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The 3-loop polynomial of knots obtained by plumbing the doubles of two knots

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Abstract

The 3-loop invariant (or, the 3-loop polynomial) of a knot is a rational form (or, a polynomial) presenting the 3-loop part of the Kontsevich invariant of knots. In this paper, we calculate the 3-loop polynomial of knots obtained by plumbing the doubles of two knots; this class of knots includes untwisted Whitehead doubles. We construct the 3-loop invariant by calculating the rational version of the Aarhus integral of a surgery presentation. As a consequence, we obtain an explicit presentation of the 3-loop polynomial for the knots.

1 Introduction

The Kontsevich invariant of knots is a very strong invariant of knots, which is universal to all quantum invariants and all Vassiliev invariants, and it is expected that the Kontsevich invariant classifies all knots. The Kontsevich invariant takes its value in the space of Jacobi diagrams. Jacobi diagrams are some kinds of uni-trivalent graphs, and they have universal properties among the pairs $(\mathfrak{g}, V)$, where $\mathfrak{g}$ is a simple Lie algebra and V is its representation. So we can "substitute" any pair of $(\mathfrak{g}, V)$ into Jacobi diagrams, and it is the reason why the Kontsevich invariant is universal to all quantum invariants. In addition, each term of the Kontsevich invariant is a Vassiliev invariant, and we can calculate it algorithmically. However, since the value of the Kontsevich invariant is presented by an infinite sum of Jacobi diagrams, it is difficult to determine all terms of Kontsevich invariant at the same time concretely. So far, a powerful method to present them is not known.

One approach to restrict the image of the Kontsevich invariant is the "loop expansion". It is conjectured in [21] that the Kontsevich invariant of a knot is expanded in the form of the loop expansion, and it is shown in [10] that the Kontsevich invariant of a knot can be expanded in this form, and it is shown in [7] that the loop expansion is a knot invariant. It is calculated by using the rational version of the Aarhus integral of a surgery presentation. A n-loop diagram is a connected open Jacobi diagram whose first Betti number is n. When we fix a loop number, each loop part is presented by some polynomials, and in particular, the 1-loop part is presented by the Alexander polynomial. The polynomial presenting the 2-loop part is called the 2-loop polynomial. The 2-loop polynomial is a 2-variable polynomial invariant of knots.

The 2-loop polynomial is calculated in many cases. For example, the 2-loop polynomial for knots with up to 7 crossings is calculated in [21], for torus knots in [11], [12], [16], for untwisted Whitehead doubles in [9], and for genus 1 knots in [17]. On the other hand, there are few examples of the calculation of the 3-loop part. The 3-loop part of the Kontsevich invariant of torus knots are calculated in [11], [12], where a cabling formula for the Kontsevich invariant is used. However, it is not easy to calculate the 3-loop part of the Kontsevich invariant in general, so the 3-loop part of other knots are not calculated so far. We can calculate the 3-loop part in the same way as the calculation of the 2-loop polynomial theoretically, but the calculation of the 3-loop part is more complicated than that of the 2-loop part.

In this paper, we formulate the 3-loop invariant (or, the 3-loop polynomial) presenting the 3-loop part of the Kontsevich invariant, we give some examples of the 3-loop polynomial of knots. More concretely, in Theorem 3.1, we calculate the 3-loop polynomial of $D(K, K')$, which are knots obtained by plumbing the doubles of two knots K (with framing 0) and K' (with framing k); this class of knots includes untwisted Whitehead doubles (Corollary 3.3). The 3-loop polynomial of these knots are relatively easy to calculate, and this theorem is one of the few examples of the calculation of the 3-loop part of knots. We can get the loop expansion of a knot by calculating the rational version of the Aarhus integral of a surgery presentation of the knot, so we construct the 3-loop polynomial of $D(K, K')$ in such a way, and we can show that its 3-loop polynomial is only depend on Vassiliev invariants of K and K' up to degree 4. As a consequence, we obtain an explicit presentation of the 3-loop polynomial of $D(K, K')$ by using Vassiliev invariants of K and K' up to degree 4. When we consider the $\mathfrak{sl}_2$ reduction of the Kontsevich invariant of knots, we can get the colored Jones polynomial of knots. In Proposition 7.2, we calculate the 3-loop part of the colored Jones polynomial. In addition, by considering the Duflo isomorphism, we can get the connected sum formula for the 3-loop invariant of any knots.

This paper is organized as follows. In Section 2, we review the Kontsevich invariant and its loop expansion. In addition, we define the 3-loop invariant (or, the 3-loop polynomial) of a knot, and we define knots $D(K, K')$, which are obtained by plumbing the doubles of two knots. In Section 3, we state the main theorem of this paper; we present the 3-loop polynomial of $D(K, K')$. As its corollary, we present the 3-loop polynomial of untwisted Whitehead doubles of knots. In Section 4, we state some properties of the 3-loop part of the Kontsevich invariant without proofs, and we state the connected sum formula for the 3-loop invariant. The proof of the connected sum formula for the 3-loop invariant is given in Appendix. In Section 5, for the proof of the main theorem, we review the rational version of the Aarhus integral. In Section 6, we prove the main theorem. In Section 7, we calculate the 3-loop part of the colored Jones polynomial. Other topics are mentioned in Appendix.

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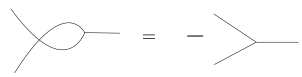
2 The Kontsevich invariant and its loop expansion

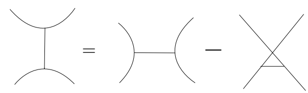
In this section, we review the Kontsevich invariant and we define the 3-loop invariant of knots. For details, see [14],[15].

Let X be an oriented compact manifold. A *Jacobi diagram* on X is an uni-trivalent graph such that univalent vertices are distinct points of X, and a cyclic order of the three edges around each trivalent vertex is fixed, in other words, each trivalent vertex is *vertex-oriented*. When we draw a Jacobi diagram on X, we draw X by thick lines and uni-trivalent graphs by thin lines, and each trivalent vertex is vertex-oriented in the counterclockwise order. Furthermore, we define the *degree* of a Jacobi diagram to be half the number of all vertices of the graph of the Jacobi diagram. We define $\mathcal{A}(X)$ to be the quotient vector space spanned by Jacobi diagrams on X subject to *the AS, IHX, and STU relations*.

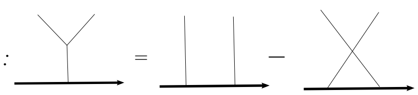
the AS relation :



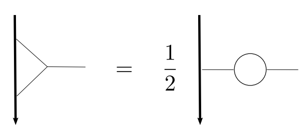
the IHX relation :

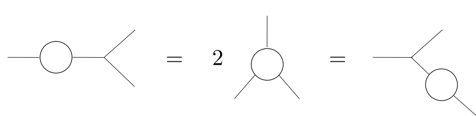


the STU relation :

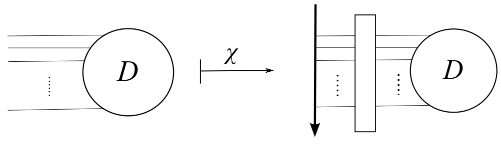


Note that we get some equations by the above relations;

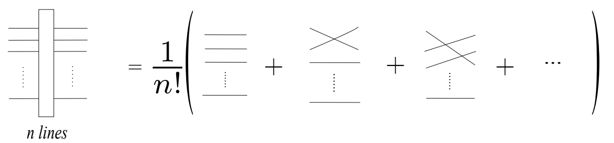




It is known, see [14],[15], that $\mathcal{A}(S^1)$ forms a commutative algebra whose product is given by connected sum of copies of S^1, and $\mathcal{A}(\downarrow)$ also forms a commutative algebra whose product is given by connecting copies of $\downarrow$. We can see that $\mathcal{A}(S^1)$ and $\mathcal{A}(\downarrow)$ are naturally isomorphic as commutative algebras by the isomorphism given by connecting two end points of $\downarrow$. An *open Jacobi diagram* is a vertex-oriented uni-trivalent graphs. We call outward pointing edges that end in a univalent vertex a leg. We define $\mathcal{B}$ to be the quotient vector space spanned by Jacobi diagrams subject to the AS, IHX relations. $\mathcal{B}$ forms a commutative algebra whose product is given by disjoint union. The *PBW isomorphism* $\chi : \mathcal{B} \to \mathcal{A}(\downarrow)$ is defined by

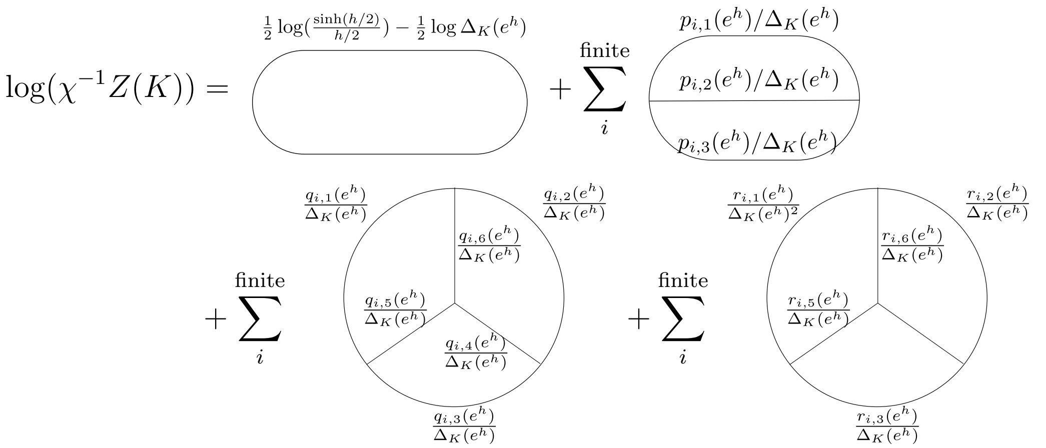


for any diagram $D \in \mathcal{B}$, where the box means the symmetrizer,



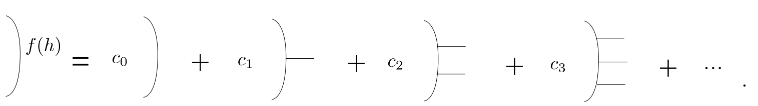
Note that PBW isomorphism is not an algebra isomorphism.

The *Kontsevich invariant* $Z(K)$ of a knot K is defined to be in $\mathcal{A}(S^1)(\cong \mathcal{A}(\downarrow))$; for details, see [14],[15]. Note that $Z(K)$ and $\chi^{-1}Z(K)$ are group-like, which means that they are exponentials of series of connected diagrams. The loop expansion of the Kontsevich invariant of knot K is a presentation of the following form ([7],[10],[14]),

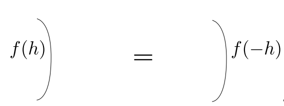


+ (terms of (> 3)-loop part),

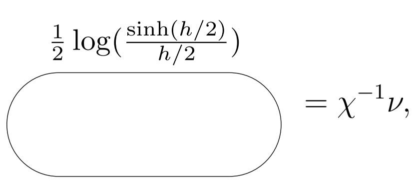
where $\Delta_K(t)$ denotes the Alexander polynomial, and $p_{i,j}(e^h), q_{i,j}(e^h), r_{i,j}(e^h)$ are polynomials in $e^{\pm h}$. For the 3-loop part of the Kontsevich invariant, see Section 4. Here, a labeling of $f(h) = c_0 + c_1 h + c_2 h^2 + c_3 h^3 + \cdots$ implies that ,



Note that



by the AS relation. Further, we note that



where we denote $\nu = Z(\text{unknot}) \in \mathcal{A}(S^1)$. Then, we define the *3-loop invariant* of K by

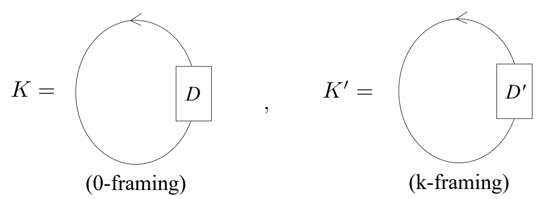
$$\begin{aligned}
&\Lambda_K(t_1, t_2, t_3, t_4) \\
&= \sum_{\substack{i \\ \tau \in \mathfrak{S}_4}} \frac{q_{i,1}(t_{\tau(1)}^{\mathrm{sgn}\tau} t_{\tau(4)}^{-\mathrm{sgn}\tau}) q_{i,2}(t_{\tau(2)}^{\mathrm{sgn}\tau} t_{\tau(4)}^{-\mathrm{sgn}\tau}) q_{i,3}(t_{\tau(3)}^{\mathrm{sgn}\tau} t_{\tau(4)}^{-\mathrm{sgn}\tau}) q_{i,4}(t_{\tau(2)}^{\mathrm{sgn}\tau} t_{\tau(3)}^{-\mathrm{sgn}\tau}) q_{i,5}(t_{\tau(3)}^{\mathrm{sgn}\tau} t_{\tau(1)}^{-\mathrm{sgn}\tau}) q_{i,6}(t_{\tau(1)}^{\mathrm{sgn}\tau} t_{\tau(2)}^{-\mathrm{sgn}\tau})}{\Delta_K(t_1 t_4^{-1}) \Delta_K(t_2 t_4^{-1}) \Delta_K(t_3 t_4^{-1}) \Delta_K(t_2 t_3^{-1}) \Delta_K(t_3 t_1^{-1}) \Delta_K(t_1 t_2^{-1})} \\
&+ \sum_{\substack{i \\ \tau \in \mathfrak{S}_4}} \frac{r_{i,1}(t_{\tau(1)}^{\mathrm{sgn}\tau} t_{\tau(4)}^{-\mathrm{sgn}\tau}) r_{i,2}(t_{\tau(2)}^{\mathrm{sgn}\tau} t_{\tau(4)}^{-\mathrm{sgn}\tau}) r_{i,3}(t_{\tau(3)}^{\mathrm{sgn}\tau} t_{\tau(4)}^{-\mathrm{sgn}\tau}) r_{i,5}(t_{\tau(3)}^{\mathrm{sgn}\tau} t_{\tau(1)}^{-\mathrm{sgn}\tau}) r_{i,6}(t_{\tau(1)}^{\mathrm{sgn}\tau} t_{\tau(2)}^{-\mathrm{sgn}\tau})}{\Delta_K(t_{\tau(1)} t_{\tau(4)}^{-1})^2 \Delta_K(t_{\tau(2)} t_{\tau(4)}^{-1}) \Delta_K(t_{\tau(3)} t_{\tau(4)}^{-1}) \Delta_K(t_{\tau(3)} t_{\tau(1)}^{-1}) \Delta_K(t_{\tau(1)} t_{\tau(2)}^{-1})} \\
&\quad \in \mathbb{Q}(t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}, t_4^{\pm 1})^1 / (\mathfrak{S}_4, t_1 t_2 t_3 t_4 = 1),
\end{aligned}$$

where $\mathbb{Q}(t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}, t_4^{\pm 1})^1$ is the ring of rational forms $\dfrac{v(t_1, t_2, t_3, t_4)}{u(t_1, t_2, t_3, t_4)}$ such that $u(1,1,1,1) = 1$. In particular, if $\Delta_K(t) = 1$, then $\Lambda_K(t_1, t_2, t_3, t_4)$ is a polynomial, so in this case, we call it the *3-loop polynomial*. For details about the 3-loop part of the Kontsevich invariant, see Section 4.1.

3 The 3-loop polynomial of $D(K, K')$

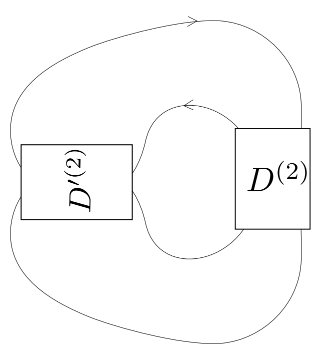
In this section, we define $D(K, K')$, which are genus 1 knots with trivial Alexander polynomial, and we state the main theorem of this paper.

Let K be a 0-framed knot, and let K' be a k-framed knot ($k \in \mathbb{Z}$). Let D, D' be 1-tangles whose closures are K, K', respectively, noting that isotopy classes of D and D' are uniquely determined by K and K'.



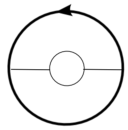
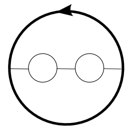
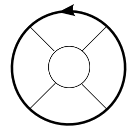
(1)

We define $D(K, K')$ to be the following knot,

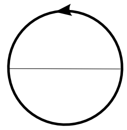
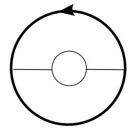


where $D^{(2)}$ and $D'^{(2)}$ are the doubles of D and D', respectively. We can obtain $D(K, K')$ by *plumbing* of the doubles of K and K', noting that $D(K, K')$ is a genus 1 knot with trivial Alexander polynomial.

The Kontsevich invariants of K and K' can be presented by

$$Z(K) = \nu \# \exp \Big(a_2$$

$$+ a_3$$

$$+ a_4$$


$$+ (\text{linear sum of diagrams with more than 4 trivalent vertices}) \Big) \in \mathcal{A}(S^1), \quad (2)$$

$$Z(K') = \nu \# \exp \Big(\frac{k}{2}$$

$$+ a'_2$$

$$+ \left(\begin{array}{l} \text{linear sum of diagrams} \\ \text{with more than} \\ \text{2 trivalent vertices} \end{array} \right) \Big) \in \mathcal{A}(S^1), \quad (3)$$

where a_i is a degree i Vassiliev invariant of K, and a'_i is a degree i Vassiliev invariant of K'.

Now, we state the main theorem of this paper. We put $u_{m,n} = t_m t_n^{-1} + t_m^{-1} t_n - 2$ and $v_{m,n} = t_m t_n^{-1} - t_m^{-1} t_n \qquad (m, n \in \{1, 2, 3, 4\})$

Theorem 3.1. *Let K and K' be knots as shown in (1), and assume that their Kontsevich invariants are presented as in (2), (3). Then, the 3-loop polynomial of $D(K, K')$ is presented by*

$$\begin{aligned}
&\Lambda_{D(K,K')}(t_1, t_2, t_3, t_4) \\
&= (-16 a_2 a'_2 - k^2 a_2 - 8 k a_3)(u_{1,2} + u_{1,3} + u_{1,4} + u_{2,3} + u_{2,4} + u_{3,4}) \\
&\quad + (-\frac{k^2 a_2}{12} + 4 k^2 a_4)(u_{1,4} u_{2,4} + u_{1,4} u_{3,4} + u_{2,4} u_{3,4} + u_{1,3} u_{2,3} + u_{1,3} u_{4,3} + u_{2,3} u_{4,3} \\
&\qquad + u_{1,2} u_{3,2} + u_{1,2} u_{4,2} + u_{3,2} u_{4,2} + u_{2,1} u_{3,1} + u_{2,1} u_{4,1} + u_{3,1} u_{4,1}) \\
&\quad + 24 k^2 a_4 (u_{1,2} u_{3,4} + u_{1,3} u_{2,4} + u_{1,4} u_{2,3}) \\
&\quad + 8 k^2 a_2^2 (u_{1,2}^2 + u_{1,3}^2 + u_{1,4}^2 + u_{2,3}^2 + u_{2,4}^2 + u_{3,4}^2) \\
&\quad - \frac{k^2 a_2}{4} (v_{1,4} v_{2,4} + v_{1,4} v_{3,4} + v_{2,4} v_{3,4} + v_{1,3} v_{2,3} + v_{1,3} v_{4,3} + v_{2,3} v_{4,3} \\
&\qquad + v_{1,2} v_{3,2} + v_{1,2} v_{4,2} + v_{3,2} v_{4,2} + v_{2,1} v_{3,1} + v_{2,1} v_{4,1} + v_{3,1} v_{4,1}) \\
&\quad \in \mathbb{Q}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}, t_4^{\pm 1}] / (\mathfrak{S}_4, t_1 t_2 t_3 t_4 = 1).
\end{aligned}$$

We prove the theorem in Section 6.

We put

$$\begin{aligned}
T_{1,3} &= t_1 t_2^3 + t_1 t_3^3 + t_2 t_1^3 + t_2 t_3^3 + t_3 t_1^3 + t_3 t_2^3 \\
&\quad + t_1^2 t_2^{-1} t_3^{-1} + t_2^2 t_1^{-1} t_3^{-1} + t_3^2 t_1^{-1} t_2^{-1} + t_1^{-2} t_2^{-3} t_3^{-3} + t_2^{-2} t_1^{-3} t_3^{-3} + t_3^{-2} t_1^{-3} t_2^{-3}, \\
T_{2,2} &= t_1^2 t_2^2 + t_2^2 t_3^2 + t_3^2 t_1^2 + t_1^{-2} t_2^{-2} + t_2^{-2} t_3^{-2} + t_3^{-2} t_1^{-2}, \\
T_{1,1,2} &= t_1 t_2 t_3^2 + t_2 t_3 t_1^2 + t_3 t_1 t_2^2 + t_1^{-1} t_2^{-1} t_3^{-2} + t_2^{-1} t_3^{-1} t_1^{-2} + t_3^{-1} t_1^{-1} t_2^{-2} \\
&\quad + t_1 t_2^{-1} + t_1 t_3^{-1} + t_2 t_1^{-1} + t_2 t_3^{-1} + t_3 t_1^{-1} + t_3 t_2^{-1},
\end{aligned}$$

$$
\begin{aligned}
T_{2,2,4} &= t_1^2t_2^2t_3^4 + t_2^2t_3^2t_1^4 + t_3^2t_1^2t_2^4 + t_1^{-2}t_2^{-2}t_3^{-4} + t_2^{-2}t_3^{-2}t_1^{-4} + t_3^{-2}t_1^{-2}t_2^{-4} \\
&\quad + t_1^2t_2^{-2} + t_1^2t_3^{-2} + t_2^2t_1^{-2} + t_2^2t_3^{-2} + t_3^2t_1^{-2} + t_3^2t_2^{-2}, \\
T_{2,3,3} &= t_1^2t_2^3t_3^3 + t_2^2t_3^3t_1^3 + t_3^2t_1^3t_2^3 + t_1t_2t_3^{-2} + t_2t_3t_1^{-2} + t_3t_1t_2^{-2} \\
&\quad + t_1^{-1}t_2^{-3} + t_1^{-1}t_3^{-3} + t_2^{-1}t_1^{-3} + t_2^{-1}t_3^{-3} + t_3^{-1}t_1^{-3} + t_3^{-1}t_2^{-3}.
\end{aligned}
$$

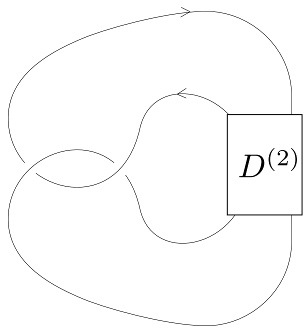
Then,

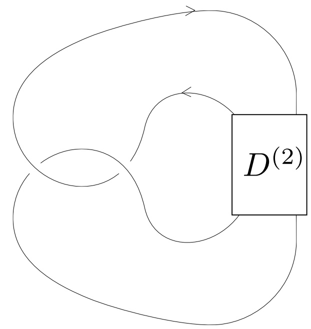
$$
\begin{aligned}
&u_{1,2} + u_{1,3} + u_{1,4} + u_{2,3} + u_{2,4} + u_{3,4} = T_{1,1,2} - 6, \\
&u_{1,4}u_{2,4} + u_{1,4}u_{3,4} + u_{2,4}u_{3,4} + u_{1,3}u_{2,3} + u_{1,3}u_{4,3} + u_{2,3}u_{4,3} \\
&\qquad + u_{1,2}u_{3,2} + u_{1,2}u_{4,2} + u_{3,2}u_{4,2} + u_{2,1}u_{3,1} + u_{2,1}u_{4,1} + u_{3,1}u_{4,1} \\
&= T_{2,3,3} + T_{1,3} - 6T_{1,1,2} + 4, \\
&u_{1,2}u_{3,4} + u_{1,3}u_{2,4} + u_{1,4}u_{2,3} = T_{2,2} - 2T_{1,1,2} + 4, \\
&u_{1,2}^2 + u_{1,3}^2 + u_{1,4}^2 + u_{2,3}^2 + u_{2,4}^2 + u_{3,4}^2 = T_{2,2,4} - 4T_{1,1,2} + 6, \\
&v_{1,4}v_{2,4} + v_{1,4}v_{3,4} + v_{2,4}v_{3,4} + v_{1,3}v_{2,3} + v_{1,3}v_{4,3} + v_{2,3}v_{4,3} \\
&\qquad + v_{1,2}v_{3,2} + v_{1,2}v_{4,2} + v_{3,2}v_{4,2} + v_{2,1}v_{3,1} + v_{2,1}v_{4,1} + v_{3,1}v_{4,1} \\
&= T_{2,3,3} + T_{1,3} - 2T_{1,1,2}.
\end{aligned}
$$

Remark 3.2. The formula of Theorem 3.1 is rewritten,

$$
\begin{aligned}
&\Lambda_{D(K,K')}(t_1, t_2, t_3, t_4) \\
&= (-16a_2a_2' - 32k^2a_2^2 - 8ka_3 - 72k^2a_4)T_{1,1,2} + 8k^2a_2^2T_{2,2,4} \\
&\quad + (-\frac{k^2a_2}{3} + 4k^2a_4)T_{2,3,3} + (-\frac{k^2a_2}{3} + 4k^2a_4)T_{1,3} + 24k^2a_4T_{2,2} \\
&\quad - 96a_2a_2' + 48k^2a_2^2 + \frac{17k^2a_2}{3} + 48ka_3 + 112k^2a_4 \\
&\quad \in \mathbb{Q}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}, t_4^{\pm 1}]/(\mathfrak{S}_4, t_1t_2t_3t_4 = 1).
\end{aligned}
$$

In particular, we can get the 3-loop polynomial of untwisted Whitehead double of K. We denote it by $Wh^{\pm}(K)$.

