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An Upper Bound on the Generic Degree of the Generalized Verschiebung for Rank Two Stable Bundles

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Abstract. In the present paper, we give an upper bound for the generic degree of the generalized Verschiebung between the moduli spaces of rank two stable bundles with trivial determinant.

Contents

Introduction

Let k be an algebraically closed field of characteristic $p > 0$ and X a smooth projective curve over k of genus $g > 1$. Denote by $X^{(1)}$ the Frobenius twist of X over k. Then, pulling-back stable bundles on $X^{(1)}$ via the relative Frobenius morphism $F_{X/k} : X \to X^{(1)}$ induces the so-called "*generalized Verschiebung*" rational map

$$\mathrm{Ver}^n_{X/k} : \mathrm{SU}^n_{X^{(1)}/k} \dashrightarrow \mathrm{SU}^n_{X/k} \tag{1}$$

between the moduli spaces of rank $n > 1$ stable bundles with trivial determinant on $X^{(1)}$ and X respectively; this can be regarded as a higher-rank variant of the Verschiebung between Jacobians.

The geometry of the rational map $\mathrm{Ver}^n_{X/k}$, i.e., the dynamics of stable bundles with respect to Frobenius pull-back, has been investigated for a long time. One motivation is the relationship with representations of the fundamental group of a curve in positive characteristic (cf., e.g., [1], [14], [26]); indeed, it is well-known that rank n vector bundles fixed by some powers of the Frobenius morphism come from continuous representations of the fundamental group in $\mathrm{GL}_n(k)$ (cf. [19]). Also, the moduli space $\mathrm{SU}^n_{X/k}$ and the generalized Verschiebung $\mathrm{Ver}^n_{X/k}$ are interesting in their own right; other studies regarding these mathematical objects (e.g., the density of Frobenius-periodic bundles, the loci of Frobenius-destabilized bundles, etc.) can be found in various literatures, e.g., [3], [9], [10], [15], [16], [18], [23], [24].

The present paper aims to address the generic degree $\deg(\mathrm{Ver}^n_{X/k})$ of $\mathrm{Ver}^n_{X/k}$ for $n = 2$. The case of $(n, g) = (2, 2)$ has already been investigated considerably. We know that, for a genus-2 curve X, the compactification of $\mathrm{SU}^2_{X/k}$ by semistable bundles is canonically isomorphic to the

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3-dimensional projective space $\mathbb{P}^3_k$ and the boundary locus can be identified with the Kummer surface associated to X. By this description, the rational map $\mathrm{Ver}^2_{X/k}$ can be expressed by polynomials of degree p (cf. [16, Proposition A.2]), which are explicitly described in the cases $p = 2$ (cf. [15]) and $p = 3$ (cf. [16]). Moreover, the scheme-theoretic base locus of $\mathrm{Ver}^2_{X/k}$ were computed in [18, Theorem 2]. This result enables us to specify the generic degree $\deg(\mathrm{Ver}^2_{X/k})$ of $\mathrm{Ver}^2_{X/k}$, i.e., we have $\deg(\mathrm{Ver}^2_{X/k}) = \frac{p^3+2p}{3}$ (cf. [23, Theorem 1.3], [18, Corollary]).

However, we have not yet reached a comprehensive understanding of $\mathrm{Ver}^n_{X/k}$ because not much seems to be known for general (p, n, g). That is, the structure of this rational map remains mysterious despite its own importance! As a step towards understanding it, for example, knowing by what value the generic degree $\deg(\mathrm{Ver}^n_{X/k})$ is bounded above will be useful information in measuring its complexity. (As far as the authors know, it seems that the relevant numerical results are still obtained only when $(n, g) = (2, 2)$.) The main result of the present paper, i.e., Theorem A below, concerns this matter and provides an upper bound of the generic degree $\deg(\mathrm{Ver}^2_{X/k})$ for infinitely many pairs (p, g).

Theorem A (= Theorem 3.2.3). *If $p + 1 > g > 1$ and $p \neq 2$, then the generic degree $\deg(\mathrm{Ver}^2_{X/k})$ of $\mathrm{Ver}^2_{X/k}$ satisfies the following inequality:*

$$\deg(\mathrm{Ver}^2_{X/k}) \leq p^{g-1} \cdot \sum_{\theta=1}^{2p-1} \frac{1}{\sin^{2g-2}(\frac{\pi\cdot\theta}{2\cdot p})} \quad \left(= \sum_{\zeta^{2p}=1, \zeta\neq 1} \frac{(-4p\zeta)^{g-1}}{(\zeta-1)^{2g-2}} \right). \tag{2}$$

We remark here that an essential ingredient in the proof of the above theorem is the correspondence between the generic fiber of $\mathrm{Ver}^2_{X/k}$ and a certain Quot scheme. This correspondence can be established by a dimension estimate resulting from Brill-Noether theory (cf. the proof of Lemma 2.3.1). We moreover use the generic étaleness of $\mathrm{Ver}^2_{X/k}$ for an ordinary X to lift the Quot scheme to characteristic 0. As a result, the required inequality is obtained by applying a formula proved by Holla (cf. [7, Theorem 4.2]), i.e., a special case of the Vafa-Intriligator formula. The latter part of the argument is entirely similar to the proof of the main theorem in [27].

Notation and Conventions. Throughout the present paper, we fix an integer $g > 1$, a prime $p > 2$, and an algebraically closed field k of characteristic p.

For each scheme T, we shall write $(\mathcal{S}ch/T)$ for the category of schemes of finite type over T. For simplicity, if R is a ring, then we shall write $(\mathcal{S}ch/R) := (\mathcal{S}ch/\mathrm{Spec}(R))$.

For a smooth projective curve X over a field K, we shall write $\Omega_{X/K}$ for the sheaf of 1-forms on X relative to K. Also, for each integer d, denote by $\mathrm{Pic}^d_{X/K}$ the Picard scheme of X/K classifying isomorphism classes of line bundles on X of degree d.

If T is a scheme and X_1, X_2 are T-schemes, then we shall denote by pr_i $(i = 1, 2)$ the i-th projection $X_1 \times_T X_2 \to X_i$.

1. Definition of the generalized Verschiebung

1.1. Let R be an integrally closed domain of finite type over k and X a smooth projective curve over R of genus g. We shall denote by

$$\mathcal{SU}^2_{X/R} : (\mathcal{S}ch/R)^{\mathrm{op}} \to (\mathcal{S}et) \tag{3}$$

the set-valued contravariant functor on the category $(\mathcal{S}ch/R)$ which, to any $T \in \mathrm{Ob}(\mathcal{S}ch/R)$, assigns the set of *equivalence* classes of T-flat families of rank 2 geometrically stable bundles on $X \times_R T$ with trivial determinant. Here, given two T-flat families of geometrically stable bundles $\mathcal{F}_1$, $\mathcal{F}_2$ on $X \times_R T$, we say that $\mathcal{F}_1$ and $\mathcal{F}_2$ are *equivalent* if $\mathcal{F}_1 \otimes \mathrm{pr}_2^*(\mathcal{N}) \cong \mathcal{F}_2$ for some line bundle $\mathcal{N}$ on T. According to [20, Theorem 0.2] and [12, Theorem 9.12], there exists a flat quasi-projective R-scheme

$$\mathrm{SU}^2_{X/R} \tag{4}$$

that corepresents universally the functor $\mathcal{SU}^2_{X/R}$. (Note that the flatness asserted in [12] is still true even when the base field is of positive characteristic.) The fiber $\mathrm{SU}^2_{X/R} \times_R \xi$ over each geometric point ξ of $\mathrm{Spec}(R)$ is irreducible, smooth, and of dimension $3g-3$ (cf., e.g., [22, Lemma A]).

