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plumbing the doubles of two knots**

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Abstract

The 3-loop polynomial of a knot is a polynomial presenting the 3-loop part of the Kontsevich invariant of knots. In this paper, we calculate the 3-loop polynomial of knots obtained by plumbing the doubles of two knots; this class of knots includes untwisted Whitehead doubles. We construct the 3-loop polynomial by calculating the rational version of the Aarhus integral of a surgery presentation. As a consequence, we obtain an explicit presentation of the 3-loop polynomial for the knots.

1 Introduction

The Kontsevich invariant of knots is a very strong invariant of knots, which is universal to all quantum invariants and all Vassiliev invariants, and it is expected that the Kontsevich invariant classifies all knots. The Kontsevich invariant takes its value in the space of Jacobi diagrams. Jacobi diagrams are some kinds of uni-trivalent graphs, and they have universal properties among the pairs (\mathfrak{g}, V) , where \mathfrak{g} is a simple Lie algebra and V is its representation. So we can “substitute” any pair of (\mathfrak{g}, V) into Jacobi diagrams, and it is the reason why the Kontsevich invariant is universal to all quantum invariants. In addition, each term of the Kontsevich invariant is a Vassiliev invariant, and we can calculate it algorithmically. However, since the value of the Kontsevich invariant is presented by an infinite sum of Jacobi diagrams, it is difficult to determine all terms of Kontsevich invariant at the same time concretely. So far, a powerful method to present them is not known.

One approach to restrict the image of the Kontsevich invariant is the “loop expansion”. It is conjectured in [21] that the Kontsevich invariant of a knot is expanded in the form of the loop expansion, and it is shown in [10] that the Kontsevich invariant of a knot can be expanded in this form, and it is shown in [7] that the loop expansion is a knot invariant. It is calculated by using the rational version of the Aarhus integral of a surgery presentation. A n -loop diagram is a connected open Jacobi diagram whose first Betti number is n . When we fix a loop number, each loop part is presented by some polynomials, and in particular, the 1-loop part is presented by the Alexander polynomial. The polynomial presenting the 2 (resp. 3)-loop part is called the 2 (resp. 3)-loop polynomial. The 2 (resp. 3)-loop polynomial is a 2 (resp. 3)-variable polynomial invariant of knots.

The 2-loop polynomial is calculated in many cases. For example, the 2-loop polynomial for knots with up to 7 crossings is calculated in [21], for torus knots in [11], [12], [16], for untwisted Whitehead doubles in [9], and for genus 1 knots in [17]. On the other hand, there are few examples of the calculation of the 3-loop polynomial of knots. The 3-loop

part of the Kontsevich invariant of torus knots are calculated in [11], [12], where a cabling formula for the Kontsevich invariant is used. However, it is not easy to calculate the 3-loop part of the Kontsevich invariant in general, so the 3-loop polynomial of other knots are not calculated so far. We can calculate the 3-loop polynomial in the same way as the calculation of the 2-loop polynomial theoretically, but the calculation of the 3-loop part is more complicated than that of the 2-loop part.

In this paper, we give some examples of the 3-loop polynomial of knots. More concretely, in Theorem 3.1, we calculate the 3-loop polynomial of $D(K, K')$, which are knots obtained by plumbing the doubles of two knots K (with framing 0) and K' (with framing k); this class of knots includes untwisted Whitehead doubles (Corollary 3.3). The 3-loop polynomial of these knots are relatively easy to calculate, and this theorem is one of the few examples of the calculation of the 3-loop polynomial of knots. We can get the loop expansion of a knot by calculating the rational version of the Aarhus integral of a surgery presentation of the knot, so we construct the 3-loop polynomial of $D(K, K')$ in such a way, and we can show that its 3-loop polynomial is only depend on Vassiliev invariants of K and K' up to degree 4. As a consequence, we obtain an explicit presentation of the 3-loop polynomial of $D(K, K')$ by using Vassiliev invariants of K and K' up to degree 4. When we consider the \mathfrak{sl}_2 reduction of the Kontsevich invariant of knots, we can get the colored Jones polynomial of knots. In Proposition 7.2, we calculate the 3-loop part of the colored Jones polynomial. In addition, by considering the Dufflo isomorphism, we can get the connected sum formula for the 3-loop polynomial of any knots.

This paper is organized as follows. In Section 2, we review the Kontsevich invariant and its loop expansion. In addition, we define the 3-loop polynomial of a knot, and we define knots $D(K, K')$, which are obtained by plumbing the doubles of two knots. In Section 3, we state the main theorem of this paper; we present the 3-loop polynomial of $D(K, K')$. As its corollary, we present the 3-loop polynomial of untwisted Whitehead doubles of knots. In Section 4, we state some properties of the 3-loop part of the Kontsevich invariant without proofs, and we state the connected sum formula for the 3-loop polynomial. The proof of the connected sum formula for the 3-loop polynomial is given in Appendix. In Section 5, for the proof of the main theorem, we review the rational version of the Aarhus integral. In Section 6, we prove the main theorem. In Section 7, we calculate the 3-loop part of the colored Jones polynomial. Other topics are mentioned in Appendix.

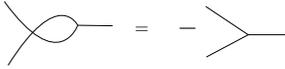
The author would like to thank Advisor Tomotada Ohtsuki for encouragement and valuable discussions and comments, and Professor Andrew Kricker for stimulating discussions and comments.

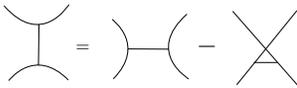
2 The Kontsevich invariant and its loop expansion

In this section, we review the Kontsevich invariant and we define the 3-loop polynomial of knots. For details, see [14],[15].

Let X be an oriented compact manifold. A *Jacobi diagram* on X is an uni-trivalent graph such that univalent vertices are distinct points of X , and a cyclic order of the three edges around each trivalent vertex is fixed, in other words, each trivalent vertex is *vertex-oriented*. When we draw a Jacobi diagram on X , we draw X by thick lines

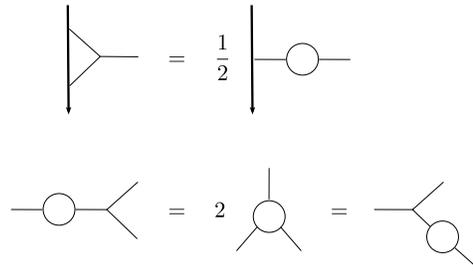
and uni-trivalent graphs by thin lines, and each trivalent vertex is vertex-oriented in the counterclockwise order. Furthermore, we define the *degree* of a Jacobi diagram to be half the number of all vertices of the graph of the Jacobi diagram. We define $\mathcal{A}(X)$ to be the quotient vector space spanned by Jacobi diagrams on X subject to *the AS, IHX, and STU relations*.

the AS relation : 

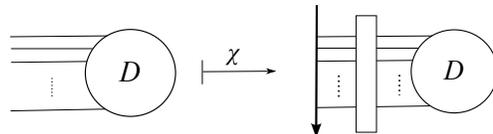
the IHX relation : 

the STU relation : 

Note that we get some equations by the above relations;



It is known, see [14],[15], that $\mathcal{A}(S^1)$ forms a commutative algebra whose product is given by connected sum of copies of S^1 , and $\mathcal{A}(\downarrow)$ also forms a commutative algebra whose product is given by connecting copies of \downarrow . We can see that $\mathcal{A}(S^1)$ and $\mathcal{A}(\downarrow)$ are naturally isomorphic as commutative algebras by the isomorphism given by connecting two end points of \downarrow . An *open Jacobi diagram* is a vertex-oriented uni-trivalent graphs. We call outward pointing edges that end in a univalent vertex a leg. We define \mathcal{B} to be the quotient vector space spanned by Jacobi diagrams subject to the AS, IHX relations. \mathcal{B} forms a commutative algebra whose product is given by disjoint union. The *PBW isomorphism* $\chi : \mathcal{B} \rightarrow \mathcal{A}(\downarrow)$ is defined by



for any diagram $D \in \mathcal{B}$, where the box means the symmetrizer,

$$\begin{array}{c} \text{---} \\ \text{---} \\ \vdots \\ \text{---} \end{array} \Big| \begin{array}{c} \text{---} \\ \text{---} \\ \vdots \\ \text{---} \end{array} = \frac{1}{n!} \left(\begin{array}{c} \text{---} \\ \text{---} \\ \vdots \\ \text{---} \end{array} + \begin{array}{c} \diagup \\ \text{---} \\ \vdots \\ \text{---} \\ \diagdown \end{array} + \begin{array}{c} \diagup \\ \diagdown \\ \text{---} \\ \vdots \\ \text{---} \\ \diagup \\ \diagdown \end{array} + \dots \right).$$

n lines

Note that PBW isomorphism is not an algebra isomorphism.

The *Kontsevich invariant* $Z(K)$ of a knot K is defined to be in $\mathcal{A}(S^1) (\cong \mathcal{A}(\downarrow))$; for details, see [14],[15]. Note that $Z(K)$ and $\chi^{-1}Z(K)$ are group-like, which means that they are exponentials of series of connected diagrams. The loop expansion of the Kontsevich invariant of knot K is a presentation of the following form ([7],[10],[14]),

$$\begin{aligned}
 \log(\chi^{-1}Z(K)) &= \begin{array}{c} \frac{1}{2} \log\left(\frac{\sinh(h/2)}{h/2}\right) - \frac{1}{2} \log \Delta_K(e^h) \\ \text{---} \end{array} \\
 &+ \sum_i^{\text{finite}} \begin{array}{c} p_{i,1}(e^h)/\Delta_K(e^h) \\ \text{---} \\ p_{i,2}(e^h)/\Delta_K(e^h) \\ \text{---} \\ p_{i,3}(e^h)/\Delta_K(e^h) \end{array} + \sum_i^{\text{finite}} \begin{array}{c} q_{i,1}(e^h)/\Delta_K(e^h) \\ q_{i,2}(e^h)/\Delta_K(e^h) \\ q_{i,3}(e^h)/\Delta_K(e^h) \\ q_{i,4}(e^h)/\Delta_K(e^h) \\ q_{i,5}(e^h)/\Delta_K(e^h) \\ q_{i,6}(e^h)/\Delta_K(e^h) \end{array} \\
 &+ (\text{terms of } (> 3)\text{-loop part}),
 \end{aligned}$$

where $\Delta_K(t)$ denotes the Alexander polynomial, and $p_{i,j}(e^h)$ and $q_{i,j}(e^h)$ are polynomials in $e^{\pm h}$. Here, a labeling of $f(h) = c_0 + c_1h + c_2h^2 + c_3h^3 + \dots$ implies that ,

$$\left. \begin{array}{c} f(h) \\ \text{---} \\ c_0 \\ \text{---} \\ c_1 \\ \text{---} \\ c_2 \\ \text{---} \\ c_3 \\ \text{---} \\ \dots \end{array} \right) = \left. \begin{array}{c} c_0 \\ \text{---} \\ c_1 \\ \text{---} \\ c_2 \\ \text{---} \\ c_3 \\ \text{---} \\ \dots \end{array} \right) + \dots$$

Note that

$$\left. \begin{array}{c} f(h) \\ \text{---} \end{array} \right) = \left. \begin{array}{c} f(-h) \\ \text{---} \end{array} \right),$$

by the AS relation. Further, we note that

$$\begin{array}{c} \frac{1}{2} \log\left(\frac{\sinh(h/2)}{h/2}\right) \\ \text{---} \end{array} = \chi^{-1}\nu,$$

where we denote $\nu = Z(\text{unknot}) \in \mathcal{A}(S^1)$. Then, we define the *3-loop polynomial* of K

by

$$\begin{aligned} & \Lambda_K(t_1, t_2, t_3, t_4) \\ &= \sum_{\substack{i \\ \tau \in \mathfrak{S}_4}} q_{i,1}(t_{\tau(1)}^{\text{sgn}\tau} t_{\tau(4)}^{-\text{sgn}\tau}) q_{i,2}(t_{\tau(2)}^{\text{sgn}\tau} t_{\tau(4)}^{-\text{sgn}\tau}) q_{i,3}(t_{\tau(3)}^{\text{sgn}\tau} t_{\tau(4)}^{-\text{sgn}\tau}) q_{i,4}(t_{\tau(2)}^{\text{sgn}\tau} t_{\tau(3)}^{-\text{sgn}\tau}) q_{i,5}(t_{\tau(3)}^{\text{sgn}\tau} t_{\tau(1)}^{-\text{sgn}\tau}) q_{i,6}(t_{\tau(1)}^{\text{sgn}\tau} t_{\tau(2)}^{-\text{sgn}\tau}) \\ & \in \mathbb{Q}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}, t_4^{\pm 1}] / (\mathfrak{S}_4, t_1 t_2 t_3 t_4 = 1). \end{aligned}$$

For details about the 3-loop part of the Kontsevich invariant, see Section 4.1.

3 The 3-loop polynomial of $D(K, K')$

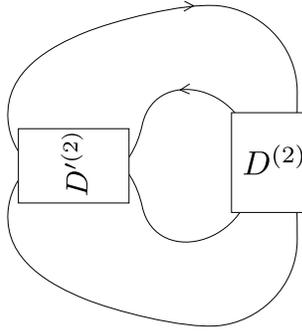
In this section, we define $D(K, K')$, which are genus 1 knots with trivial Alexander polynomial, and we state the main theorem of this paper.

Let K be a 0-framed knot, and let K' be a k -framed knot ($k \in \mathbb{Z}$). Let D, D' be 1-tangles whose closures are K, K' , respectively, noting that isotopy classes of D and D' are uniquely determined by K and K' .

$$K = \begin{array}{c} \text{---} \curvearrowleft \\ \text{---} \text{---} \text{---} \\ \text{---} \curvearrowright \end{array} \quad \begin{array}{c} \boxed{D} \\ \text{---} \end{array} \quad , \quad K' = \begin{array}{c} \text{---} \curvearrowleft \\ \text{---} \text{---} \text{---} \\ \text{---} \curvearrowright \end{array} \quad \begin{array}{c} \boxed{D'} \\ \text{---} \end{array} \quad (1)$$

(0-framing) (k-framing)

We define $D(K, K')$ to be the following knot,



where $D^{(2)}$ and $D'^{(2)}$ are the doubles of D and D' , respectively. We can obtain $D(K, K')$ by *plumbing* of the doubles of K and K' , noting that $D(K, K')$ is a genus 1 knot with trivial Alexander polynomial.