$Wh^+(K) =$ 

, $Wh^-(K) =$ 

Here, D is a 1-tangle whose closure is K as shown in (1).

Corollary 3.3. *The 3-loop polynomial of $Wh^{\pm}(K)$ is presented by*

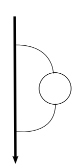
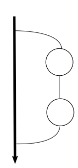
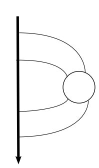
$$\begin{aligned}
&\Lambda_{Wh^{\pm}(K)}(t_1,t_2,t_3,t_4)\\
&= (-a_2 \pm 8a_3)(u_{1,2}+u_{1,3}+u_{1,4}+u_{2,3}+u_{2,4}+u_{3,4})\\
&\quad + (-\frac{a_2}{12}+4a_4)(u_{1,4}u_{2,4}+u_{1,4}u_{3,4}+u_{2,4}u_{3,4}+u_{1,3}u_{2,3}+u_{1,3}u_{4,3}+u_{2,3}u_{4,3}\\
&\qquad + u_{1,2}u_{3,2}+u_{1,2}u_{4,2}+u_{3,2}u_{4,2}+u_{2,1}u_{3,1}+u_{2,1}u_{4,1}+u_{3,1}u_{4,1})\\
&\quad + 24a_4(u_{1,2}u_{3,4}+u_{1,3}u_{2,4}+u_{1,4}u_{2,3})\\
&\quad + 8a_2^2(u_{1,2}^2+u_{1,3}^2+u_{1,4}^2+u_{2,3}^2+u_{2,4}^2+u_{3,4}^2)\\
&\quad - \frac{a_2}{4}(v_{1,4}v_{2,4}+v_{1,4}v_{3,4}+v_{2,4}v_{3,4}+v_{1,3}v_{2,3}+v_{1,3}v_{4,3}+v_{2,3}v_{4,3}\\
&\qquad + v_{1,2}v_{3,2}+v_{1,2}v_{4,2}+v_{3,2}v_{4,2}+v_{2,1}v_{3,1}+v_{2,1}v_{4,1}+v_{3,1}v_{4,1})\\
&\quad \in \mathbb{Q}[t_1^{\pm1},t_2^{\pm1},t_3^{\pm1},t_4^{\pm1}]/(\mathfrak{S}_4, t_1t_2t_3t_4=1).
\end{aligned}$$

The corollary immediately follows from Theorem 3.1.

Remark 3.4. As in Remark 3.2, the formula of Theorem 3.3 is rewritten,

$$\begin{aligned}
&\Lambda_{Wh^{\pm}(K)}(t_1,t_2,t_3,t_4)\\
&= (-32a_2^2 \pm 8a_3 - 72a_4)T_{1,1,2} + 8a_2^2 T_{2,2,4}\\
&\quad + (-\frac{a_2}{3}+4a_4)T_{2,3,3} + (-\frac{a_2}{3}+4a_4)T_{1,3} + 24a_4T_{2,2}\\
&\quad + 48a_2^2 + \frac{17a_2}{3} \mp 48ka_3 + 112a_4\\
&\quad \in \mathbb{Q}[t_1^{\pm1},t_2^{\pm1},t_3^{\pm1},t_4^{\pm1}]/(\mathfrak{S}_4, t_1t_2t_3t_4=1).
\end{aligned}$$

Remark 3.5. It is known ([24]) that $Z(K)$ is presented by

$Z(K) = \nu \# \exp\Big(-\frac{1}{2}c_2$  $-\frac{1}{24}j_3$  $+\frac{1}{24}(-12c_4+6c_2^2-c_2)$ 

$+$ (linear sum of diagrams with more than 4 trivalent vertices)$\Big)$,

where c_n are coefficient of the Conway polynomial $\nabla_K(z)=\sum c_nz^n$ and j_n are coefficient of the Jones polynomial $J_K(e^t)=\sum j_nt^n$. Note that the Conway polynomial is defined by $\nabla_K(t^{\frac{1}{2}}-t^{-\frac{1}{2}})=\Delta_K(t)$. Therefore we can get

$$a_2 = -\frac{1}{2}c_2, \quad a_3 = -\frac{1}{24}j_3, \quad a_4 = \frac{1}{24}(-12c_4+6c_2^2-c_2)$$

Example 3.6. As an example, we calculate the 3-loop polynomial of the untwisted Whitehead double of $(2, 2n+1)$ torus knots, $T(2, 2n+1)$. We consider the untwisted Whitehead double of them, $Wh^{\pm}\big(T(2, 2n+1)\big)$. It can be shown by using the skein relation that

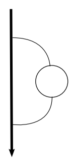
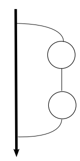
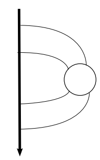
$$\nabla_{T(2,2n+1)}(z) = \sum_{j=0}^{n} \binom{n+j}{2j} z^{2j}, \quad J_{T(2,2n+1)}(t) = \frac{t^n - t^{n+3} - t^{3n+2} + t^{3n+3}}{1 - t^2},$$

and so

$$c_2 = \frac{n(n+1)}{2}, \quad c_4 = \frac{(n+2)(n+1)n(n-1)}{24}, \quad j_3 = -n(n+1)(2n+1).$$

From this, we get

$Z\big(T(2, 2n+1)\big)$

$$= \nu \# \exp\Big(- \frac{n(n+1)}{4}$$

$$+ \frac{n(n+1)(2n+1)}{24}$$

$$+ \frac{n(n+1)(2n^2+2n+1)}{48}$$


$$+ \text{(linear sum of diagrams with more than 4 trivalent vertices)}\Big).$$

Therefore

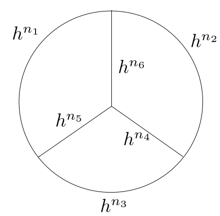
$$\begin{aligned}
&\Lambda_{Wh^{\pm}(T(2,2n+1))}(t_1, t_2, t_3, t_4) \\
&= \frac{n(n+1)}{12}\big(3 \pm (8n+4)\big)(u_{1,2} + u_{1,3} + u_{1,4} + u_{2,3} + u_{2,4} + u_{3,4}) \\
&\quad + \frac{n(n+1)(8n^2+8n+5)}{48}(u_{1,4}u_{2,4} + u_{1,4}u_{3,4} + u_{2,4}u_{3,4} + u_{1,3}u_{2,3} + u_{1,3}u_{4,3} + u_{2,3}u_{4,3} \\
&\qquad + u_{1,2}u_{3,2} + u_{1,2}u_{4,2} + u_{3,2}u_{4,2} + u_{2,1}u_{3,1} + u_{2,1}u_{4,1} + u_{3,1}u_{4,1}) \\
&\quad + \frac{n(n+1)(2n^2+2n+1)}{2}(u_{1,2}u_{3,4} + u_{1,3}u_{2,4} + u_{1,4}u_{2,3}) \\
&\quad + \frac{n^2(n+1)^2}{2}(u_{1,2}^2 + u_{1,3}^2 + u_{1,4}^2 + u_{2,3}^2 + u_{2,4}^2 + u_{3,4}^2) \\
&\quad + \frac{n(n+1)}{16}(v_{1,4}v_{2,4} + v_{1,4}v_{3,4} + v_{2,4}v_{3,4} + v_{1,3}v_{2,3} + v_{1,3}v_{4,3} + v_{2,3}v_{4,3} \\
&\qquad + v_{1,2}v_{3,2} + v_{1,2}v_{4,2} + v_{3,2}v_{4,2} + v_{2,1}v_{3,1} + v_{2,1}v_{4,1} + v_{3,1}v_{4,1}) \\
&\quad \in \mathbb{Q}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}, t_4^{\pm 1}]/(\mathfrak{S}_4, t_1t_2t_3t_4 = 1).
\end{aligned}$$

4 Some properties of the 3-loop part of the Kontsevich invariant and the 3-loop invariant

4.1 The 3-loop part of Kontsevich invariant

In this section, we state some properties of the 3-loop part of the Kontsevich invariant. We omit proofs, and for details, see [18].

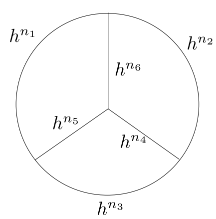
Let $\mathcal{B}_{\mathrm{conn}}$ be the subspace of $\mathcal{B}$ spanned by connected diagrams, and let $\mathcal{B}_{\mathrm{conn}}^{(\text{3-loop})}$ be the subspace of $\mathcal{B}_{\mathrm{conn}}$ spanned by 3-loop open Jacobi diagrams. It is known ([18]) that any elements in $\mathcal{B}_{\mathrm{conn}}^{(\text{3-loop})}$ can be presented by the linear combination of diagrams of the following form,



(4)

such that $n_1+n_2+n_3+n_4+n_5+n_6$ is an even number (If it is an odd number, the Jacobi diagram equal to 0). We can correspond the diagram (4) to $h_1^{n_1}h_2^{n_2}h_3^{n_3}h_4^{n_4}h_5^{n_5}h_6^{n_6}$. These variables satisfy that

$$h_1 - h_2 - h_6 = 0, \quad h_1 - h_3 + h_5 = 0, \quad h_4 + h_5 + h_6 = 0. \tag{5}$$

Here, we regard 

as a tetrahedron, and introduce new variables corresponding with faces of the tetrahedron,

$$x_1 = h_1 - h_5 + h_6, \quad x_2 = h_2 + h_4 - h_6, \quad x_3 = h_3 - h_4 + h_5, \quad x_4 = -h_1 - h_2 - h_3. \tag{6}$$

By definition, these variables satisfy that $x_1 + x_2 + x_3 + x_4 = 0$. Therefore, we get the following isomorphism,

$$\mathcal{B}_{\mathrm{conn}}^{(\text{3-loop})} \cong \left(\mathbb{C}[x_1, x_2, x_3, x_4]/(x_1 + x_2 + x_3 + x_4 = 0)\right)/\mathfrak{S}_4,$$

where the action of $\tau \in \mathfrak{S}_4$ takes a polynomial $f(x_1, x_2, x_3, x_4)$ to
$f((\mathrm{sgn}\tau)x_{\tau(1)}, (\mathrm{sgn}\tau)x_{\tau(2)}, (\mathrm{sgn}\tau)x_{\tau(3)}, (\mathrm{sgn}\tau)x_{\tau(4)})$. Therefore,

$$\mathcal{B}_{\mathrm{conn}}^{(\text{3-loop})} \cong \mathbb{C}[\sigma_2, \sigma_3^2, \sigma_4]$$

where σ_i is the elementary symmetric polynomial of degree i. By (5), (6),

$$h_1 = \frac{x_1 - x_4}{4}, \quad h_2 = \frac{x_2 - x_4}{4}, \quad h_3 = \frac{x_3 - x_4}{4},$$
$$h_4 = \frac{x_2 - x_3}{4}, \quad h_5 = \frac{x_3 - x_1}{4}, \quad h_6 = \frac{x_1 - x_2}{4}$$

so, we can correspond the diagram (4) to

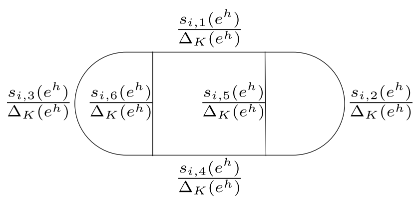
$$\left(\frac{x_1 - x_4}{4}\right)^{n_1} \left(\frac{x_2 - x_4}{4}\right)^{n_2} \left(\frac{x_3 - x_4}{4}\right)^{n_3} \left(\frac{x_2 - x_3}{4}\right)^{n_4} \left(\frac{x_3 - x_1}{4}\right)^{n_5} \left(\frac{x_1 - x_2}{4}\right)^{n_6}.$$

Note that there is a injective map

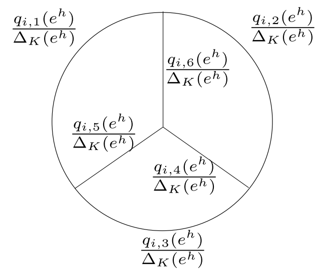
$$(\mathbb{C}(t_1^{\pm 1}, t_2^{\pm 2}, t_3^{\pm 3}, t_4^{\pm 4})^1/(t_1 t_2 t_3 t_4 = 1))/\mathfrak{S}_4 \hookrightarrow (\mathbb{C}[[x_1, x_2, x_3, x_4]]/(x_1 + x_2 + x_3 + x_4 = 0))/\mathfrak{S}_4$$
$$t_i \mapsto e^{x_i/4}$$

where $(\mathbb{C}[[x_1, x_2, x_3, x_4]]/(x_1 + x_2 + x_3 + x_4 = 0))/\mathfrak{S}_4$ is the completion of $(\mathbb{C}[x_1, x_2, x_3, x_4]/(x_1 + x_2 + x_3 + x_4 = 0))/\mathfrak{S}_4$ with respect to the degree.

Next, by using the IHX relation, we can deform a 3-loop graphs with labeling of $\left(\text{polynomial}|_{t=e^h}/\Delta_K(e^h)\right)$ into one of the following two types,

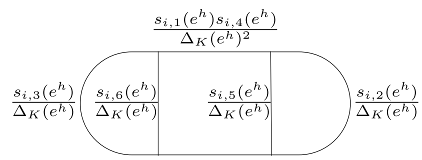


,



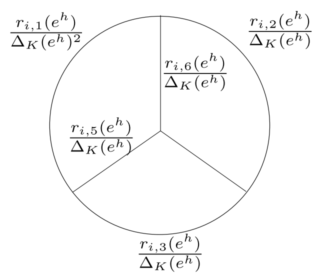
.

If $\dfrac{s_{i,1}(e^h)}{\Delta_K(e^h)} = 1$ or $\dfrac{s_{i,4}(e^h)}{\Delta_K(e^h)} = 1$, the first one can be deformed into a sum of the second ones by the IHX relation , and otherwise, it can be deformed into



,

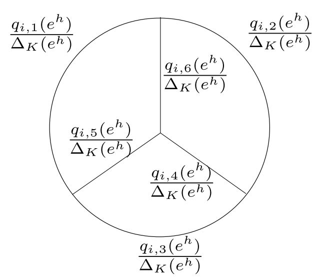
and this can be deformed into a sum of



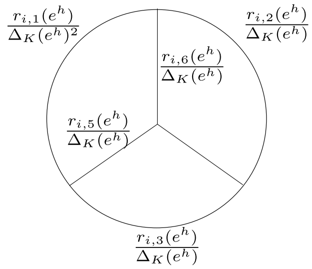
by the IHX relation.

Therefore we put $t_i = e^{x_i/4} \quad (i = 1, 2, 3, 4)$, then we can get the definition of the 3-loop invariant as in Section 2.

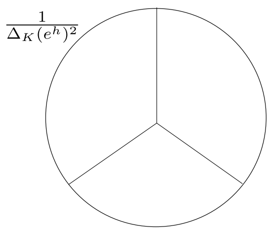
Remark 4.1. In general, the form of $\sum_{i}^{\text{finite}}$

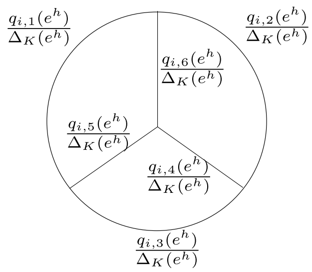


and the form of

$\sum_{i}^{\text{finite}}$ 

cannot transfer to each other. For example,

if 

$= \sum_{i}^{\text{finite}}$ 

holds for $\Delta_K(t) = t + t^{-1} - 1$ and some polynomials $q_{i,j}(t)$, then we have

$$4\left(\frac{1}{\Delta_K(t_1t_4^{-1})^2} + \frac{1}{\Delta_K(t_2t_4^{-1})^2} + \frac{1}{\Delta_K(t_3t_4^{-1})^2} + \frac{1}{\Delta_K(t_2t_3^{-1})^2} + \frac{1}{\Delta_K(t_3t_1^{-1})^2} + \frac{1}{\Delta_K(t_1t_2^{-1})^2}\right)$$
$$= \frac{s(t_1,t_2,t_3,t_4)}{\Delta_K(t_1t_4^{-1})\Delta_K(t_2t_4^{-1})\Delta_K(t_3t_4^{-1})\Delta_K(t_2t_3^{-1})\Delta_K(t_3t_1^{-1})\Delta_K(t_1t_2^{-1})},$$

then,

$$\begin{aligned}&\Delta_K(t_2t_4^{-1})^2\Delta_K(t_3t_4^{-1})^2\cdots\Delta_K(t_1t_2^{-1})^2 + \Delta_K(t_1t_4^{-1})^2\Delta_K(t_3t_4^{-1})^2\cdots\Delta_K(t_1t_2^{-1})^2\\ &+\cdots+\Delta_K(t_1t_4^{-1})^2\Delta_K(t_2t_4^{-1})^2\cdots\Delta_K(t_3t_1^{-1})^2\\ &=\Delta_K(t_1t_4^{-1})\Delta_K(t_2t_4^{-1})\Delta_K(t_3t_4^{-1})\Delta_K(t_2t_3^{-1})\Delta_K(t_3t_1^{-1})\Delta_K(t_1t_2^{-1})\cdot\frac{1}{4}s(t_1,t_2,t_3,t_4),\end{aligned} \tag{7}$$

where $s(t_1,t_2,t_3,t_4)$ is a polynomial. However, when we substitute $t_1 = 1, t_2 = \frac{1}{2}, t_3 = 1-\sqrt{-3}, t_4 = \frac{1+\sqrt{-3}}{2}$, the right hand side of (7) is equal to 0, but the left hand side is not.

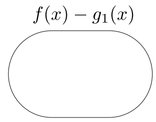
4.2 A connected sum formula for the 3-loop invariant

In this section, we state a connected sum formula for the 3-loop invariant.

Let K_1, K_2 be 0-framing knots, and let $K_1\#K_2$ be their connected sum. We denote

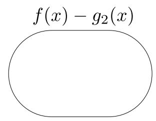
$$f(x) = \frac{1}{2}\log\frac{\sinh(x/2)}{x/2}, \quad g_j(x) = \frac{1}{2}\log\Delta_{K_j}(e^x),$$
$$g(x) = \frac{1}{2}\log\Delta_{K_1\#K_2}(e^x) = g_1(x) + g_2(x),$$

where $\Delta_{K_j}(t)$ is the Alexander polynomial of K_j. We denote

$$\chi^{-1}Z(K_1) = \exp\Bigg($$ 

$$+\gamma_1^{(2)} + \gamma_1^{(3)}$$

$$+ \text{(linear sum of diagrams with more than 4 trivalent vertices)}\Bigg),$$

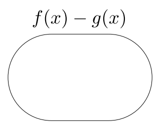
$$\chi^{-1}Z(K_2) = \exp\Bigg($$ 

$$+\gamma_2^{(2)} + \gamma_2^{(3)}$$

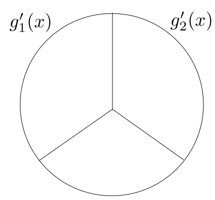
$$+ \text{(linear sum of diagrams with more than 4 trivalent vertices)}\Bigg),$$

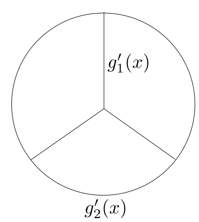
where $\gamma_j^{(2)}, \gamma_j^{(3)}$ is the 2, 3-loop part of K_j, respectively.