1.2. Next, write $X^{(1)}$ for the Frobenius twist of X over R and $F_{X/R} : X \to X^{(1)}$ for the relative Frobenius morphism of X over R. Let

$$\mathcal{SU}^{2,\circledcirc}_{X^{(1)}/R} \tag{5}$$

be the subfunctor of $\mathcal{SU}^2_{X^{(1)}/R}$ classifying geometrically stable bundles whose pull-back under $F_{X/R}$ is geometrically stable. Recall (cf. [8, Proposition 2.3.1]) that the property of being geometrically stable is an open condition in flat families. Hence, there exists an open subscheme

$$\mathrm{SU}^{2,\circledcirc}_{X^{(1)}/R} \tag{6}$$

of $\mathrm{SU}^2_{X^{(1)}/R}$ that corepresents universally the functor $\mathcal{SU}^{2,\circledcirc}_{X^{(1)}/R}$ (cf. [23, Theorem A.6]). The assignment $[\mathcal{F}] \mapsto [F^*_{X/R}(\mathcal{F})]$ (for each $[\mathcal{F}] \in \mathcal{SU}^{2,\circledcirc}_{X^{(1)}/R}$) determines a natural transformation

$$\mathcal{V}er^{2,\circledcirc}_{X/R} : \mathcal{SU}^{2,\circledcirc}_{X^{(1)}/R} \to \mathcal{SU}^2_{X/R} \tag{7}$$

of functors, which induces a dominant morphism

$$\mathrm{Ver}^{2,\circledcirc}_{X/R} : \mathrm{SU}^{2,\circledcirc}_{X^{(1)}/R} \to \mathrm{SU}^2_{X/R} \tag{8}$$

between flat R-schemes of the same relative dimension (cf. [22], [23, Theorem A.6]). The formation of $\mathrm{Ver}^{2,\circledcirc}_{X/R}$ is compatible, in an evident sense, with restriction to each geometric point of R.

Now, let us specialize the situation to the case where $R = k$. Because of the facts mentioned above, one may define the generic degree of $\mathrm{Ver}^{2,\circledcirc}_{X/k}$, which we denote by

$$\deg(\mathrm{Ver}^2_{X/k}), \tag{9}$$

and this value does not depend on the choice of X. Here, denote by $\mathcal{M}_g$ the moduli stack classifying smooth projective curves over k of genus g. Since $\mathcal{M}_g$ is an irreducible DM stack over k (cf. [2, § 5]), it makes sense to speak of a "*general*" curve, i.e., a curve that determines a point of $\mathcal{M}_g$ that lies outside a certain fixed closed substack not equal to $\mathcal{M}_g$ itself. In particular, in order to specify the value $\deg(\mathrm{Ver}^2_{X/k})$, *we are always free to replace X with a general curve in $\mathcal{M}_g$.*

2. Relationship with Quot schemes

2.1. We recall the notion of a Quot scheme as follows. Let T be a noetherian scheme, Y a smooth projective curve over T of genus g and $\mathcal{E}$ a vector bundle on Y. For each integers r, d with $r \geq 0$, we shall denote by

$$\mathcal{Q}uot^{r,d}_{\mathcal{E}/Y/T} : (\mathcal{S}ch/T)^{\mathrm{op}} \to (\mathcal{S}et) \tag{10}$$

the set-valued contravariant functor on the category $(\mathcal{S}ch/T)$ which, to any $f : T' \to T$, associates the set of isomorphism classes of $\mathcal{O}_{Y\times_T T'}$-linear injections $s : \mathcal{F} \hookrightarrow (\mathrm{id}_Y \times f)^*(\mathcal{E})$ such that

- the cokernel $\mathrm{Coker}(s)$ is flat over T' (which, by the fact that Y/T is smooth of relative dimension 1, implies that $\mathcal{F}$ is *locally free*), and
- $\mathcal{F}$ is of rank r and degree d.

It is known (cf. [4, Theorem 5.14]) that $\mathcal{Q}uot^{r,d}_{\mathcal{E}/Y/T}$ may be represented by a proper scheme over T. By abuse of notation, write $\mathcal{Q}uot^{r,d}_{\mathcal{E}/Y/T}$ for the scheme that represents the functor $\mathcal{Q}uot^{r,d}_{\mathcal{E}/Y/T}$.

2.2. Let X be a smooth projective curve over k of genus g which is general in $\mathcal{M}_g$. In particular, we may assume that X is *ordinary*. (Recall that the locus of $\mathcal{M}_g$ classifying ordinary curves is open and dense.) Moreover, we assume that $p+1 > g\,(>1)$. (This assumption will be applied in Lemma 2.3.1 and Proposition 3.2.1.) Let us take a geometric generic point $\eta : \mathrm{Spec}(K) \to \mathrm{SU}^2_{X/k}$ (where K denotes an algebraically closed field over k) of $\mathrm{SU}^2_{X/k}$; this point classifies a rank 2 stable bundle $\mathcal{E}$ on $X_K := X \times_k K$ with $\det(\mathcal{E}) \cong \mathcal{O}_{X_K}$. Write $X^{(1)}_K$ for the Frobenius twist of X_K over K and $F : X_K \to X^{(1)}_K$ for the relative Frobenius morphism. If $\mathcal{G}$ is an $\mathcal{O}_{X^{(1)}_K}$-module and $\mathcal{H}$ is an $\mathcal{O}_{X_K}$-module, then the adjunction relation "$F^*(-) \dashv F_*(-)$" gives a natural bijection

$$\mathrm{ad} : \mathrm{Hom}_{\mathcal{O}_{X_K}}(F^*(\mathcal{G}), \mathcal{H}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_{X^{(1)}_K}}(\mathcal{G}, F_*(\mathcal{H})), \tag{11}$$

which is functorial with respect to both $\mathcal{G}$ and $\mathcal{H}$.

Since F is finite and faithfully flat of degree p, the direct image $F_*(\mathcal{E})$ forms a vector bundle on $X^{(1)}_K$ of rank $2p$. Now, consider the Quot scheme

$$\mathcal{Q} := \mathcal{Q}uot^{2,0}_{F_*(\mathcal{E})/X^{(1)}_K/K}. \tag{12}$$

This K-scheme has the closed subscheme

$$\mathcal{Q}^{\mathrm{triv}} \ \left(\text{resp., } \mathcal{Q}^{\mathrm{triv},F}\right) \tag{13}$$

classifying injections $s : \mathcal{F} \hookrightarrow F_*(\mathcal{E})$ with $\det(\mathcal{F}) \cong \mathcal{O}_{X^{(1)}_K}$ (resp., $F^*(\det(\mathcal{F})) \cong \mathcal{O}_{X_K}$). In particular, there exists a natural closed immersion $\mathcal{Q}^{\mathrm{triv}} \hookrightarrow \mathcal{Q}^{\mathrm{triv},F}$.

2.3. Since X has been assumed to be ordinary, $\mathrm{Ver}^2_{X/k}$ is generically étale (cf. [22, Corollary 2.1.1]). Hence, the fiber product $\mathrm{SU}^{2,\circledcirc}_{X^{(1)}/k} \times_{\mathrm{SU}^2_{X/k},\eta} K$ is isomorphic to the disjoint union of finitely many copies of $\mathrm{Spec}(K)$. Let us take a K-rational point $\widetilde{\eta}$ of $\mathrm{SU}^{2,\circledcirc}_{X^{(1)}/k} \times_{\mathrm{SU}^2_{X/k},\eta} K$; this point corresponds to a rank 2 stable bundle $\mathcal{F}$ on $X^{(1)}_K$ with trivial determinant. The point $\widetilde{\eta}$ is, by definition, mapped to η by $\mathrm{Ver}^2_{X/k}$, meaning that there exists an isomorphism $t : F^*(\mathcal{F}) \xrightarrow{\sim} \mathcal{E}$. The morphism $\mathrm{ad}(t) : \mathcal{F} \to F_*(\mathcal{E})$ is injective because the composite

$$F^*(\mathcal{F}) \xrightarrow{F^*(\mathrm{ad}(t))} F^*(F_*(\mathcal{E})) \xrightarrow{\mathrm{ad}^{-1}(\mathrm{id}_{F_*(\mathcal{E})})} \mathcal{E} \tag{14}$$

coincides with t and F is faithfully flat. By the stability of $\mathcal{E}$ (which implies that $\mathrm{End}_{\mathcal{O}_{X_K}}(\mathcal{E}) = k$), the K-rational point of $\mathcal{Q}^{\mathrm{triv}}$ classifying $\mathrm{ad}(t)$ does not depend on the choice of t. Hence, the assignment $\mathcal{F} \mapsto \mathrm{ad}(t)$ determines a well-defined K-morphism

$$\mathrm{SU}^{2,\circledcirc}_{X^{(1)}/k} \times_{\mathrm{SU}^2_{X/k},\eta} K \to \mathcal{Q}^{\mathrm{triv}}. \tag{15}$$