The Kontsevich invariants of K and K' can be presented by

$$\begin{aligned} Z(K) = \nu \# \exp \left(a_2 \begin{array}{c} \curvearrowleft \\ \text{---} \text{---} \text{---} \\ \curvearrowright \end{array} + a_3 \begin{array}{c} \curvearrowleft \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \curvearrowright \end{array} + a_4 \begin{array}{c} \curvearrowleft \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \curvearrowright \end{array} \right. \\ \left. + (\text{linear sum of diagrams with more than 4 trivalent vertices}) \right) \in \mathcal{A}(S^1), \quad (2) \end{aligned}$$

$$Z(K') = \nu \# \exp \left(\frac{k}{2} \begin{array}{c} \text{---} \\ \bigcirc \end{array} + a'_2 \begin{array}{c} \text{---} \\ \bigcirc \\ \text{---} \\ \bigcirc \end{array} + \left(\begin{array}{l} \text{linear sum of diagrams} \\ \text{with more than} \\ \text{2 trivalent vertices} \end{array} \right) \right) \in \mathcal{A}(S^1), \quad (3)$$

where a_i is a degree i Vassiliev invariant of K , and a'_i is a degree i Vassiliev invariant of K' .

Now, we state the main theorem of this paper. We put $u_{m,n} = t_m t_n^{-1} + t_m^{-1} t_n - 2$ and $v_{m,n} = t_m t_n^{-1} - t_m^{-1} t_n$ ($m, n \in \{1, 2, 3, 4\}$)

Theorem 3.1. *Let K and K' be knots as shown in (1), and assume that their Kontsevich invariants are presented as in (2), (3). Then, the 3-loop polynomial of $D(K, K')$ is presented by*

$$\begin{aligned} \Lambda_{D(K, K')}(t_1, t_2, t_3, t_4) &= (-16a_2 a'_2 - k^2 a_2 - 8ka_3)(u_{1,2} + u_{1,3} + u_{1,4} + u_{2,3} + u_{2,4} + u_{3,4}) \\ &+ \left(-\frac{k^2 a_2}{12} + 4k^2 a_4\right)(u_{1,4} u_{2,4} + u_{1,4} u_{3,4} + u_{2,4} u_{3,4} + u_{1,3} u_{2,3} + u_{1,3} u_{4,3} + u_{2,3} u_{4,3} \\ &\quad + u_{1,2} u_{3,2} + u_{1,2} u_{4,2} + u_{3,2} u_{4,2} + u_{2,1} u_{3,1} + u_{2,1} u_{4,1} + u_{3,1} u_{4,1}) \\ &+ 24k^2 a_4(u_{1,2} u_{3,4} + u_{1,3} u_{2,4} + u_{1,4} u_{2,3}) \\ &+ 8k^2 a_2^2(u_{1,2}^2 + u_{1,3}^2 + u_{1,4}^2 + u_{2,3}^2 + u_{2,4}^2 + u_{3,4}^2) \\ &- \frac{k^2 a_2}{4}(v_{1,4} v_{2,4} + v_{1,4} v_{3,4} + v_{2,4} v_{3,4} + v_{1,3} v_{2,3} + v_{1,3} v_{4,3} + v_{2,3} v_{4,3} \\ &\quad + v_{1,2} v_{3,2} + v_{1,2} v_{4,2} + v_{3,2} v_{4,2} + v_{2,1} v_{3,1} + v_{2,1} v_{4,1} + v_{3,1} v_{4,1}) \\ &\in \mathbb{Q}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}, t_4^{\pm 1}] / (\mathfrak{S}_4, t_1 t_2 t_3 t_4 = 1). \end{aligned}$$

We prove the theorem in Section 6.

We put

$$\begin{aligned} T_{1,3} &= t_1 t_2^3 + t_1 t_3^3 + t_2 t_1^3 + t_2 t_3^3 + t_3 t_1^3 + t_3 t_2^3 \\ &\quad + t_1^2 t_2^{-1} t_3^{-1} + t_2^2 t_1^{-1} t_3^{-1} + t_3^2 t_1^{-1} t_2^{-1} + t_1^{-2} t_2^{-3} t_3^{-3} + t_2^{-2} t_1^{-3} t_3^{-3} + t_3^{-2} t_1^{-3} t_2^{-3}, \\ T_{2,2} &= t_1^2 t_2^2 + t_2^2 t_3^2 + t_3^2 t_1^2 + t_1^{-2} t_2^{-2} + t_2^{-2} t_3^{-2} + t_3^{-2} t_1^{-2}, \\ T_{1,1,2} &= t_1 t_2 t_3^2 + t_2 t_3 t_1^2 + t_3 t_1 t_2^2 + t_1^{-1} t_2^{-1} t_3^{-2} + t_2^{-1} t_3^{-1} t_1^{-2} + t_3^{-1} t_1^{-1} t_2^{-2} \\ &\quad + t_1 t_2^{-1} + t_1 t_3^{-1} + t_2 t_1^{-1} + t_2 t_3^{-1} + t_3 t_1^{-1} + t_3 t_2^{-1}, \\ T_{2,2,4} &= t_1^2 t_2^2 t_3^4 + t_2^2 t_3^2 t_1^4 + t_3^2 t_1^2 t_2^4 + t_1^{-2} t_2^{-2} t_3^{-4} + t_2^{-2} t_3^{-2} t_1^{-4} + t_3^{-2} t_1^{-2} t_2^{-4} \\ &\quad + t_1^2 t_2^{-2} + t_1^2 t_3^{-2} + t_2^2 t_1^{-2} + t_2^2 t_3^{-2} + t_3^2 t_1^{-2} + t_3^2 t_2^{-2}, \\ T_{2,3,3} &= t_1^2 t_2^3 t_3^3 + t_2^2 t_3^3 t_1^3 + t_3^2 t_1^3 t_2^3 + t_1 t_2 t_3^{-2} + t_2 t_3 t_1^{-2} + t_3 t_1 t_2^{-2} \\ &\quad + t_1^{-1} t_2^{-3} + t_1^{-1} t_3^{-3} + t_2^{-1} t_1^{-3} + t_2^{-1} t_3^{-3} + t_3^{-1} t_1^{-3} + t_3^{-1} t_2^{-3}. \end{aligned}$$

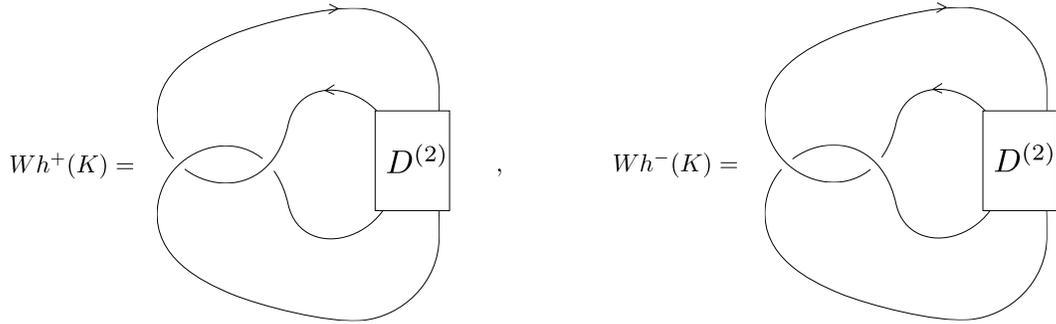
Then,

$$\begin{aligned}
u_{1,2} + u_{1,3} + u_{1,4} + u_{2,3} + u_{2,4} + u_{3,4} &= T_{1,1,2} - 6, \\
u_{1,4}u_{2,4} + u_{1,4}u_{3,4} + u_{2,4}u_{3,4} + u_{1,3}u_{2,3} + u_{1,3}u_{4,3} + u_{2,3}u_{4,3} \\
&\quad + u_{1,2}u_{3,2} + u_{1,2}u_{4,2} + u_{3,2}u_{4,2} + u_{2,1}u_{3,1} + u_{2,1}u_{4,1} + u_{3,1}u_{4,1} \\
&= T_{2,3,3} + T_{1,3} - 6T_{1,1,2} + 4, \\
u_{1,2}u_{3,4} + u_{1,3}u_{2,4} + u_{1,4}u_{2,3} &= T_{2,2} - 2T_{1,1,2} + 4, \\
u_{1,2}^2 + u_{1,3}^2 + u_{1,4}^2 + u_{2,3}^2 + u_{2,4}^2 + u_{3,4}^2 &= T_{2,2,4} - 4T_{1,1,2} + 6, \\
v_{1,4}v_{2,4} + v_{1,4}v_{3,4} + v_{2,4}v_{3,4} + v_{1,3}v_{2,3} + v_{1,3}v_{4,3} + v_{2,3}v_{4,3} \\
&\quad + v_{1,2}v_{3,2} + v_{1,2}v_{4,2} + v_{3,2}v_{4,2} + v_{2,1}v_{3,1} + v_{2,1}v_{4,1} + v_{3,1}v_{4,1} \\
&= T_{2,3,3} + T_{1,3} - 2T_{1,1,2}.
\end{aligned}$$

Remark 3.2. The formula of Theorem 3.1 is rewritten,

$$\begin{aligned}
&\Lambda_{D(K,K')}(t_1, t_2, t_3, t_4) \\
&= (-16a_2a'_2 - 32k^2a_2^2 - 8ka_3 - 72k^2a_4)T_{1,1,2} + 8k^2a_2^2T_{2,2,4} \\
&\quad + \left(-\frac{k^2a_2}{3} + 4k^2a_4\right)T_{2,3,3} + \left(-\frac{k^2a_2}{3} + 4k^2a_4\right)T_{1,3} + 24k^2a_4T_{2,2} \\
&\quad - 96a_2a'_2 + 48k^2a_2^2 + \frac{17k^2a_2}{3} + 48ka_3 + 112k^2a_4 \\
&\in \mathbb{Q}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}, t_4^{\pm 1}]/(\mathfrak{S}_4, t_1t_2t_3t_4 = 1).
\end{aligned}$$

In particular, we can get the 3-loop polynomial of untwisted Whitehead double of K . We denote it by $Wh^\pm(K)$.



Here, D is a 1-tangle whose closure is K as shown in (1).

Corollary 3.3. *The 3-loop polynomial of $Wh^\pm(K)$ is presented by*

$$\begin{aligned}
& \Lambda_{Wh^\pm(K)}(t_1, t_2, t_3, t_4) \\
&= (-a_2 \pm 8a_3)(u_{1,2} + u_{1,3} + u_{1,4} + u_{2,3} + u_{2,4} + u_{3,4}) \\
&\quad + \left(-\frac{a_2}{12} + 4a_4\right)(u_{1,4}u_{2,4} + u_{1,4}u_{3,4} + u_{2,4}u_{3,4} + u_{1,3}u_{2,3} + u_{1,3}u_{4,3} + u_{2,3}u_{4,3} \\
&\quad\quad + u_{1,2}u_{3,2} + u_{1,2}u_{4,2} + u_{3,2}u_{4,2} + u_{2,1}u_{3,1} + u_{2,1}u_{4,1} + u_{3,1}u_{4,1}) \\
&\quad + 24a_4(u_{1,2}u_{3,4} + u_{1,3}u_{2,4} + u_{1,4}u_{2,3}) \\
&\quad + 8a_2^2(u_{1,2}^2 + u_{1,3}^2 + u_{1,4}^2 + u_{2,3}^2 + u_{2,4}^2 + u_{3,4}^2) \\
&\quad - \frac{a_2}{4}(v_{1,4}v_{2,4} + v_{1,4}v_{3,4} + v_{2,4}v_{3,4} + v_{1,3}v_{2,3} + v_{1,3}v_{4,3} + v_{2,3}v_{4,3} \\
&\quad\quad + v_{1,2}v_{3,2} + v_{1,2}v_{4,2} + v_{3,2}v_{4,2} + v_{2,1}v_{3,1} + v_{2,1}v_{4,1} + v_{3,1}v_{4,1}) \\
&\in \mathbb{Q}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}, t_4^{\pm 1}]/(\mathfrak{S}_4, t_1 t_2 t_3 t_4 = 1).
\end{aligned}$$

The corollary immediately follows from Theorem 3.1.

Remark 3.4. As in Remark 3.2, the formula of Theorem 3.3 is rewritten,

$$\begin{aligned}
& \Lambda_{Wh^\pm(K)}(t_1, t_2, t_3, t_4) \\
&= (-32a_2^2 \pm 8a_3 - 72a_4)T_{1,1,2} + 8a_2^2 T_{2,2,4} \\
&\quad + \left(-\frac{a_2}{3} + 4a_4\right)T_{2,3,3} + \left(-\frac{a_2}{3} + 4a_4\right)T_{1,3} + 24a_4 T_{2,2} \\
&\quad + 48a_2^2 + \frac{17a_2}{3} \mp 48ka_3 + 112a_4 \\
&\in \mathbb{Q}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}, t_4^{\pm 1}]/(\mathfrak{S}_4, t_1 t_2 t_3 t_4 = 1).
\end{aligned}$$

Remark 3.5. It is known ([24]) that $Z(K)$ is presented by

$$\begin{aligned}
Z(K) = \nu \# \exp \left(-\frac{1}{2}c_2 \left[\text{diagram 1} \right] - \frac{1}{24}j_3 \left[\text{diagram 2} \right] + \frac{1}{24}(-12c_4 + 6c_2^2 - c_2) \left[\text{diagram 3} \right] \right. \\
\left. + (\text{linear sum of diagrams with more than 4 trivalent vertices}) \right),
\end{aligned}$$

where c_n are coefficient of the Conway polynomial $\nabla_K(z) = \sum c_n z^n$ and j_n are coefficient of the Jones polynomial $J_K(e^t) = \sum j_n t^n$. Note that the Conway polynomial is defined by $\nabla_K(t^{\frac{1}{2}} - t^{-\frac{1}{2}}) = \Delta_K(t)$. Therefore we can get

$$a_2 = -\frac{1}{2}c_2, \quad a_3 = -\frac{1}{24}j_3, \quad a_4 = \frac{1}{24}(-12c_4 + 6c_2^2 - c_2)$$

Example 3.6. As an example, we calculate the 3-loop polynomial of the untwisted Whitehead double of $(2, 2n+1)$ torus knots, $T(2, 2n+1)$. We consider the untwisted Whitehead double of them, $Wh^\pm(T(2, 2n+1))$. It can be shown by using the skein relation that

$$\nabla_{T(2,2n+1)}(z) = \sum_{j=0}^n \binom{n+j}{2j} z^{2j}, \quad J_{T(2,2n+1)}(t) = \frac{t^n - t^{n+3} - t^{3n+2} + t^{3n+3}}{1-t^2},$$

and so

$$c_2 = \frac{n(n+1)}{2}, \quad c_4 = \frac{(n+2)(n+1)n(n-1)}{24}, \quad j_3 = -n(n+1)(2n+1).$$