Proposition 4.2. *We can get*

$$\chi^{-1}Z(K_1\#K_2) = \exp\Bigg($$ 

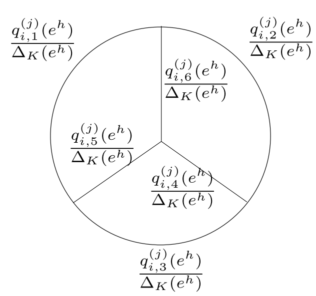
$$+\gamma_1^{(2)} + \gamma_2^{(2)} + \gamma_1^{(3)} + \gamma_2^{(3)}$$

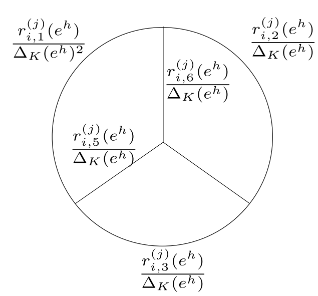
$$+\frac{1}{6}$$ 

$$+\frac{1}{6}$$ 

$$+ \textit{(linear sum of diagrams with more than 4 trivalent vertices)}\Bigg).$$

When the 3-loop part of K_j is presented by

$$Z^{(3\text{-loop})}(K_j) = \sum_i$$ 

$$+\sum_i$$ 

,

the 3-loop invariant of K_j is given by

$$\begin{aligned}
&\Lambda_{K_j}(t_1,t_2,t_3,t_4)\\
&=\sum_{\substack{i\\ \tau\in\mathfrak{S}_4}}\frac{q^{(j)}_{i,1}(t^{\mathrm{sgn}\tau}_{\tau(1)}t^{-\mathrm{sgn}\tau}_{\tau(4)})q^{(j)}_{i,2}(t^{\mathrm{sgn}\tau}_{\tau(2)}t^{-\mathrm{sgn}\tau}_{\tau(4)})q^{(j)}_{i,3}(t^{\mathrm{sgn}\tau}_{\tau(3)}t^{-\mathrm{sgn}\tau}_{\tau(4)})q^{(j)}_{i,4}(t^{\mathrm{sgn}\tau}_{\tau(2)}t^{-\mathrm{sgn}\tau}_{\tau(3)})q^{(j)}_{i,5}(t^{\mathrm{sgn}\tau}_{\tau(3)}t^{-\mathrm{sgn}\tau}_{\tau(1)})q^{(j)}_{i,6}(t^{\mathrm{sgn}\tau}_{\tau(1)}t^{-\mathrm{sgn}\tau}_{\tau(2)})}{\Delta_K(t_1t_4^{-1})\Delta_K(t_2t_4^{-1})\Delta_K(t_3t_4^{-1})\Delta_K(t_2t_3^{-1})\Delta_K(t_3t_1^{-1})\Delta_K(t_1t_2^{-1})}\\
&+\sum_{\substack{i\\ \tau\in\mathfrak{S}_4}}\frac{r^{(j)}_{i,1}(t^{\mathrm{sgn}\tau}_{\tau(1)}t^{-\mathrm{sgn}\tau}_{\tau(4)})r^{(j)}_{i,2}(t^{\mathrm{sgn}\tau}_{\tau(2)}t^{-\mathrm{sgn}\tau}_{\tau(4)})r^{(j)}_{i,3}(t^{\mathrm{sgn}\tau}_{\tau(3)}t^{-\mathrm{sgn}\tau}_{\tau(4)})r^{(j)}_{i,5}(t^{\mathrm{sgn}\tau}_{\tau(3)}t^{-\mathrm{sgn}\tau}_{\tau(1)})r^{(j)}_{i,6}(t^{\mathrm{sgn}\tau}_{\tau(1)}t^{-\mathrm{sgn}\tau}_{\tau(2)})}{\Delta_K(t_{\tau(1)}t^{-1}_{\tau(4)})^2\Delta_K(t_{\tau(2)}t^{-1}_{\tau(4)})\Delta_K(t_{\tau(3)}t^{-1}_{\tau(4)})\Delta_K(t_{\tau(3)}t^{-1}_{\tau(1)})\Delta_K(t_{\tau(1)}t^{-1}_{\tau(2)})}\\
&\quad\in\mathbb{Q}(t_1^{\pm1},t_2^{\pm1},t_3^{\pm1},t_4^{\pm1})^1/(\mathfrak{S}_4,t_1t_2t_3t_4=1)\quad (j=1,2),
\end{aligned}$$

as in Section 2.

Corollary 4.3. *We get the 3-loop invariant of $K_1\#K_2$ as follows,*

$$\begin{aligned}
&\Lambda_{K_1\#K_2}(t_1,t_2,t_3,t_4)=\Lambda_{K_1}(t_1,t_2,t_3,t_4)+\Lambda_{K_2}(t_1,t_2,t_3,t_4)\\
&+\sum_{\substack{i\\ \tau\in\mathfrak{S}_4}}\frac{t_{\tau(1)}t_{\tau(2)}t^{-2}_{\tau(4)}\Delta'_{K_1}(t_{\tau(1)}t^{-1}_{\tau(4)})\Delta'_{K_1}(t_{\tau(2)}t^{-1}_{\tau(4)})}{24\Delta_{K_1}(t_{\tau(1)}t^{-1}_{\tau(4)})\Delta_{K_1}(t_{\tau(2)}t^{-1}_{\tau(4)})}+\sum_{\substack{i\\ \tau\in\mathfrak{S}_4}}\frac{t_{\tau(1)}t^{-1}_{\tau(2)}t_{\tau(3)}t^{-1}_{\tau(4)}\Delta'_{K_1}(t_{\tau(3)}t^{-1}_{\tau(4)})\Delta'_{K_1}(t_{\tau(1)}t^{-1}_{\tau(2)})}{24\Delta_{K_1}(t_{\tau(3)}t^{-1}_{\tau(4)})\Delta_{K_1}(t_{\tau(1)}t^{-1}_{\tau(2)})}\\
&\quad\in\mathbb{Q}(t_1^{\pm1},t_2^{\pm1},t_3^{\pm1},t_4^{\pm1})^1/(\mathfrak{S}_4,t_1t_2t_3t_4=1).
\end{aligned}$$

In particular, if $\Delta_{K_1}(t)=\Delta_{K_2}(t)=1$, then

$$\Lambda_{K_1\#K_2}(t_1,t_2,t_3,t_4)=\Lambda_{K_1}(t_1,t_2,t_3,t_4)+\Lambda_{K_2}(t_1,t_2,t_3,t_4)\in\mathbb{Q}[t_1^{\pm1},t_2^{\pm1},t_3^{\pm1},t_4^{\pm1}]/(\mathfrak{S}_4,t_1t_2t_3t_4=1).$$

For the proof of Proposition 4.2, see Appendix.
For example, we can get

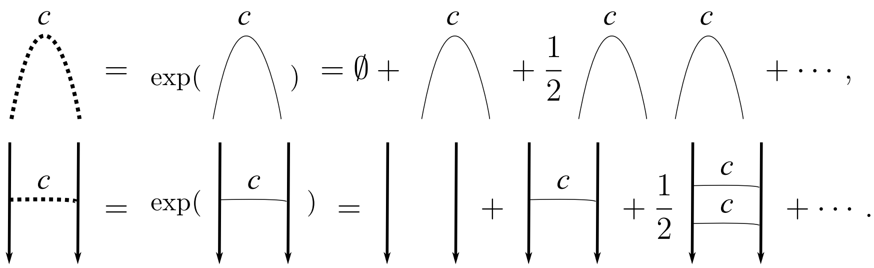
$$\Lambda_{D(K_1,K'_1)\#D(K_2,K'_2)}(t_1,t_2,t_3,t_4)=\Lambda_{D(K_1,K'_1)}(t_1,t_2,t_3,t_4)+\Lambda_{D(K_2,K'_2)}(t_1,t_2,t_3,t_4),$$

where K_j is a 0-framing knot and K'_j is a k_j-framing knot ($k_j\in\mathbb{Z}$, $j=1,2$), and $D(K_j,K'_j)$ is the knot defined in Section 3. In a similar way, we can get the 3-loop polynomial of $\sum_{j=1}^{g}D(K_j,K'_j)$.

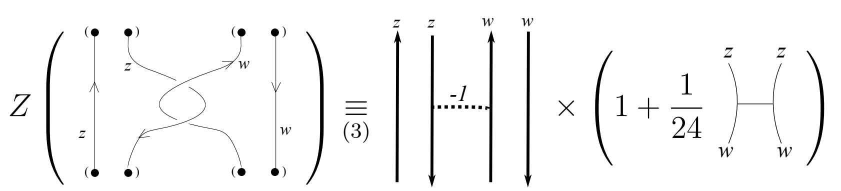
5 The rational version of the Aarhus integral and a computation of the loop expansion

In this section, we review how to compute the loop expansion. Along this, we calculate the 3-loop invariant. For details, see for example, [7], [10].

In the following of this paper, we represent "exponential" by , for example;



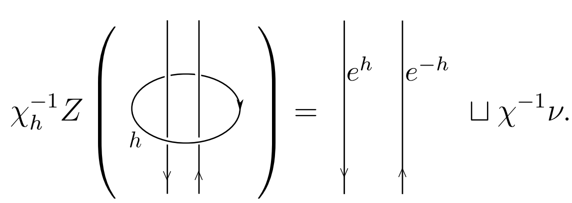
Further, we write $\alpha \underset{(m+1)}{\equiv} \beta$ (α and β are Jacobi diagrams) if $\alpha - \beta$ can be presented by a linear sum of Jacobi diagrams with more than m trivalent vertices, where we do not count trivalent vertices generated by attached power series. When $m = 4$, we write "$\equiv$" instead of "$\underset{(5)}{\equiv}$". If a uni-trivalent graph has m trivalent vertices, we can put it anywhere modulo "$\underset{(m+1)}{\equiv}$", so we can write them separately, for example,



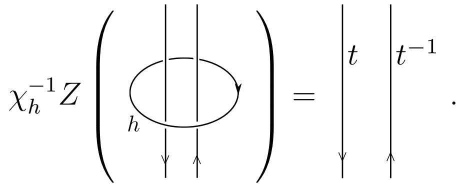
Let K be a 0-framed knot in S^3. It is known that K has a *surgery presentation* $K_0 \cup L$, such that K_0 is isotopic to the unknot with 0 framing and L is a (l-components) framed link, and the linking number of K_0 and the each component of L is equal to 0, and the pair obtained from the pair (S^3, K_0) by surgery along L is homeomorphic to (S^3, K). We can obtain the loop expansion of the Kontsevich invariant of K from the Kontsevich invariant of $K_0 \cup L$, in the following way ([10]). Let $\mathcal{A}(*_X)$ be the space of open Jacobi diagrams whose legs are labeled by elements of a set X.

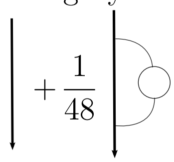
Step 1 Compute the $\chi_h^{-1} Z(K_0 \cup L)$

We label the component corresponding to K_0 by h, and label the components corresponding to L by the set $X = \{x_1, x_2, \cdots, x_l\}$. Then, we compute $\chi_h^{-1} Z(K_0 \cup L)$, where $\chi_h : \mathcal{A}(*_h \sqcup \coprod_X S^1) \to \mathcal{A}(\downarrow_h \sqcup \coprod_X S^1) \cong \mathcal{A}(S^1_h \sqcup \coprod_X S^1)$. As in [5], we note that



We put $t = e^h$, and we write again (omitting $\chi^{-1}\nu$ for simplicity, because it does not contribute to the 3-loop part),



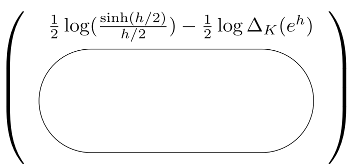
Step 2 Compute the $\chi^{-1}\check{Z}(K_0 \cup L)$
$\chi_h^{-1}\check{Z}(K_0 \cup L)$ is obtained from $\chi_h^{-1}Z(K_0 \cup L)$ by connected-summing by ν to each component labeling by a component of X, where we denote $\nu = Z(\text{unknot}) \in \mathcal{A}(\downarrow)$. Note that $\nu \underset{(3)}{\equiv}$ 

. Then, we compute $\chi_X^{-1}\chi_h^{-1}\check{Z}(K_0 \cup L)$, where we choose a disjoint union of the unknot and a string link $K_0 \cup \check{L}$ whose closure is isotopic to $K_0 \cup L$, and $\chi_X : \mathcal{A}(*_X) \to \mathcal{A}(\bigsqcup_X \downarrow)$. We denote it by $\chi^{-1}\check{Z}(K_0 \cup L)$.

Step 3 Compute the rational version of the *Aarhus integral* (see [1], [2], [3], [10])
The Kontsevich invariant of K is computed by the rational version of the Aarhus integral as follows,

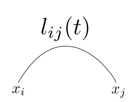
$$\chi^{-1}Z(K) = \chi^{-1}Z^{LMO}(S^3, K)$$

$$= \exp\left(\right) \sqcup \frac{\langle\langle \chi^{-1}\check{Z}(K_0 \cup L)\rangle\rangle}{\langle\langle \chi^{-1}\check{Z}(U_+)\rangle\rangle^{\sigma_+}\langle\langle \chi^{-1}\check{Z}(U_-)\rangle\rangle^{\sigma_-}},$$



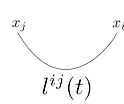
where $U_\pm$ denotes the unknot with ± 1 framing, and σ_+ and σ_- are the number of the positive and negative eigenvalues of the linking matrix of L. "$\langle\langle \quad \rangle\rangle$" is defined as follows. It is known that $\chi^{-1}\check{Z}(K_0 \cup L)$ is presented by

$$\chi^{-1}\check{Z}(K_0 \cup L) = \exp\Big(\frac{1}{2}\sum_{x_i, x_j \in X} \quad \Big) \sqcup P\big(\chi^{-1}\check{Z}(K_0 \cup L)\big), \tag{8}$$



where $\big(l_{ij}(t)\big)$ is an equivariant linking matrix of $L \subset S^3 \backslash K_0$ satisfying that $l_{ji}(t) = l_{ij}(t^{-1})$, and $P\big(\chi^{-1}\check{Z}(K_0 \cup L)\big)$ is a sum of diagrams which have at least one trivalent vertex on each component. Then,

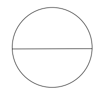
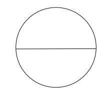
$$\langle\langle \chi^{-1}\check{Z}(K_0 \cup L)\rangle\rangle = \Big\langle \exp\Big(-\frac{1}{2}\sum_{x_i, x_j \in X} \quad \Big), P\big(\chi^{-1}\check{Z}(K_0 \cup L)\big)\Big\rangle, \tag{9}$$



where $\left(l^{ij}(t)\right) = \left(l_{ij}(t)\right)^{-1}$, and $\langle \quad , \quad \rangle$ is defined by

$$\langle C_1, C_2 \rangle = \left(\begin{array}{l} \text{sum of all ways gluing the } x\text{-marked legs of } C_1 \\ \text{to the } x\text{-marked legs of } C_2 \text{ for all } x \in X \end{array} \right). \tag{10}$$

For details, see [1]. Note ([6]) that

$$\langle\!\langle \chi^{-1} \check{Z}(U_\pm) \rangle\!\rangle = \langle \chi^{-1}\nu, \chi^{-1}\nu \rangle^{-1} \exp\left(\mp \frac{1}{16}\right.$$

$$\left.\right) \underset{(3)}{\equiv} \exp\left(\mp \frac{1}{16}\right.$$

$$\left.\right). \tag{11}$$

From this, we can compute the loop expansion of the Kontsevich invariant of K, and we get

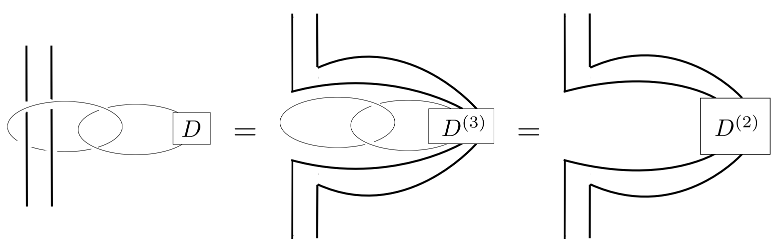
$$\begin{aligned} &\text{the 3-loop part of } \chi^{-1} Z(K) \\ &= \text{the 3-loop part of } \frac{\langle\!\langle \chi^{-1} \check{Z}(K_0 \cup L) \rangle\!\rangle}{\langle\!\langle \chi^{-1} \check{Z}(U_+) \rangle\!\rangle^{\sigma_+} \langle\!\langle \chi^{-1} \check{Z}(U_-) \rangle\!\rangle^{\sigma_-}}. \end{aligned}$$

Then, we obtain the 3-loop invariant $\Lambda_K(t_1, t_2, t_3, t_4)$.

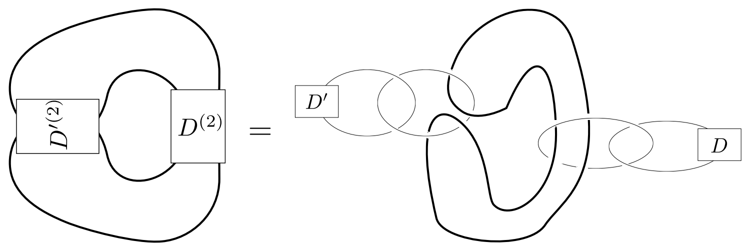
6 The proof of Theorem 3.1

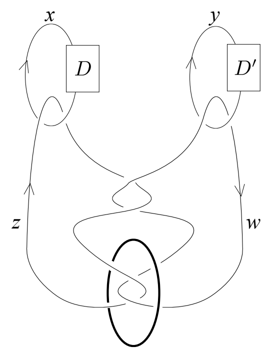
In this section, we prove Theorem 3.1.

Proof of Theorem 3.1. By handle slide, we can show that

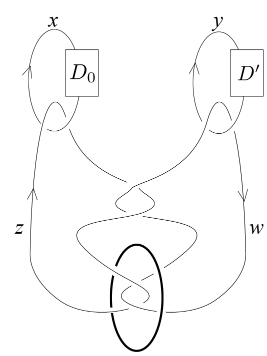


(surgery along the link drawn by thin lines), so we get the following surgery presentation,



$=$  $= K_0 \cup L,$

where K_0 is depicted by a thick line, and L is depicted by thin lines. We put

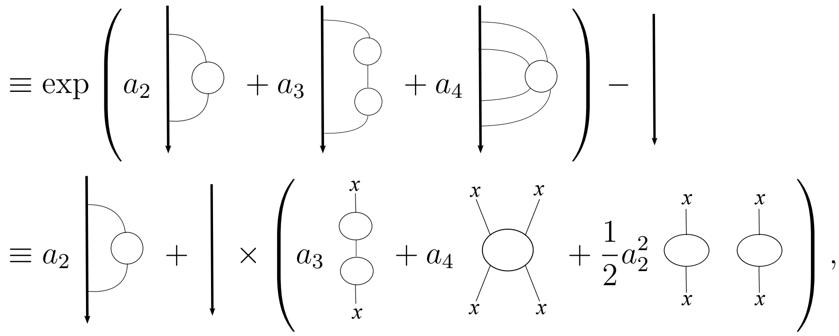
$D_0 =$, and $K_0 \cup L_0 =$  .

We can see that $K_0 \cup L_0$ is equal to the surgery presentation of $D(\text{unknot}, K') = \text{unknot}$, so its 3-loop part is equals to 0. In addition, linking matrices of L and L_0 are equal, and α (in (18) below) are common for L and L_0. Thus, we get

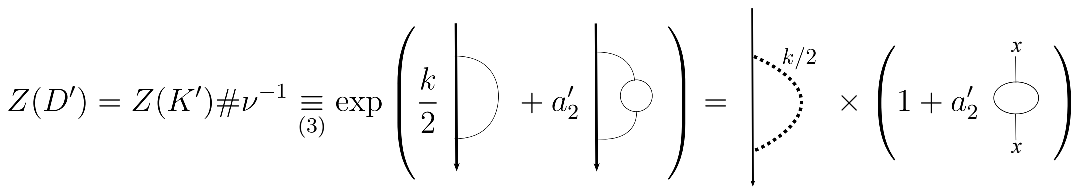
$$\begin{aligned}
&\text{the 3-loop part of } \frac{\langle\!\langle \chi^{-1}\check{Z}\big(K_0 \cup L)\rangle\!\rangle}{\langle\!\langle \chi^{-1}\check{Z}(U_+)\rangle\!\rangle^{\sigma_+}\langle\!\langle \chi^{-1}\check{Z}(U_-)\rangle\!\rangle^{\sigma_-}} \\
&= \text{the 3-loop part of } \frac{\langle\!\langle \chi^{-1}\check{Z}\big(K_0 \cup L) - \chi^{-1}\check{Z}\big(K_0 \cup L_0)\rangle\!\rangle}{\langle\!\langle \chi^{-1}\check{Z}(U_+)\rangle\!\rangle^{\sigma_+}\langle\!\langle \chi^{-1}\check{Z}(U_-)\rangle\!\rangle^{\sigma_-}}.
\end{aligned}$$

Then we calculate $\chi^{-1}\check{Z}(K_0 \cup L) - \chi^{-1}\check{Z}(K_0 \cup L_0)$. First, we calculate $\chi_h^{-1}Z(K_0 \cup L) - \chi_h^{-1}Z(K_0 \cup L_0)$ by decomposing into the following parts. We note that each term of the formula of (12) below has at least 2 trivalent vertices, so it is sufficient to calculate other parts modulo "$\underset{(3)}{\equiv}$".

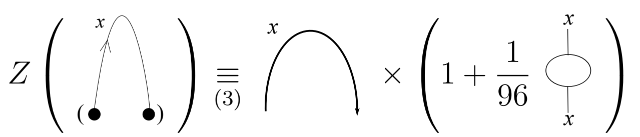
$$Z(D) - Z(D_0) = Z(K)\#\nu^{-1} - Z(D_0)$$



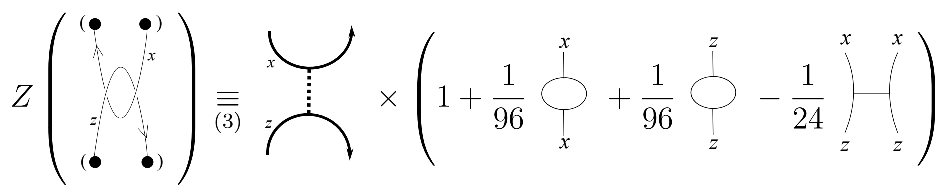
(12)



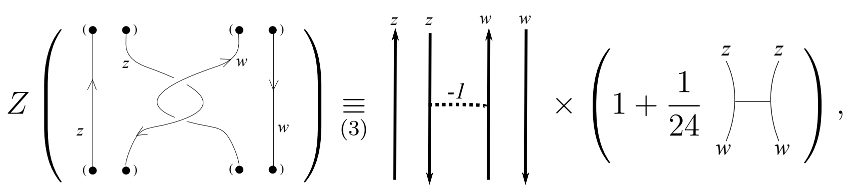
(13)



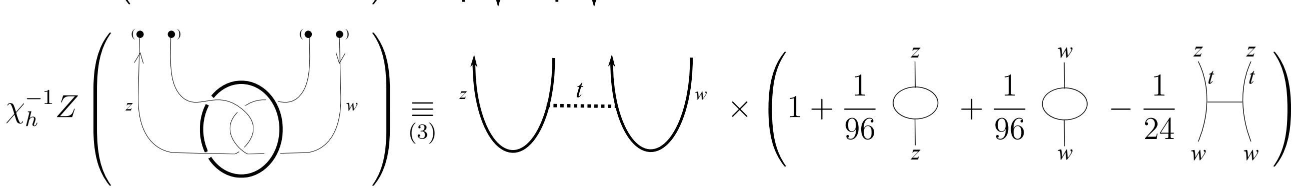
(14)



(15)



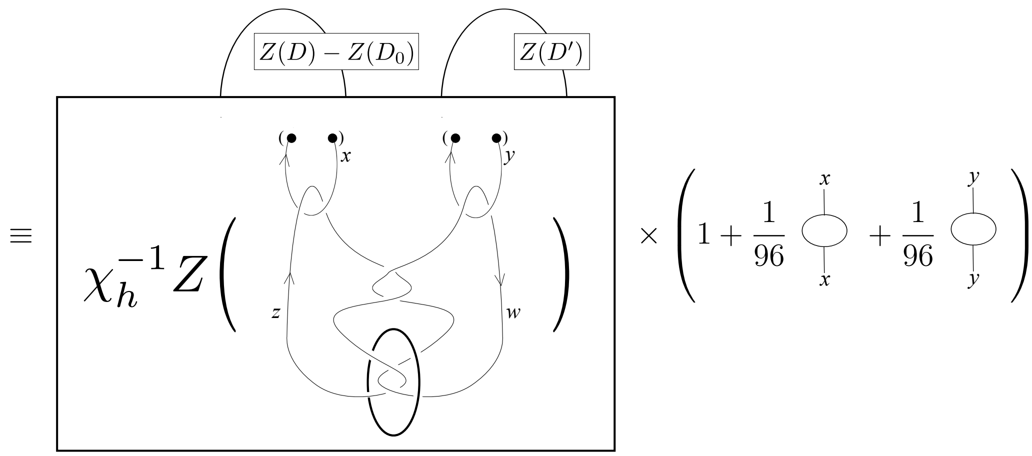
(16)