Lemma 2.3.1. *(Recall that we have assumed that $p+1 > g > 1$.) Let us take an $\mathcal{O}_{X_K}$-linear injection $s : \mathcal{F} \hookrightarrow F_*(\mathcal{E})$ such that $\mathcal{F}$ is a rank 2 vector bundle. We shall write $d := \deg(\mathcal{F})$. Then, the inequality $d \leq 0$ holds, meaning that the maximal degree of rank 2 vector bundles embedded into $F_*(\mathcal{E})$ is at most 0. Moreover, s is classified by $\mathcal{Q}^{\mathrm{triv},F}$ if and only if the morphism $\mathrm{ad}^{-1}(s) : F^*(\mathcal{F}) \to \mathcal{E}$ is an isomorphism.*

Proof. First, we shall consider the former assertion. Suppose that $d > 0$. By comparing the respective degrees of $F^*(\mathcal{F})$ and $\mathcal{E}$, the (nonzero) morphism $\mathrm{ad}^{-1}(s)$ cannot be an isomorphism at the generic point of X_K. Hence, since $\mathcal{E}$ is locally free of rank 2 and X_K is a smooth curve over K, the subsheaf $\mathrm{Im}(\mathrm{ad}^{-1}(s))$ of $\mathcal{E}$ forms a line bundle. Write $\nabla_{\mathrm{can}} : F^*(\mathcal{F}) \to \Omega_{X_K/K} \otimes F^*(\mathcal{F})$ for the connection on $F^*(\mathcal{F})$ determined uniquely by the condition that the sections of the subsheaf $F^{-1}(\mathcal{F})$ are horizontal (cf. [11, Theorem (5.1)]).

In the following, we shall prove the claim that the line subbundle $\mathrm{Ker}(\mathrm{ad}^{-1}(s))$ ($\subseteq F^*(\mathcal{F})$) is not closed under ∇_{can}. Suppose, on the contrary, that $\mathrm{Ker}(\mathrm{ad}^{-1}(s))$ is closed under ∇_{can}. Since the restriction of ∇_{can} to $\mathrm{Ker}(\mathrm{ad}^{-1}(s))$ has vanishing p-curvature, $\mathrm{Ker}(\mathrm{ad}^{-1}(s))$ is isomorphic to $F^*(\mathcal{U})$ for some line bundle $\mathcal{U}$ on $X_K^{(1)}$. Moreover, the resulting (horizontal) composite $F^*(\mathcal{U}) \xrightarrow{\sim} \mathrm{Ker}(\mathrm{ad}^{-1}(s)) \hookrightarrow F^*(\mathcal{F})$ comes, via pull-back by F, from an injection $\mathcal{U} \hookrightarrow \mathcal{F}$. The composite $F^*(\mathcal{U}) \hookrightarrow F^*(\mathcal{F}) \xrightarrow{\mathrm{ad}^{-1}(s)} \mathcal{E}$ is identical to the zero map, so the corresponding map $\mathcal{U} \hookrightarrow \mathcal{F} \overset{s}{\hookrightarrow} F_*(\mathcal{E})$ (via ad) must be the zero map. This contradicts the injectivity of s, and hence, completes the proof of the claim.

Next, observe that $F^*(\mathcal{F})$ may be regarded as an extension of $\mathrm{Im}(\mathrm{ad}^{-1}(s))$ by $\mathrm{Im}(\mathrm{ad}^{-1}(s))^\vee \otimes \det(F^*(\mathcal{F}))$; let us fix an isomorphism $\mathrm{Im}(\mathrm{ad}^{-1}(s))^\vee \otimes \det(F^*(\mathcal{F})) \xrightarrow{\sim} \mathrm{Ker}(\mathrm{ad}^{-1}(s))$. By the claim proved above, the following composite turns out to be injective:

$$\begin{aligned} \mathrm{Im}(\mathrm{ad}^{-1}(s))^\vee \otimes \det(F^*(\mathcal{F})) &\xrightarrow{\sim} \mathrm{Ker}(\mathrm{ad}^{-1}(s)) \\ &\hookrightarrow F^*(\mathcal{F}) \\ &\xrightarrow{\nabla_{\mathrm{can}}} \Omega_{X_K/K} \otimes F^*(\mathcal{F}) \\ &\twoheadrightarrow \Omega_{X_K/K} \otimes \mathrm{Im}(\mathrm{ad}^{-1}(s)) \\ &\xrightarrow{\sim} \mathrm{pr}_1^*(\Omega_{X/k}) \otimes \mathrm{Im}(\mathrm{ad}^{-1}(s)). \end{aligned} \tag{16}$$

This composite may be verified to be $\mathcal{O}_{X_K}$-linear, and hence, determines a nonzero global section

$$q \in \Gamma(X_K, \mathrm{Im}(\mathrm{ad}^{-1}(s))^{\otimes 2} \otimes \det(F^*(\mathcal{F}))^\vee \otimes \mathrm{pr}_1^*(\Omega_{X/k})). \tag{17}$$

Let us fix a line subbundle $\mathcal{N}$ of $\mathcal{E}$ containing the subsheaf $\mathrm{Im}(\mathrm{ad}^{-1}(s))$. (Hence, $\mathcal{E}$ is obtained as an extension of $\mathcal{N}^\vee$ by $\mathcal{N}$.) The section q may be regarded, via the inclusion $\mathrm{Im}(\mathrm{ad}^{-1}(s))^{\otimes 2} \hookrightarrow$

$\mathcal{N}^{\otimes 2}$, as a (nonzero) global section of $\mathcal{N}^{\otimes 2} \otimes \det(F^*(\mathcal{F}))^\vee \otimes \mathrm{pr}_1^*(\Omega_{X/k})$. On the other hand, according to [17, Proposition 3.1], the degree of a line subbundle of $\mathcal{E}$ is at most $-\frac{g}{2}$ (resp., $-\frac{g-1}{2}$) if g is even (resp., odd). This implies

$$\begin{aligned} \deg(\mathcal{N}^{\otimes 2} \otimes \det(F^*(\mathcal{F}))^\vee \otimes \mathrm{pr}_1^*(\Omega_{X/k})) &= 2 \cdot \deg(\mathcal{N}) - pd + 2g - 2 \\ &\leq 2 \cdot \left(-\frac{g-1}{2}\right) - pd + 2g - 2 \\ &< 0, \end{aligned} \tag{18}$$

where the last inequality follows from the assumptions $p+1 > g$ and $d > 0$. However, this is a contradiction because $q \neq 0$. Consequently, we have $d \leq 0$, as desired.