From this, we get

$$\begin{aligned} & Z(T(2, 2n+1)) \\ &= \nu \# \exp \left(-\frac{n(n+1)}{4} \left[\text{diagram 1} \right] + \frac{n(n+1)(2n+1)}{24} \left[\text{diagram 2} \right] + \frac{n(n+1)(2n^2+2n+1)}{48} \left[\text{diagram 3} \right] \right. \\ & \quad \left. + (\text{linear sum of diagrams with more than 4 trivalent vertices}) \right). \end{aligned}$$

Therefore

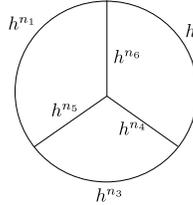
$$\begin{aligned} & \Lambda_{Wh^\pm(T(2,2n+1))}(t_1, t_2, t_3, t_4) \\ &= \frac{n(n+1)}{12} (3 \pm (8n+4)) (u_{1,2} + u_{1,3} + u_{1,4} + u_{2,3} + u_{2,4} + u_{3,4}) \\ & \quad + \frac{n(n+1)(8n^2+8n+5)}{48} (u_{1,4}u_{2,4} + u_{1,4}u_{3,4} + u_{2,4}u_{3,4} + u_{1,3}u_{2,3} + u_{1,3}u_{4,3} + u_{2,3}u_{4,3} \\ & \quad \quad + u_{1,2}u_{3,2} + u_{1,2}u_{4,2} + u_{3,2}u_{4,2} + u_{2,1}u_{3,1} + u_{2,1}u_{4,1} + u_{3,1}u_{4,1}) \\ & \quad + \frac{n(n+1)(2n^2+2n+1)}{2} (u_{1,2}u_{3,4} + u_{1,3}u_{2,4} + u_{1,4}u_{2,3}) \\ & \quad + \frac{n^2(n+1)^2}{2} (u_{1,2}^2 + u_{1,3}^2 + u_{1,4}^2 + u_{2,3}^2 + u_{2,4}^2 + u_{3,4}^2) \\ & \quad + \frac{n(n+1)}{16} (v_{1,4}v_{2,4} + v_{1,4}v_{3,4} + v_{2,4}v_{3,4} + v_{1,3}v_{2,3} + v_{1,3}v_{4,3} + v_{2,3}v_{4,3} \\ & \quad \quad + v_{1,2}v_{3,2} + v_{1,2}v_{4,2} + v_{3,2}v_{4,2} + v_{2,1}v_{3,1} + v_{2,1}v_{4,1} + v_{3,1}v_{4,1}) \\ & \in \mathbb{Q}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}, t_4^{\pm 1}] / (\mathfrak{S}_4, t_1 t_2 t_3 t_4 = 1). \end{aligned}$$

4 Some properties of the 3-loop part of the Kontsevich invariant and the 3-loop polynomial

4.1 The 3-loop part of Kontsevich invariant

In this section, we state some properties of the 3-loop part of the Kontsevich invariant. We omit proofs, and for details, see [18].

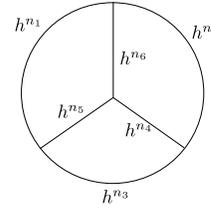
Let $\mathcal{B}_{\text{conn}}$ be the subspace of \mathcal{B} spanned by connected diagrams, and let $\mathcal{B}_{\text{conn}}^{(3\text{-loop})}$ be the subspace of $\mathcal{B}_{\text{conn}}$ spanned by 3-loop open Jacobi diagrams. It is known ([18]) that any elements in $\mathcal{B}_{\text{conn}}^{(3\text{-loop})}$ can be presented by the linear combination of diagrams of the following form,



$$(4)$$

such that $n_1 + n_2 + n_3 + n_4 + n_5 + n_6$ is an even number (If it is an odd number, the Jacobi diagram equal to 0). We can correspond the diagram (4) to $h_1^{n_1} h_2^{n_2} h_3^{n_3} h_4^{n_4} h_5^{n_5} h_6^{n_6}$. These variables satisfy that

$$h_1 - h_2 - h_6 = 0, \quad h_1 - h_3 + h_5 = 0, \quad h_4 + h_5 + h_6 = 0. \quad (5)$$

Here, we regard  as a tetrahedron, and introduce new variables corresponding with faces of the tetrahedron,

$$x_1 = h_1 - h_5 + h_6, \quad x_2 = h_2 + h_4 - h_6, \quad x_3 = h_3 - h_4 + h_5, \quad x_4 = -h_1 - h_2 - h_3. \quad (6)$$

By definition, these variables satisfy that $x_1 + x_2 + x_3 + x_4 = 0$. Therefore, we get the following isomorphism,

$$\mathcal{B}_{\text{conn}}^{(3\text{-loop})} \cong (\mathbb{C}[x_1, x_2, x_3, x_4] / (x_1 + x_2 + x_3 + x_4 = 0)) / \mathfrak{S}_4,$$

where the action of $\tau \in \mathfrak{S}_4$ takes a polynomial $f(x_1, x_2, x_3, x_4)$ to $f((\text{sgn}\tau)x_{\tau(1)}, (\text{sgn}\tau)x_{\tau(2)}, (\text{sgn}\tau)x_{\tau(3)}, (\text{sgn}\tau)x_{\tau(4)})$. Therefore,

$$\mathcal{B}_{\text{conn}}^{(3\text{-loop})} \cong \mathbb{C}[\sigma_2, \sigma_3^2, \sigma_4]$$

where σ_i is the elementary symmetric polynomial of degree i . By (5), (6),

$$\begin{aligned} h_1 &= \frac{x_1 - x_4}{4}, & h_2 &= \frac{x_2 - x_4}{4}, & h_3 &= \frac{x_3 - x_4}{4}, \\ h_4 &= \frac{x_2 - x_3}{4}, & h_5 &= \frac{x_3 - x_1}{4}, & h_6 &= \frac{x_1 - x_2}{4} \end{aligned}$$

so, we can correspond the diagram (4) to

$$\left(\frac{x_1 - x_4}{4}\right)^{n_1} \left(\frac{x_2 - x_4}{4}\right)^{n_2} \left(\frac{x_3 - x_4}{4}\right)^{n_3} \left(\frac{x_2 - x_3}{4}\right)^{n_4} \left(\frac{x_3 - x_1}{4}\right)^{n_5} \left(\frac{x_1 - x_2}{4}\right)^{n_6}.$$

Note that there is a injection map

$$\begin{aligned} (\mathbb{C}[t_1^{\pm 1}, t_2^{\pm 2}, t_3^{\pm 3}, t_4^{\pm 4}]/(t_1 t_2 t_3 t_4 = 1))/\mathfrak{S}_4 &\hookrightarrow (\mathbb{C}[[x_1, x_2, x_3, x_4]]/(x_1 + x_2 + x_3 + x_4 = 0))/\mathfrak{S}_4 \\ t_i &\mapsto e^{x_i/4} \end{aligned}$$

where $(\mathbb{C}[[x_1, x_2, x_3, x_4]]/(x_1 + x_2 + x_3 + x_4 = 0))/\mathfrak{S}_4$ is the completion of $(\mathbb{C}[x_1, x_2, x_3, x_4]/(x_1 + x_2 + x_3 + x_4 = 0))/\mathfrak{S}_4$ with respect to the degree. Therefore we put $t_i = e^{x_i/4}$ ($i = 1, 2, 3, 4$), then we can get the definition of the 3-loop polynomial as in Section 2.

4.2 A connected sum formula for the 3-loop polynomial

In this section, we state a connected sum formula for the 3-loop polynomial.

Let K_1, K_2 be 0-framing knots, and let $K_1 \# K_2$ be their connected sum. We denote

$$\begin{aligned} f(x) &= \frac{1}{2} \log \frac{\sinh(x/2)}{x/2}, \quad g_j(x) = \frac{1}{2} \log \Delta_{K_j}(e^x), \\ g(x) &= \frac{1}{2} \log \Delta_{K_1 \# K_2}(e^x) = g_1(x) + g_2(x), \end{aligned}$$

where $\Delta_{K_j}(t)$ is the Alexander polynomial of K_j . We denote

$$\begin{aligned} \chi^{-1} Z(K_1) &= \exp \left(\left(\overset{f(x) - g_1(x)}{\text{Diagram}} + \gamma_1^{(2)} + \gamma_1^{(3)} \right. \right. \\ &\quad \left. \left. + (\text{linear sum of diagrams with more than 4 trivalent vertices}) \right), \\ \chi^{-1} Z(K_2) &= \exp \left(\left(\overset{f(x) - g_2(x)}{\text{Diagram}} + \gamma_2^{(2)} + \gamma_2^{(3)} \right. \right. \\ &\quad \left. \left. + (\text{linear sum of diagrams with more than 4 trivalent vertices}) \right), \end{aligned}$$

where $\gamma_j^{(2)}, \gamma_j^{(3)}$ is the 2, 3-loop part of K_j , respectively.

Proposition 4.1. *We can get*

$$\begin{aligned}
\chi^{-1}Z(K_1\#K_2) = \exp \left(\right. & \left. \begin{array}{c} f(x) - g(x) \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \end{array} \right) + \gamma_1^{(2)} + \gamma_2^{(2)} + \gamma_1^{(3)} + \gamma_2^{(3)} \\
& + \frac{1}{6} \begin{array}{c} g'_1(x) \quad g'_2(x) \\ \text{---} \quad \text{---} \\ \text{---} \\ \text{---} \end{array} + \frac{1}{6} \begin{array}{c} g'_1(x) \\ \text{---} \\ \text{---} \\ g'_2(x) \end{array} \\
& + (\text{linear sum of diagrams with more than 4 trivalent vertices}).
\end{aligned}$$

When the 3-loop part of K_j is presented by

$$Z^{(3\text{-loop})}(K_j) = \sum_i \begin{array}{c} \frac{q_{i,1}^{(j)}(e^h)}{\Delta_K(e^h)} \quad \frac{q_{i,2}^{(j)}(e^h)}{\Delta_K(e^h)} \\ \text{---} \quad \text{---} \\ \frac{q_{i,6}^{(j)}(e^h)}{\Delta_K(e^h)} \\ \text{---} \\ \frac{q_{i,5}^{(j)}(e^h)}{\Delta_K(e^h)} \quad \frac{q_{i,4}^{(j)}(e^h)}{\Delta_K(e^h)} \\ \text{---} \\ \frac{q_{i,3}^{(j)}(e^h)}{\Delta_K(e^h)} \end{array},$$

the 3-loop polynomial of K_j is given by

$$\begin{aligned}
& \Lambda_{K_j}(t_1, t_2, t_3, t_4) \\
& = \sum_{\substack{i \\ \tau \in \mathfrak{S}_4}} q_{i,1}^{(j)}(t_{\tau(1)}^{\text{sgn}\tau} t_{\tau(4)}^{-\text{sgn}\tau}) q_{i,2}^{(j)}(t_{\tau(2)}^{\text{sgn}\tau} t_{\tau(4)}^{-\text{sgn}\tau}) q_{i,3}^{(j)}(t_{\tau(3)}^{\text{sgn}\tau} t_{\tau(4)}^{-\text{sgn}\tau}) q_{i,4}^{(j)}(t_{\tau(2)}^{\text{sgn}\tau} t_{\tau(3)}^{-\text{sgn}\tau}) q_{i,5}^{(j)}(t_{\tau(3)}^{\text{sgn}\tau} t_{\tau(1)}^{-\text{sgn}\tau}) q_{i,6}^{(j)}(t_{\tau(1)}^{\text{sgn}\tau} t_{\tau(2)}^{-\text{sgn}\tau}) \\
& \in \mathbb{Q}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}, t_4^{\pm 1}] / (\mathfrak{S}_4, t_1 t_2 t_3 t_4 = 1) \quad (j = 1, 2),
\end{aligned}$$

as in Section 2.

Corollary 4.2. *We get the 3-loop polynomial of $K_1 \# K_2$ as follows,*

$$\begin{aligned}
& \Lambda_{K_1 \# K_2}(t_1, t_2, t_3, t_4) \\
&= \sum_{\substack{i \\ \tau \in \mathfrak{S}_4}} q_{i,1}^{(1)}(t_{\tau(1)} t_{\tau(4)}^{-1}) q_{i,2}^{(1)}(t_{\tau(2)} t_{\tau(4)}^{-1}) q_{i,3}^{(1)}(t_{\tau(3)} t_{\tau(4)}^{-1}) q_{i,4}^{(1)}(t_{\tau(2)} t_{\tau(3)}^{-1}) q_{i,5}^{(1)}(t_{\tau(3)} t_{\tau(1)}^{-1}) q_{i,6}^{(1)}(t_{\tau(1)} t_{\tau(2)}^{-1}) \\
&\quad \times \Delta_{K_2}(t_{\tau(1)} t_{\tau(4)}^{-1}) \Delta_{K_2}(t_{\tau(2)} t_{\tau(4)}^{-1}) \Delta_{K_2}(t_{\tau(3)} t_{\tau(4)}^{-1}) \Delta_{K_2}(t_{\tau(2)} t_{\tau(3)}^{-1}) \Delta_{K_2}(t_{\tau(3)} t_{\tau(1)}^{-1}) \Delta_{K_2}(t_{\tau(1)} t_{\tau(2)}^{-1}) \\
&+ \sum_{\substack{i \\ \tau \in \mathfrak{S}_4}} q_{i,1}^{(2)}(t_{\tau(1)} t_{\tau(4)}^{-1}) q_{i,2}^{(2)}(t_{\tau(2)} t_{\tau(4)}^{-1}) q_{i,3}^{(2)}(t_{\tau(3)} t_{\tau(4)}^{-1}) q_{i,4}^{(2)}(t_{\tau(2)} t_{\tau(3)}^{-1}) q_{i,5}^{(2)}(t_{\tau(3)} t_{\tau(1)}^{-1}) q_{i,6}^{(2)}(t_{\tau(1)} t_{\tau(2)}^{-1}) \\
&\quad \times \Delta_{K_1}(t_{\tau(1)} t_{\tau(4)}^{-1}) \Delta_{K_1}(t_{\tau(2)} t_{\tau(4)}^{-1}) \Delta_{K_1}(t_{\tau(3)} t_{\tau(4)}^{-1}) \Delta_{K_1}(t_{\tau(2)} t_{\tau(3)}^{-1}) \Delta_{K_1}(t_{\tau(3)} t_{\tau(1)}^{-1}) \Delta_{K_1}(t_{\tau(1)} t_{\tau(2)}^{-1}) \\
&+ \sum_{\tau \in \mathfrak{S}_4} \frac{1}{24} t_{\tau(1)} t_{\tau(2)} t_{\tau(4)}^{-2} \\
&\quad \times \Delta'_{K_1}(t_{\tau(1)} t_{\tau(4)}^{-1}) \Delta_{K_1}(t_{\tau(2)} t_{\tau(4)}^{-1}) \Delta_{K_1}(t_{\tau(3)} t_{\tau(4)}^{-1}) \Delta_{K_1}(t_{\tau(2)} t_{\tau(3)}^{-1}) \Delta_{K_1}(t_{\tau(3)} t_{\tau(1)}^{-1}) \Delta_{K_1}(t_{\tau(1)} t_{\tau(2)}^{-1}) \\
&\quad \times \Delta_{K_2}(t_{\tau(1)} t_{\tau(4)}^{-1}) \Delta'_{K_2}(t_{\tau(2)} t_{\tau(4)}^{-1}) \Delta_{K_2}(t_{\tau(3)} t_{\tau(4)}^{-1}) \Delta_{K_2}(t_{\tau(2)} t_{\tau(3)}^{-1}) \Delta_{K_2}(t_{\tau(3)} t_{\tau(1)}^{-1}) \Delta_{K_2}(t_{\tau(1)} t_{\tau(2)}^{-1}) \\
&+ \sum_{\tau \in \mathfrak{S}_4} \frac{1}{24} t_{\tau(1)} t_{\tau(2)}^{-1} t_{\tau(3)} t_{\tau(4)}^{-1} \\
&\quad \times \Delta_{K_1}(t_{\tau(1)} t_{\tau(4)}^{-1}) \Delta_{K_1}(t_{\tau(2)} t_{\tau(4)}^{-1}) \Delta'_{K_1}(t_{\tau(3)} t_{\tau(4)}^{-1}) \Delta_{K_1}(t_{\tau(2)} t_{\tau(3)}^{-1}) \Delta_{K_1}(t_{\tau(3)} t_{\tau(1)}^{-1}) \Delta_{K_1}(t_{\tau(1)} t_{\tau(2)}^{-1}) \\
&\quad \times \Delta_{K_2}(t_{\tau(1)} t_{\tau(4)}^{-1}) \Delta_{K_2}(t_{\tau(2)} t_{\tau(4)}^{-1}) \Delta_{K_2}(t_{\tau(3)} t_{\tau(4)}^{-1}) \Delta_{K_2}(t_{\tau(2)} t_{\tau(3)}^{-1}) \Delta_{K_2}(t_{\tau(3)} t_{\tau(1)}^{-1}) \Delta'_{K_2}(t_{\tau(1)} t_{\tau(2)}^{-1}) \\
&\in \mathbb{Q}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}, t_4^{\pm 1}] / (\mathfrak{S}_4, t_1 t_2 t_3 t_4 = 1).
\end{aligned}$$

