}(q)+1}\ \sum_{\gamma\in\mathcal{M}(p,q)}[\gamma/e_f]\otimes\gamma_* g^f_{p,q}(1) \\ &= -\sum_{\gamma\in\mathcal{M}(\mathrm{NP},p_1)}[\gamma/e_f]\otimes\gamma'(\partial_3^{11})^{-1} \\ &\quad -\sum_{i,j\geq 2}\sum_{\gamma\in\mathcal{M}(p_i,q_j)}[\gamma/e_f]\otimes\gamma'\frac{1}{\det A_2}A_2^{ij} \\ &\quad -\sum_{\gamma\in\mathcal{M}(q_1,\mathrm{SP})}[\gamma/e_f]\otimes\gamma'(\partial_1^{11})^{-1}. \end{aligned}$$

Lemma 7.12.

$$\begin{aligned} & \overline{d}'(M,e) \\ = & -\sum_{\gamma\in\mathcal{M}(\mathrm{NP},p_1)}[\gamma/e_f]\otimes\frac{\gamma'}{\partial_3^{11}} - \sum_{i,j\geq 2}\sum_{\gamma\in\mathcal{M}(p_i,q_j)}[\gamma/e_f]\otimes\frac{\gamma'}{\det A_2}A_2^{ij} - \sum_{\gamma\in\mathcal{M}(q_1,\mathrm{SP})}[\gamma/e_f]\otimes\frac{\gamma'}{\partial_1^{11}}. \end{aligned}$$

7.7 The proof of Theorem

As a direct consequence of Lemma 7.11 and Lemma 7.12, we have

$$D'(\mathrm{Tor}(M,e)) = -\overline{d'}(M,e).$$

8 $d'(M,e)$ from the point of view of the self-linking homology class

For a 2-component oriented link $K_1 \sqcup K_2 \subset S^3$, the linking number $\mathrm{lk}_{S^3}(K_1, K_2) \in \mathbb{Z}$ was defined as follows: Take a 2-chain $\widetilde{K_1} \in C_2(S^3, K_1; \mathbb{Z})$ satisfying $\partial\widetilde{K_1} = K_1$ (we note that we can take a Seifert surface of K_1 as a $\widetilde{K_1}$). Then $\mathrm{lk}_{S^3}(K_1, K_2)$ is defined by $\mathrm{lk}_{S^3}(K_1, K_2) = [\widetilde{K_1}] \cap [K_2] \in H_0(K_2; \mathbb{Z}) = \mathbb{Z}$. Let v be a normal vector filed of K_1 in S^3. The self-linking number

$$\mathrm{self.lk}_{S^3}(K_1, v) \in \mathbb{Z}$$

of a framed knot (K_1, v) is defined as a linking number of K_1 and $v(K_1)$. Here $v(K_1)$ is a copy of K_1 given by pushing K_1 along v.

We now turn to $d'(M,e)$. The description $d'(M,e) = [\Sigma \cap \Delta]$ given in Section 4.4 forms an intersection of a "Seifert 4-chain" Σ of $v_e(\Delta)$ and Δ. Actually, $d'(M,e)$ can be formulated as a "self-linking 1-dimensional homology class" of an embedded 3-manifold in a 6-manifold with a local coefficient.

8.1 A self-linking homology class of an embedded 3-manifold

Let Y_1, Y_2 be closed 3-manifolds embedded in a closed 6-manifold X satisfying $Y_1 \cap Y_2 = \emptyset \subset X$. Let E be a local system on X corresponding to a representation $\pi_1(X) \to \mathrm{Aut}E_0$, where E_0 is a vector space. We assume that they satisfy the following conditions:

- $E|_{Y_1}$ and $E|_{Y_2}$ are trivial. (Then $H_3(Y_i, E) \cong H_3(Y_i, \mathbb{Z}) \otimes_{\mathbb{Z}} E_0$ for each $i = 1, 2$.)
- There is a special element $1 \in E_0$ so that $[Y_1] \otimes 1 = [Y_2] \otimes 1 = 0 \in H_3(X; E)$.

Since $[Y_1] \otimes 1 = 0 \in H_3(X : E)$, there exists a 4-chain $\widetilde{Y_1} \in C_4(X; E)$ satisfying $\partial\widetilde{Y_1} = Y_1$. The intersection of $[\widetilde{Y_1}] \in H_4(X, Y_1; E)$ and $[Y_2] \in H_3(Y_2; \mathbb{Z})$ gives a 1-dimensional homology class $[\widetilde{Y_1}] \cap [Y_2] = [\widetilde{Y_1} \cap Y_2] \in H_1(Y_2; E) = H_1(Y_2; \mathbb{Z}) \otimes_{\mathbb{Z}} E_0$.

Lemma 8.1. *Let $\widetilde{Y_1}, \widetilde{Y_1}' \in C_4(X; E)$ be 4-chains satisfying $\partial\widetilde{Y_1} = \partial\widetilde{Y_1}' = Y_1$. Then $[\widetilde{Y_1}] \cap [Y_2] = [\widetilde{Y_1}'] \cap [Y_2] \in H_1(Y; \mathbb{Z}) \otimes_{\mathbb{Z}} E_0$.*