Next, we shall prove the latter assertion. The "if" part is clear from $\det(\mathcal{E}) \cong \mathcal{O}_X$, so it suffices to consider the "only if" part. Here, we assume that s is classified by $\mathcal{Q}^{\mathrm{triv},F}$, but the vector bundle $\mathrm{Im}(\mathrm{ad}^{-1}(s))\,(\neq \{0\})$ is of rank 1. Just as in the proof of the former assertion, we can obtain a nonzero global section

$$q \in \Gamma(X_K, \mathrm{Im}(\mathrm{ad}^{-1}(s))^{\otimes 2} \otimes \mathrm{pr}_1^*(\Omega_{X/k})) \tag{19}$$

of the line bundle $\mathrm{Im}(\mathrm{ad}^{-1}(s))^{\otimes 2} \otimes \mathrm{pr}_1^*(\Omega_{X/k})$ (where we recall that $\det(F^*(\mathcal{F})) \cong \mathcal{O}_{X_K}$). Moreover, by taking a line subbundle $\mathcal{N}$ of $\mathcal{E}$ containing $\mathrm{Im}(\mathrm{ad}^{-1}(s))$, we may regard q as a (nonzero) global section of $\mathcal{N}^{\otimes 2} \otimes \mathrm{pr}_1^*(\Omega_{X/k})$. By a well-known fact of Brill-Noether theory (cf., e.g., [5], [21]), the existence of such a section q implies that the scheme-theoretic image of the morphism $\mathrm{Spec}(K) \to \mathrm{Pic}_{X/k}^{2g-2+2\cdot\deg(\mathcal{N})}$ classifying $\mathcal{N}^{\otimes 2} \otimes \mathrm{pr}_1^*(\Omega_{X/k})$ is of dimension $\leq 2g-2+2\cdot\deg(\mathcal{N})$ (because of the assumption that X is general). On the other hand, the morphism $\mathrm{Pic}_{X/k}^{\deg(\mathcal{N})} \to \mathrm{Pic}_{X/k}^{2g-2+2\cdot\deg(\mathcal{N})}$ determined by $[\mathcal{M}] \mapsto [\mathcal{M}^{\otimes 2} \otimes \Omega_{X/k}]$ is finite. It follows that the scheme-theoretic image of the morphism $\mathrm{Spec}(K) \to \mathrm{Pic}_{X/k}^{\deg(\mathcal{N})}$ classifying $\mathcal{N}$ is of dimension $\leq 2g-2+2\cdot\deg(\mathcal{N})$. However, it contradicts the fact proved in Lemma 2.3.2 below. Consequently, the locally free sheaf $\mathrm{Im}(\mathrm{ad}^{-1}(s))$ must be of rank 2. Since $\deg(F^*(\mathcal{F})) = \deg(\mathcal{E}) = 0$, $\mathrm{ad}^{-1}(s)$ turns out to be an isomorphism. This completes the proof of the "only if" part, as desired. □

The following lemma was applied in the proof of the previous assertion.

Lemma 2.3.2. *Let us keep the notation in the proof of the latter assertion of Lemma 2.3.1. Then, the scheme-theoretic image I of the morphism $\mathrm{Spec}(K) \to \mathrm{Pic}_{X/k}^{\deg(\mathcal{N})}$ classifying $\mathcal{N}$ is of dimension $\geq 2g-1+2\cdot\deg(\mathcal{N})$.*

Proof. Define D to be the moduli space classifying isomorphism classes of short exact sequences:

$$\mathfrak{f}_0 : 0 \to \mathcal{N}_0 \to \mathcal{E}_0 \to \mathcal{N}_0^\vee \to 0 \tag{20}$$

with $[\mathcal{N}_0] \in I$ and $[\mathcal{E}_0] \in \mathrm{SU}_{X/k}^2$. Let us take an arbitrary exact sequence $\mathfrak{f}_0$ as above. By the definition of stability and the properness of X/k, every nonzero endomorphism of $\mathcal{E}_0$ is an isomorphism, and hence,

$$h^0((\mathcal{N}_0^\vee)^\vee \otimes \mathcal{N}_0) = h^0((\mathcal{E}_0/\mathcal{N}_0)^\vee \otimes \mathcal{N}_0) = 0. \tag{21}$$

Thus, by the Riemann-Roch theorem, we obtain

$$h^1((\mathcal{N}_0^\vee)^\vee \otimes \mathcal{N}_0) = g - 1 - 2\cdot\deg(\mathcal{N}_0) = g - 1 - 2\cdot\deg(\mathcal{N}). \tag{22}$$

This implies that any fiber of the projection $D \to I$ given by $[\mathfrak{f}_0] \mapsto [\mathcal{N}_0]$ is of dimension $\leq (g-1-2\cdot\deg(\mathcal{N}))-1 = g-2-2\cdot\deg(\mathcal{N})$. Hence, if $\dim(I) < 2g-1+2\cdot\deg(\mathcal{N})$, we have

$$
\begin{aligned}
\dim(D) &\leq \dim(I) + g - 2 - 2\cdot\deg(\mathcal{N}) \\
&< (2g-1+2\cdot\deg(\mathcal{N})) + g - 2 - 2\cdot\deg(\mathcal{N}) \\
&= 3g-3 \left(= \dim(\mathrm{SU}^2_{X/k})\right).
\end{aligned}
\tag{23}
$$

This is a contradiction because the existence of the extension $0 \to \mathcal{N} \to \mathcal{E} \to \mathcal{N}^\vee \to 0$ implies that the morphism $D \to \mathrm{SU}^2_{X/k}$ given by $[\mathfrak{f}_0] \mapsto [\mathcal{E}_0]$ must be dominant. Thus, we obtain the inequality $\dim(I) \geq 2g-1+2\cdot\deg(\mathcal{N})$, as desired. $\square$

Remark 2.3.3. By an argument entirely similar to the proof of the latter assertion of Lemma 2.3.1, we can verify the following assertion: if $s : \mathcal{F} \hookrightarrow F_*(\mathcal{E})$ is an injection classified by a K-rational point of $\mathcal{Q}$, then s is classified by $\mathcal{Q}^{\mathrm{triv},F}$ if and only if $\det(\mathcal{F})$ descends to a line bundle on X.

Proposition 2.3.4. *The morphism (15) is an isomorphism. In particular, $\mathcal{Q}^{\mathrm{triv}}$ is finite and étale over K, and the following equality holds:*

$$
\deg(\mathrm{Ver}^2_{X/k}) = \deg(\mathcal{Q}^{\mathrm{triv}}/K). \tag{24}
$$

Proof. First, let us consider the former assertion. According to Lemma 2.3.1 above, the morphism (15) induces a bijection between the respective sets of K-rational points. In particular, $\mathcal{Q}^{\mathrm{triv}}$ is finite over K. Let $u : U \to \mathrm{Spec}(K)$ be a K-scheme defined as a connected component of $\mathcal{Q}^{\mathrm{triv}}$. Then, the reduced scheme U_{red} associated to U is $\mathrm{Spec}(K)$. The K-scheme U classifies an injection

$$
s_U : \mathcal{F}_U \to (\mathrm{id}_{X_K} \times u)^*(F_*(\mathcal{E})) \ \ (= (F \times \mathrm{id}_U)_*((\mathrm{id}_{X_K} \times u)^*(\mathcal{E}))) \tag{25}
$$

on $X_K \times_K U$. By Lemma 2.3.1 again, the morphism

$$
(F \times \mathrm{id}_U)^*(\mathcal{F}_U) \to (\mathrm{id}_{X_K} \times u)^*(\mathcal{E}) \tag{26}
$$

corresponding to s_U via the adjunction relation "$(F \times \mathrm{id}_U)^*(-) \dashv (F \times \mathrm{id}_U)_*(-)$" is surjective when restricted to $X_K \times_K U_{\mathrm{red}}$ $(= X_K)$. By Nakayama's lemma and the fact that both $(F \times \mathrm{id}_U)^*(\mathcal{F}_U)$ and $(\mathrm{id}_{X_K} \times u)^*(\mathcal{E}))$ are locally free, (26) turns out to be an isomorphism. In particular, $\mathcal{F}_U$ determines a K-morphism $U \to \mathrm{SU}^{2,\circledcirc}_{X^{(1)}/k} \times_{\mathrm{SU}^2_{X/k},\eta} K$. By applying this discussion to the various connected components of $\mathcal{Q}^{\mathrm{triv}}$, we obtain a morphism $\mathcal{Q}^{\mathrm{triv}} \to \mathrm{SU}^{2,\circledcirc}_{X^{(1)}/k} \times_{\mathrm{SU}^2_{X/k},\eta} K$. One may verify that this morphism determines, by construction, the inverse to (15). This completes the proof of the former assertion. The latter assertion follows directly from the former assertion together with the fact that $\mathrm{SU}^{2,\circledcirc}_{X^{(1)}/k} \times_{\mathrm{SU}^2_{X/k},\eta} K$ is finite and étale over K. $\square$

Also, we obtain the following proposition.