In particular, if $\Delta_{K_1}(t) = \Delta_{K_2}(t) = 1$, then

$$\Lambda_{K_1 \# K_2}(t_1, t_2, t_3, t_4) = \Lambda_{K_1}(t_1, t_2, t_3, t_4) + \Lambda_{K_2}(t_1, t_2, t_3, t_4).$$

For the proof of Proposition 4.1, see Appendix.

For example, we can get

$$\Lambda_{D(K_1, K'_1) \# D(K_2, K'_2)}(t_1, t_2, t_3, t_4) = \Lambda_{D(K_1, K'_1)}(t_1, t_2, t_3, t_4) + \Lambda_{D(K_2, K'_2)}(t_1, t_2, t_3, t_4),$$

where K_j is a 0-framing knot and K'_j is a k_j -framing knot ($k_j \in \mathbb{Z}$, $j = 1, 2$), and $D(K_j, K'_j)$ is the knot defined in Section 3. In a similar way, we can get the 3-loop polynomial of $\sum_{j=1}^g D(K_j, K'_j)$.

5 The rational version of the Aarhus integral and a computation of the loop expansion

In this section, we review how to compute the loop expansion. Along this, we calculate the 3-loop polynomial. For details, see for example, [7], [10].

where $(l^{ij}(t)) = (l_{ij}(t))^{-1}$, and $\langle \quad, \quad \rangle$ is defined by

$$\langle C_1, C_2 \rangle = \left(\begin{array}{l} \text{sum of all ways gluing the } x\text{-marked legs of } C_1 \\ \text{to the } x\text{-marked legs of } C_2 \text{ for all } x \in X \end{array} \right). \quad (9)$$

For details, see [1]. Note ([6]) that

$$\langle\langle \chi^{-1} \check{Z}(U_{\pm}) \rangle\rangle = \langle \chi^{-1} \nu, \chi^{-1} \nu \rangle^{-1} \exp \left(\mp \frac{1}{16} \bigcirc \right) \stackrel{(3)}{=} \exp \left(\mp \frac{1}{16} \bigcirc \right). \quad (10)$$

From this, we can compute the loop expansion of the Kontsevich invariant of K , and we get

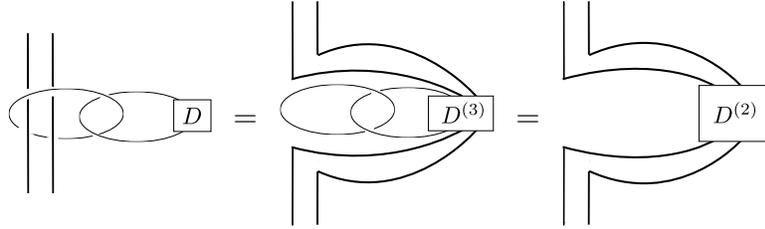
$$\begin{aligned} & \text{the 3-loop part of } \chi^{-1} Z(K) \\ &= \text{the 3-loop part of } \frac{\langle\langle \chi^{-1} \check{Z}(K_0 \cup L) \rangle\rangle}{\langle\langle \chi^{-1} \check{Z}(U_+) \rangle\rangle^{\sigma_+} \langle\langle \chi^{-1} \check{Z}(U_-) \rangle\rangle^{\sigma_-}}. \end{aligned}$$

Then, we obtain the 3-loop polynomial $\Lambda_K(t_1, t_2, t_3, t_4)$.

6 The proof of Theorem 3.1

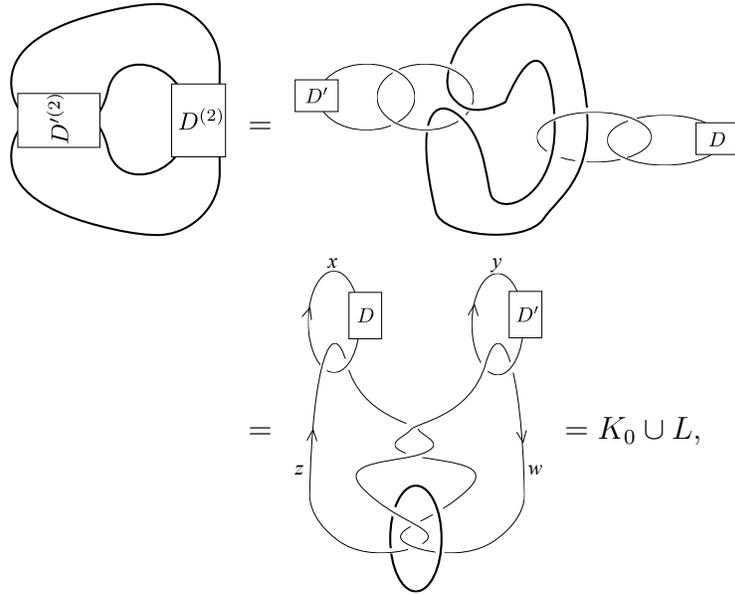
In this section, we prove Theorem 3.1.

Proof of Theorem 3.1. By handle slide, we can show that

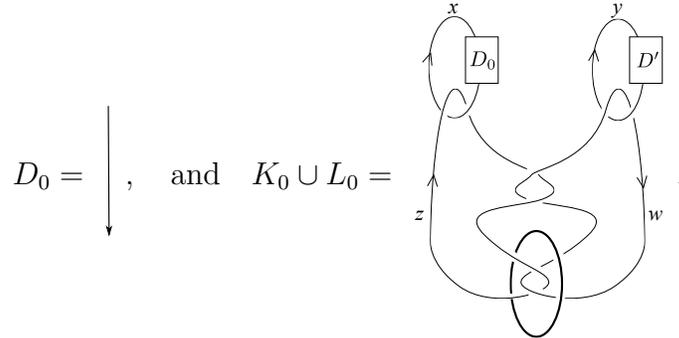


(surgery along the link drawn by thin lines),

so we get the following surgery presentation,



where K_0 is depicted by a thick line, and L is depicted by thin lines. We put



We can see that $K_0 \cup L_0$ is equal to the surgery presentation of $D(\text{unknot}, K') = \text{unknot}$, so its 3-loop part is equals to 0. In addition, linking matrices of L and L_0 are equal, and α (in (17) below) are common for L and L_0 . Thus, we get

$$\begin{aligned} & \text{the 3-loop part of } \frac{\langle\langle \chi^{-1} \check{Z}(K_0 \cup L) \rangle\rangle}{\langle\langle \chi^{-1} \check{Z}(U_+) \rangle\rangle^{\sigma_+} \langle\langle \chi^{-1} \check{Z}(U_-) \rangle\rangle^{\sigma_-}} \\ &= \text{the 3-loop part of } \frac{\langle\langle \chi^{-1} \check{Z}(K_0 \cup L) - \chi^{-1} \check{Z}(K_0 \cup L_0) \rangle\rangle}{\langle\langle \chi^{-1} \check{Z}(U_+) \rangle\rangle^{\sigma_+} \langle\langle \chi^{-1} \check{Z}(U_-) \rangle\rangle^{\sigma_-}}. \end{aligned}$$

Then we calculate $\chi^{-1} \check{Z}(K_0 \cup L) - \chi^{-1} \check{Z}(K_0 \cup L_0)$. First, we calculate $\chi_h^{-1} Z(K_0 \cup L) - \chi_h^{-1} Z(K_0 \cup L_0)$ by decomposing into the following parts. We note that each term of the formula of (11) below has at least 2 trivalent vertices, so it is sufficient to calculate other

parts modulo “ $\equiv_{(3)}$ ”.

$$\begin{aligned}
Z(D) - Z(D_0) &= Z(K)\#v^{-1} - Z(D_0) \\
&\equiv \exp \left(a_2 \left[\text{diagram} \right] + a_3 \left[\text{diagram} \right] + a_4 \left[\text{diagram} \right] \right) - \left[\text{diagram} \right] \\
&\equiv a_2 \left[\text{diagram} \right] + \left[\text{diagram} \right] \times \left(a_3 \left[\text{diagram} \right] + a_4 \left[\text{diagram} \right] + \frac{1}{2} a_2^2 \left[\text{diagram} \right] \left[\text{diagram} \right] \right), \quad (11)
\end{aligned}$$

$$\begin{aligned}
Z(D') = Z(K')\#v^{-1} &\equiv_{(3)} \exp \left(\frac{k}{2} \left[\text{diagram} \right] + a'_2 \left[\text{diagram} \right] \right) = \left[\text{diagram} \right]^{k/2} \times \left(1 + a'_2 \left[\text{diagram} \right] \right), \\
& \quad (12)
\end{aligned}$$

$$\begin{aligned}
Z \left(\left[\text{diagram} \right] \right) &\equiv_{(3)} \left[\text{diagram} \right] \times \left(1 + \frac{1}{96} \left[\text{diagram} \right] \right), \quad (13)
\end{aligned}$$

$$\begin{aligned}
Z \left(\left[\text{diagram} \right] \right) &\equiv_{(3)} \left[\text{diagram} \right] \times \left(1 + \frac{1}{96} \left[\text{diagram} \right] + \frac{1}{96} \left[\text{diagram} \right] - \frac{1}{24} \left[\text{diagram} \right] \left[\text{diagram} \right] \right), \\
& \quad (14)
\end{aligned}$$

$$\begin{aligned}
Z \left(\left[\text{diagram} \right] \right) &\equiv_{(3)} \left[\text{diagram} \right] \times \left(1 + \frac{1}{24} \left[\text{diagram} \right] \left[\text{diagram} \right] \right), \quad (15)
\end{aligned}$$

$$\begin{aligned}
\chi_h^{-1} Z \left(\left[\text{diagram} \right] \right) &\equiv_{(3)} \left[\text{diagram} \right] \times \left(1 + \frac{1}{96} \left[\text{diagram} \right] + \frac{1}{96} \left[\text{diagram} \right] - \frac{1}{24} \left[\text{diagram} \right] \left[\text{diagram} \right] \right). \\
& \quad (16)
\end{aligned}$$

where

$$\beta'_0 = a_2 \begin{array}{c} x \\ \circ \\ x \end{array} \sqcup \beta_0$$

$$\beta_1 = a_2 a'_2 \begin{array}{c} x \\ \circ \\ x \end{array} \begin{array}{c} y \\ \circ \\ y \end{array} + a_3 \begin{array}{c} x \\ \circ \\ \circ \\ x \end{array} + a_4 \begin{array}{c} x & x \\ \diagdown & \diagup \\ \circ & \\ \diagup & \diagdown \\ x & x \end{array} + \frac{1}{2} a_2^2 \begin{array}{c} x \\ \circ \\ x \end{array} \begin{array}{c} x \\ \circ \\ x \end{array}.$$