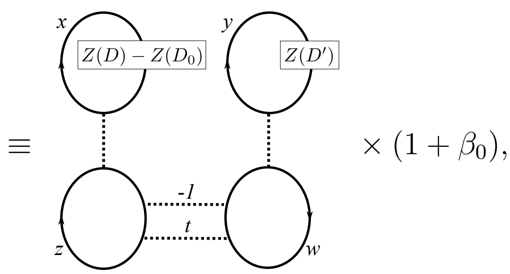


(17)

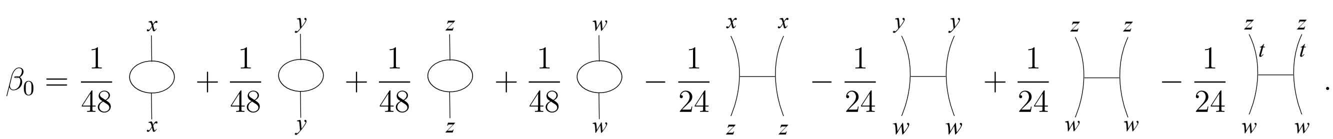
Then, by (14), (15), (16), (17), we get

$$\chi_h^{-1}Z(K_0 \cup L) - \chi_h^{-1}Z(K_0 \cup L_0)$$



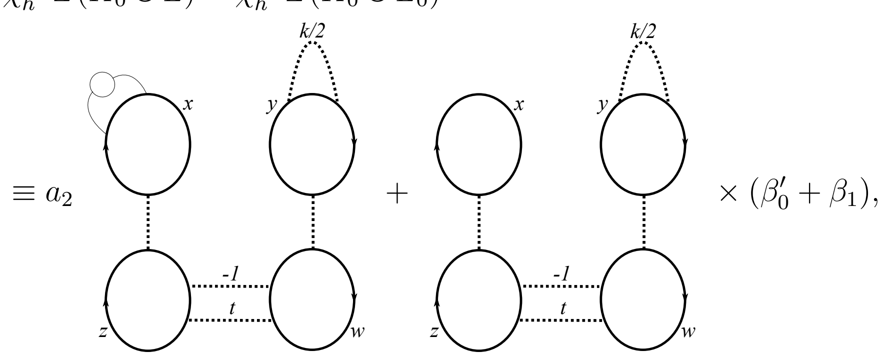


where

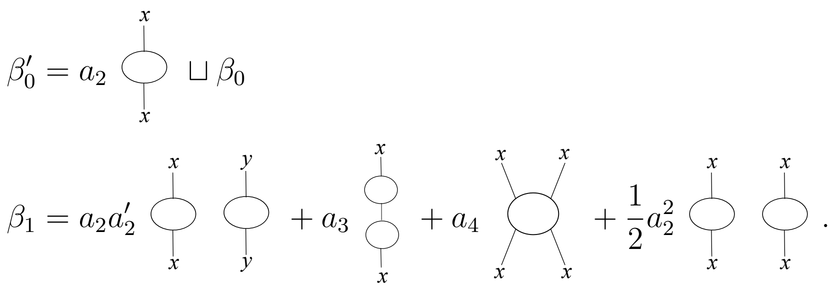


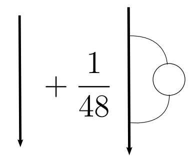
Hence, by (12), (13),

$$\chi_h^{-1}Z(K_0 \cup L) - \chi_h^{-1}Z(K_0 \cup L_0)$$

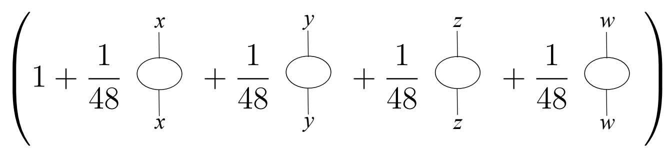


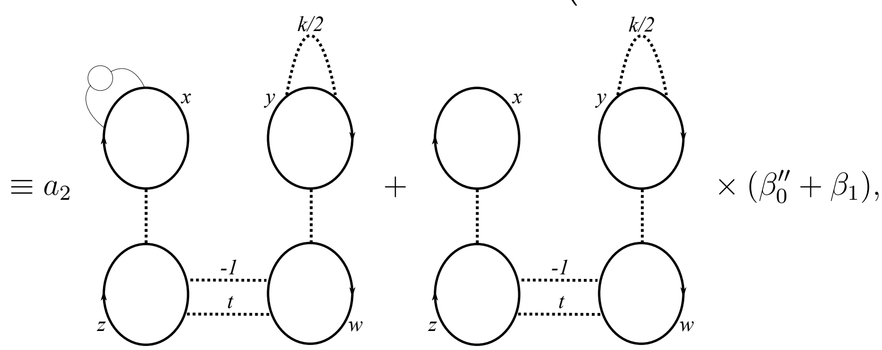
where



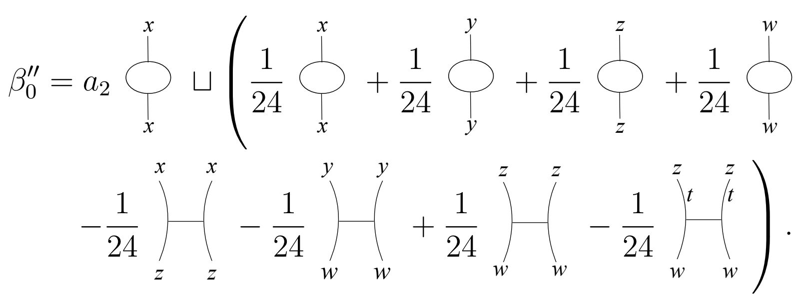
Next, we calculate $\chi_h^{-1}\check{Z}(K_0 \cup L) - \chi_h^{-1}\check{Z}(K_0 \cup L_0)$. Recall that $\nu \underset{(3)}{\equiv}$  , so

$$\chi_h^{-1}\check{Z}(K_0 \cup L) - \chi_h^{-1}\check{Z}(K_0 \cup L_0)$$
$$= \left(\chi_h^{-1}Z(K_0 \cup L) - \chi_h^{-1}Z(K_0 \cup L_0)\right)\#\nu^{\otimes 4}$$
$$\equiv \left(\chi_h^{-1}Z(K_0 \cup L) - \chi_h^{-1}Z(K_0 \cup L_0)\right) \times$$



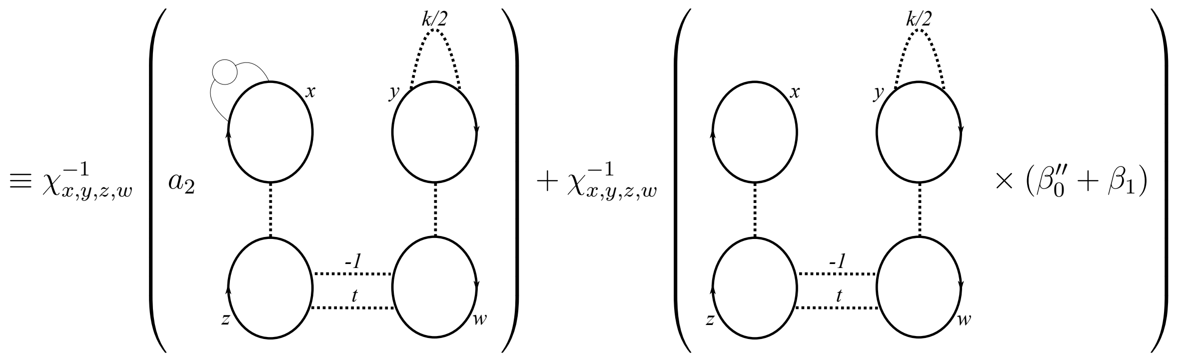


where

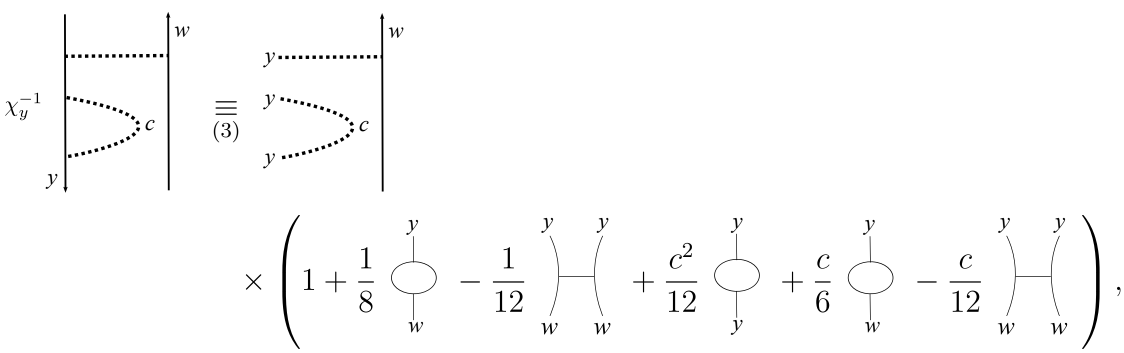


Then, we get

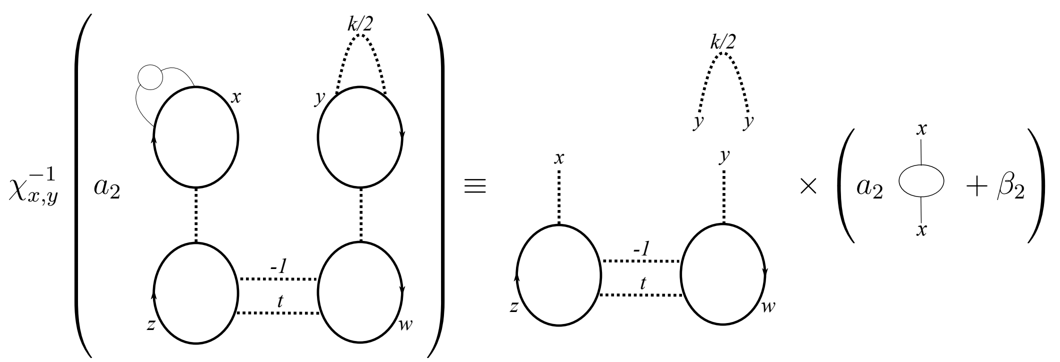
$$\chi^{-1}\check{Z}(K_0 \cup L) - \chi^{-1}\check{Z}(K_0 \cup L_0)$$
$$= \chi_{x,y,z,w}^{-1}\left(\chi_h^{-1}\check{Z}(K_0 \cup L) - \chi_h^{-1}\check{Z}(K_0 \cup L_0)\right)$$



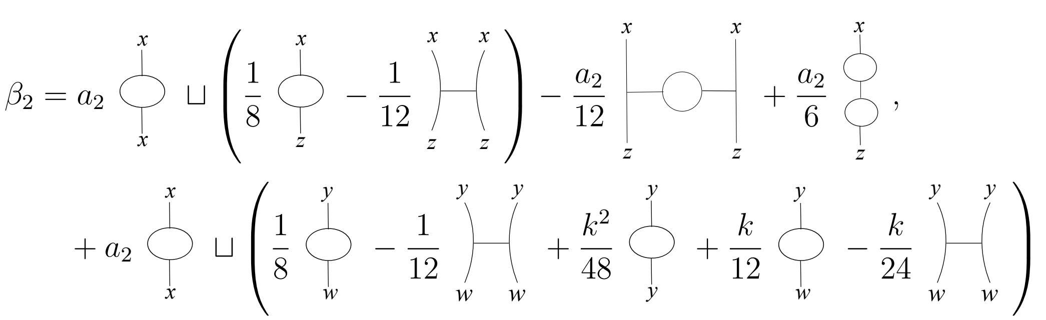
It is known [17] that



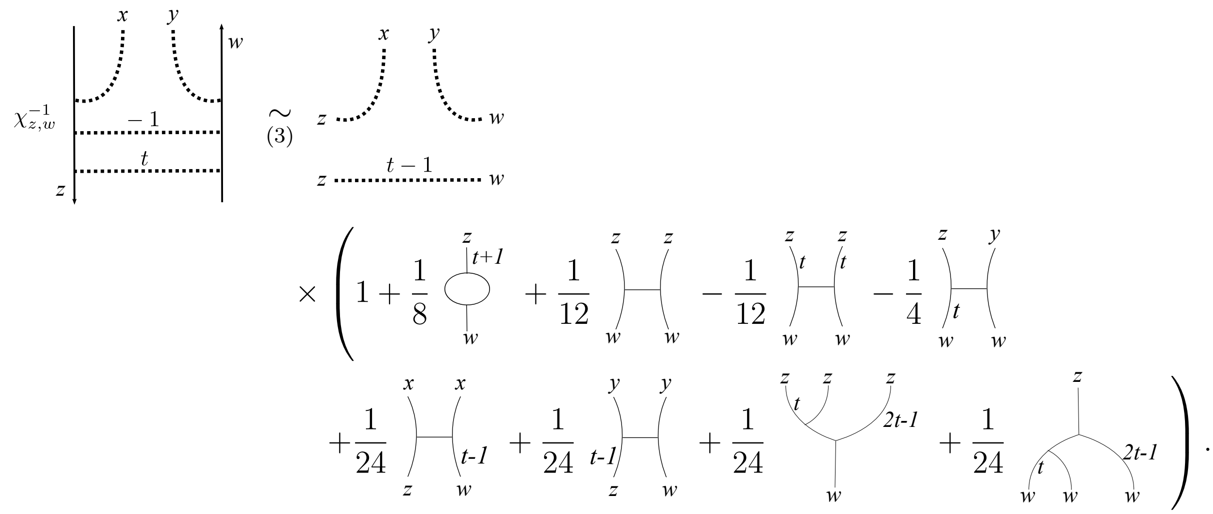
where c is a scalar. Hence, by the above formula and Lemma 6.2 below,



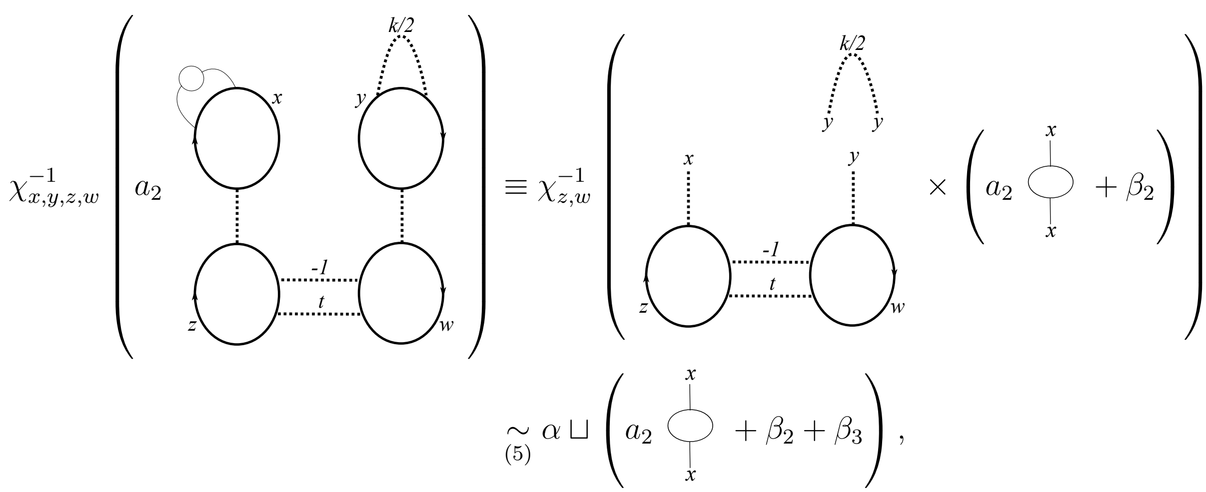
where



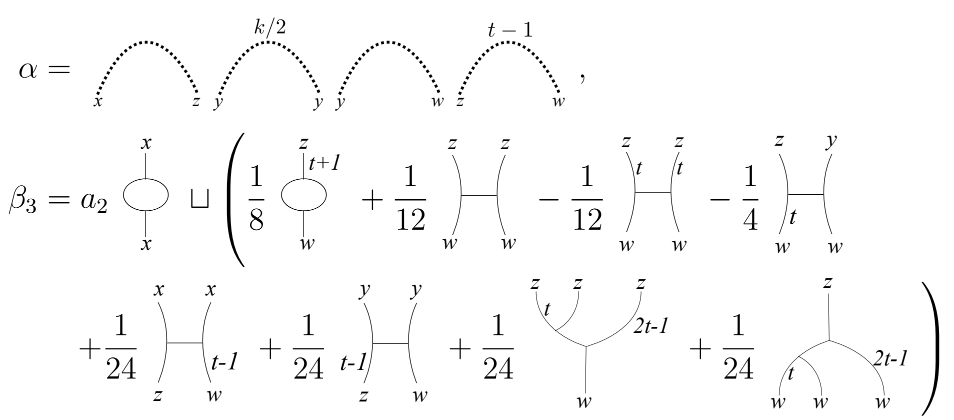
Moreover, it is known [17] that



For the notation " $\underset{(m+1)}{\sim}$ ", see Remark 6.1 below. Therefore we get



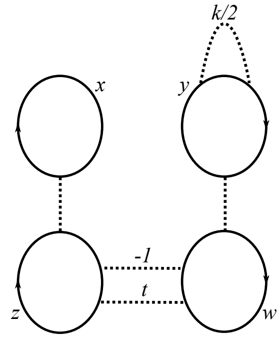
where



(18)

In addition, we get

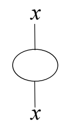
$$\chi^{-1}_{x,y,z,w}\Bigg($$



$$\times\,(\beta_0''+\beta_1)\Bigg) \equiv \alpha \sqcup (\beta_0''+\beta_1).$$

Thus, we obtain

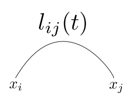
$$\chi^{-1}\check{Z}(K_0\cup L)-\chi^{-1}\check{Z}(K_0\cup L_0)\equiv \alpha\sqcup\Big(a_2$$



$$+\beta_0''+\beta_1+\beta_2+\beta_3\Big),$$

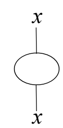
where, putting $x_1=x, x_2=y, x_3=z, x_4=w$, the formula (8) is written in the following form,

$$\exp\Big(\frac{1}{2}\sum_{x_i,x_j\in X}$$



$$\Big)=\alpha,$$

$$P\big(\chi^{-1}\check{Z}(K_0\cup L)-\chi^{-1}\check{Z}(K_0\cup L_0)\big)\equiv a_2$$



$$+\beta_0''+\beta_1+\beta_2+\beta_3.$$

The equivariant linking matrix $\big(l_{ij}(t)\big)$ of $L\subset S^3\backslash K_0$ (and $L_0\subset S^3\backslash K_0$) is given by

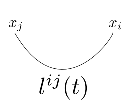
$$\big(l_{ij}(t)\big)=\begin{pmatrix}0&0&1&0\\0&k&0&1\\1&0&0&t-1\\0&1&t^{-1}-1&0\end{pmatrix}.$$

Hence,

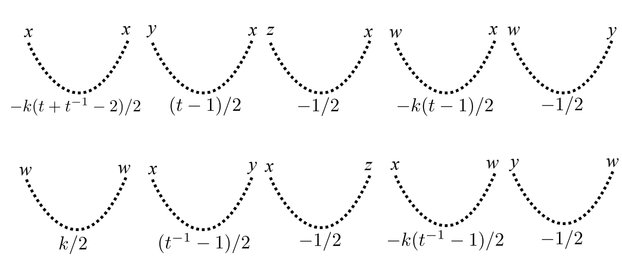
$$-\frac{1}{2}\big(l^{ij}(t)\big)=\frac{1}{2}\begin{pmatrix}-k(t+t^{-1}-2)&t-1&-1&-k(t-1)\\t^{-1}-1&0&0&-1\\-1&0&0&0\\-k(t^{-1}-1)&-1&0&k\end{pmatrix}.$$

We put

$$\hat{\alpha}=\exp\Big(-\frac{1}{2}\sum_{x_i,x_j\in X}$$

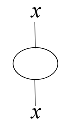


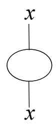
$$\Big)=$$



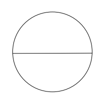
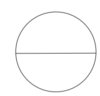
.

Then, we obtain the 3-loop part of $\chi^{-1}Z\big(D(K,K')\big)$ from above Kontsevich invariants using the rational version of the Aarhus integral. By (9), we get

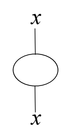
$$\langle\langle \chi^{-1}\check{Z}(K_0 \cup L) - \chi^{-1}\check{Z}(K_0 \cup L_0)\rangle\rangle \equiv \Big\langle \hat{\alpha}, a_2$$  $$+\beta_0'' + \beta_1 + \beta_2 + \beta_3\Big\rangle.$$

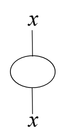
Note that $\Big\langle \hat{\alpha}, a_2$  $+\beta_0''+\beta_1+\beta_2+\beta_3\Big\rangle$ contains only diagrams with at least 2 trivalent vertices. So, we calculate the normalization term modulo "$\underset{(3)}{\equiv}$". In our case, $\sigma_+ = \sigma_- = 2$, so we get by (11)

$$\langle\!\langle \chi^{-1}\check{Z}(U_+)\rangle\!\rangle^{\sigma_+}\langle\!\langle \chi^{-1}\check{Z}(U_-)\rangle\!\rangle^{\sigma_-} = \langle\!\langle \chi^{-1}\check{Z}(U_+)\rangle\!\rangle^{2}\langle\!\langle \chi^{-1}\check{Z}(U_-)\rangle\!\rangle^{2}$$

$$\underset{(3)}{\equiv} \exp\left(-\frac{1}{8}\right.$$  $$\left.\right)\exp\left(\frac{1}{8}\right.$$  $$\left.\right) = 1.$$

Then, we calculate the Aarhus integral as follows,

$$\frac{\langle\!\langle \chi^{-1}\check{Z}\big(K_0 \cup L) - \chi^{-1}\check{Z}\big(K_0 \cup L_0)\rangle\!\rangle}{\langle\!\langle \chi^{-1}\check{Z}(U_+)\rangle\!\rangle^{\sigma_+}\langle\!\langle \chi^{-1}\check{Z}(U_-)\rangle\!\rangle^{\sigma_-}} \equiv \Big\langle \hat{\alpha}, a_2$$  $$+\beta_0'' + \beta_1 + \beta_2 + \beta_3\Big\rangle.$$

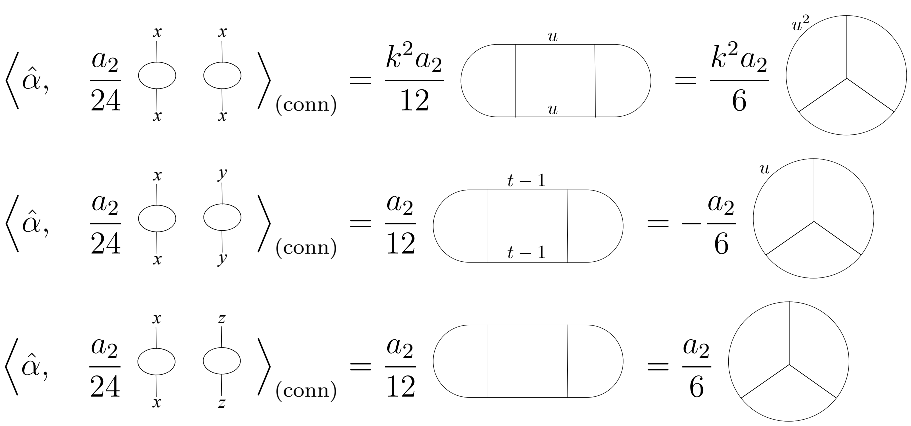
Note that $\Big\langle \hat{\alpha}, a_2$  $\Big\rangle$ part is in the 2-loop part. Therefore, we get

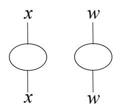
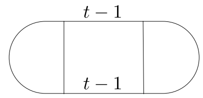
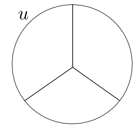
$$\text{3-loop part of } \chi^{-1}Z\big(D(K,K')\big) = \langle \hat{\alpha}, (\beta_0'' + \beta_1 + \beta_2 + \beta_3)\rangle_{(\mathrm{conn})},$$

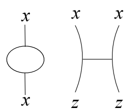
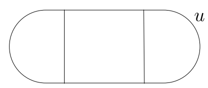
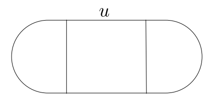
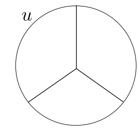
where we denote the connected part of $\langle\ \ ,\ \ \rangle$ by $\langle\ \ ,\ \ \rangle_{(\mathrm{conn})}$.