Proposition 2.3.5. *The Quot scheme $\mathcal{Q}$ decomposes into the disjoint union $\mathcal{Q} = \mathcal{Q}^{\mathrm{triv},F} \sqcup \mathcal{R}$ for some K-scheme $\mathcal{R}$.*

Proof. Denote by $\mathcal{R}$ the closed subscheme of $\mathcal{Q}$ classifying injections $s : \mathcal{F} \hookrightarrow F_*(\mathcal{E})$ with $\mathrm{Coker}(\mathrm{ad}^{-1}(s)) \neq \{0\}$. It follows from Lemma 2.3.1 (and its proof) that $\mathcal{Q}^{\mathrm{triv},F}$ coincides with the complement of $\mathcal{R}$ in $\mathcal{Q}$, so it is open in $\mathcal{Q}$. This completes the proof of the assertion. $\square$

2.4. Next, we shall consider the relationship between $\mathcal{Q}^{\mathrm{triv}}$ and $\mathcal{Q}^{\mathrm{triv},F}$. Denote by

$$\mathrm{Ker}(\mathrm{Ver}^1_{X_K/K}) \tag{27}$$

the scheme-theoretic inverse image of the point $[\mathcal{O}_{X_K}] \in \mathrm{Pic}^0_{X_K/K}$ via the classical Verschiebung map $\mathrm{Ver}^1_{X_K/K} : \mathrm{Pic}^0_{X_K^{(1)}/K} \to \mathrm{Pic}^0_{X_K/K}$, i.e., the morphism given by $[\mathcal{N}] \mapsto [F^*(\mathcal{N})]$. It is well-known that $\mathrm{Ker}(\mathrm{Ver}^1_{X_K/K})$ is finite and faithfully flat over K of degree p^g. Moreover, since X has been assumed to be ordinary, it is étale over K, i.e., isomorphic to the disjoint union of p^g copies of $\mathrm{Spec}(K)$.

Proposition 2.4.1. *There exists a natural isomorphism*

$$\mathcal{Q}^{\mathrm{triv}} \times_K \mathrm{Ker}(\mathrm{Ver}^1_{X_K/K}) \xrightarrow{\sim} \mathcal{Q}^{\mathrm{triv},F} \tag{28}$$

over K. In particular, $\mathcal{Q}^{\mathrm{triv},F}$ is isomorphic to the disjoint union of finitely many copies of $\mathrm{Spec}(K)$*, and the following equality holds:*

$$\deg(\mathcal{Q}^{\mathrm{triv}}/K) = \frac{1}{p^g} \cdot \deg(\mathcal{Q}^{\mathrm{triv},F}/K). \tag{29}$$

Proof. First, we construct an action of $\mathrm{Ker}(\mathrm{Ver}^1_{X_K/K})$ on $\mathcal{Q}$. Let us take a pair $([s],[\mathcal{L}])$ classified by a morphism $T \to \mathcal{Q} \times_K \mathrm{Ker}(\mathrm{Ver}^1_{X_K/K})$, where T denotes a K-scheme and s denotes an injection $s : \mathcal{F} \hookrightarrow \mathrm{pr}_1^*(F_*(\mathcal{E}))\ (= (F \times \mathrm{id}_T)_*(\mathrm{pr}_1^*(\mathcal{E})))$ on $X \times_K T$. Choose a representative $\mathcal{L}$ of $[\mathcal{L}]$. After possibly tensoring it with a suitable line bundle pulled back from T, we may assume that there is an isomorphism $\iota : (F \times \mathrm{id}_T)^*(\mathcal{L}) \xrightarrow{\sim} \mathcal{O}_{X \times_K T}$. Then, the following composite injection determines a T-rational point $[s \diamond \mathcal{L}]$ of $\mathcal{Q}$:

$$\begin{aligned} s \diamond \mathcal{L} : \mathcal{F} \otimes \mathcal{L}^{\otimes \frac{p+1}{2}} &\xrightarrow{s \otimes \mathrm{id}} (F \times \mathrm{id}_T)_*(\mathrm{pr}_1^*(\mathcal{E})) \otimes \mathcal{L}^{\otimes \frac{p+1}{2}} \\ &\xrightarrow{\sim} (F \times \mathrm{id}_T)_*(\mathrm{pr}_1^*(\mathcal{E}) \otimes (F \times \mathrm{id}_T)^*(\mathcal{L}^{\otimes \frac{p+1}{2}})) \\ &\xrightarrow{\sim} (F \times \mathrm{id}_T)_*(\mathrm{pr}_1^*(\mathcal{E}) \otimes (F \times \mathrm{id}_T)^*(\mathcal{L})^{\otimes \frac{p+1}{2}}) \\ &\xrightarrow{\sim} (F \times \mathrm{id}_T)_*(\mathrm{pr}_1^*(\mathcal{E})), \end{aligned} \tag{30}$$

where the second and last arrows are the isomorphisms induced by the projection formula and ι respectively. Note that this T-rational point does not depend on the choices of the representative $\mathcal{L}$ and the isomorphism ι. The resulting assignment $([s],[\mathcal{L}]) \mapsto [s \diamond \mathcal{L}]$ is functorial with respect to T, and hence, defines a well-defined action

$$\mathcal{Q} \times \mathrm{Ker}(\mathrm{Ver}^1_{X_K/K}) \to \mathcal{Q}. \tag{31}$$

Note that if $\mathcal{F}$ and $\mathcal{L}$ are as above, then

$$\det(\mathcal{F} \otimes \mathcal{L}^{\otimes \frac{p+1}{2}}) \cong \det(\mathcal{F}) \otimes \mathcal{L}^{\otimes 2 \cdot \frac{p+1}{2}} \cong \det(\mathcal{F}) \otimes \mathcal{L}^{\otimes (p+1)} \cong \det(\mathcal{F}) \otimes \mathcal{L}. \tag{32}$$

Hence, the action (31) restricts to a morphism

$$\mathcal{Q}^{\mathrm{triv}} \times_K \mathrm{Ker}(\mathrm{Ver}^1_{X_K/K}) \to \mathcal{Q}^{\mathrm{triv},F}. \tag{33}$$

On the other hand, (32) also implies that the assignment $[s] \mapsto ([s \diamond \det(\mathcal{F})^\vee], [\det(\mathcal{F})])$ determines the inverse to (33). This completes the proof of this proposition. □

3. Computation via the Vafa-Intriligator formula

By combining Propositions 2.3.4 and 2.4.1, we obtain the following equalities:

$$\deg(\mathrm{Ver}^2_{X/k}) = \deg(\mathcal{Q}^{\mathrm{triv}}/K) = \frac{1}{p^g} \cdot \deg(\mathcal{Q}^{\mathrm{triv},F}/K). \tag{34}$$

Therefore, to give a bound of $\deg(\mathrm{Ver}^2_{X/k})$, it suffices to estimate the value $\deg(\mathcal{Q}^{\mathrm{triv},F}/K)$.