Next, we calculate $\chi_h^{-1} \check{Z}(K_0 \cup L) - \chi_h^{-1} \check{Z}(K_0 \cup L_0)$. Recall that $\nu \stackrel{(3)}{\equiv} \left| \begin{array}{c} \\ \\ \\ \end{array} \right| + \frac{1}{48} \left| \begin{array}{c} \\ \\ \\ \end{array} \right|$, so

$$\begin{aligned} & \chi_h^{-1} \check{Z}(K_0 \cup L) - \chi_h^{-1} \check{Z}(K_0 \cup L_0) \\ &= (\chi_h^{-1} Z(K_0 \cup L) - \chi_h^{-1} Z(K_0 \cup L_0)) \# \nu^{\otimes 4} \\ &\equiv (\chi_h^{-1} Z(K_0 \cup L) - \chi_h^{-1} Z(K_0 \cup L_0)) \times \left(1 + \frac{1}{48} \begin{array}{c} x \\ \circ \\ x \end{array} + \frac{1}{48} \begin{array}{c} y \\ \circ \\ y \end{array} + \frac{1}{48} \begin{array}{c} z \\ \circ \\ z \end{array} + \frac{1}{48} \begin{array}{c} w \\ \circ \\ w \end{array} \right) \end{aligned}$$

$$\equiv a_2 \begin{array}{c} \begin{array}{c} \circ \\ \\ \\ \end{array} \begin{array}{c} x \\ \circ \\ x \end{array} \\ \vdots \\ \begin{array}{c} \circ \\ \\ \\ \end{array} \begin{array}{c} z \\ \circ \\ z \end{array} \end{array} \begin{array}{c} \begin{array}{c} \\ \\ \\ \end{array} \begin{array}{c} y \\ \circ \\ y \end{array} \\ \vdots \\ \begin{array}{c} \circ \\ \\ \\ \end{array} \begin{array}{c} w \\ \circ \\ w \end{array} \end{array} + \begin{array}{c} \begin{array}{c} \circ \\ \\ \\ \end{array} \begin{array}{c} x \\ \circ \\ x \end{array} \\ \vdots \\ \begin{array}{c} \circ \\ \\ \\ \end{array} \begin{array}{c} z \\ \circ \\ z \end{array} \end{array} \begin{array}{c} \begin{array}{c} \\ \\ \\ \end{array} \begin{array}{c} y \\ \circ \\ y \end{array} \\ \vdots \\ \begin{array}{c} \circ \\ \\ \\ \end{array} \begin{array}{c} w \\ \circ \\ w \end{array} \end{array} \times (\beta''_0 + \beta_1),$$

where

$$\begin{aligned} \beta''_0 &= a_2 \begin{array}{c} x \\ \circ \\ x \end{array} \sqcup \left(\frac{1}{24} \begin{array}{c} x \\ \circ \\ x \end{array} + \frac{1}{24} \begin{array}{c} y \\ \circ \\ y \end{array} + \frac{1}{24} \begin{array}{c} z \\ \circ \\ z \end{array} + \frac{1}{24} \begin{array}{c} w \\ \circ \\ w \end{array} \right. \\ &\quad \left. - \frac{1}{24} \begin{array}{c} x \\ \\ \\ z \end{array} \begin{array}{c} x \\ \\ \\ z \end{array} - \frac{1}{24} \begin{array}{c} y \\ \\ \\ w \end{array} \begin{array}{c} y \\ \\ \\ w \end{array} + \frac{1}{24} \begin{array}{c} z \\ \\ \\ w \end{array} \begin{array}{c} z \\ \\ \\ w \end{array} - \frac{1}{24} \begin{array}{c} z \\ \\ \\ w \end{array} \begin{array}{c} z \\ \\ \\ w \end{array} \right). \end{aligned}$$

Then, we get

$$\begin{aligned} & \chi^{-1} \check{Z}(K_0 \cup L) - \chi^{-1} \check{Z}(K_0 \cup L_0) \\ &= \chi_{x,y,z,w}^{-1} (\chi_h^{-1} \check{Z}(K_0 \cup L) - \chi_h^{-1} \check{Z}(K_0 \cup L_0)) \end{aligned}$$

$$\equiv \chi_{x,y,z,w}^{-1} \left(a_2 \left(\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} \right) + \chi_{x,y,z,w}^{-1} \left(\begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} \right) \times (\beta_0'' + \beta_1) \right).$$

It is known [17] that

$$\chi_y^{-1} \left(\begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \end{array} \right) \equiv (3) \left(\begin{array}{c} \text{Diagram 7} \\ \text{Diagram 8} \end{array} \right) \times \left(1 + \frac{1}{8} \begin{array}{c} y \\ \circ \\ w \end{array} - \frac{1}{12} \begin{array}{c} y \\ \text{---} \\ w \end{array} \begin{array}{c} y \\ \text{---} \\ w \end{array} + \frac{c^2}{12} \begin{array}{c} y \\ \circ \\ y \end{array} + \frac{c}{6} \begin{array}{c} y \\ \circ \\ w \end{array} - \frac{c}{12} \begin{array}{c} y \\ \text{---} \\ w \end{array} \begin{array}{c} y \\ \text{---} \\ w \end{array} \right),$$

where c is a scalar. Hence, by the above formula and Lemma 6.2 below,

$$\chi_{x,y}^{-1} \left(\begin{array}{c} \text{Diagram 9} \\ \text{Diagram 10} \end{array} \right) \equiv \begin{array}{c} \text{Diagram 11} \\ \text{Diagram 12} \end{array} \times \left(a_2 \begin{array}{c} x \\ \circ \\ x \end{array} + \beta_2 \right),$$

where

$$\beta_2 = a_2 \begin{array}{c} x \\ \circ \\ x \end{array} \sqcup \left(\frac{1}{8} \begin{array}{c} x \\ \circ \\ z \end{array} - \frac{1}{12} \begin{array}{c} x \\ \text{---} \\ z \end{array} \begin{array}{c} x \\ \text{---} \\ z \end{array} \right) - \frac{a_2}{12} \begin{array}{c} x \\ \text{---} \\ z \end{array} \begin{array}{c} x \\ \text{---} \\ z \end{array} + \frac{a_2}{6} \begin{array}{c} x \\ \circ \\ \circ \\ z \end{array},$$

$$+ a_2 \begin{array}{c} x \\ \circ \\ x \end{array} \sqcup \left(\frac{1}{8} \begin{array}{c} y \\ \circ \\ w \end{array} - \frac{1}{12} \begin{array}{c} y \\ \text{---} \\ w \end{array} \begin{array}{c} y \\ \text{---} \\ w \end{array} + \frac{k^2}{48} \begin{array}{c} y \\ \circ \\ y \end{array} + \frac{k}{12} \begin{array}{c} y \\ \circ \\ w \end{array} - \frac{k}{24} \begin{array}{c} y \\ \text{---} \\ w \end{array} \begin{array}{c} y \\ \text{---} \\ w \end{array} \right).$$

In addition, we get

$$\chi_{x,y,z,w}^{-1} \left(\begin{array}{c} \text{Diagram with four circles } x, y, z, w \text{ and links } k/2, -l, t \\ \times (\beta_0'' + \beta_1) \end{array} \right) \equiv \alpha \sqcup (\beta_0'' + \beta_1).$$

Thus, we obtain

$$\chi^{-1} \check{Z}(K_0 \cup L) - \chi^{-1} \check{Z}(K_0 \cup L_0) \equiv \alpha \sqcup \left(a_2 \begin{array}{c} x \\ \circ \\ x \end{array} + \beta_0'' + \beta_1 + \beta_2 + \beta_3 \right),$$

where, putting $x_1 = x, x_2 = y, x_3 = z, x_4 = w$, the formula (7) is written in the following form,

$$\exp \left(\frac{1}{2} \sum_{x_i, x_j \in X} \overset{l_{ij}(t)}{\curvearrowright}_{x_i, x_j} \right) = \alpha,$$

$$P(\chi^{-1} \check{Z}(K_0 \cup L) - \chi^{-1} \check{Z}(K_0 \cup L_0)) \equiv a_2 \begin{array}{c} x \\ \circ \\ x \end{array} + \beta_0'' + \beta_1 + \beta_2 + \beta_3.$$

The equivariant linking matrix $(l_{ij}(t))$ of $L \subset S^3 \setminus K_0$ (and $L_0 \subset S^3 \setminus K_0$) is given by

$$(l_{ij}(t)) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & k & 0 & 1 \\ 1 & 0 & 0 & t-1 \\ 0 & 1 & t^{-1}-1 & 0 \end{pmatrix}.$$

Hence,

$$-\frac{1}{2} (l^{ij}(t)) = \frac{1}{2} \begin{pmatrix} -k(t+t^{-1}-2) & t-1 & -1 & -k(t-1) \\ t^{-1}-1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ -k(t^{-1}-1) & -1 & 0 & k \end{pmatrix}.$$

We put

$$\hat{\alpha} = \exp \left(-\frac{1}{2} \sum_{x_i, x_j \in X} \overset{x_j}{\curvearrowright}_{x_i}^{x_j} \right) = \begin{array}{c} \text{Diagram with arcs between } x, y, z, w \text{ and weights } -k(t+t^{-1}-2)/2, (t-1)/2, -1/2, -k(t-1)/2, -1/2, \\ \text{Diagram with arcs between } w, x, y, z \text{ and weights } k/2, (t^{-1}-1)/2, -1/2, -k(t^{-1}-1)/2, -1/2 \end{array}.$$

Then, we obtain the 3-loop part of $\chi^{-1}Z(D(K, K'))$ from above Kontsevich invariants using the rational version of the Aarhus integral. By (8), we get

$$\langle\langle \chi^{-1}\check{Z}(K_0 \cup L) - \chi^{-1}\check{Z}(K_0 \cup L_0) \rangle\rangle \equiv \left\langle \hat{\alpha}, a_2 \begin{array}{c} x \\ \bigcirc \\ x \end{array} + \beta_0'' + \beta_1 + \beta_2 + \beta_3 \right\rangle.$$

Note that $\left\langle \hat{\alpha}, a_2 \begin{array}{c} x \\ \bigcirc \\ x \end{array} + \beta_0'' + \beta_1 + \beta_2 + \beta_3 \right\rangle$ contains only diagrams with at least 2 trivalent vertices. So, we calculate the normalization term modulo $\stackrel{(3)}{\equiv}$. In our case, $\sigma_+ = \sigma_- = 2$, so we get by (10)

$$\begin{aligned} \langle\langle \chi^{-1}\check{Z}(U_+) \rangle\rangle^{\sigma_+} \langle\langle \chi^{-1}\check{Z}(U_-) \rangle\rangle^{\sigma_-} &= \langle\langle \chi^{-1}\check{Z}(U_+) \rangle\rangle^2 \langle\langle \chi^{-1}\check{Z}(U_-) \rangle\rangle^2 \\ &\stackrel{(3)}{\equiv} \exp\left(-\frac{1}{8} \bigcirc\right) \exp\left(\frac{1}{8} \bigcirc\right) = 1. \end{aligned}$$

Then, we calculate the Aarhus integral as follows,

$$\frac{\langle\langle \chi^{-1}\check{Z}(K_0 \cup L) - \chi^{-1}\check{Z}(K_0 \cup L_0) \rangle\rangle}{\langle\langle \chi^{-1}\check{Z}(U_+) \rangle\rangle^{\sigma_+} \langle\langle \chi^{-1}\check{Z}(U_-) \rangle\rangle^{\sigma_-}} \equiv \left\langle \hat{\alpha}, a_2 \begin{array}{c} x \\ \bigcirc \\ x \end{array} + \beta_0'' + \beta_1 + \beta_2 + \beta_3 \right\rangle.$$

Note that $\left\langle \hat{\alpha}, a_2 \begin{array}{c} x \\ \bigcirc \\ x \end{array} \right\rangle$ part is in the 2-loop part. Therefore, we get

$$\text{3-loop part of } \chi^{-1}Z(D(K, K')) = \langle \hat{\alpha}, (\beta_0'' + \beta_1 + \beta_2 + \beta_3) \rangle_{(\text{conn})},$$

where we denote the connected part of $\langle \ , \ \rangle$ by $\langle \ , \ \rangle_{(\text{conn})}$.

Then, we calculate each term of $\langle \hat{\alpha}, (\beta_0'' + \beta_1 + \beta_2 + \beta_3) \rangle_{(\text{conn})}$. We denote $u = t + t^{-1} - 2$ and $v = t - t^{-1}$.

We calculate $\langle \hat{\alpha}, \beta_0'' \rangle_{(\text{conn})}$, as follows.

$$\begin{aligned} \left\langle \hat{\alpha}, \frac{a_2}{24} \begin{array}{c} x \\ \bigcirc \\ x \end{array} \begin{array}{c} x \\ \bigcirc \\ x \end{array} \right\rangle_{(\text{conn})} &= \frac{k^2 a_2}{12} \begin{array}{c} u \\ \text{---} \\ u \end{array} = \frac{k^2 a_2}{6} \begin{array}{c} u^2 \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \\ \left\langle \hat{\alpha}, \frac{a_2}{24} \begin{array}{c} x \\ \bigcirc \\ x \end{array} \begin{array}{c} y \\ \bigcirc \\ y \end{array} \right\rangle_{(\text{conn})} &= \frac{a_2}{12} \begin{array}{c} t-1 \\ \text{---} \\ t-1 \end{array} = -\frac{a_2}{6} \begin{array}{c} u \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \\ \left\langle \hat{\alpha}, \frac{a_2}{24} \begin{array}{c} x \\ \bigcirc \\ x \end{array} \begin{array}{c} z \\ \bigcirc \\ z \end{array} \right\rangle_{(\text{conn})} &= \frac{a_2}{12} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} = \frac{a_2}{6} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \end{aligned}$$

$$\begin{aligned}
\langle \hat{\alpha}, \frac{a_2}{24} \begin{array}{c} x \\ \circ \\ x \end{array} \begin{array}{c} w \\ \circ \\ w \end{array} \rangle_{(\text{conn})} &= \frac{k^2 a_2}{12} \begin{array}{c} t-1 \\ \text{---} \\ t-1 \end{array} = -\frac{k^2 a_2}{6} \begin{array}{c} u \\ \text{---} \\ \text{---} \\ \text{---} \\ u \end{array} \\
\langle \hat{\alpha}, -\frac{a_2}{24} \begin{array}{c} x \\ \circ \\ x \end{array} \begin{array}{c} x \\ \text{---} \\ z \end{array} \begin{array}{c} x \\ \text{---} \\ z \end{array} \rangle_{(\text{conn})} &= \frac{k a_2}{12} \begin{array}{c} u \\ \text{---} \\ \text{---} \\ \text{---} \\ u \end{array} - \frac{k a_2}{6} \begin{array}{c} u \\ \text{---} \\ \text{---} \\ \text{---} \\ u \end{array} = -\frac{k a_2}{6} \begin{array}{c} u \\ \text{---} \\ \text{---} \\ \text{---} \\ u \end{array} \\
\langle \hat{\alpha}, -\frac{a_2}{24} \begin{array}{c} x \\ \circ \\ x \end{array} \begin{array}{c} y \\ \text{---} \\ w \end{array} \begin{array}{c} y \\ \text{---} \\ w \end{array} \rangle_{(\text{conn})} &= \frac{k a_2}{12} \begin{array}{c} t-1 \\ \text{---} \\ t-1 \end{array} = -\frac{k a_2}{6} \begin{array}{c} u \\ \text{---} \\ \text{---} \\ \text{---} \\ u \end{array} \\
\langle \hat{\alpha}, \frac{a_2}{24} \begin{array}{c} x \\ \circ \\ x \end{array} \begin{array}{c} z \\ \text{---} \\ w \end{array} \begin{array}{c} z \\ \text{---} \\ w \end{array} \rangle_{(\text{conn})} &= \frac{k a_2}{12} \begin{array}{c} u \\ \text{---} \\ \text{---} \\ \text{---} \\ u \end{array} = \frac{k a_2}{6} \begin{array}{c} u \\ \text{---} \\ \text{---} \\ \text{---} \\ u \end{array} \\
\langle \hat{\alpha}, -\frac{a_2}{24} \begin{array}{c} x \\ \circ \\ x \end{array} \begin{array}{c} z \\ \text{---} \\ w \end{array} \begin{array}{c} z \\ \text{---} \\ w \end{array} \rangle_{(\text{conn})} &= -\frac{k a_2}{12} \begin{array}{c} t \\ \text{---} \\ t \end{array} = -\frac{k a_2}{6} \begin{array}{c} u \\ \text{---} \\ \text{---} \\ \text{---} \\ u \end{array}
\end{aligned}$$