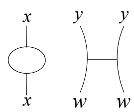
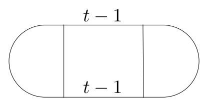
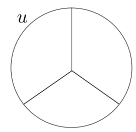
Then, we calculate each term of $\langle \hat{\alpha}, (\beta_0''+\beta_1+\beta_2+\beta_3)\rangle_{(\mathrm{conn})}$. We denote $u = t+t^{-1}-2$ and $v = t - t^{-1}$.

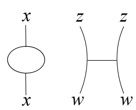
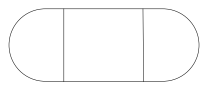
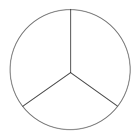
We calculate $\langle \hat{\alpha}, \beta_0''\rangle_{(\mathrm{conn})}$, as follows.

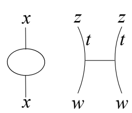
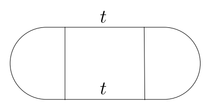
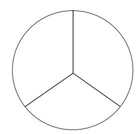


$\left\langle \hat{\alpha}, \quad \frac{a_2}{24}\right.$  $\left.\right\rangle_{(\mathrm{conn})} = \frac{k^2 a_2}{12}$  $= -\frac{k^2 a_2}{6}$ 

$\left\langle \hat{\alpha}, \quad -\frac{a_2}{24}\right.$  $\left.\right\rangle_{(\mathrm{conn})} = \frac{k a_2}{12}$  $- \frac{k a_2}{6}$  $= -\frac{k a_2}{6}$ 

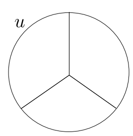
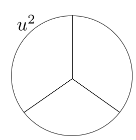
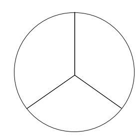
$\left\langle \hat{\alpha}, \quad -\frac{a_2}{24}\right.$  $\left.\right\rangle_{(\mathrm{conn})} = \frac{k a_2}{12}$  $= -\frac{k a_2}{6}$ 

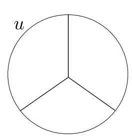
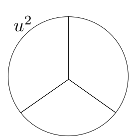
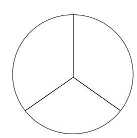
$\left\langle \hat{\alpha}, \quad \frac{a_2}{24}\right.$  $\left.\right\rangle_{(\mathrm{conn})} = \frac{k a_2}{12}$  $= \frac{k a_2}{6}$ 

$\left\langle \hat{\alpha}, \quad -\frac{a_2}{24}\right.$  $\left.\right\rangle_{(\mathrm{conn})} = -\frac{k a_2}{12}$  $= -\frac{k a_2}{6}$ 

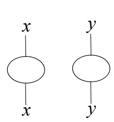
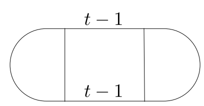
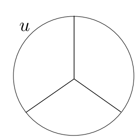
Hence,

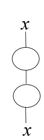
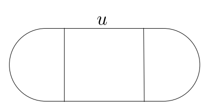
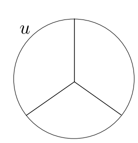
$\langle \hat{\alpha}, \beta_0'' \rangle_{(conn)}$

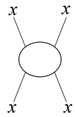
$= (-\frac{a_2}{6} - \frac{k^2 a_2}{6} - \frac{k a_2}{6} - \frac{k a_2}{6})$  $+ \frac{k^2 a_2}{6}$  $+ (\frac{a_2}{6} + \frac{k a_2}{6} - \frac{k a_2}{6})$ 

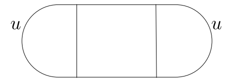
$= (-\frac{a_2}{6} - \frac{k^2 a_2}{6} - \frac{k a_2}{3})$  $+ \frac{k^2 a_2}{6}$  $+ \frac{a_2}{6}$  . (19)

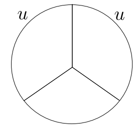
We calculate $\langle \hat{\alpha}, \beta_1 \rangle_{(\mathrm{conn})}$, as follows.

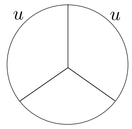
$\left\langle \hat{\alpha}, \quad a_2 a_2'\right.$  $\left.\right\rangle_{(\mathrm{conn})} = 2 a_2 a_2'$  $= -4 a_2 a_2'$ 

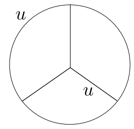
$\left\langle \hat{\alpha}, \quad a_3\right.$  $\left.\right\rangle_{(\mathrm{conn})} = -k a_3$  $= -2k a_3$ 

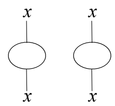
$$\left\langle \hat{\alpha}, \quad a_4 \right.$$ 

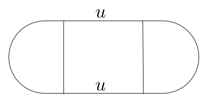
$$\left. \right\rangle_{\rm (conn)} = 3k^2 a_4$$ 

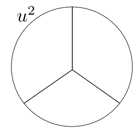
$$- k^2 a_4$$ 

$$= 2k^2 a_4$$ 

$$+ 3k^2 a_4$$ 

$$\left\langle \hat{\alpha}, \quad \frac{a_2^2}{2} \right.$$ 

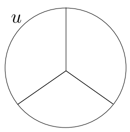
$$\left. \right\rangle_{\rm (conn)} = k^2 a_2^2$$ 

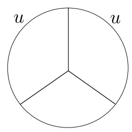
$$= 2k^2 a_2^2$$ 

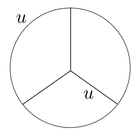
.

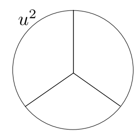
Hence,

$\langle \hat{\alpha}, \beta_1 \rangle_{(conn)}$

$$= (-4a_2 a_2' - 2ka_3)$$ 

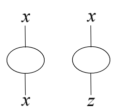
$$+ 2k^2 a_4$$ 

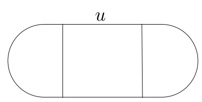
$$+ 3k^2 a_4$$ 

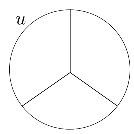
$$+ 2k^2 a_2^2$$ 

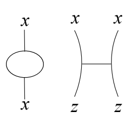
. (20)

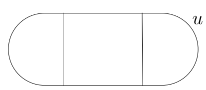
We calculate $\langle \hat{\alpha}, \beta_2 \rangle_{\rm (conn)}$, as follows.

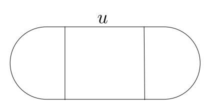
$$\left\langle \hat{\alpha}, \quad \frac{a_2}{8} \right.$$ 

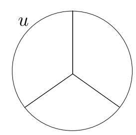
$$\left. \right\rangle_{\rm (conn)} = \frac{ka_2}{4}$$ 

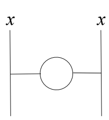
$$= \frac{ka_2}{2}$$ 

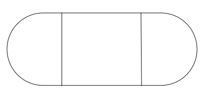
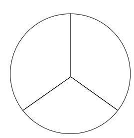
$$\left\langle \hat{\alpha}, \quad -\frac{a_2}{12} \right.$$ 

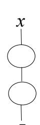
$$\left. \right\rangle_{\rm (conn)} = \frac{ka_2}{6}$$ 

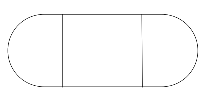
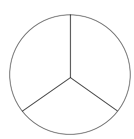
$$- \frac{ka_2}{3}$$ 

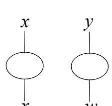
$$= -\frac{ka_2}{3}$$ 

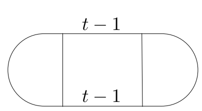
$$\left\langle \hat{\alpha}, \quad -\frac{a_2}{12} \right.$$ 

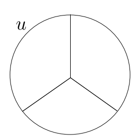
$$\left. \right\rangle_{\rm (conn)} = \frac{a_2}{12}$$ 
$$= \frac{a_2}{6}$$ 

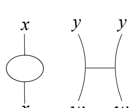
$$\left\langle \hat{\alpha}, \quad \frac{a_2}{6} \right.$$ 

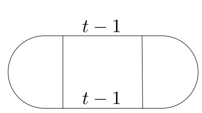
$$\left. \right\rangle_{\rm (conn)} = -\frac{a_2}{6}$$ 
$$= -\frac{a_2}{3}$$ 

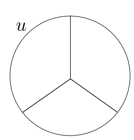
$$\left\langle \hat{\alpha}, \quad \frac{a_2}{8} \right.$$ 

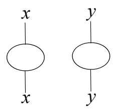
$$\left. \right\rangle_{\rm (conn)} = -\frac{ka_2}{4}$$ 

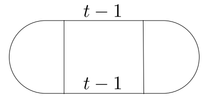
$$= \frac{ka_2}{2}$$ 

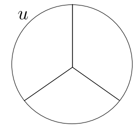
$$\left\langle \hat{\alpha}, \quad -\frac{a_2}{12} \right.$$ 

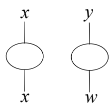
$$\left. \right\rangle_{\rm (conn)} = \frac{ka_2}{6}$$ 

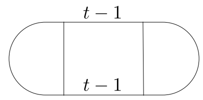
$$= -\frac{ka_2}{3}$$ 

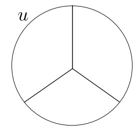
$\Big\langle \hat{\alpha},\quad \frac{k^2 a_2}{48}$ 

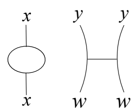
$\Big\rangle_{\text{(conn)}} = \frac{k^2 a_2}{24}$ 

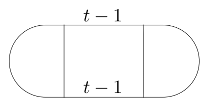
$= -\frac{k^2 a_2}{12}$ 

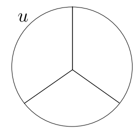
$\Big\langle \hat{\alpha},\quad \frac{k a_2}{12}$ 

$\Big\rangle_{\text{(conn)}} = -\frac{k^2 a_2}{6}$ 

$= \frac{k^2 a_2}{3}$ 

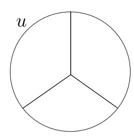
$\Big\langle \hat{\alpha},\quad -\frac{k a_2}{24}$ 

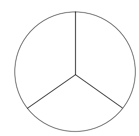
$\Big\rangle_{\text{(conn)}} = \frac{k^2 a_2}{12}$ 

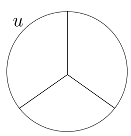
$= -\frac{k^2 a_2}{6}$ 

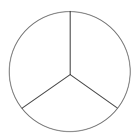
Hence,

$\langle \hat{\alpha}, \beta_2 \rangle_{(conn)}$

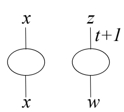
$= (\frac{k a_2}{2} - \frac{k a_2}{3} + \frac{k a_2}{2} - \frac{k a_2}{3} - \frac{k^2 a_2}{12} + \frac{k^2 a_2}{3} - \frac{k^2 a_2}{6})$ 

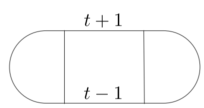
$+ (\frac{a_2}{6} - \frac{a_2}{3})$ 

$= (\frac{k a_2}{3} + \frac{k^2 a_2}{12})$ 

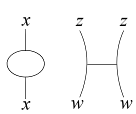
$- \frac{a_2}{6}$ 
$.$ (21)

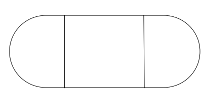
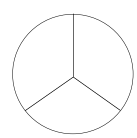
We calculate $\langle \hat{\alpha}, \beta_3 \rangle_{\text{(conn)}}$, as follows.

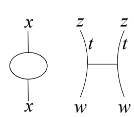
$\Big\langle \hat{\alpha},\quad \frac{a_2}{8}$ 

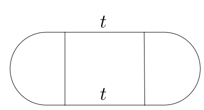
$\Big\rangle_{\text{(conn)}} = \frac{k a_2}{4}$ 

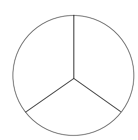
$= 0$

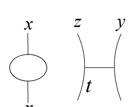
$\Big\langle \hat{\alpha},\quad \frac{a_2}{12}$ 

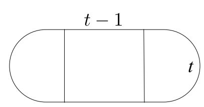
$\Big\rangle_{\text{(conn)}} = \frac{k a_2}{6}$ 
$= \frac{k a_2}{3}$ 

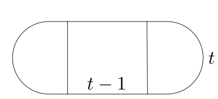
$\Big\langle \hat{\alpha},\quad -\frac{a_2}{12}$ 

$\Big\rangle_{\text{(conn)}} = -\frac{k a_2}{6}$ 

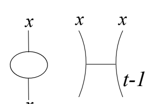
$= -\frac{k a_2}{3}$ 

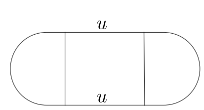
$\Big\langle \hat{\alpha},\quad -\frac{a_2}{4}$ 

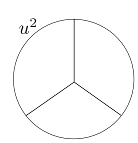
$\Big\rangle_{\text{(conn)}} = \frac{k a_2}{2}$ 

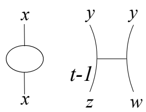
$- \frac{k a_2}{2}$ 

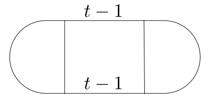
$= 0$

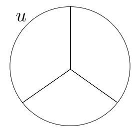
$\Big\langle \hat{\alpha},\quad \frac{a_2}{24}$ 

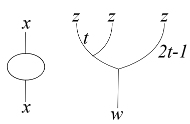
$\Big\rangle_{\text{(conn)}} = -\frac{k^2 a_2}{12}$ 

$= -\frac{k^2 a_2}{6}$ 

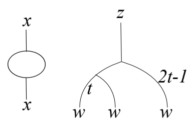
$$\left\langle \hat{\alpha}, \quad \frac{a_2}{24}\right.$$ 

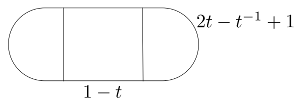
$$\left.\right\rangle_{\text{(conn)}} = -\frac{a_2}{12}$$ 

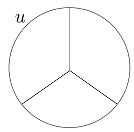
$$= \frac{a_2}{6}$$ 

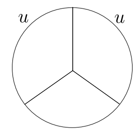
$$\left\langle \hat{\alpha}, \quad \frac{a_2}{24}\right.$$ 

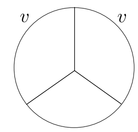
$$\left.\right\rangle_{\text{(conn)}} = 0$$

$$\left\langle \hat{\alpha}, \quad \frac{a_2}{24}\right.$$ 

$$\left.\right\rangle_{\text{(conn)}} = \frac{k^2 a_2}{12}$$ 

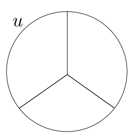
$$= -\frac{k^2 a_2}{6}$$ 

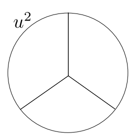
$$-\frac{k^2 a_2}{24}$$ 

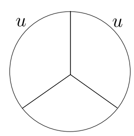
$$-\frac{k^2 a_2}{8}$$ 

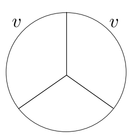
Hence,

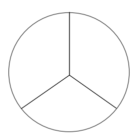
$\langle \hat{\alpha}, \beta_3 \rangle_{(conn)}$

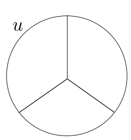
$$= \left(\frac{a_2}{6} - \frac{k^2 a_2}{6}\right)$$ 

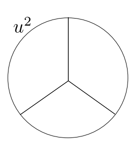
$$-\frac{k^2 a_2}{6}$$ 

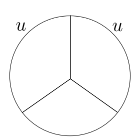
$$-\frac{k^2 a_2}{24}$$ 

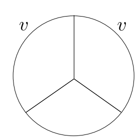
$$-\frac{k^2 a_2}{8}$$ 

$$+\left(\frac{k a_2}{3} - \frac{k a_2}{3}\right)$$ 

$$= \left(\frac{a_2}{6} - \frac{k^2 a_2}{6}\right)$$ 

$$-\frac{k^2 a_2}{6}$$ 

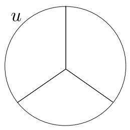
$$-\frac{k^2 a_2}{24}$$ 

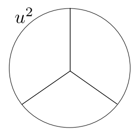
$$-\frac{k^2 a_2}{8}$$ 

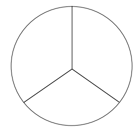
$$. \quad (22)$$

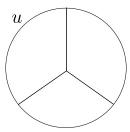
By (19), (20), (21), (22), we get

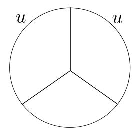
$$\langle \hat{\alpha}, (\beta_0'' + \beta_1 + \beta_2 + \beta_3) \rangle_{(conn)}$$
$$= \langle \hat{\alpha}, \beta_0'' \rangle_{(conn)} + \langle \hat{\alpha}, \beta_1 \rangle_{(conn)} + \langle \hat{\alpha}, \beta_2 \rangle_{(conn)} + \langle \hat{\alpha}, \beta_3 \rangle_{(conn)}$$

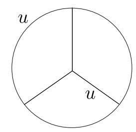
$$= (-\frac{a_2}{6} - \frac{k^2 a_2}{6} - \frac{k a_2}{3})$$ 

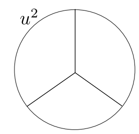
$$+ \frac{k^2 a_2}{6}$$ 

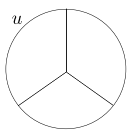
$$+ \frac{a_2}{6}$$ 

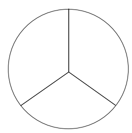
$$+ (-4 a_2 a_2' - 2 k a_3)$$ 

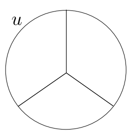
$$+ (2k^2 a_4 - \frac{k^2 a_2}{24})$$ 

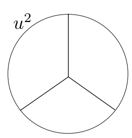
$$+ 3k^2 a_4$$ 

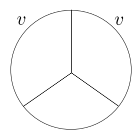
$$+ 2k^2 a_2^2$$ 

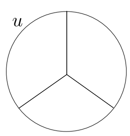
$$+ (\frac{k a_2}{3} + \frac{k^2 a_2}{12})$$ 

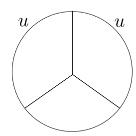
$$- \frac{a_2}{6}$$ 

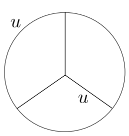
$$+ (\frac{a_2}{6} - \frac{k^2 a_2}{6})$$ 

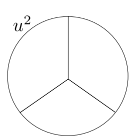
$$- \frac{k^2 a_2}{6}$$ 

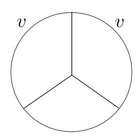
$$- \frac{k^2 a_2}{8}$$ 

$$= (-4 a_2 a_2' - \frac{k^2 a_2}{4} - 2 k a_3)$$ 

$$+ (-\frac{k^2 a_2}{24} + 2k^2 a_4)$$ 

$$+ 3k^2 a_4$$ 

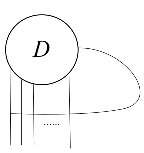
$$+ 2k^2 a_2$$ 

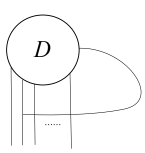
$$- \frac{k^2 a_2}{8}$$ 

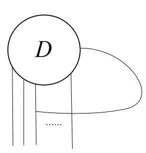
.

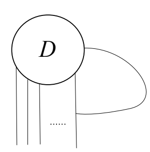
By the definition of the 3-loop polynomial, we get the required formula. □

Remark 6.1. The symbol "$\sim$" means the *link relation*, see [1], [2], [3],



$+$


$+$


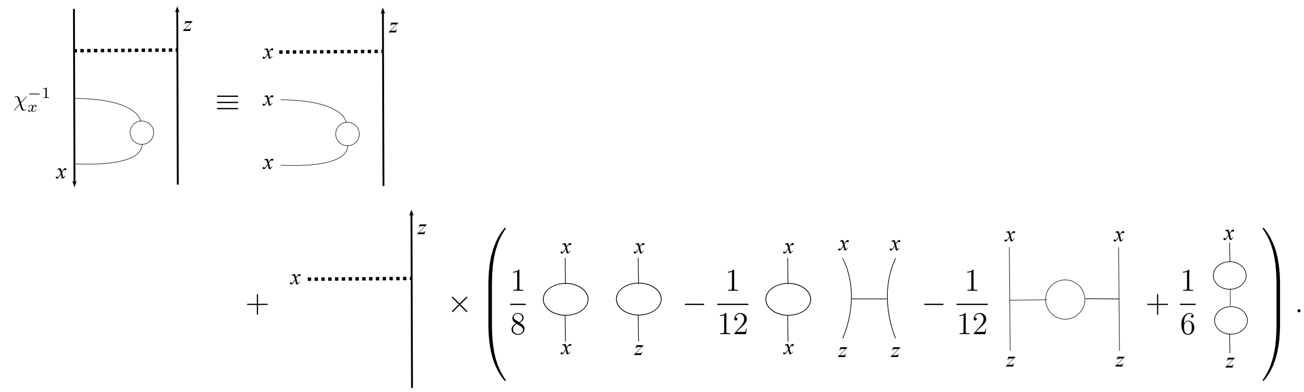
$+ \cdots +$


$\sim 0,$

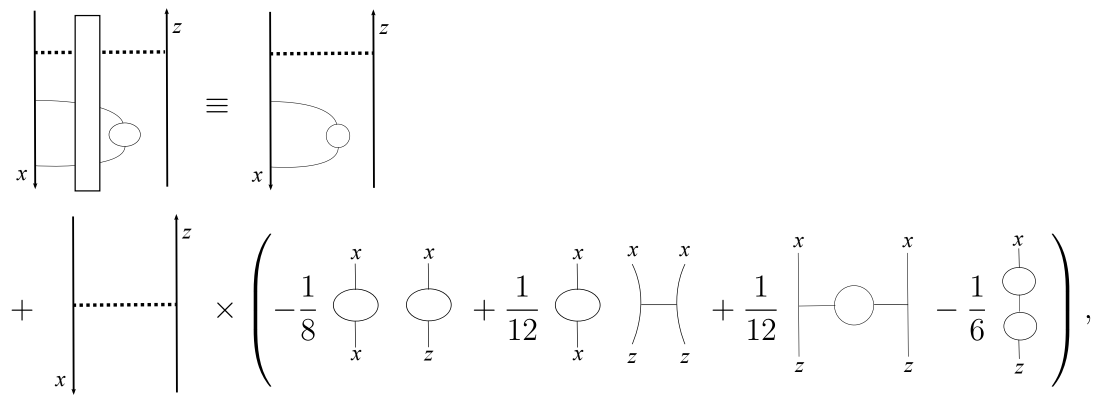
and "$\underset{(m+1)}{\sim}$" is the equivalent relation which is generated by $\underset{(m+1)}{\equiv}$ and $\sim$. It is known that under the link relation, the result of the Aarhus integral does not change ([6]).