3.1. In this subsection, we review a numerical formula concerning the degree of a certain Quot scheme over the field of complex numbers $\mathbb{C}$. Let C be a smooth projective curve over $\mathbb{C}$ of genus g. Let r, n, and d be integers with $1 \leq r \leq n$, and let $\mathcal{G}$ be a vector bundle on C of rank n and degree d. Then, we define invariants

$$\begin{aligned} e_{\max}(\mathcal{G}, r) &:= \max\left\{\deg(\mathcal{F}) \in \mathbb{Z} \,\middle|\, \mathcal{F} \text{ is a subbundle of } \mathcal{G} \text{ of rank } r \right\}, \\ s_r(\mathcal{G}) &:= d \cdot r - n \cdot e_{\max}(\mathcal{G}, r). \end{aligned} \tag{35}$$

Denote by $\mathrm{U}_C^{n,d}$ the moduli space of stable bundles on C of rank n and degree d. Since $\mathrm{U}_C^{n,d}$ is irreducible (cf., e.g., [22, Lemma A]), it makes sense to speak of a "general" stable bundle in $\mathrm{U}_C^{n,d}$, i.e., a stable bundle that corresponds to a point of the scheme $\mathrm{U}_C^{n,d}$ that lies outside a certain fixed closed subscheme. If $\mathcal{G}$ is a general stable bundle in $\mathrm{U}_C^{n,d}$, then it holds (cf. [6], [13, § 1]) that $s_r(\mathcal{G}) = r(n-r)(g-1)+\epsilon$, where ϵ is the integer uniquely determined by the equality just before and $0 \leq \epsilon < n$. Also, the number ϵ coincides (cf. [7, § 1]) with the dimension of every irreducible component of the Quot scheme $\mathcal{Q}uot^{r,e_{\max}(\mathcal{G},r)}_{\mathcal{G}/C/\mathbb{C}}$. If, moreover, the equality $s_r(\mathcal{G}) = r(n-r)(g-1)$ holds (i.e., $\dim(\mathcal{Q}uot^{r,e_{\max}(\mathcal{G},r)}_{\mathcal{G}/C/\mathbb{C}}) = 0$), then $\mathcal{Q}uot^{r,e_{\max}(\mathcal{G},r)}_{\mathcal{G}/C/\mathbb{C}}$ is étale over $\mathbb{C}$ (cf. [7, Proposition 4.1]). Finally, under this particular assumption, a formula for the degree of this Quot scheme was given by Holla as follows.

Theorem 3.1.1. *Let C be a smooth projective curve over $\mathbb{C}$ of genus g and $\mathcal{G}$ a general stable bundle in $\mathrm{U}_C^{n,d}$. Write (a,b) for the unique pair of integers such that $d = an - b$ with $0 \leq b < n$. Also, we suppose that the equality $s_r(\mathcal{G}) = r(n-r)(g-1)$ (equivalently, $e_{\max}(\mathcal{G}, r) = (dr - r(n-r)(g-1))/n$) holds. Then, the degree $\deg(\mathcal{Q}uot^{r,e_{\max}(\mathcal{G},r)}_{\mathcal{G}/C/\mathbb{C}}/\mathbb{C})$ of $\mathcal{Q}uot^{r,e_{\max}(\mathcal{G},r)}_{\mathcal{G}/C/\mathbb{C}}$ over $\mathbb{C}$ is calculated by the following formula.*

$$\frac{(-1)^{(r-1)(br-(g-1)r^2)/n} n^{r(g-1)}}{r!} \cdot \sum_{\zeta_1, \cdots, \zeta_r} \frac{\prod_{i=1}^r \zeta_i^{b-g+1}}{\prod_{i \neq j}(\zeta_i - \zeta_j)^{g-1}}, \tag{36}$$

where the sum is taken over the set of r-tuples $(\zeta_1, \cdots, \zeta_r) \in \mathbb{C}^r$ of mutually distinct n-th roots of unity in $\mathbb{C}$.

Proof. The assertion follows from [7, Theorem 4.2], where the "k" (resp., "r") corresponds to our r (resp., n). □

3.2. With the notation in the previous section, we relate the above formula to the degree of the related Quot schemes, and then, give an upper bound of the value $\deg(\mathcal{Q}^{\mathrm{triv},F}/K)$.

Proposition 3.2.1. *(Recall that we have assumed that $p+1>g>1$.) We have the following inequality:*

$$\deg(\mathcal{Q}^{\mathrm{triv},F}/K) \leq p^{2g-1}\cdot \sum_{\theta=1}^{2p-1} \frac{1}{\sin^{2g-2}\left(\frac{\pi\cdot\theta}{2\cdot p}\right)}. \tag{37}$$

Proof. Denote by W the ring of Witt vectors with coefficients in K and L the fraction field of W. Since $\dim(X_K^{(1)})=1$, which implies $H^2(X_K^{(1)}, \Omega^\vee_{X_K^{(1)}/K})=0$, it follows from well-known generalities on deformation theory that $X_K^{(1)}$ may be lifted to a smooth projective curve $X_W^{(1)}$ over W of genus g. In a similar vein, the equality $H^2(X_K^{(1)}, \mathcal{E}nd_{\mathcal{O}_{X_K^{(1)}}}(F_*(\mathcal{E})))=0$ implies that $F_*(\mathcal{E})$ may be lifted to a vector bundle $\mathcal{V}_W$ on $X_W^{(1)}$. Now let v be a K-rational point of $\mathcal{Q}^{\mathrm{triv},F}$, which classifies an injection $s:\mathcal{F}\hookrightarrow F_*(\mathcal{E})$. By Proposition 2.3.5, the tangent space of $\mathcal{Q}^{\mathrm{triv},F}$ at v may be identified with the tangent space of $\mathcal{Q}$ at the same point, so it is isomorphic to the K-vector space $\mathrm{Hom}_{\mathcal{O}_{X_K^{(1)}}}(\mathcal{F}, \mathrm{Coker}(s))$. Also, the obstruction to lifting v to any first order thickening of $\mathrm{Spec}(K)$ is given by an element of $\mathrm{Ext}^1_{\mathcal{O}_{X_K^{(1)}}}(\mathcal{F}, \mathrm{Coker}(s))$. On the other hand, the étaleness of $\mathcal{Q}^{\mathrm{triv},F}/K$ (cf. Proposition 2.4.1) implies the equality $\mathrm{Hom}_{\mathcal{O}_{X^{(1)}}}(\mathcal{F}, \mathrm{Coker}(s))=0$, and hence, we have $\mathrm{Ext}^1_{\mathcal{O}_{X^{(1)}}}(\mathcal{F}, \mathrm{Coker}(s))=0$ by Lemma 3.2.2 below. Thus, it follows that v may be lifted uniquely to a W-rational point of $\mathcal{Q}uot^{2,0}_{\mathcal{V}_W/X_W^{(1)}/W}$. In particular, there exists an open and closed subscheme $\mathcal{Q}_W^{\mathrm{triv},F}$ of $\mathcal{Q}uot^{2,0}_{\mathcal{V}_W/X_W^{(1)}/W}$ whose special fiber coincides with $\mathcal{Q}^{\mathrm{triv},F}$. Here, it follows from a routine argument that L may be supposed to be a subfield of $\mathbb{C}$. Write $X_{\mathbb{C}}^{(1)}$ for the base-change of $X_W^{(1)}$ via the morphism $\mathrm{Spec}(\mathbb{C})\to\mathrm{Spec}(W)$ induced by the composite embedding $W\hookrightarrow L\hookrightarrow\mathbb{C}$, and $\mathcal{V}_{\mathbb{C}}$ for the pull-back of $\mathcal{V}_W$ via the natural morphism $X_{\mathbb{C}}^{(1)}\to X_W^{(1)}$. Then the degree of $\mathcal{V}_{\mathbb{C}}$ coincides with the degree of $F_*(\mathcal{E})$, so $\mathcal{V}_{\mathbb{C}}$ is a vector bundle of degree $\deg(\mathcal{V}_{\mathbb{C}})=2\cdot(p-1)(g-1)$ (cf. the proof of Lemma 3.2.2 below). Since $F_*(\mathcal{E})$ is stable (cf. [25, Theorem 2.2]), one may verify from the definition of stability and the properness of Quot schemes (cf. [4, Theorem 5.14]) that $\mathcal{V}_{\mathbb{C}}$ is a stable vector bundle. By the former assertion of Lemma 2.3.1, together with the properness of $\mathcal{Q}_W^{\mathrm{triv},F}/W$, $\mathcal{Q}uot^{2,0}_{\mathcal{V}_{\mathbb{C}}/X_{\mathbb{C}}^{(1)}/\mathbb{C}}$ classifies maximal subbundles of $\mathcal{V}_{\mathbb{C}}$. One may assume, without loss of generality, that the deformation $\mathcal{V}_{\mathbb{C}}$ is sufficiently general in $\mathrm{U}^{2p,2(p-1)(g-1)}_{X_{\mathbb{C}}^{(1)}}$ so that the dimension of any component in $\mathcal{Q}uot^{2,0}_{\mathcal{V}_{\mathbb{C}}/X_{\mathbb{C}}^{(1)}/\mathbb{C}}$ is the same (cf. § 3.1). In particular, if we write $\mathcal{Q}_{\mathbb{C}}^{\mathrm{triv},F}:=\mathcal{Q}_W^{\mathrm{triv},F}\times_W\mathbb{C}$, then the finiteness of $\mathcal{Q}^{\mathrm{triv},F}/K$ implies that $\mathcal{Q}_{\mathbb{C}}^{\mathrm{triv},F}$, hence also $\mathcal{Q}uot^{2,0}_{\mathcal{V}_{\mathbb{C}}/X_{\mathbb{C}}^{(1)}/\mathbb{C}}$, is 0-dimensional. Thus, we have