Hence,

$$\begin{aligned}
&\langle \hat{\alpha}, \beta_0'' \rangle_{(\text{conn})} \\
&= \left(-\frac{a_2}{6} - \frac{k^2 a_2}{6} - \frac{k a_2}{6} - \frac{k a_2}{6} \right) \begin{array}{c} u \\ \text{---} \\ \text{---} \\ \text{---} \\ u \end{array} + \frac{k^2 a_2}{6} \begin{array}{c} u^2 \\ \text{---} \\ \text{---} \\ \text{---} \\ u^2 \end{array} + \left(\frac{a_2}{6} + \frac{k a_2}{6} - \frac{k a_2}{6} \right) \begin{array}{c} u \\ \text{---} \\ \text{---} \\ \text{---} \\ u \end{array} \\
&= \left(-\frac{a_2}{6} - \frac{k^2 a_2}{6} - \frac{k a_2}{3} \right) \begin{array}{c} u \\ \text{---} \\ \text{---} \\ \text{---} \\ u \end{array} + \frac{k^2 a_2}{6} \begin{array}{c} u^2 \\ \text{---} \\ \text{---} \\ \text{---} \\ u^2 \end{array} + \frac{a_2}{6} \begin{array}{c} u \\ \text{---} \\ \text{---} \\ \text{---} \\ u \end{array}. \tag{18}
\end{aligned}$$

We calculate $\langle \hat{\alpha}, \beta_1 \rangle_{(\text{conn})}$, as follows.

$$\begin{aligned}
\langle \hat{\alpha}, a_2 a_2' \begin{array}{c} x \\ \circ \\ x \end{array} \begin{array}{c} y \\ \circ \\ y \end{array} \rangle_{(\text{conn})} &= 2 a_2 a_2' \begin{array}{c} t-1 \\ \text{---} \\ t-1 \end{array} = -4 a_2 a_2' \begin{array}{c} u \\ \text{---} \\ \text{---} \\ \text{---} \\ u \end{array} \\
\langle \hat{\alpha}, a_3 \begin{array}{c} x \\ \circ \\ \circ \\ \circ \\ x \end{array} \rangle_{(\text{conn})} &= -k a_3 \begin{array}{c} u \\ \text{---} \\ \text{---} \\ \text{---} \\ u \end{array} = -2 k a_3 \begin{array}{c} u \\ \text{---} \\ \text{---} \\ \text{---} \\ u \end{array}
\end{aligned}$$

$$\begin{aligned}
\langle \hat{\alpha}, \frac{k^2 a_2}{48} \begin{array}{c} x \\ \circ \\ x \end{array} \begin{array}{c} y \\ \circ \\ y \end{array} \rangle_{(\text{conn})} &= \frac{k^2 a_2}{24} \begin{array}{c} t-1 \\ \text{---} \\ t-1 \end{array} = -\frac{k^2 a_2}{12} \begin{array}{c} u \\ \circ \\ \text{---} \\ \circ \\ u \end{array} \\
\langle \hat{\alpha}, \frac{k a_2}{12} \begin{array}{c} x \\ \circ \\ x \end{array} \begin{array}{c} y \\ \circ \\ w \end{array} \rangle_{(\text{conn})} &= -\frac{k^2 a_2}{6} \begin{array}{c} t-1 \\ \text{---} \\ t-1 \end{array} = \frac{k^2 a_2}{3} \begin{array}{c} u \\ \circ \\ \text{---} \\ \circ \\ u \end{array} \\
\langle \hat{\alpha}, -\frac{k a_2}{24} \begin{array}{c} x \\ \circ \\ x \end{array} \begin{array}{c} y \\ \text{---} \\ w \end{array} \begin{array}{c} y \\ \text{---} \\ w \end{array} \rangle_{(\text{conn})} &= \frac{k^2 a_2}{12} \begin{array}{c} t-1 \\ \text{---} \\ t-1 \end{array} = -\frac{k^2 a_2}{6} \begin{array}{c} u \\ \circ \\ \text{---} \\ \circ \\ u \end{array}
\end{aligned}$$

Hence,

$$\begin{aligned}
&\langle \hat{\alpha}, \beta_2 \rangle_{(\text{conn})} \\
&= \left(\frac{k a_2}{2} - \frac{k a_2}{3} + \frac{k a_2}{2} - \frac{k a_2}{3} - \frac{k^2 a_2}{12} + \frac{k^2 a_2}{3} - \frac{k^2 a_2}{6} \right) \begin{array}{c} u \\ \circ \\ \text{---} \\ \circ \\ u \end{array} + \left(\frac{a_2}{6} - \frac{a_2}{3} \right) \begin{array}{c} \circ \\ \text{---} \\ \circ \end{array} \\
&= \left(\frac{k a_2}{3} + \frac{k^2 a_2}{12} \right) \begin{array}{c} u \\ \circ \\ \text{---} \\ \circ \\ u \end{array} - \frac{a_2}{6} \begin{array}{c} \circ \\ \text{---} \\ \circ \end{array}. \tag{20}
\end{aligned}$$

We calculate $\langle \hat{\alpha}, \beta_3 \rangle_{(\text{conn})}$, as follows.

$$\begin{aligned}
\langle \hat{\alpha}, \frac{a_2}{8} \begin{array}{c} x \\ \circ \\ x \end{array} \begin{array}{c} z \\ \text{---} \\ w \end{array} \begin{array}{c} z \\ \text{---} \\ w \end{array} \rangle_{(\text{conn})} &= \frac{k a_2}{4} \begin{array}{c} t+1 \\ \text{---} \\ t-1 \end{array} = 0 \\
\langle \hat{\alpha}, \frac{a_2}{12} \begin{array}{c} x \\ \circ \\ x \end{array} \begin{array}{c} z \\ \text{---} \\ w \end{array} \begin{array}{c} z \\ \text{---} \\ w \end{array} \rangle_{(\text{conn})} &= \frac{k a_2}{6} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} = \frac{k a_2}{3} \begin{array}{c} \circ \\ \text{---} \\ \circ \end{array} \\
\langle \hat{\alpha}, -\frac{a_2}{12} \begin{array}{c} x \\ \circ \\ x \end{array} \begin{array}{c} z \\ \text{---} \\ w \end{array} \begin{array}{c} z \\ \text{---} \\ w \end{array} \rangle_{(\text{conn})} &= -\frac{k a_2}{6} \begin{array}{c} t \\ \text{---} \\ t \end{array} = -\frac{k a_2}{3} \begin{array}{c} \circ \\ \text{---} \\ \circ \end{array} \\
\langle \hat{\alpha}, -\frac{a_2}{4} \begin{array}{c} x \\ \circ \\ x \end{array} \begin{array}{c} z \\ \text{---} \\ w \end{array} \begin{array}{c} y \\ \text{---} \\ w \end{array} \rangle_{(\text{conn})} &= \frac{k a_2}{2} \begin{array}{c} t-1 \\ \text{---} \\ t \end{array} - \frac{k a_2}{2} \begin{array}{c} \text{---} \\ \text{---} \\ t-1 \end{array} = 0 \\
\langle \hat{\alpha}, \frac{a_2}{24} \begin{array}{c} x \\ \circ \\ x \end{array} \begin{array}{c} x \\ \text{---} \\ z \end{array} \begin{array}{c} x \\ \text{---} \\ w \end{array} \rangle_{(\text{conn})} &= -\frac{k^2 a_2}{12} \begin{array}{c} u \\ \text{---} \\ u \end{array} = -\frac{k^2 a_2}{6} \begin{array}{c} u^2 \\ \circ \\ \text{---} \\ \circ \\ u^2 \end{array}
\end{aligned}$$

$$\begin{aligned}
\langle \hat{\alpha}, \frac{a_2}{24} \begin{array}{c} x \\ \circ \\ x \end{array} \begin{array}{c} y \\ t-1 \\ z \end{array} \begin{array}{c} y \\ w \end{array} \rangle_{(\text{conn})} &= -\frac{a_2}{12} \begin{array}{c} t-1 \\ \text{---} \\ t-1 \end{array} = \frac{a_2}{6} \begin{array}{c} u \\ \circ \\ \diagup \quad \diagdown \end{array} \\
\langle \hat{\alpha}, \frac{a_2}{24} \begin{array}{c} x \\ \circ \\ x \end{array} \begin{array}{c} z \\ t \\ z \\ 2t-1 \\ w \end{array} \rangle_{(\text{conn})} &= 0 \\
\langle \hat{\alpha}, \frac{a_2}{24} \begin{array}{c} x \\ \circ \\ x \end{array} \begin{array}{c} z \\ t \\ w \\ 2t-1 \\ w \end{array} \rangle_{(\text{conn})} &= \frac{k^2 a_2}{12} \begin{array}{c} 2t-t^{-1}+1 \\ \text{---} \\ 1-t \end{array} \\
&= -\frac{k^2 a_2}{6} \begin{array}{c} u \\ \circ \\ \diagup \quad \diagdown \end{array} - \frac{k^2 a_2}{24} \begin{array}{c} u \\ \circ \\ \diagup \quad \diagdown \end{array} - \frac{k^2 a_2}{8} \begin{array}{c} v \\ \circ \\ \diagup \quad \diagdown \end{array}
\end{aligned}$$

Hence,

$$\begin{aligned}
&\langle \hat{\alpha}, \beta_3 \rangle_{(\text{conn})} \\
&= \left(\frac{a_2}{6} - \frac{k^2 a_2}{6} \right) \begin{array}{c} u \\ \circ \\ \diagup \quad \diagdown \end{array} - \frac{k^2 a_2}{6} \begin{array}{c} u^2 \\ \circ \\ \diagup \quad \diagdown \end{array} - \frac{k^2 a_2}{24} \begin{array}{c} u \\ \circ \\ \diagup \quad \diagdown \end{array} \\
&\quad - \frac{k^2 a_2}{8} \begin{array}{c} v \\ \circ \\ \diagup \quad \diagdown \end{array} + \left(\frac{k a_2}{3} - \frac{k a_2}{3} \right) \begin{array}{c} \circ \\ \diagup \quad \diagdown \end{array} \\
&= \left(\frac{a_2}{6} - \frac{k^2 a_2}{6} \right) \begin{array}{c} u \\ \circ \\ \diagup \quad \diagdown \end{array} - \frac{k^2 a_2}{6} \begin{array}{c} u^2 \\ \circ \\ \diagup \quad \diagdown \end{array} - \frac{k^2 a_2}{24} \begin{array}{c} u \\ \circ \\ \diagup \quad \diagdown \end{array} - \frac{k^2 a_2}{8} \begin{array}{c} v \\ \circ \\ \diagup \quad \diagdown \end{array}. \quad (21)
\end{aligned}$$

By (18), (19), (20), (21), we get

$$\begin{aligned}
& \langle \hat{\alpha}, (\beta_0'' + \beta_1 + \beta_2 + \beta_3) \rangle_{(conn)} \\
&= \langle \hat{\alpha}, \beta_0'' \rangle_{(conn)} + \langle \hat{\alpha}, \beta_1 \rangle_{(conn)} + \langle \hat{\alpha}, \beta_2 \rangle_{(conn)} + \langle \hat{\alpha}, \beta_3 \rangle_{(conn)} \\
&= \left(-\frac{a_2}{6} - \frac{k^2 a_2}{6} - \frac{k a_2}{3}\right) \text{Diagram}_1 + \frac{k^2 a_2}{6} \text{Diagram}_2 + \frac{a_2}{6} \text{Diagram}_3 \\
&\quad + (-4a_2 a_2' - 2ka_3) \text{Diagram}_4 + \left(2k^2 a_4 - \frac{k^2 a_2}{24}\right) \text{Diagram}_5 + 3k^2 a_4 \text{Diagram}_6 + 2k^2 a_2^2 \text{Diagram}_7 \\
&\quad + \left(\frac{k a_2}{3} + \frac{k^2 a_2}{12}\right) \text{Diagram}_8 - \frac{a_2}{6} \text{Diagram}_9 \\
&\quad + \left(\frac{a_2}{6} - \frac{k^2 a_2}{6}\right) \text{Diagram}_{10} - \frac{k^2 a_2}{6} \text{Diagram}_{11} - \frac{k^2 a_2}{8} \text{Diagram}_{12} \\
&= \left(-4a_2 a_2' - \frac{k^2 a_2}{4} - 2ka_3\right) \text{Diagram}_{13} + \left(-\frac{k^2 a_2}{24} + 2k^2 a_4\right) \text{Diagram}_{14} \\
&\quad + 3k^2 a_4 \text{Diagram}_{15} + 2k^2 a_2 \text{Diagram}_{16} - \frac{k^2 a_2}{8} \text{Diagram}_{17}.
\end{aligned}$$

By the definition of the 3-loop polynomial, we get the required formula. \square

Remark 6.1. The symbol “ \sim ” means the *link relation*, see [1], [2], [3],

$$\text{Diagram}_1 + \text{Diagram}_2 + \text{Diagram}_3 + \dots + \text{Diagram}_4 \sim 0,$$

and “ \sim ” is the equivalent relation which is generated by $\equiv_{(m+1)}$ and \sim . It is known that under the link relation, the result of the Aarhus integral does not change ([6]).

Lastly, we prove the lemma used in the above calculation.