Proof. Since

$$([\widetilde{Y_1}] \cap [Y_2]) \otimes 1 = [\widetilde{Y_1}] \cap ([Y_2] \otimes 1) \in H_1(Y_2; \mathbb{Z}) \otimes_{\mathbb{Z}} E_0 \otimes_{\mathbb{Z}} E_0$$

and the homomorphism

$$H_1(Y_2; E)(= H_1(Y_2) \otimes_{\mathbb{Z}} E_0) \to (H_1(Y_2; \mathbb{Z}) \otimes_{\mathbb{Z}} E_0 \otimes_{\mathbb{Z}} E_0 =) H_1(Y_2; E \otimes_{\mathbb{Z}} E)$$

given by

$$x \otimes_{\mathbb{Z}} a \mapsto x \otimes_{\mathbb{Z}} a \otimes_{\mathbb{Z}} 1$$

for $x \in H_1(Y_2;\mathbb{Z})$ and $a \in E_0$ is injective, it is sufficient to show that

$$[\widetilde{Y_1}] \cap ([Y_2] \otimes 1) = [\widetilde{Y_1}'] \cap ([Y_2] \otimes 1) \in H_1(Y_2; E \otimes_{\mathbb{Z}} E).$$

$[\widetilde{Y_1}] \cap ([Y_2] \otimes 1) - [\widetilde{Y_1}'] \cap ([Y_2] \otimes 1) = [\widetilde{Y_1} - \widetilde{Y_1}'] \cap ([Y_2] \otimes 1)$ is in the image of the morphism

$$\cap : H_4(X; E) \otimes_{\mathbb{Z}} H_3(Y_2; E) \to H_1(Y_2; E \otimes E).$$

The assumption $[Y_2] \otimes 1 = 0$ says that $[\widetilde{Y_1} - \widetilde{Y_1}'] \cap ([Y_2] \otimes 1) = 0$. □

This lemma implies that *a linking homology class* is well defined:

Definition 8.2.

$$\mathrm{lk}(Y_1, Y_2) = [\widetilde{Y_1}] \cap [Y_2] \in H_1(Y_2; E).$$

If there is a non-vanishing vector field v of the normal bundle of Y_1 in X, we can formulate a self-linking homology class of the framed 3-manifold (Y_1, v). Let $v(Y_1) \subset X$ is a parallel of Y_1 given by pushing Y_1 along v_1.

Definition 8.3. The *self-linking homology class* of (Y_1, v) is defined to be

$$\mathrm{self.lk}(Y_1, v) = \mathrm{lk}(v(Y_1), Y_1).$$

8.2 $d'(M, e)$ as a self-linking homology class

The geometric description given in Proposition 4.2 implies that $d'(M, e)$ is a self-linking homology class of (Δ, v_e):

$$d'(M, e) = \mathrm{self.lk}(\Delta, v_e).$$

Thus we obtain a description of $\mathrm{Tor}(M, e)$ from the point of view of a self-linking homology class.

Theorem 2.

$$D(\mathrm{Tor}(M, e)) = \mathrm{self.lk}(\Delta, v_e).$$

9 Lescop's invariant and $d'(M, e)$

Let M be a closed oriented 3-manifold with $b_1(M) = \mathrm{rk} H_1(M;\mathbb{Z}) = 1$. In this section we review the Lescop's invariant $I_\Delta \in Q(H)$ introduced in [14] as $d'(M, e_0)$ for a special

Euler structure e_0. Then we give an alternative proof of Lescop's theorem ([14, Theorem 4.7]) on the relation between I_Δ and the normalized Alexander polynomial $\Delta(t)$.

Let $\tau : TM \to M \times \mathbb{R}^3$ be a framing. Take a vector $v \in S^2 \subset \mathbb{R}^3$. We have a non-vanishing vector field $\tau^{-1}(v)$ on M. Let e_τ be the Euler structure determined by $\tau^{-1}(v)$. In our language, the invariant I_Δ is defined to be an element of $Q(H)$ characterized by the following equation:

$$d'(M, e_\tau) = [K] \otimes I_\Delta \in H_1(M;\mathbb{Z}) \otimes Q(H).$$

Here $K \subset M$ is an oriented knot satisfying $[K] = 1 \in H_1(M;\mathbb{Z})$.

Proposition 9.1 ([14, Theorem 4.7])**.** *Let M be a closed oriented 3-manifold with $b_1(M) = 1$. Let $K \subset M$ be an oriented knot satisfying $[K] = 1 \in H_1(M;\mathbb{Z})$. Let e_τ be an Euler structure given by a framing $\tau : TM \to M \times \mathbb{R}^3$. Then*

$$I_\Delta = \frac{1+t}{1-t} + \frac{t\Delta(t)'}{\Delta(t)}.$$

, where t is generator of H given as $[K] = t$. The Alexander polynomial $\Delta(t) \in Q(H)$ is normalized as $\Delta(-t) = \Delta(t)$ and $\Delta(1) = 1$.

Proof. It is known (see [17, Section 5.2] of Chapter II) that the normalized Alexander polynomial can be computed form the Reidemeister-Turaev torsion as

$$\Delta(t) = \mathrm{Tor}(M, e_\tau)(t-1)(t^{-1}-1).$$

Thus

$$\begin{aligned} d'(M, e_\tau) &= D(\mathrm{Tor}(M, e_\tau)) \\ &= D\left((t-1)^{-1}(t^{-1}-1)^{-1}\Delta(t)\right) \\ &= [t] \otimes t\frac{d}{dt}\log\left|\left((t-1)^{-1}(t^{-1}-1)^{-1}\Delta(t)\right)\right| \\ &= [t] \otimes \left(\frac{1+t}{1-t} + \frac{t\Delta'(t)}{\Delta(t)}\right). \end{aligned}$$

This implies that $I_\Delta = \frac{1+t}{1-t} + \frac{t\Delta'(t)}{\Delta(t)}$. □

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