$$\deg(\mathcal{Q}^{\mathrm{triv},F}/K)=\deg(\mathcal{Q}_W^{\mathrm{triv},F}/W)=\deg(\mathcal{Q}_{\mathbb{C}}^{\mathrm{triv},F}/\mathbb{C})\leq \deg(\mathcal{Q}uot^{2,0}_{\mathcal{V}_{\mathbb{C}}/X_{\mathbb{C}}^{(1)}/\mathbb{C}}/\mathbb{C}). \tag{38}$$

If, moreover, (a,b) is the unique pair of integers satisfying $\deg(\mathcal{V}_{\mathbb{C}})=2p\cdot a-b$ with $0\leq b<2p$, then it follows from the hypothesis $p+1>g$ that $a=g-1$ and $b=2(g-1)$. Thus, since $\mathcal{V}_{\mathbb{C}}$ is assumed to be general, we can apply Theorem 3.1.1 in the case where the data "$(C,\mathcal{G},n,d,r,a,b,e_{\max}(\mathcal{G},r))$" is taken to be

$$(X_{\mathbb{C}}^{(1)}, \mathcal{V}_{\mathbb{C}}, 2p, 2(p-1)(g-1), 2, g-1, 2(g-1), 0) \tag{39}$$

and obtain the following sequence of equalities

$$
\begin{aligned}
(40)\qquad & \deg_{\mathbb{C}}(\mathcal{Q}uot^{2,0}_{\mathcal{V}_{\mathbb{C}}/X^{(1)}_{\mathbb{C}}/\mathbb{C}}) \\
&= \frac{(-1)^{(2-1)(2(g-1)2-(g-1)2^2)/2p}(2p)^{2(g-1)}}{2!} \cdot \sum_{\rho_1,\rho_2} \frac{\prod_{i=1}^2 \rho_i^{2(g-1)-g+1}}{\prod_{i\neq j}(\rho_i-\rho_j)^{g-1}} \\
&= (-1)^{g-1}\cdot 2^{2g-2}\cdot p^{2g-1}\cdot \sum_{\zeta^{2p}=1,\zeta\neq 1} \frac{\zeta^{g-1}}{(\zeta-1)^{2g-2}} \\
&= 2^{g-1}\cdot p^{2g-1}\cdot \sum_{\zeta^{2p}=1,\zeta\neq 1} \frac{1}{(1-\frac{\zeta+\zeta^{-1}}{2})^{g-1}} \\
&= p^{2g-1}\cdot \sum_{\theta=1}^{2p-1} \frac{1}{\sin^{2g-2}(\frac{\pi\cdot\theta}{2\cdot p})}.
\end{aligned}
$$

Thus, the assertion follows from (38) and (40). □

The following lemma was applied in the proof of the previous proposition.

Lemma 3.2.2. *Let $s : \mathcal{F} \hookrightarrow F_*(\mathcal{E})$ be an injection classified by a K-rational point of $\mathcal{Q}^{\mathrm{triv},F}$. Write $\mathcal{G} := \mathrm{Coker}(s)$. Then $\mathcal{G}$ is a vector bundle on $X^{(1)}_K$, and the following equality holds:*

$$
(41)\qquad \dim_K(\mathrm{Hom}_{\mathcal{O}_{X^{(1)}_K}}(\mathcal{F},\mathcal{G})) = \dim_K(\mathrm{Ext}^1_{\mathcal{O}_{X^{(1)}_K}}(\mathcal{F},\mathcal{G})).
$$

Proof. First, we verify that $\mathcal{G}$ is a vector bundle. Recall (cf. Lemma 2.3.1) that the composite

$$
(42)\qquad F^*(\mathcal{F}) \xrightarrow{F^*(s)} F^*(F_*(\mathcal{E})) \xrightarrow{\mathrm{ad}^{-1}(\mathrm{id}_{F_*(\mathcal{E})})} \mathcal{E}
$$

is an isomorphism. Hence, the composite $\mathrm{Ker}(\mathrm{ad}^{-1}(\mathrm{id}_{F_*(\mathcal{E})})) \hookrightarrow F^*(F_*(\mathcal{E})) \twoheadrightarrow F^*(\mathcal{G})$ is an isomorphism, so $F^*(\mathcal{G})$ is a vector bundle. By the faithful flatness of F, $\mathcal{G}$ turns out to be a vector bundle on $X^{(1)}_K$, as desired.

Next we shall prove (41). Since F is finite, we have an equality of Euler characteristics $\chi(F_*(\mathcal{E})) = \chi(\mathcal{E}) = 2(1-g)$. Since $\mathrm{rk}(\mathcal{H}om_{\mathcal{O}_{X^{(1)}_K}}(\mathcal{F},\mathcal{G})) = 2\cdot(2p-2)$, it follows from the Riemann-Roch theorem that

$$
(43)\qquad \deg(F_*(\mathcal{E})) = \chi(F_*(\mathcal{E})) - \mathrm{rk}(F_*(\mathcal{E}))(1-g) = \chi(\mathcal{E}) - 2p\cdot(1-g) = 2\cdot(p-1)(g-1),
$$

and that

$$
\begin{aligned}
(44)\qquad \deg(\mathcal{H}om_{\mathcal{O}_{X^{(1)}_K}}(\mathcal{F},\mathcal{G})) &= 2\cdot\deg(\mathcal{G}) - (2p-2)\cdot\deg(\mathcal{F}) \\
&= 2\cdot\deg(F_*(\mathcal{E})) - 0 \\
&= 4\cdot(p-1)(g-1).
\end{aligned}
$$

Finally, by applying the Riemann-Roch theorem again, we obtain

$$
\begin{aligned}
(45)\qquad & \dim_K(\mathrm{Hom}_{\mathcal{O}_{X^{(1)}_K}}(\mathcal{F},\mathcal{G})) - \dim_K(\mathrm{Ext}^1_{\mathcal{O}_{X^{(1)}_K}}(\mathcal{F},\mathcal{G})) \\
&= \deg(\mathcal{H}om_{\mathcal{O}_{X^{(1)}_K}}(\mathcal{F},\mathcal{G})) + \mathrm{rk}(\mathcal{H}om_{\mathcal{O}_{X^{(1)}_K}}(\mathcal{F},\mathcal{G}))(1-g) \\
&= 4\cdot(p-1)(g-1) + 2\cdot(2p-2)(1-g) \\
&= 0,
\end{aligned}
$$

thus completing the proof of this lemma. □

By applying the results obtained so far, we conclude the main result of the present paper.