We show this. It is shown by a mirror image of Lemma 5.1 of [17] that

$$\left\{ \begin{array}{c} \text{Diagram 1} \\ \vdots \\ \text{Diagram } n \end{array} \right\}_x \equiv \left\{ \begin{array}{c} \text{Diagram 2} \\ \vdots \\ \text{Diagram } n \end{array} \right\}_x - \frac{n-1}{2} \left\{ \begin{array}{c} \text{Diagram 3} \\ \vdots \\ \text{Diagram } n \end{array} \right\}_x + \frac{(n-1)(n-2)}{6} \left\{ \begin{array}{c} \text{Diagram 4} \\ \vdots \\ \text{Diagram } n \end{array} \right\}_x. \quad (23)$$

By applying (23) (replacing n with $n+2$) to the left-hand side of (22), we get

$$\left\{ \begin{array}{c} \text{Diagram 1} \\ \vdots \\ \text{Diagram } n \end{array} \right\}_x \equiv \left\{ \begin{array}{c} \text{Diagram 2} \\ \vdots \\ \text{Diagram } n \end{array} \right\}_x - \frac{n+1}{2} \left\{ \begin{array}{c} \text{Diagram 3} \\ \vdots \\ \text{Diagram } n \end{array} \right\}_x + \frac{n(n+1)}{6} \left\{ \begin{array}{c} \text{Diagram 4} \\ \vdots \\ \text{Diagram } n \end{array} \right\}_x. \quad (24)$$

The first term of the right-hand side of (24) is calculated by applying (23) (replacing n with $n+1$) as follows,

$$\begin{aligned}
 \left\{ \begin{array}{c} \text{Diagram 1} \\ \vdots \\ \text{Diagram } n \end{array} \right\}_x &\equiv \left\{ \begin{array}{c} \text{Diagram 2} \\ \vdots \\ \text{Diagram } n \end{array} \right\}_x - \frac{n}{2} \left\{ \begin{array}{c} \text{Diagram 3} \\ \vdots \\ \text{Diagram } n \end{array} \right\}_x + \frac{n(n-1)}{6} \left\{ \begin{array}{c} \text{Diagram 4} \\ \vdots \\ \text{Diagram } n \end{array} \right\}_x \\
 &\equiv \left\{ \begin{array}{c} \text{Diagram 2} \\ \vdots \\ \text{Diagram } n \end{array} \right\}_x - \frac{n}{4} \left\{ \begin{array}{c} \text{Diagram 5} \\ \vdots \\ \text{Diagram } n \end{array} \right\}_x \times \begin{array}{c} x \\ \circ \\ z \end{array} \\
 &\quad + \frac{n(n-1)}{6} \left\{ \begin{array}{c} \text{Diagram 6} \\ \vdots \\ \text{Diagram } n \end{array} \right\}_x \times \begin{array}{c} x \quad x \\ | \quad | \\ \circ \\ | \quad | \\ z \quad z \end{array}, \quad (25)
 \end{aligned}$$

since it is shown that

$$\left\{ \begin{array}{c} \text{Diagram 1} \\ \vdots \\ \text{Diagram } n \end{array} \right\}_x \equiv \frac{1}{2} \left\{ \begin{array}{c} \text{Diagram 2} \\ \vdots \\ \text{Diagram } n \end{array} \right\}_x \equiv \frac{1}{2} \left\{ \begin{array}{c} \text{Diagram 3} \\ \vdots \\ \text{Diagram } n \end{array} \right\}_x \times \begin{array}{c} x \\ \circ \\ z \end{array},$$

$$n \left\{ \begin{array}{c} \text{Diagram 1} \\ \vdots \\ \text{Diagram 2} \end{array} \right\} \equiv n \left\{ \begin{array}{c} \text{Diagram 3} \\ \vdots \\ \text{Diagram 4} \end{array} \right\} \equiv n-2 \left\{ \begin{array}{c} \text{Diagram 5} \\ \vdots \\ \text{Diagram 6} \end{array} \right\} \times \begin{array}{c} x \\ | \\ \text{Diagram 7} \\ | \\ z \end{array}.$$

Further, it is shown that by Lemma 5.2 in [17] that

$$n \left\{ \begin{array}{c} \text{Diagram 1} \\ \vdots \\ \text{Diagram 2} \end{array} \right\} \stackrel{(3)}{\equiv} n \left\{ \begin{array}{c} \text{Diagram 3} \\ \vdots \\ \text{Diagram 4} \end{array} \right\} - \frac{n(n-1)}{8} n-1 \left\{ \begin{array}{c} \text{Diagram 5} \\ \vdots \\ \text{Diagram 6} \end{array} \right\} + \frac{n(n-1)(n-2)}{12} n-1 \left\{ \begin{array}{c} \text{Diagram 7} \\ \vdots \\ \text{Diagram 8} \end{array} \right\}.$$

(26)

By applying (26) to (25), we obtain that

$$\begin{aligned} n \left\{ \begin{array}{c} \text{Diagram 1} \\ \vdots \\ \text{Diagram 2} \end{array} \right\} &\equiv n \left\{ \begin{array}{c} \text{Diagram 3} \\ \vdots \\ \text{Diagram 4} \end{array} \right\} - \frac{n(n-1)}{8} n-2 \left\{ \begin{array}{c} \text{Diagram 5} \\ \vdots \\ \text{Diagram 6} \end{array} \right\} \times \begin{array}{c} x \\ | \\ \text{Diagram 7} \\ | \\ x \end{array} \begin{array}{c} z \\ | \\ \text{Diagram 8} \\ | \\ z \end{array} \\ &+ \frac{n(n-1)(n-2)}{12} n-3 \left\{ \begin{array}{c} \text{Diagram 9} \\ \vdots \\ \text{Diagram 10} \end{array} \right\} \times \begin{array}{c} x \\ | \\ \text{Diagram 11} \\ | \\ x \end{array} \begin{array}{c} x \\ | \\ \text{Diagram 12} \\ | \\ z \end{array} \begin{array}{c} x \\ | \\ \text{Diagram 13} \\ | \\ z \end{array} \\ &- \frac{n}{4} n-1 \left\{ \begin{array}{c} \text{Diagram 14} \\ \vdots \\ \text{Diagram 15} \end{array} \right\} \times \begin{array}{c} x \\ | \\ \text{Diagram 16} \\ | \\ z \end{array} + \frac{n(n-1)}{6} n-2 \left\{ \begin{array}{c} \text{Diagram 17} \\ \vdots \\ \text{Diagram 18} \end{array} \right\} \times \begin{array}{c} x \\ | \\ \text{Diagram 19} \\ | \\ z \end{array} \begin{array}{c} x \\ | \\ \text{Diagram 20} \\ | \\ z \end{array}. \end{aligned}$$

(27)

The second term of the right-hand side of (24) is calculated as follows,

$$\begin{aligned} -\frac{n+1}{2} n \left\{ \begin{array}{c} \text{Diagram 1} \\ \vdots \\ \text{Diagram 2} \end{array} \right\} &\equiv -\frac{n}{2} n \left\{ \begin{array}{c} \text{Diagram 3} \\ \vdots \\ \text{Diagram 4} \end{array} \right\} \\ &\equiv -\frac{n}{2} n \left\{ \begin{array}{c} \text{Diagram 5} \\ \vdots \\ \text{Diagram 6} \end{array} \right\} + \frac{n(n-1)}{4} n \left\{ \begin{array}{c} \text{Diagram 7} \\ \vdots \\ \text{Diagram 8} \end{array} \right\}, \end{aligned}$$

where we obtain the first equivalence by classifying the strand which is connected to the bottom strand at the right hand side of the symmetrizer of the first term, and we obtain the second equivalence by applying (23) to the left symmetrizer of the diagram of the right-hand side of the first line. Hence,

$$-\frac{n+1}{2} \left\{ \begin{array}{c} \text{Diagram 1} \\ \vdots \\ \text{Diagram } n \end{array} \right\} \equiv \frac{n}{4} \left\{ \begin{array}{c} \text{Diagram 2} \\ \vdots \\ \text{Diagram } n-1 \end{array} \right\} \times \begin{array}{c} x \\ \circ \\ z \end{array} - \frac{n(n-1)}{4} \left\{ \begin{array}{c} \text{Diagram 3} \\ \vdots \\ \text{Diagram } n-2 \end{array} \right\} \times \begin{array}{c} x \\ | \\ \circ \\ | \\ z \end{array} \quad (28)$$

The third term of the right-hand side of (24) is calculated as follows. By classifying the strand which is connected to the bottom strand at the right hand side of the symmetrizer, the connected component of this strand is shown, as follows,

$$\begin{array}{c} x \\ \text{---} \\ \text{Diagram 1} \end{array} \equiv 0, \quad \begin{array}{c} x \\ \text{---} \\ \text{Diagram 2} \end{array} \equiv - \begin{array}{c} x \\ \circ \\ z \end{array}, \quad (n-1) \begin{array}{c} x \\ \text{---} \\ \text{Diagram 3} \end{array} \equiv (n-1) \begin{array}{c} x \\ | \\ \circ \\ | \\ z \end{array}.$$

Then, we get

$$\frac{n(n+1)}{6} \left\{ \begin{array}{c} \text{Diagram 4} \\ \vdots \\ \text{Diagram } n \end{array} \right\} \equiv -\frac{n}{6} \left\{ \begin{array}{c} \text{Diagram 5} \\ \vdots \\ \text{Diagram } n-1 \end{array} \right\} \times \begin{array}{c} x \\ \circ \\ z \end{array} + \frac{n(n-1)}{6} \left\{ \begin{array}{c} \text{Diagram 6} \\ \vdots \\ \text{Diagram } n-2 \end{array} \right\} \times \begin{array}{c} x \\ | \\ \circ \\ | \\ z \end{array} \quad (29)$$

Thus, by applying (27), (28), (29) to (24), we obtain (22). Therefore, we obtain the required formula of the lemma. \square

7 The \mathfrak{sl}_2 reduction of the 3-loop polynomial

7.1 The review of the loop expansion of the colored Jones polynomial

In this section, we briefly review the colored Jones polynomial and the loop expansion of it. For details, see [22], [23].

The colored Jones polynomial $J_n(K; t)$ is the polynomial invariant of knots, which is obtained by

$$J_n(K; t) = \frac{V_n(K; t)}{V_n(\text{the unknot}; t)} = \frac{t^{1/2} - t^{-1/2}}{t^{n/2} - t^{-n/2}} \cdot V_n(K; t),$$

where $V_n(K; t)$ is obtained by $V_n(K; e^{-h}) = W_{\mathfrak{sl}_2, V_n}(Z(K))$, and $W_{\mathfrak{sl}_2, V_n}$ denotes the weight system derived from the Lie algebra \mathfrak{sl}_2 and its irreducible representation V_n . For details, see [14], [15], and [13]. It is known, see Conjecture 1.2 of [22], Theorem 1.2 of [23], Proposition 3.1 of [20], that $J_n(K; t)$ can be presented in the following form,

$$J_n(K; e^h) = \sum_{l \geq 0} h^l \sum_{k \geq 0} d_{l,k}(nh)^k = \sum_{l \geq 0} h^l \frac{P_l(e^{nh})}{\Delta_K(e^{nh})^{2l+1}}$$

for some $P_l(t) \in \mathbb{Q}[t^{\pm 1}]$. This is called the loop expansion of the colored Jones polynomial.

The 3-loop part of the colored Jones polynomial is given by $\frac{P_2(e^{nh})}{\Delta_K(e^{nh})^5}$.

7.2 The 3-loop part of the colored Jones polynomial

In this section, we consider the 3-loop part of the colored Jones polynomial.

For a knot K , $\Lambda_K(t^{\frac{1}{2}}, t^{\frac{1}{2}}, t^{-\frac{1}{2}}, t^{-\frac{1}{2}})$ is a symmetric polynomial in $t^{\pm 1}$ divisible by $t - 1$ (since $\Lambda_K(1, 1, 1, 1) = 0$) and, hence, divisible by $(t - 1)^2$. We define the *reduced 3-loop polynomial* by

$$\hat{\Lambda}_K(t) = \frac{\Lambda_K(t^{\frac{1}{2}}, t^{\frac{1}{2}}, t^{-\frac{1}{2}}, t^{-\frac{1}{2}})}{(t^{1/2} - t^{-1/2})^2},$$

which is symmetric polynomial in $t^{\pm 1}$.

Example 7.1. The reduced 3-loop polynomial of $D(K, K')$ is presented by

$$\begin{aligned} & \hat{\Lambda}_{D(K, K')}(t) \\ &= \left(-\frac{4k^2 a_2}{3} + 64k^2 a_4 + 32k^2 a_2^2\right)(t + t^{-1}) - \frac{16k^2 a_2}{3} - 64k^2 a_2^2 - 64a_2 a_2' - 32ka_3 - 128k^2 a_4. \end{aligned}$$

We denote the reduced 2-loop polynomial by $\hat{\Theta}_K(t) = \frac{\Theta_K(t, t^{-1}, 1)}{(t^{1/2} - t^{-1/2})^2}$ defined in [16], where $\Theta_K(t_1, t_2, t_3)$ is the 2-loop polynomial of K . For definition of the 2-loop polynomial, see for example [14], [17].

Proposition 7.2. *The 3-loop part of the colored Jones polynomial $\frac{P_2(e^{nh})}{\Delta_K(e^{nh})^5}$ is presented by*

$$\begin{aligned} \frac{P_2(t)}{\Delta_K(t)^5} &= \frac{(t^{1/2} - t^{-1/2})^2}{\Delta_K(t)^5} \hat{\Lambda}_K(t) + \frac{(t^{1/2} - t^{-1/2})^4}{2\Delta_K(t)^5} \hat{\Theta}_K(t)^2 \\ &+ \frac{\Delta_K'(t)t^2}{3(t-1)\Delta_K(t)^2} + \frac{\Delta_K''(t)t^2}{6\Delta_K(t)^2} - \frac{\Delta_K'(t)t^2}{3\Delta_K(t)^3}. \end{aligned}$$

Further, by (32) and Lemma 7.4 below, we obtain

$$\begin{aligned}
& \sum_i \left(\frac{q_{i,1}(e^h)}{\Delta_K(e^h)} \frac{q_{i,2}(e^h)}{\Delta_K(e^h)} \frac{q_{i,3}(e^h)}{\Delta_K(e^h)} \frac{q_{i,4}(e^h)}{\Delta_K(e^h)} \frac{q_{i,5}(e^h)}{\Delta_K(e^h)} \frac{q_{i,6}(e^h)}{\Delta_K(e^h)} \right) \\
& \equiv_{\text{s}l_2} \sum_i 4h^2 \left(q_{i,3}(1)q_{i,6}(1) \left(\frac{q_{i,1}(e^h)}{\Delta_K(e^h)} \frac{q_{i,2}(e^h)}{\Delta_K(e^h)} \frac{q_{i,4}(e^h)}{\Delta_K(e^h)} \frac{q_{i,5}(e^h)}{\Delta_K(e^h)} \right) + q_{i,2}(1)q_{i,5}(1) \left(\frac{q_{i,1}(e^h)}{\Delta_K(e^h)} \frac{q_{i,3}(e^h)}{\Delta_K(e^h)} \frac{q_{i,4}(e^h)}{\Delta_K(e^h)} \frac{q_{i,6}(e^h)}{\Delta_K(e^h)} \right) \right. \\
& \quad \left. + q_{i,1}(1)q_{i,4}(1) \left(\frac{q_{i,2}(e^h)}{\Delta_K(e^h)} \frac{q_{i,3}(e^h)}{\Delta_K(e^h)} \frac{q_{i,5}(e^h)}{\Delta_K(e^h)} \frac{q_{i,6}(e^h)}{\Delta_K(e^h)} \right) \right) \\
& \equiv_{\text{s}l_2} \sum_i \frac{4h^2}{\Delta_K(\hat{t})^4} (q_{i,1}(\hat{t})q_{i,2}(\hat{t})q_{i,3}(1)q_{i,4}(\hat{t})q_{i,5}(\hat{t}^{-1})q_{i,6}(1) + q_{i,1}(\hat{t}^{-1})q_{i,2}(\hat{t}^{-1})q_{i,3}(1)q_{i,4}(\hat{t}^{-1})q_{i,5}(\hat{t})q_{i,6}(1) \\
& \quad + q_{i,1}(\hat{t})q_{i,2}(1)q_{i,3}(\hat{t})q_{i,4}(\hat{t}^{-1})q_{i,5}(1)q_{i,6}(\hat{t}) + q_{i,1}(\hat{t}^{-1})q_{i,2}(1)q_{i,3}(\hat{t}^{-1})q_{i,4}(\hat{t})q_{i,5}(1)q_{i,6}(\hat{t}^{-1}) \\
& \quad + q_{i,1}(1)q_{i,2}(\hat{t})q_{i,3}(\hat{t})q_{i,4}(1)q_{i,5}(\hat{t})q_{i,6}(\hat{t}^{-1}) + q_{i,1}(1)q_{i,2}(\hat{t}^{-1})q_{i,3}(\hat{t}^{-1})q_{i,4}(1)q_{i,5}(\hat{t}^{-1})q_{i,6}(\hat{t})) \\
& \equiv_{\text{s}l_2} \frac{\tilde{\Lambda}_K(\hat{t})}{\Delta_K(\hat{t})^4} h^2.
\end{aligned}$$