Lastly, we prove the lemma used in the above calculation.

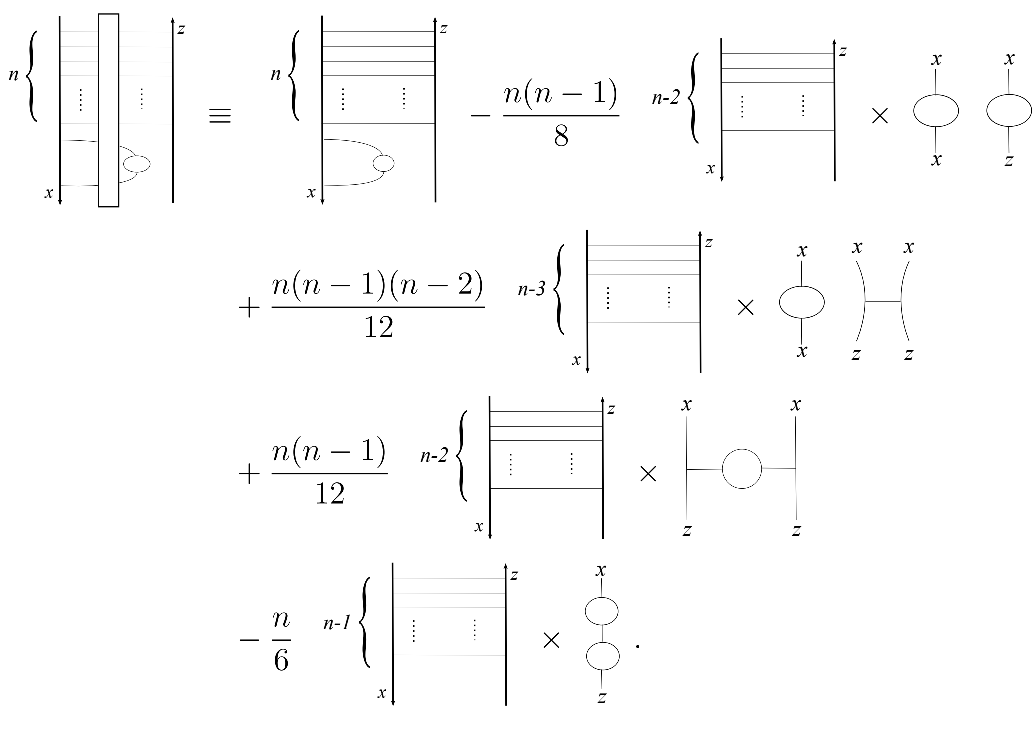
Lemma 6.2.



Proof. The required formula follows from the equation below,

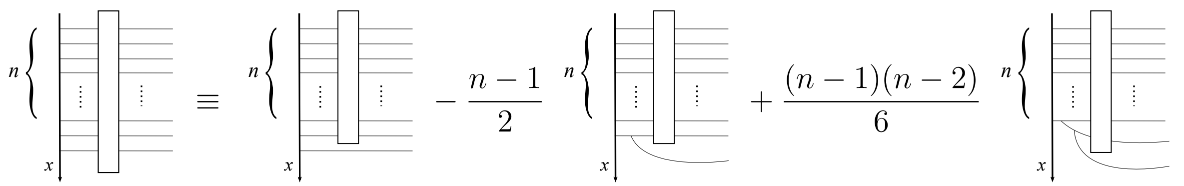


and in order to prove this, it is sufficient to show that the following formula,



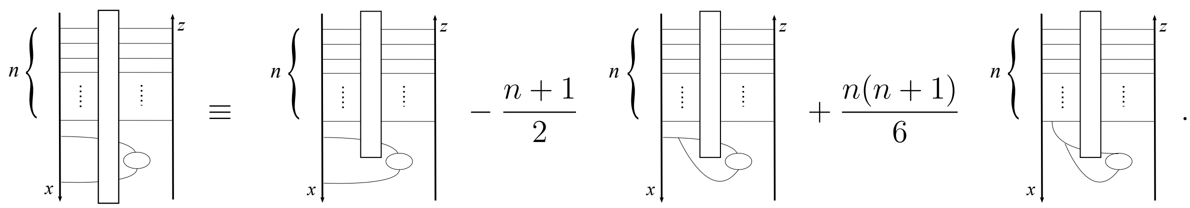
(23)

We show this. It is shown by a mirror image of Lemma 5.1 of [17] that



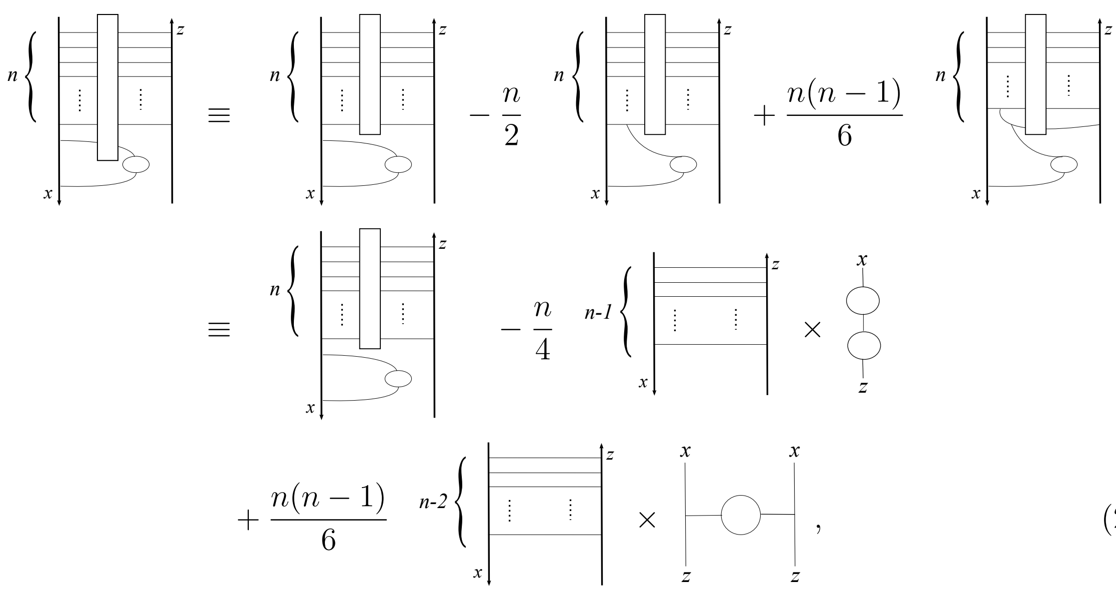
(24)

By applying (24) (replacing n with $n+2$) to the left-hand side of (23), we get



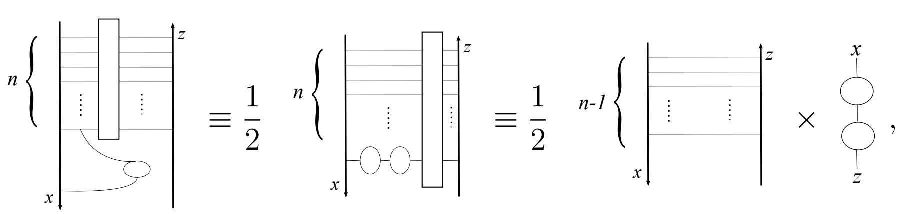
(25)

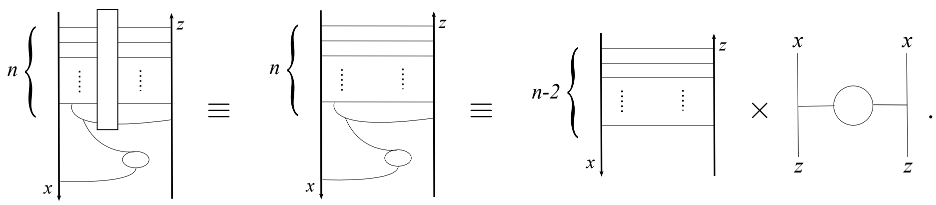
The first term of the right-hand side of (25) is calculated by applying (24) (replacing n with $n+1$) as follows,



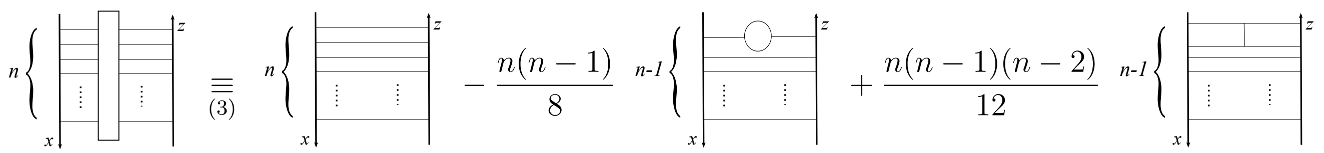
(26)

since it is shown that



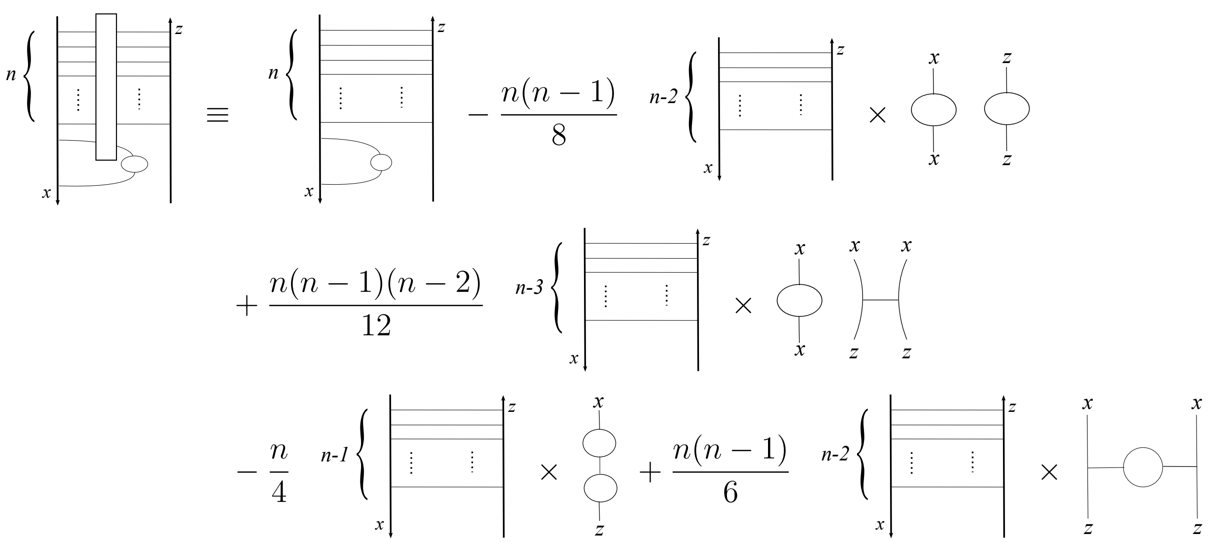


Further, it is shown that by Lemma 5.2 in [17] that



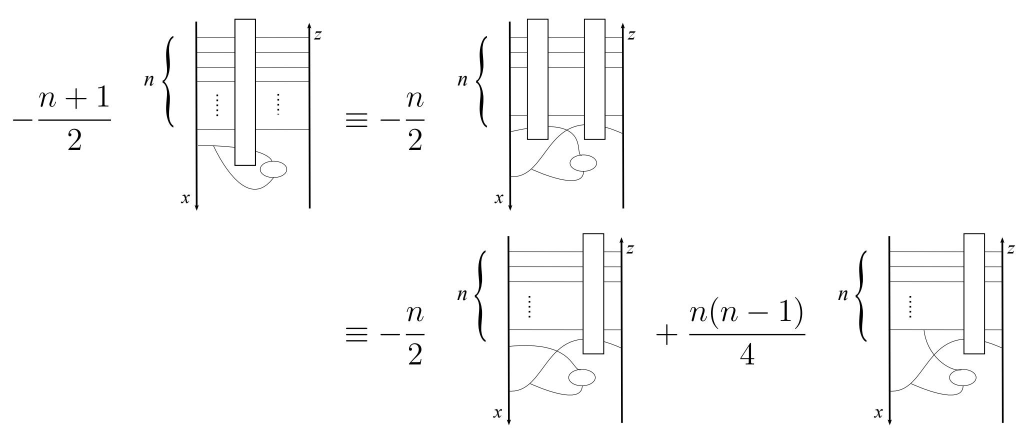
(27)

By applying (27) to (26), we obtain that

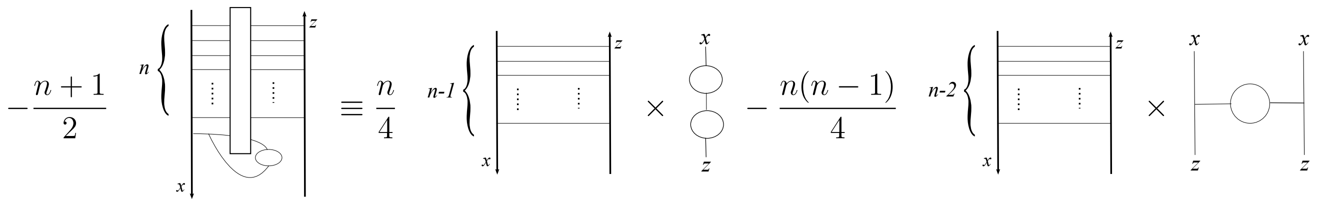


(28)

The second term of the right-hand side of (25) is calculated as follows,

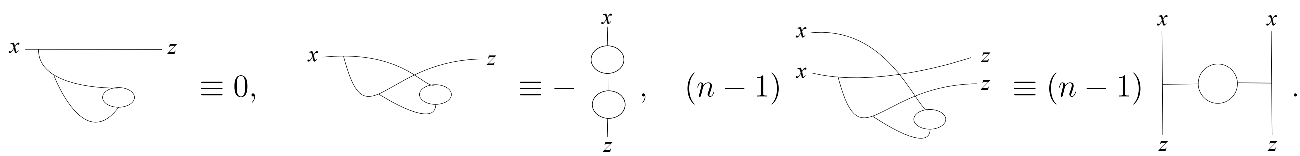


where we obtain the first equivalence by classifying the strand which is connected to the bottom strand at the right hand side of the symmetrizer of the first term, and we obtain the second equivalence by applying (24) to the left symmetrizer of the diagram of the right-hand side of the first line. Hence,

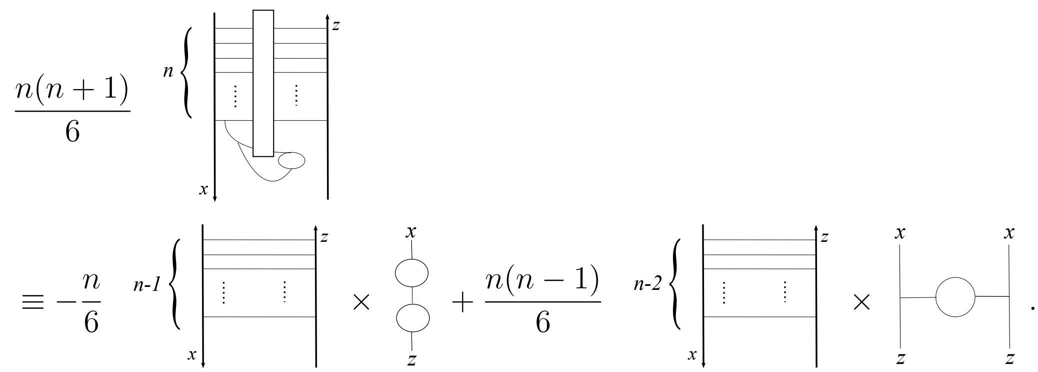


(29)

The third term of the right-hand side of (25) is calculated as follows. By classifying the strand which is connected to the bottom strand at the right hand side of the symmetrizer, the connected component of this strand is shown, as follows,



Then, we get



(30)

Thus, by applying (28), (29), (30) to (25), we obtain (23). Therefore, we obtain the required formula of the lemma. □

7 The $\mathfrak{sl}_2$ reduction of the 3-loop invariant

7.1 The review of the loop expansion of the colored Jones polynomial

In this section, we briefly review the colored Jones polynomial and the loop expansion of it. For details, see [22], [23].

The colored Jones polynomial $J_n(K;t)$ is the polynomial invariant of knots, which is obtained by

$$J_n(K;t) = \frac{V_n(K;t)}{V_n(\text{the unknot};t)} = \frac{t^{1/2} - t^{-1/2}}{t^{n/2} - t^{-n/2}} \cdot V_n(K;t),$$

where $V_n(K;t)$ is obtained by $V_n(K;e^{-h}) = W_{\mathfrak{sl}_2,V_n}\big(Z(K)\big)$, and $W_{\mathfrak{sl}_2,V_n}$ denotes the weight system derived from the Lie algebra $\mathfrak{sl}_2$ and its irreducible representation V_n. For details, see [14], [15], and [13]. It is known, see Conjecture 1.2 of [22], Theorem 1.2 of [23], Proposition 3.1 of [20], that $J_n(K;t)$ can be presented in the following form,

$$J_n(K;e^h) = \sum_{l\geq 0} h^l \sum_{k\geq 0} d_{l,k}(nh)^k = \sum_{l\geq 0} h^l \frac{P_l(e^{nh})}{\Delta_K(e^{nh})^{2l+1}}$$

for some $P_l(t) \in \mathbb{Q}[t^{\pm 1}]$. This is called the loop expansion of the colored Jones polynomial. The 3-loop part of the colored Jones polynomial is given by $\dfrac{P_2(e^{nh})}{\Delta_K(e^{nh})^5}$.

7.2 The 3-loop part of the colored Jones polynomial

In this section, we consider the 3-loop part of the colored Jones polynomial.

For a knot K, $\Lambda_K(t^{\frac{1}{2}}, t^{\frac{1}{2}}, t^{-\frac{1}{2}}, t^{-\frac{1}{2}})$ is a symmetric polynomial in $t^{\pm 1}$ divisible by $t-1$ (since $\Lambda_K(1,1,1,1) = 0$) and, hence, divisible by $(t-1)^2$. We define the *reduced 3-loop invariant* by

$$\hat{\Lambda}_K(t) = \frac{\Lambda_K(t^{\frac{1}{2}}, t^{\frac{1}{2}}, t^{-\frac{1}{2}}, t^{-\frac{1}{2}})}{(t^{1/2}-t^{-1/2})^2} \in \mathbb{Q}(t^{\pm 1})^1,$$

which is symmetricl in $t^{\pm 1}$. If $\Delta_K(t) = 1$, then this is a polynomial, so we call it the *reduced 3-loop polynomial*.

Example 7.1. The reduced 3-loop polynomial of $D(K, K')$ is presented by

$$\hat{\Lambda}_{D(K,K')}(t)$$
$$= (-\frac{4k^2a_2}{3} + 64k^2a_4 + 32k^2a_2^2)(t+t^{-1}) - \frac{16k^2a_2}{3} - 64k^2a_2^2 - 64a_2a_2' - 32ka_3 - 128k^2a_4.$$

We denote the reduced 2-loop polynomial by $\hat{\Theta}_K(t) = \dfrac{\Theta_K(t, t^{-1}, 1)}{(t^{1/2}-t^{-1/2})^2}$ defined in [16], where $\Theta_K(t_1, t_2, t_3)$ is the 2-loop polynomial of K. For definition of the 2-loop polynomial, see for example [14], [17].

Proposition 7.2. *The 3-loop part of the colored Jones polynomial* $\dfrac{P_2(e^{nh})}{\Delta_K(e^{nh})^5}$ *is presented by*

$$\frac{P_2(t)}{\Delta_K(t)^5} = (t^{1/2}-t^{-1/2})^2 \frac{\hat{\Lambda}_K(t)}{\Delta_K(t)} + \frac{(t^{1/2}-t^{-1/2})^4}{2\Delta_K(t)^5}\hat{\Theta}_K(t)^2$$
$$+ \frac{\Delta_K'(t)t^2}{3(t-1)\Delta_K(t)^2} + \frac{\Delta_K''(t)t^2}{6\Delta_K(t)^2} - \frac{\Delta_K'(t)^2t^2}{3\Delta_K(t)^3}.$$

Proof. In this proof, we write $\alpha \equiv_{\mathfrak{sl}_2} \beta$ if $W_{\mathfrak{sl}_2,V_n}(\alpha) = W_{\mathfrak{sl}_2,V_n}(\beta)$. It is shown in the proof of Proposition 3.1 in [20] that $\displaystyle\sum_{l\geq 0} \frac{P_l(t)}{\Delta_K(t)^{2l+1}} h^l$ is given by

$$\sum_{l\geq 0} \frac{P_l(t)}{\Delta_K(t)^{2l+1}} h^l = \frac{1}{t^{1/2}-t^{-1/2}} \cdot \frac{\mathcal{D}}{[\mathcal{D}]} \left((t^{1/2}-t^{-1/2}) \sum_{l\geq 0} \frac{\hat{P}_l(t)}{\Delta_K(t)^{2l+1}} h^l \right), \tag{31}$$

where

$$\mathcal{D} = 2t\frac{d}{dt},$$

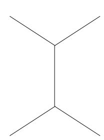
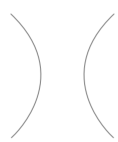
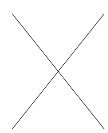
$$\frac{\mathcal{D}}{[\mathcal{D}]} = \frac{e^{h/2}-e^{-h/2}}{e^{h\mathcal{D}/2}-e^{-h\mathcal{D}/2}}\mathcal{D} = 1 + \frac{h^2}{24}(1-\mathcal{D}^2) + (\text{higher terms}),$$

$$\chi^{-1}Z(K) \sqcup (\chi^{-1}\nu)^{-1} \equiv_{\mathfrak{sl}_2} \sum_{l\geq 0} \frac{\hat{P}_l(\hat{t})}{\Delta_K(\hat{t})^{2l+1}} h^l, \quad \text{for some } \hat{P}(\hat{t}) \in \mathbb{Q}[\hat{t}^{\pm 1}].$$

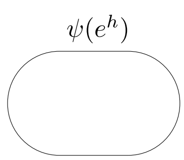
Here, $\hat{t} = e^{\sqrt{2C}h}$, and C denotes the Casimir element of $\mathfrak{sl}_2$ whose eigenvalue on V_n is equal to $\dfrac{n^2-1}{2}$. At first, we calculate $\hat{P}_l(\hat{t})$. We put

$$\tilde{\Lambda}_K(t) = (t^{1/2}-t^{-1/2})^2 \hat{\Lambda}_K(t), \quad \tilde{\Theta}_K(t) = (t^{1/2}-t^{-1/2})^2 \hat{\Theta}_K(t).$$

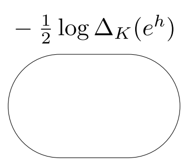
By using the equivalence below,

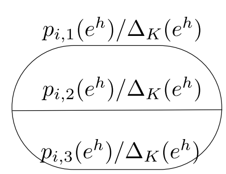
 $\equiv_{\mathfrak{sl}_2} 2h \Big($  $-$  $\Big)$, (32)

it is shown in [16], [20] that



$\equiv_{\mathfrak{sl}_2} \psi(\hat{t}) + \psi(\hat{t}^{-1}) + \psi(1)$ (33)