Theorem 3.2.3 (= Theorem A). *Let X be a smooth projective curve over k of genus g with $p+1 > g > 1$ and $p \neq 2$. Then, the following inequality holds:*

$$\deg(\mathrm{Ver}^2_{X/k}) \leq p^{g-1} \cdot \sum_{\theta=1}^{2p-1} \frac{1}{\sin^{2g-2}(\frac{\pi\cdot\theta}{2\cdot p})} \quad \left(= \sum_{\zeta^{2p}=1, \zeta\neq 1} \frac{(-4p\zeta)^{g-1}}{(\zeta-1)^{2g-2}} \right). \tag{46}$$

Proof. By the italicized comment at the end of § 1, one may assume, without loss of generality, that X is sufficiently general for which the above discussions work. Then, by the discussion at the beginning of § 3 and Proposition 3.2.1, we have

$$\deg(\mathrm{Ver}^2_{X/k}) = \frac{1}{p^g} \cdot \deg(\mathcal{Q}^{\mathrm{triv},F}/K) \leq p^{g-1} \cdot \sum_{\theta=1}^{2p-1} \frac{1}{\sin^{2g-2}\left(\frac{\pi\cdot\theta}{2\cdot p}\right)}. \tag{47}$$

This completes the proof of the theorem. □

Similarly to the discussion in [27, § 6.2, (2)], we can describe the right-hand side of (46) as a polynomial with respect to p of degree $3g-3$. For example, (46) reads

$$\deg(\mathrm{Ver}^2_{X/k}) \leq \frac{4p^3 - p}{3} \text{ if } g = 2, \quad \text{and} \quad \deg(\mathrm{Ver}^2_{X/k}) \leq \frac{16p^6 + 40p^4 - 11p^2}{45} \text{ if } g = 3. \tag{48}$$

By comparing with the explicit computation of $\deg(\mathrm{Ver}^2_{X/k})$ for $g = 2$ obtained already (cf. Introduction), we see that (46) is not optimal. In particular, (by considering the discussion in the proof of Proposition 3.2.1) the K-scheme $\mathcal{R}$ $\left(= \mathcal{Q} \setminus \mathcal{Q}^{\mathrm{triv},F}\right)$ in Proposition 2.3.5 for any general genus-2 curve turns out to be nonempty. Hence, the comment in Remark 2.3.3 implies the following assertion.

Corollary 3.2.4. *Let X be a smooth projective curve over k of genus 2 and $\eta : \mathrm{Spec}(K) \to \mathrm{SU}^2_{X/k}$ (where K denotes an algebraically closed field over k) a geometric generic point of $\mathrm{SU}^2_{X/k}$. Denote by $\mathcal{E}$ the rank 2 stable bundle on $X_K := X \times_k K$ classified by η and by F the relative Frobenius morphism $X_K \to X_K^{(1)}$ of X_K over K. Suppose that X is general in $\mathcal{M}_2$. Then, the direct image $F_*(\mathcal{E})$ of $\mathcal{E}$ via F admits an $\mathcal{O}_{X_K^{(1)}}$-submodule $\mathcal{F}$ which is a rank 2 vector bundle of degree 0 and whose determinant $\det(\mathcal{F})$ does not descend to a line bundle on X.*

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References

[1] H. Brenner, A. Kaid, On deep Frobenius descent and flat bundles, *Math. Res. Lett.* **15** (2008), pp. 1101-1115.

[2] P. Deligne, D. Mumford, The irreducibility of the space of curves of given genus, *Publ. Math. I.H.E.S.* **36** (1969), pp. 75-110.

[3] L. Ducrohet, V. Mehta, Density of vector bundles periodic under the action of Frobenius, *Bull. Sci. Math.* **134** (2010), pp. 454-460.
[4] B. Fantechi, L. Güttsche, L. Illusie, S. Kleiman, N. Nitsure, A. Vistoli, Angelo, *Fundamental algebraic geometry. Grothendieck's FGA explained,* Mathematical Surveys and Monographs, **123** AMS (2005).
[5] D. Gieseker, Stable curves and special divisors: Petri's conjecture, *Invent. Math.* **66** (1982), pp. 251-275.
[6] A. Hirschowitz, Problémes de Brill-Noether en rang supérieur, *C. R. Math. Acad. Sci. Paris*, **338** (2004), pp. 869-879.
[7] Y. Holla, Counting maximal subbundles via Gromov-Witten invariants, *Math. Ann.* **328** (2004), pp. 121-133.
[8] D. Huybrechts, M. Lehn, *The geometry of moduli spaces of sheaves*, Aspects of Mathematics, **31** Friedr. Vieweg & Sohn, Braunschweig, (1997).
[9] K. Joshi, C. Pauly, Hitchin-Mochizuki morphism, opers and Frobenius-destabilized vector bundles over curves, *Adv. Math.* **274** (2015), pp. 39-75.
[10] K. Joshi, S. Ramanan, E. Z. Xia, J. K. Yu, On vector bundles destabilized by Frobenius pull-back, *Composito Mathematica* **142** (2006), pp. 616-630.
[11] N. M. Katz, Nilpotent connections and the monodromy theorem: Applications of a result of Turrittin, *Publ. Math. I.H.E.S.* **39** (1970), pp. 175-232.
[12] J. T. A. Lang, *Relative moduli spaces of semi-stable sheaves on families of curves*, Herbert Utz Verlag, Muenchen, (2001).
[13] H. Lange, P. Newstead, Maximal subbundles and Gromov-Witten invariants, *A Tribute to C. S. Seshadri*, Trend Math. Birkhäuser, Basel (2003), pp. 310-322.
[14] Y. Laszlo, A non-trivial family of bundles fixed by the square of Frobenius, *C. R. Math. Acad. Sci. Paris* **333** (2001), pp. 651-656.
[15] Y. Laszlo, C. Pauly, The action of the Frobenius maps on rank 2 vector bundles in characteristic 2, *Journal of Algebraic Geometry* **11** (2002), pp. 129-143.
[16] Y. Laszlo, C. Pauly, The Frobenius map, rank 2 vector bundles and Kummer's quartic surface in characteristic 2 and 3, *Advances in Mathematics* **185** (2004), pp. 246-269.
[17] H. Lange, M.S. Narasimhan, Maximal subbundles of rank two vector bundles on curves, *Math. Ann.* **266**, (1983), pp. 55-72.
[18] H. Lange, C. Pauly, On Frobenius-destabilized rank-2 vector bundles over curves, *Comment. Math. Helv.* **83** (2008), pp. 179-209.
[19] H. Lange, U. Stuhler, Vektorbündel auf Kurven und Darstellungen der algebraischen Fundamentalgruppe, *Math. Z.* **156** (1977), pp. 73-83.
[20] A. Langer, Semistable sheaves in positive characteristic, *Ann. Math.* **159** (2004), pp. 251-276.
[21] R. Lazarsfeld, Brill-Noether-Petri without degenerations, *J. Diff. Geometry* **23** (1986), pp. 299-307.
[22] V. B. Mehta, S. Subramanian, Nef line bundles which are not ample, *Math. Zeit.* **219** (1995), pp. 235-244.
[23] B. Osserman, The generalized Verschiebung map for curves of genus 2, *Math. Ann.* **336** (2006), pp. 963-986.
[24] B. Osserman, Frobenius-unstable bundles and p-curvature, *Transactions of the Amer. Math. Soc.* **360** (2008), pp. 273-305.
[25] X. Sun, Direct images of bundles under Frobenius morphism, *Invent. Math.* **173** (2008), pp. 427-447.
[26] Y. Yang, The fixed point locus of the Verschiebung on $\mathcal{M}_X(2,0)$ for genus-2 curves X in characteristic 2, *Canadian Math. Bulletin* **57** (2014), pp. 439-448.
[27] Y. Wakabayashi, An explicit formula for the generic number of dormant indigenous bundles, *Publ. Res. Inst. Math. Sci.* **50** (2014), pp. 383-409.

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