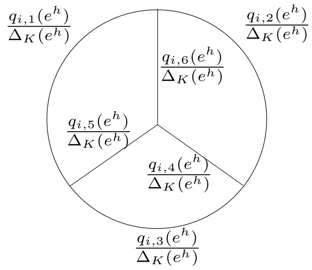
exp $\Big($ 

$\Big) \equiv_{\mathfrak{sl}_2} \dfrac{1}{\Delta_K(\hat{t})}$, $\quad \displaystyle\sum_i$ 

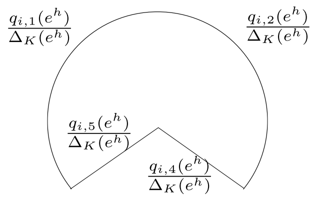
$\equiv_{\mathfrak{sl}_2} \dfrac{\tilde{\Theta}_K(\hat{t})}{\Delta_K(\hat{t})^2} h,$

Further, by (33) and Lemma 7.4 below, we obtain

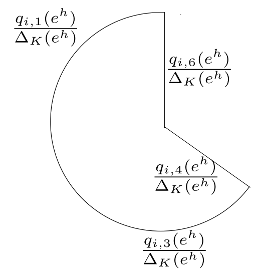
$$\sum_i$$



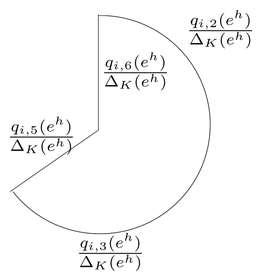
$$\equiv_{\mathfrak{sl}_2} \sum_i 4h^2 \Bigg(q_{i,3}(1) q_{i,6}(1)$$



$$+ q_{i,2}(1) q_{i,5}(1)$$



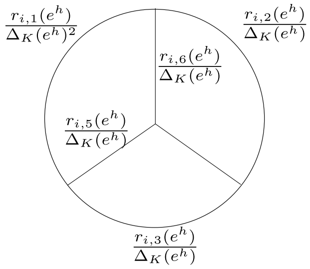
$$+ q_{i,1}(1) q_{i,4}(1)$$

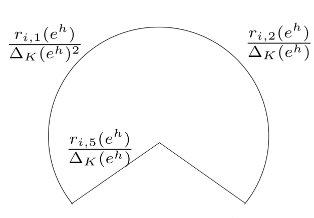


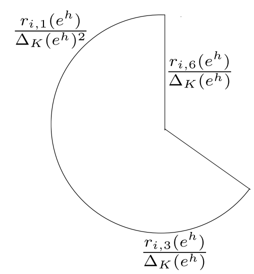
$$\Bigg)$$

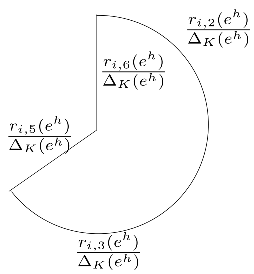
$$\begin{aligned}
&\equiv_{\mathfrak{sl}_2} \sum_i \frac{4h^2}{\Delta_K(\hat{t})^4} \big(q_{i,1}(\hat{t}) q_{i,2}(\hat{t}) q_{i,3}(1) q_{i,4}(\hat{t}) q_{i,5}(\hat{t}^{-1}) q_{i,6}(1) + q_{i,1}(\hat{t}^{-1}) q_{i,2}(\hat{t}^{-1}) q_{i,3}(1) q_{i,4}(\hat{t}^{-1}) q_{i,5}(\hat{t}) q_{i,6}(1) \\
&\quad + q_{i,1}(\hat{t}) q_{i,2}(1) q_{i,3}(\hat{t}) q_{i,4}(\hat{t}^{-1}) q_{i,5}(1) q_{i,6}(\hat{t}) + q_{i,1}(\hat{t}^{-1}) q_{i,2}(1) q_{i,3}(\hat{t}^{-1}) q_{i,4}(\hat{t}) q_{i,5}(1) q_{i,6}(\hat{t}^{-1}) \\
&\quad + q_{i,1}(1) q_{i,2}(\hat{t}) q_{i,3}(\hat{t}) q_{i,4}(1) q_{i,5}(\hat{t}) q_{i,6}(\hat{t}^{-1}) + q_{i,1}(1) q_{i,2}(\hat{t}^{-1}) q_{i,3}(\hat{t}^{-1}) q_{i,4}(1) q_{i,5}(\hat{t}^{-1}) q_{i,6}(\hat{t}) \big).
\end{aligned}$$

and

$$\sum_i$$ 

$$\equiv_{\mathfrak{sl}_2} \sum_i 4h^2 \Bigg(r_{i,3}(1) r_{i,6}(1)$$ 

$$+ r_{i,2}(1) r_{i,5}(1)$$ 

$$+ r_{i,1}(1)$$ 

$$\Bigg)$$

$$\begin{aligned}
&\equiv_{\mathfrak{sl}_2} \sum_i \frac{4h^2}{\Delta_K(\hat{t})^4} \big(r_{i,1}(\hat{t}) r_{i,2}(\hat{t}) r_{i,3}(1) r_{i,5}(\hat{t}^{-1}) r_{i,6}(1) + r_{i,1}(\hat{t}^{-1}) r_{i,2}(\hat{t}^{-1}) r_{i,3}(1) r_{i,5}(\hat{t}) r_{i,6}(1) \\
&\quad + r_{i,1}(\hat{t}) r_{i,2}(1) r_{i,3}(\hat{t}) r_{i,5}(1) r_{i,6}(\hat{t}) + r_{i,1}(\hat{t}^{-1}) r_{i,2}(1) r_{i,3}(\hat{t}^{-1}) r_{i,5}(1) r_{i,6}(\hat{t}^{-1}) \\
&\quad + r_{i,1}(1) r_{i,2}(\hat{t}) r_{i,3}(\hat{t}) r_{i,5}(\hat{t}) r_{i,6}(\hat{t}^{-1}) + r_{i,1}(1) r_{i,2}(\hat{t}^{-1}) r_{i,3}(\hat{t}^{-1}) r_{i,5}(\hat{t}^{-1}) r_{i,6}(\hat{t}) \big).
\end{aligned}$$

Note that $q_{i,1}(1) q_{i,2}(1) q_{i,3}(1) q_{i,4}(1) q_{i,5}(1) q_{i,6}(1) = r_{i,1}(1) r_{i,2}(1) r_{i,3}(1) r_{i,5}(1) r_{i,6}(1) = 0$. Hence, we get

$$\begin{aligned}
\sum_{l \geq 0} \frac{\hat{P}_l(\hat{t})}{\Delta_K(\hat{t})^{2l+1}} h^l &\equiv_{\mathfrak{sl}_2} \frac{1}{\Delta_K(\hat{t})} \left(1 + \frac{\tilde{\Theta}_K(\hat{t})}{\Delta_K(\hat{t})^2} h + \tilde{\Lambda}_K(\hat{t}) h^2 + \frac{\tilde{\Theta}_K(\hat{t})^2}{2\Delta_K(\hat{t})^4} h^2 + (\text{higher terms}) \right) \\
&= \frac{1}{\Delta_K(\hat{t})} + \frac{\tilde{\Theta}_K(\hat{t})}{\Delta_K(\hat{t})^3} h + \left(\frac{\tilde{\Lambda}_K(\hat{t})}{\Delta_K(\hat{t})} + \frac{\tilde{\Theta}_K(\hat{t})^2}{2\Delta_K(\hat{t})^5} \right) h^2 + (\text{higher terms}).
\end{aligned}$$

Thus, by (31), we obtain

$$\begin{aligned}
\sum_{l \geq 0} \frac{P_l(t)}{\Delta_K(t)^{2l+1}} h^l = \frac{1}{t^{1/2} - t^{-1/2}} \cdot \frac{\mathcal{D}}{[\mathcal{D}]} \Bigg((t^{1/2} - t^{-1/2}) \Big(\frac{1}{\Delta_K(t)} + \frac{\tilde{\Theta}_K(t)}{\Delta_K(t)^3} h \\
+ \left(\frac{\tilde{\Lambda}_K(t)}{\Delta_K(t)} + \frac{\tilde{\Theta}_K(t)^2}{2\Delta_K(t)^5} \right) h^2 + (\text{higher terms}) \Big) \Bigg)
\end{aligned}$$

$$= \frac{1}{\Delta_K(t)} + \frac{\tilde{\Theta}_K(t)}{\Delta_K(t)^3}h + \left(\frac{\tilde{\Lambda}_K(t)}{\Delta_K(t)} + \frac{\tilde{\Theta}_K(t)^2}{2\Delta_K(t)^5}\right)h^2$$

$$+ \frac{h^2}{24}\cdot\frac{1}{t^{1/2}-t^{-1/2}}(1-\mathcal{D}^2)\Big(\frac{t^{1/2}-t^{-1/2}}{\Delta_K(t)}\Big) + \text{(higher terms)}. \qquad (34)$$

Here, the last term of the right-hand side of (34) is calculated as follows,

$$\frac{1}{t^{1/2}-t^{-1/2}}(1-\mathcal{D}^2)\Big(\frac{t^{1/2}-t^{-1/2}}{\Delta_K(t)}\Big) = \frac{1}{t^{1/2}-t^{-1/2}}\left(1-\left(2t\frac{d}{dt}\right)^2\right)\Big(\frac{t^{1/2}-t^{-1/2}}{\Delta_K(t)}\Big)$$

$$= \frac{8\Delta_K'(t)t^2}{(t-1)\Delta_K(t)^2} + \frac{4\Delta_K''(t)t^2}{\Delta_K(t)^2} - \frac{8\Delta_K'(t)^2t^2}{\Delta_K(t)^3}. \qquad (35)$$

By applying (35) to (34), we get

$$\sum_{l\geq 0}\frac{P_l(t)}{\Delta_K(t)^{2l+1}}h^l = \frac{1}{\Delta_K(t)} + \frac{\tilde{\Theta}_K(t)}{\Delta_K(t)^3}h$$

$$+ \left(\frac{\tilde{\Lambda}_K(t)}{\Delta_K(t)} + \frac{\tilde{\Theta}_K(t)^2}{2\Delta_K(t)^5} + \frac{\Delta_K'(t)t^2}{3(t-1)\Delta_K(t)^2} + \frac{\Delta_K''(t)t^2}{6\Delta_K(t)^2} - \frac{\Delta_K'(t)^2t^2}{3\Delta_K(t)^3}\right)h^2$$

$$+ \text{(higher terms)}.$$

Therefore, considering the h^2 terms, we obtain the required formula. □

We recall that the Conway polynomial $\nabla_K(z)$ is defined by $\nabla_K(t^{1/2}-t^{-1/2}) = \Delta_K(t)$.

Remark 7.3. The formula of Proposition 7.2 is rewritten in terms of the Conway polynomial as

$$\frac{P_2(t)}{\Delta_K(t)^5} = \frac{z^2}{\nabla_K(z)}\hat{\Lambda}_K(t) + \frac{z^4}{2\nabla_K(z)^5}\hat{\Theta}_K(t)^2$$

$$+ \frac{3z^2+8}{24z}\frac{\nabla_K'(z)}{\nabla_K(z)^2} + \frac{z^2+4}{24}\frac{\nabla_K''(z)}{\nabla_K(z)^2} - \frac{z^2+4}{12}\frac{\nabla_K'(z)^2}{\nabla_K(z)^3},$$

where $z = t^{1/2} - t^{-1/2}$.

Proof. For $z = t^{1/2} - t^{-1/2}$, we get

$$2t\frac{dz}{dt} = t^{1/2} + t^{-1/2}, \quad \mathcal{D} = 2t\frac{d}{dt} = (t^{1/2}+t^{-1/2})\frac{d}{dz},$$

$$\mathcal{D}\left(\frac{t^{1/2}-t^{-1/2}}{\Delta_K(t)}\right) = 2t\frac{d}{dt}\frac{t^{1/2}-t^{-1/2}}{\Delta_K(t)} = (t^{1/2}+t^{-1/2})\frac{d}{dz}\frac{z}{\nabla_K(z)},$$

$$\mathcal{D}^2\left(\frac{t^{1/2}-t^{-1/2}}{\Delta_K(t)}\right) = 2t\frac{d}{dt}\left((t^{1/2}+t^{-1/2})\frac{d}{dz}\frac{z}{\nabla_K(z)}\right)$$

$$= (t^{1/2}-t^{-1/2})\frac{d}{dz}\frac{z}{\nabla_K(z)} + (t^{1/2}+t^{-1/2})^2(\frac{d}{dz})^2\frac{z}{\nabla_K(z)}$$

$$= (t^{1/2} - t^{-1/2}) \frac{d}{dz} \frac{z}{\nabla_K(z)} + (z^2+4) \left(\frac{d}{dz} \right)^2 \frac{z}{\nabla_K(z)}.$$

Thus, we obtain

$$\begin{aligned} \frac{1}{t^{1/2} - t^{-1/2}} \mathcal{D}^2 \left(\frac{t^{1/2} - t^{-1/2}}{\Delta_K(t)} \right) &= \frac{d}{dz} \frac{z}{\nabla_K(z)} + \frac{z^2+4}{z} \left(\frac{d}{dz} \right)^2 \frac{z}{\nabla_K(z)} \\ &= \frac{1}{\nabla_K(z)} - \frac{3z^2+8}{z} \frac{\nabla'_K(z)}{\nabla_K(z)^2} - (z^2+4) \frac{\nabla''_K(z)}{\nabla_K(z)^2} + 2(z^2+4) \frac{\nabla'_K(z)^2}{\nabla_K(z)^3}. \end{aligned}$$

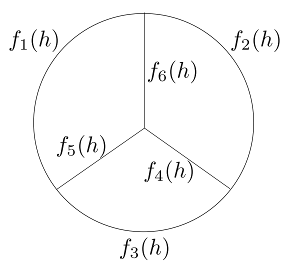
Therefore, the formula (35) can be rewritten as

$$\frac{1}{t^{1/2} - t^{-1/2}} (1 - \mathcal{D}^2) \left(\frac{t^{1/2} - t^{-1/2}}{\Delta_K(t)} \right) = \frac{3z^2+8}{z} \frac{\nabla'_K(z)}{\nabla_K(z)^2} + (z^2+4) \frac{\nabla''_K(z)}{\nabla_K(z)^2} - 2(z^2+4) \frac{\nabla'_K(z)^2}{\nabla_K(z)^3}.$$

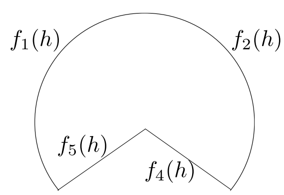
By applying this to (34), we obtain the required formula. Hence, the formula of Proposition 7.2 is rewritten as the formula of the remark. □

We prove the lemma used in the proof of Proposition 7.2.

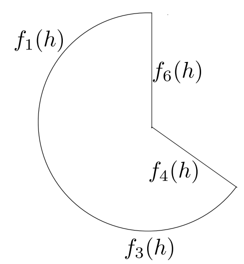
Lemma 7.4.



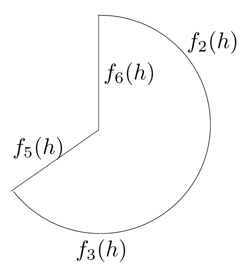
$\equiv_{\mathfrak{sl}_2} 4h^2 \Bigg(f_3(0) f_6(0)$



$+ f_2(0) f_5(0)$



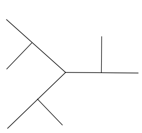
$+ f_1(0) f_4(0)$



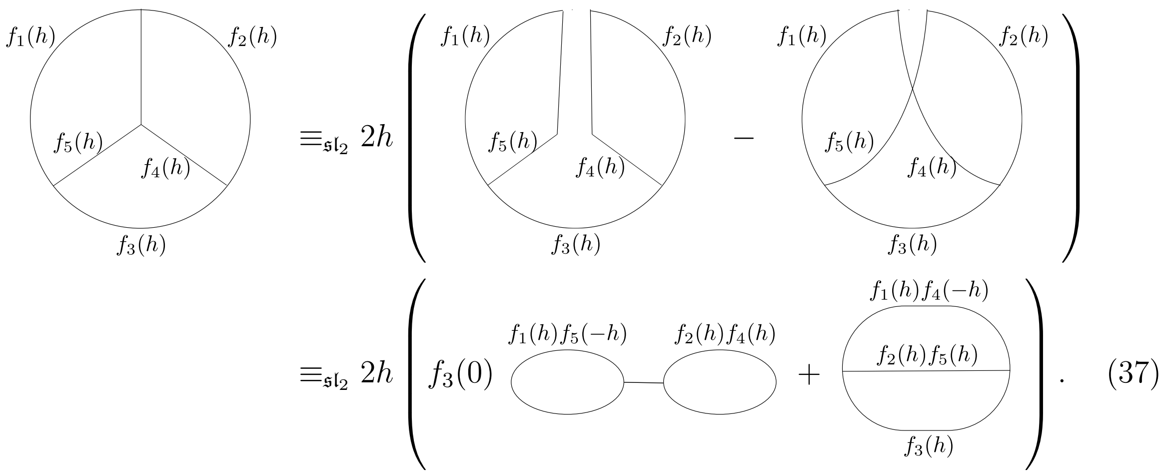
$\Bigg)$,

where $f_i(h)$ is a power series in h.

Proof. By using (32), we get the following equivalences,

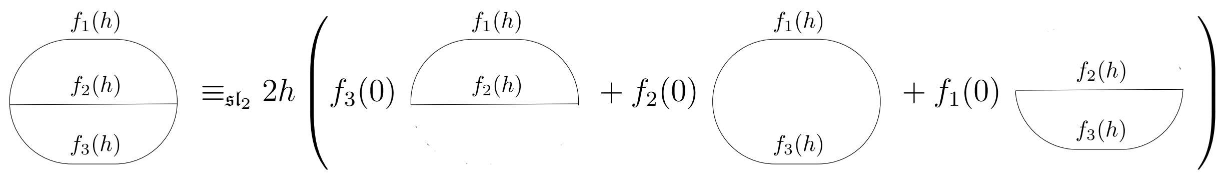


$\equiv_{\mathfrak{sl}_2} 0,$ (36)



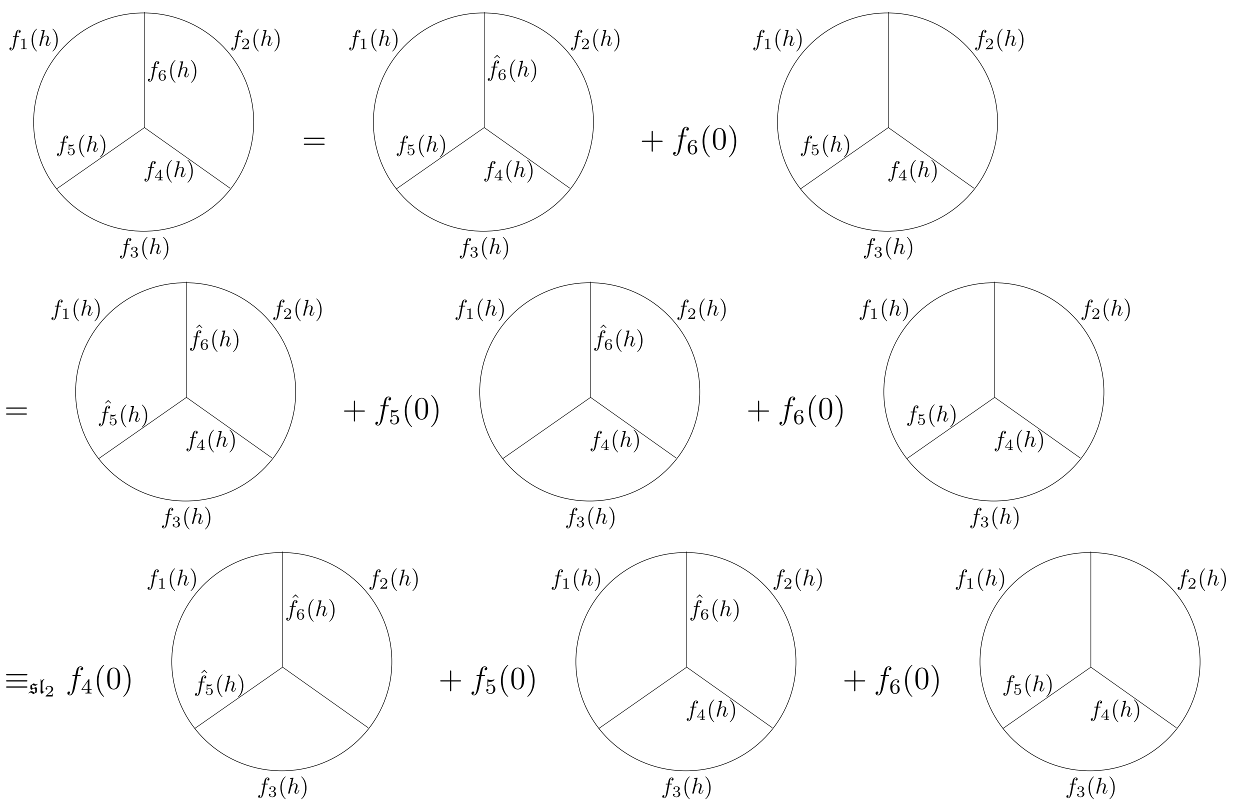
(37)

Further, it is shown by the formula after Lemma 6.2 in [16] that



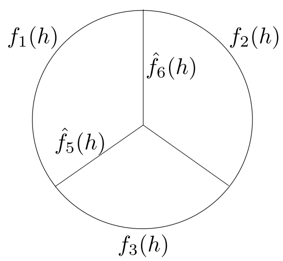
(38)

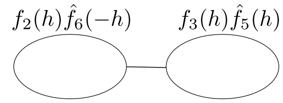
We put $\hat{f}_i(h) = f_i(h) - f_i(0)$. Then, by (36),

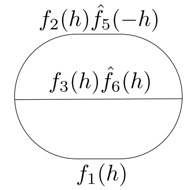


(39)

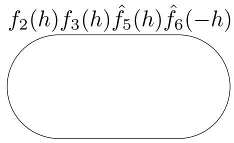
By (37),(38), the first term of the right-hand side of (39) is calculated as follows,

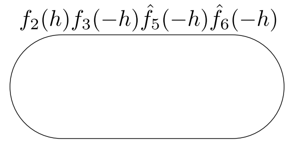
$f_4(0)$ 

$\equiv_{\mathfrak{sl}_2} 2hf_4(0) \Bigg(f_1(0)$ 

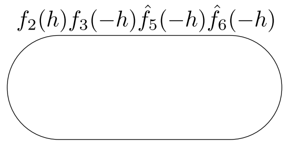
$+$ 

$\Bigg)$

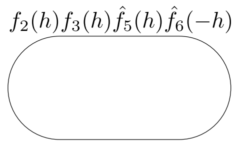
$\equiv_{\mathfrak{sl}_2} 2hf_4(0) \Bigg(2hf_1(0) \Big($ 

$-$ 

$\Big)$

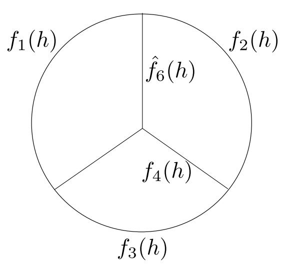
$+2hf_1(0)\Big($ 

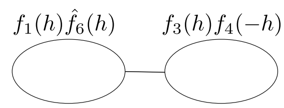
$\Big)\Bigg)$

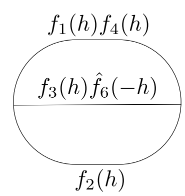
$\equiv_{\mathfrak{sl}_2} 4h^2 f_1(0)f_4(0)$ 

. (40)

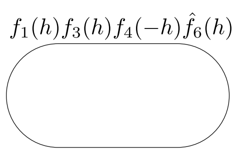
By (37),(38), the second term of the right-hand side of (39) is calculated as follows,

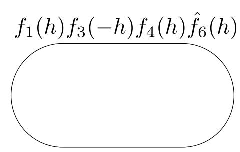
$f_5(0)$ 

$\equiv_{\mathfrak{sl}_2} 2hf_5(0) \Bigg(f_2(0)$ 

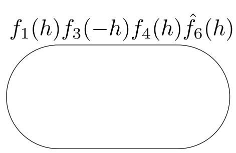
$+$ 

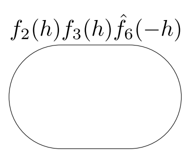
$\Bigg)$

$\equiv_{\mathfrak{sl}_2} 2hf_5(0) \Bigg(2hf_2(0) \Big($ 

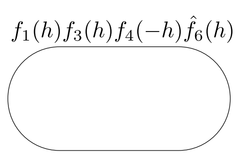
$-$ 

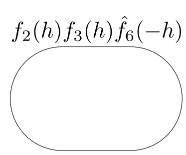
$\Big)$

$+2h\Big(f_2(0)$ 

$+ f_1(0)f_4(0)$ 

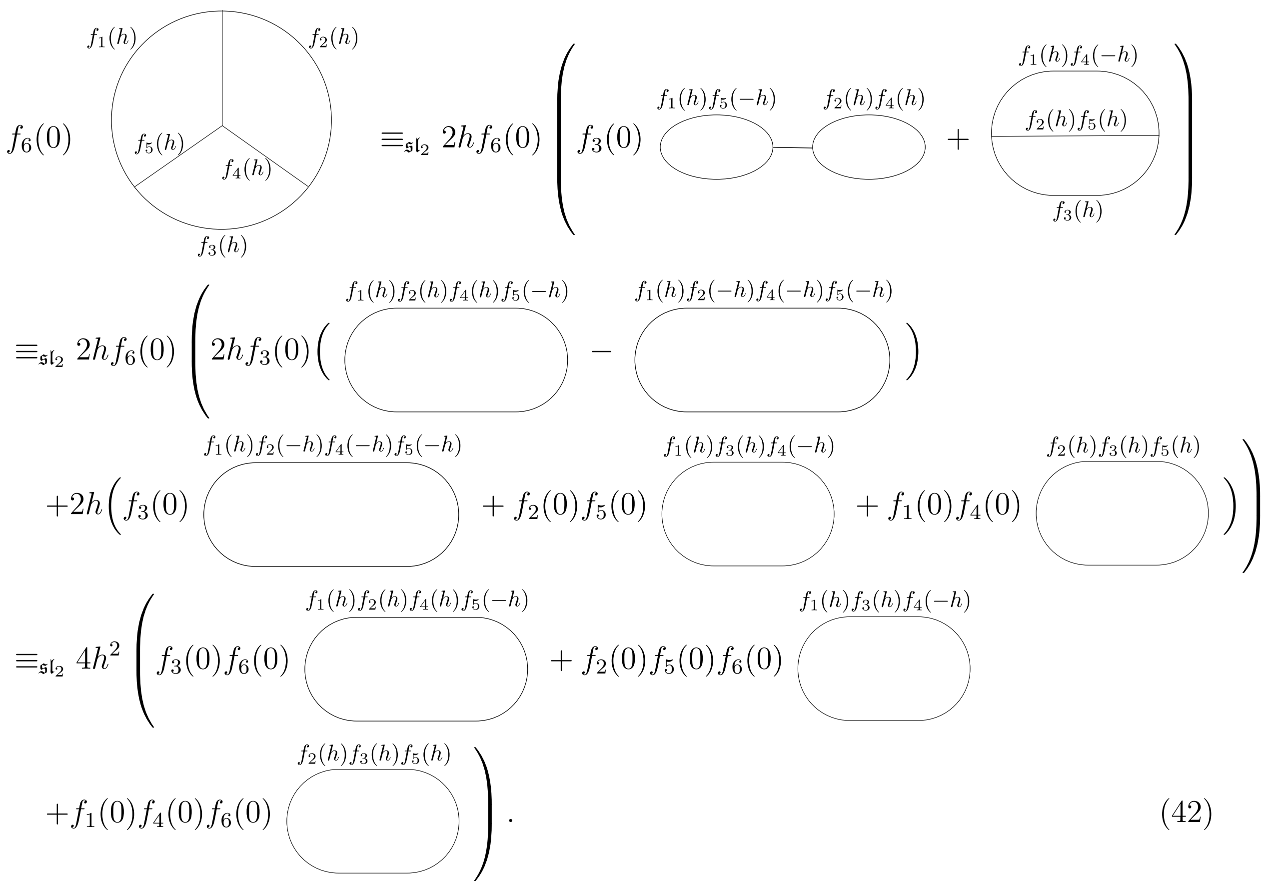
$\Big)\Bigg)$

$\equiv_{\mathfrak{sl}_2} 4h^2 \Bigg(f_2(0)f_5(0)$ 

$+ f_1(0)f_4(0)f_5(0)$ 

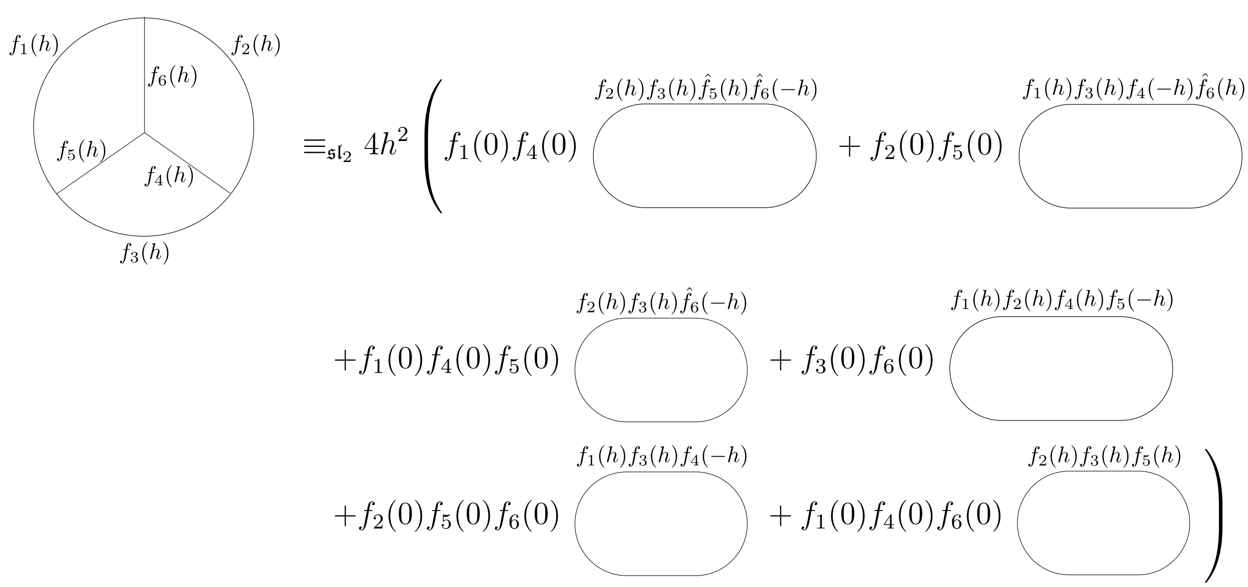
$\Bigg)$. (41)

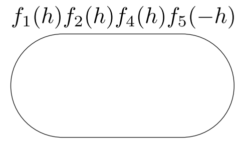
By (37),(38), the third term of the right-hand side of (39) is calculated as follows,

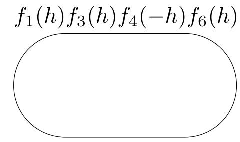


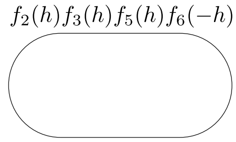
(42)

Thus, by applying (40), (41), (42) to (39), we obtain



$$\equiv_{\mathfrak{sl}_2} 4h^2 \Bigg(f_3(0)f_6(0)$$ 

$$+ f_2(0)f_5(0)$$ 

$$+f_1(0)f_4(0)$$ 

$$\Bigg).$$

Therefore, we obtain the lemma. □

Appendix

A The Duflo isomorphism and the proof of the connected sum formula for the 3-loop invariant

In this section, we review the *Duflo isomorphism* ([4], [5]), and we prove Proposition 4.2 (the connected sum formula for the 3-loop invariant) in Section 4.2. For the notation, see Section 4.2.

Let D' be a diagram which have at least one trivalent vertex on each component. Then we define $\partial_{D'} : \mathcal{B} \to \mathcal{B}$ by

$$\partial_{D'}(D) = \begin{cases} 0 & \textit{if } D' \textit{ has more legs than } D, \\ \textit{the sum of all ways of gluing} & \\ \textit{all the legs of } D' \textit{ to some} & \textit{otherwise} \\ \textit{(or all) legs of } D & \end{cases}$$

Duflo isomorphism $\Upsilon : \mathcal{B} \to \mathcal{B}$ is defined by

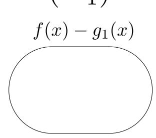
$$\Upsilon = \chi \circ \partial_\Omega,$$

where we denote $\Omega = \chi^{-1}\nu$. It is known [5] that Υ is an algebra isomorphism. Note that $\Upsilon^{-1}Z(K)$ is group-like.

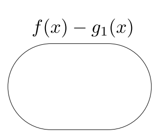
Then, we prove Proposition 4.2.

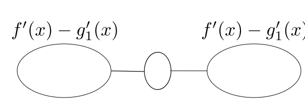
Proof of Proposition 4.2.

$$\Upsilon^{-1}Z(K_1) = \partial_{\Omega^{-1}}\chi^{-1}Z(K_1)$$

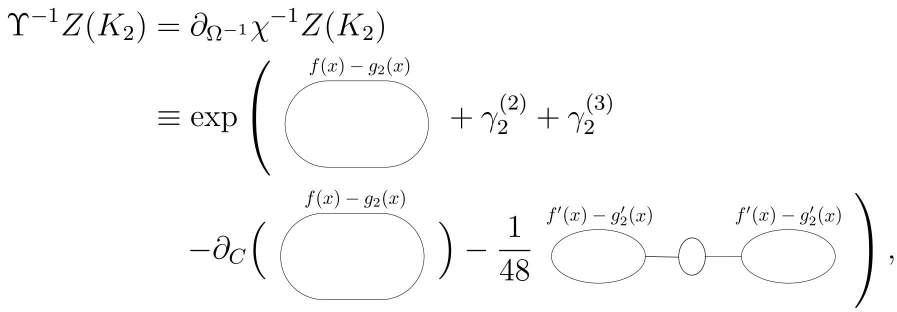
$$\equiv \exp \Bigg($$ 

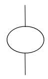
$$+ \gamma_1^{(2)} + \gamma_1^{(3)}$$

$$-\partial_C\Big($$ 

$$\Big) - \frac{1}{48}$$ 

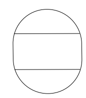
$$\Bigg),$$



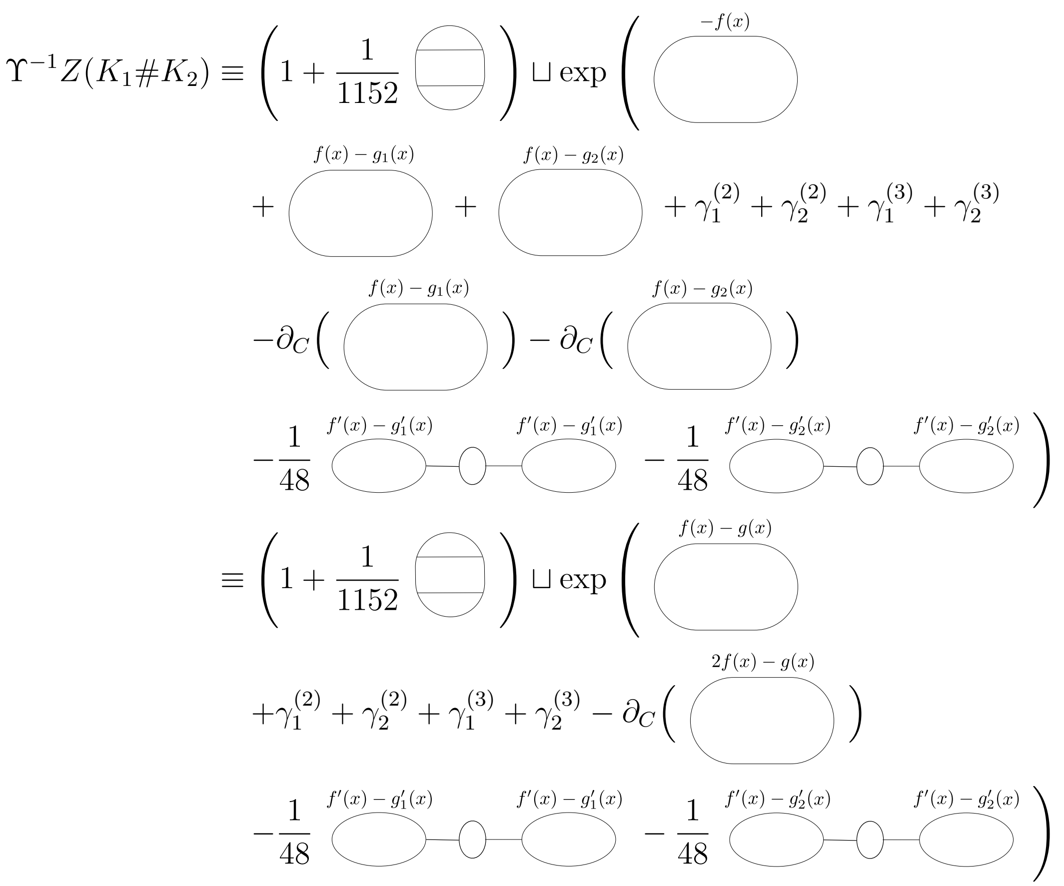
where $C = \frac{1}{48}$  $\in \mathcal{B}$. Since Υ is an algebra map,

$$\Upsilon^{-1}Z(K_1\#K_2) = \Upsilon^{-1}\big(Z(K_1)\#Z(K_2)\#\nu^{-1}\big) = \Upsilon^{-1}Z(K_1)\sqcup\Upsilon^{-1}Z(K_2)\sqcup\Upsilon^{-1}\nu^{-1}.$$

Here, as in [8],

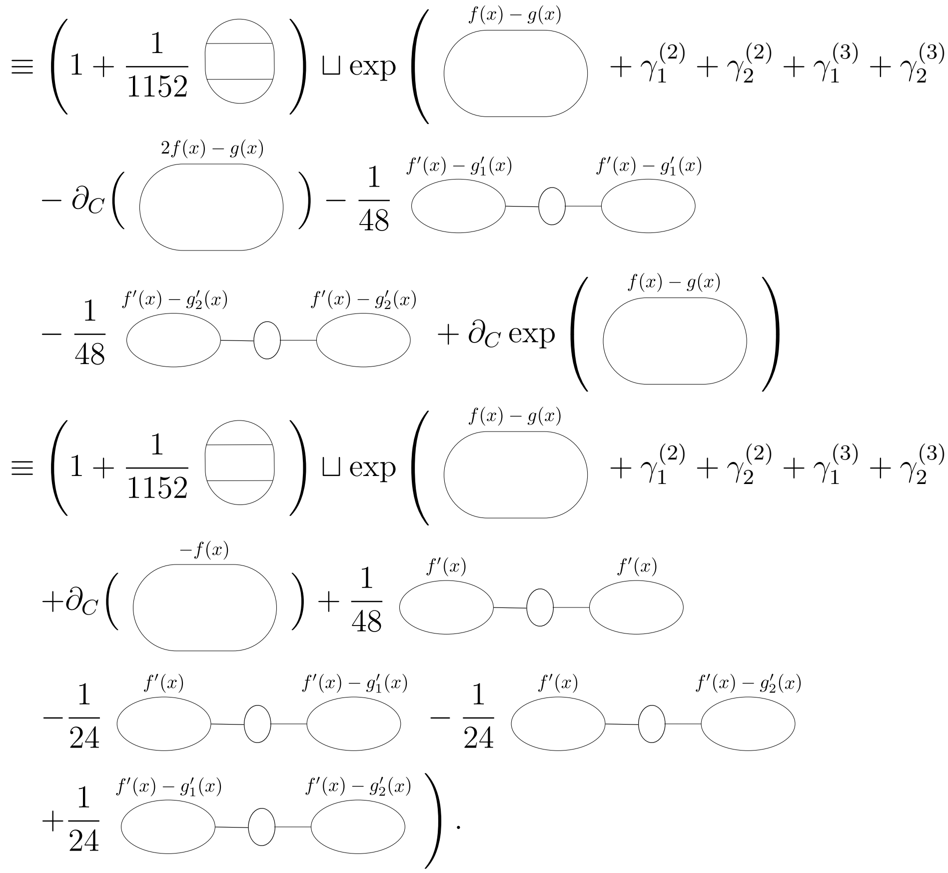
$\Upsilon^{-1}\nu = \langle\Omega,\Omega\rangle^{-1}\Omega$, so $\Upsilon^{-1}\nu^{-1} = \langle\Omega,\Omega\rangle\Omega^{-1} \equiv \Big(1+\frac{1}{1152}$  $\Big)\sqcup\Omega^{-1}$.

Thus, we get

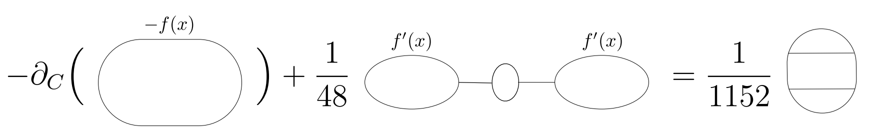


Therefore,

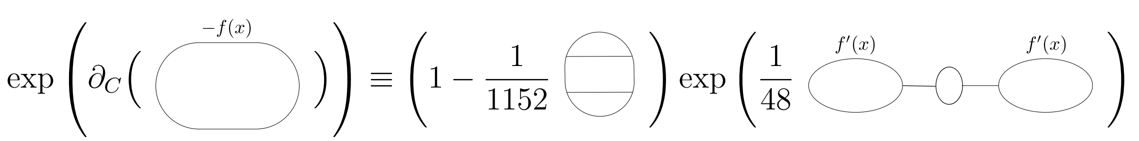
$$\chi^{-1}Z(K_1\#K_2) = \partial_\Omega \Upsilon^{-1} Z(K_1\#K_2)$$



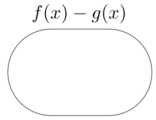
Here, as in [4], [5], $\partial_\Omega(\Omega) = \langle \Omega, \Omega \rangle \Omega$, so we get



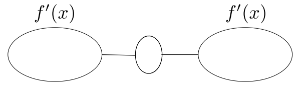
This implies that

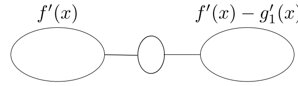


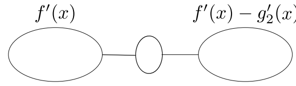
Therefore,

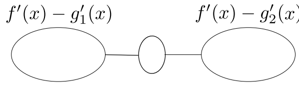
$$\chi^{-1}Z(K_1\#K_2) \equiv \exp\Bigg($$


$$+\gamma_1^{(2)}+\gamma_2^{(2)}+\gamma_1^{(3)}+\gamma_2^{(3)}$$

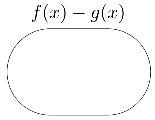
$$+\frac{1}{24}$$


$$-\frac{1}{24}$$


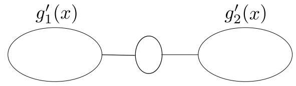
$$-\frac{1}{24}$$


$$+\frac{1}{24}$$


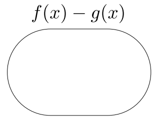
$$\Bigg)$$

$$= \exp\Bigg($$


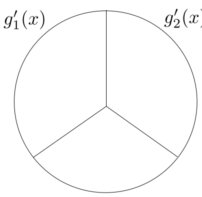
$$+\gamma_1^{(2)}+\gamma_2^{(2)}+\gamma_1^{(3)}+\gamma_2^{(3)}$$

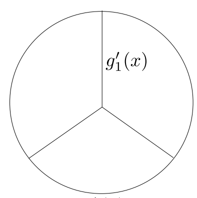
$$+\frac{1}{24}$$


$$\Bigg)$$

$$= \exp\Bigg($$


$$+\gamma_1^{(2)}+\gamma_2^{(2)}+\gamma_1^{(3)}+\gamma_2^{(3)}$$

$$+\frac{1}{6}$$


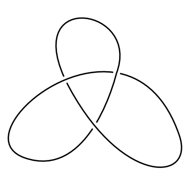
$$+\frac{1}{6}$$


$$\Bigg).$$

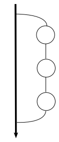
Hence, we obtain the proposition. □

B A Vassiliev invariant of degree 4 of the untwisted Whitehead double of the trefoil knot

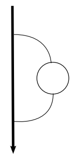
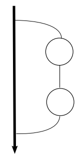
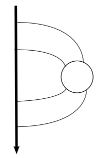
In this section, we calculate a Vassiliev invariant of degree 4 of the untwisted Whitehead double of the trefoil knot concretely We calculate it in two ways; one is by using a calculator and the other is by using our main theorem. Thereby, we verify the result of the main theorem.

The trefoil knot is $T(2,3) =$  , and we give it 0-framing.

By using a calculator, we can get its degree 4 part of the Kontsevich invariant,

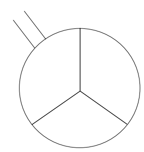
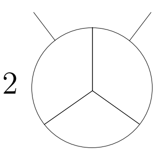
$$\log\Big(Z\big(Wh^{\pm}(T(2,3))\big)\#\nu^{-1}\Big)^{(\text{degree } 4)} = \big(\frac{1}{8} \mp \frac{1}{4}\big)$$  . (43)

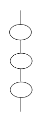
On the other hand, a straightforward calculation shows that (see Example 3.6)

$$Z\big(T(2,3)\big)\#\nu^{-1} \equiv \exp\Bigg(-\frac{1}{2}$$  $$+\frac{1}{4}$$  $$+\frac{5}{24}$$  $$\Bigg).$$

Note that $\Delta_{Wh^{\pm}\left(T(2,3)\right)}(t) = 1$. So by Theorem 3.1, we get

$$\log\Big(\chi^{-1}\big(Z(Wh^{\pm}(T(2,3)))\#\nu^{-1}\big)\Big)^{(\text{degree } 4)}$$

$$= \Big(-4\cdot(-\frac{1}{2})\cdot 0 - \frac{(\pm 1)^2\cdot(-1/2)}{4} - 2\cdot(\pm 1)\cdot\frac{1}{4}\Big)$$  $$-\frac{(\pm 1)^2\cdot(-1/2)}{8}\cdot 2\cdot 2$$ 

$$= \big(\frac{1}{8} \mp \frac{1}{4}\big)$$  .

This matches to (43).

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