Note that $q_{i,1}(1)q_{i,2}(1)q_{i,3}(1)q_{i,4}(1)q_{i,5}(1)q_{i,6}(1) = 0$. Hence, we get

$$\begin{aligned}
\sum_{l \geq 0} \frac{\hat{P}_l(\hat{t})}{\Delta_K(\hat{t})^{2l+1}} h^l & \equiv_{\text{s}l_2} \frac{1}{\Delta_K(\hat{t})} \left(1 + \frac{\tilde{\Theta}_K(\hat{t})}{\Delta_K(\hat{t})^2} h + \frac{\tilde{\Lambda}_K(\hat{t})}{\Delta_K(\hat{t})^4} h^2 + \frac{\tilde{\Theta}_K(\hat{t})^2}{2\Delta_K(\hat{t})^4} h^2 + (\text{higher terms}) \right) \\
& = \frac{1}{\Delta_K(\hat{t})} + \frac{\tilde{\Theta}_K(\hat{t})}{\Delta_K(\hat{t})^3} h + \left(\frac{\tilde{\Lambda}_K(\hat{t})}{\Delta_K(\hat{t})^5} + \frac{\tilde{\Theta}_K(\hat{t})^2}{2\Delta_K(\hat{t})^5} \right) h^2 + (\text{higher terms}).
\end{aligned}$$

Thus, by (30), we obtain

$$\begin{aligned}
\sum_{l \geq 0} \frac{P_l(t)}{\Delta_K(t)^{2l+1}} h^l &= \frac{1}{t^{1/2} - t^{-1/2}} \cdot \frac{\mathcal{D}}{[\mathcal{D}]} \left((t^{1/2} - t^{-1/2}) \left(\frac{1}{\Delta_K(t)} + \frac{\tilde{\Theta}_K(t)}{\Delta_K(t)^3} h \right. \right. \\
&\quad \left. \left. + \left(\frac{\tilde{\Lambda}_K(t)}{\Delta_K(t)^5} + \frac{\tilde{\Theta}_K(t)^2}{2\Delta_K(t)^5} \right) h^2 + (\text{higher terms}) \right) \right) \\
&= \frac{1}{\Delta_K(t)} + \frac{\tilde{\Theta}_K(t)}{\Delta_K(t)^3} h + \left(\frac{\tilde{\Lambda}_K(t)}{\Delta_K(t)^5} + \frac{\tilde{\Theta}_K(t)^2}{2\Delta_K(t)^5} \right) h^2 \\
&\quad + \frac{h^2}{24} \cdot \frac{1}{t^{1/2} - t^{-1/2}} (1 - \mathcal{D}^2) \left(\frac{t^{1/2} - t^{-1/2}}{\Delta_K(t)} \right) + (\text{higher terms}). \quad (33)
\end{aligned}$$

Here, the last term of the right-hand side of (33) is calculated as follows,

$$\begin{aligned}
\frac{1}{t^{1/2} - t^{-1/2}} (1 - \mathcal{D}^2) \left(\frac{t^{1/2} - t^{-1/2}}{\Delta_K(t)} \right) &= \frac{1}{t^{1/2} - t^{-1/2}} \left(1 - \left(2t \frac{d}{dt} \right)^2 \right) \left(\frac{t^{1/2} - t^{-1/2}}{\Delta_K(t)} \right) \\
&= \frac{8\Delta'_K(t)t^2}{(t-1)\Delta_K(t)^2} + \frac{4\Delta''_K(t)t^2}{\Delta_K(t)^2} - \frac{8\Delta'_K(t)^2 t^2}{\Delta_K(t)^3}. \quad (34)
\end{aligned}$$

By applying (34) to (33), we get

$$\begin{aligned}
\sum_{l \geq 0} \frac{P_l(t)}{\Delta_K(t)^{2l+1}} h^l &= \frac{1}{\Delta_K(t)} + \frac{\tilde{\Theta}_K(t)}{\Delta_K(t)^3} h \\
&\quad + \left(\frac{\tilde{\Lambda}_K(t)}{\Delta_K(t)^5} + \frac{\tilde{\Theta}_K(t)^2}{2\Delta_K(t)^5} + \frac{\Delta'_K(t)t^2}{3(t-1)\Delta_K(t)^2} + \frac{\Delta''_K(t)t^2}{6\Delta_K(t)^2} - \frac{\Delta'_K(t)^2 t^2}{3\Delta_K(t)^3} \right) h^2 \\
&\quad + (\text{higher terms}).
\end{aligned}$$

Therefore, considering the h^2 terms, we obtain the required formula. \square

We recall that the Conway polynomial $\nabla_K(z)$ is defined by $\nabla_K(t^{1/2} - t^{-1/2}) = \Delta_K(t)$.

Remark 7.3. The formula of Proposition 7.2 is rewritten in terms of the Conway polynomial as

$$\begin{aligned}
\frac{P_2(t)}{\Delta_K(t)^5} &= \frac{z^2}{\nabla_K(z)^5} \hat{\Lambda}_K(t) + \frac{z^4}{2\nabla_K(z)^5} \hat{\Theta}_K(t)^2 \\
&\quad + \frac{3z^2 + 8 \frac{\nabla'_K(z)}{\nabla_K(z)^2}}{24z} + \frac{z^2 + 4 \frac{\nabla''_K(z)}{\nabla_K(z)^2}}{24} - \frac{z^2 + 4 \frac{\nabla'_K(z)^2}{\nabla_K(z)^3}}{12}
\end{aligned}$$

where $z = t^{1/2} - t^{-1/2}$.

Proof. For $z = t^{1/2} - t^{-1/2}$, we get

$$2t \frac{dz}{dt} = t^{1/2} + t^{-1/2}, \quad \mathcal{D} = 2t \frac{d}{dt} = (t^{1/2} + t^{-1/2}) \frac{d}{dz},$$

$$\begin{aligned}
\mathcal{D} \left(\frac{t^{1/2} - t^{-1/2}}{\Delta_K(t)} \right) &= 2t \frac{d}{dt} \frac{t^{1/2} - t^{-1/2}}{\Delta_K(t)} = (t^{1/2} + t^{-1/2}) \frac{d}{dz} \frac{z}{\nabla_K(z)}, \\
\mathcal{D}^2 \left(\frac{t^{1/2} - t^{-1/2}}{\Delta_K(t)} \right) &= 2t \frac{d}{dt} \left((t^{1/2} + t^{-1/2}) \frac{d}{dz} \frac{z}{\nabla_K(z)} \right) \\
&= (t^{1/2} - t^{-1/2}) \frac{d}{dz} \frac{z}{\nabla_K(z)} + (t^{1/2} + t^{-1/2})^2 \left(\frac{d}{dz} \right)^2 \frac{z}{\nabla_K(z)} \\
&= (t^{1/2} - t^{-1/2}) \frac{d}{dz} \frac{z}{\nabla_K(z)} + (z^2 + 4) \left(\frac{d}{dz} \right)^2 \frac{z}{\nabla_K(z)}.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
\frac{1}{t^{1/2} - t^{-1/2}} \mathcal{D}^2 \left(\frac{t^{1/2} - t^{-1/2}}{\Delta_K(t)} \right) &= \frac{d}{dz} \frac{z}{\nabla_K(z)} + \frac{z^2 + 4}{z} \left(\frac{d}{dz} \right)^2 \frac{z}{\nabla_K(z)} \\
&= \frac{1}{\nabla_K(z)} - \frac{3z^2 + 8}{z} \frac{\nabla'_K(z)}{\nabla_K(z)^2} - (z^2 + 4) \frac{\nabla''_K(z)}{\nabla_K(z)^2} + 2(z^2 + 4) \frac{\nabla'_K(z)^2}{\nabla_K(z)^3}.
\end{aligned}$$

Therefore, the formula (34) can be rewritten as

$$\frac{1}{t^{1/2} - t^{-1/2}} (1 - \mathcal{D}^2) \left(\frac{t^{1/2} - t^{-1/2}}{\Delta_K(t)} \right) = \frac{3z^2 + 8}{z} \frac{\nabla'_K(z)}{\nabla_K(z)^2} + (z^2 + 4) \frac{\nabla''_K(z)}{\nabla_K(z)^2} - 2(z^2 + 4) \frac{\nabla'_K(z)^2}{\nabla_K(z)^3}.$$

By applying this to (33), we obtain the required formula. Hence, the formula of Proposition 7.2 is rewritten as the formula of the remark. \square

We prove the lemma used in the proof of Proposition 7.2.

Lemma 7.4.

$$\begin{aligned}
& \text{Circle with sectors } f_1(h), f_2(h), f_3(h), f_4(h), f_5(h), f_6(h) \\
& \equiv_{st_2} 4h^2 \left(f_3(0)f_6(0) \begin{array}{c} \text{Circle with wedge removed} \\ f_1(h), f_2(h), f_5(h), f_4(h) \end{array} + f_2(0)f_5(0) \begin{array}{c} \text{Circle with wedge removed} \\ f_1(h), f_6(h), f_4(h), f_3(h) \end{array} \right. \\
& \left. + f_1(0)f_4(0) \begin{array}{c} \text{Circle with wedge removed} \\ f_6(h), f_2(h), f_5(h), f_3(h) \end{array} \right),
\end{aligned}$$

where $f_i(h)$ is a power series in h .

Proof. By using (31), we get the following equivalences,

$$\begin{array}{c} \diagup \\ \diagdown \\ \diagup \\ \diagdown \end{array} \equiv_{\text{sl}_2} 0, \quad (35)$$

$$\begin{aligned} & \begin{array}{c} f_1(h) \quad f_2(h) \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ f_5(h) \quad f_4(h) \\ \diagdown \quad \diagup \\ f_3(h) \end{array} \equiv_{\text{sl}_2} 2h \left(\begin{array}{c} f_1(h) \quad f_2(h) \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ f_5(h) \quad f_4(h) \\ \diagdown \quad \diagup \\ f_3(h) \end{array} - \begin{array}{c} f_1(h) \quad f_2(h) \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ f_5(h) \quad f_4(h) \\ \diagdown \quad \diagup \\ f_3(h) \end{array} \right) \\ & \equiv_{\text{sl}_2} 2h \left(f_3(0) \begin{array}{c} f_1(h)f_5(-h) \quad f_2(h)f_4(h) \\ \text{---} \\ \text{---} \end{array} + \begin{array}{c} f_1(h)f_4(-h) \\ \text{---} \\ f_2(h)f_5(h) \\ \text{---} \\ f_3(h) \end{array} \right). \quad (36) \end{aligned}$$

Further, it is shown by the formula after Lemma 6.2 in [16] that

$$\begin{array}{c} f_1(h) \\ \text{---} \\ f_2(h) \\ \text{---} \\ f_3(h) \end{array} \equiv_{\text{sl}_2} 2h \left(f_3(0) \begin{array}{c} f_1(h) \\ \text{---} \\ f_2(h) \end{array} + f_2(0) \begin{array}{c} f_1(h) \\ \text{---} \\ f_3(h) \end{array} + f_1(0) \begin{array}{c} f_2(h) \\ \text{---} \\ f_3(h) \end{array} \right). \quad (37)$$

We put $\hat{f}_i(h) = f_i(h) - f_i(0)$. Then, by (35),

$$\begin{aligned} & \begin{array}{c} f_1(h) \quad f_2(h) \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ f_5(h) \quad f_4(h) \\ \diagdown \quad \diagup \\ f_3(h) \end{array} = \begin{array}{c} f_1(h) \quad f_2(h) \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ f_5(h) \quad f_4(h) \\ \diagdown \quad \diagup \\ f_3(h) \end{array} + f_6(0) \begin{array}{c} f_1(h) \quad f_2(h) \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ f_5(h) \quad f_4(h) \\ \diagdown \quad \diagup \\ f_3(h) \end{array} \\ & = \begin{array}{c} f_1(h) \quad f_2(h) \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ \hat{f}_5(h) \quad f_4(h) \\ \diagdown \quad \diagup \\ f_3(h) \end{array} + f_5(0) \begin{array}{c} f_1(h) \quad f_2(h) \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ f_5(h) \quad f_4(h) \\ \diagdown \quad \diagup \\ f_3(h) \end{array} + f_6(0) \begin{array}{c} f_1(h) \quad f_2(h) \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ f_5(h) \quad f_4(h) \\ \diagdown \quad \diagup \\ f_3(h) \end{array} \end{aligned}$$

$$\begin{aligned}
& \equiv_{\mathfrak{sl}_2} f_4(0) \left(\begin{array}{c} f_1(h) \quad f_2(h) \\ | \quad | \\ \hat{f}_6(h) \\ | \\ \hat{f}_5(h) \\ / \quad \backslash \\ f_3(h) \end{array} \right) + f_5(0) \left(\begin{array}{c} f_1(h) \quad f_2(h) \\ | \quad | \\ \hat{f}_6(h) \\ | \\ f_4(h) \\ / \quad \backslash \\ f_3(h) \end{array} \right) + f_6(0) \left(\begin{array}{c} f_1(h) \quad f_2(h) \\ | \quad | \\ f_5(h) \\ | \\ f_4(h) \\ / \quad \backslash \\ f_3(h) \end{array} \right) . \\
& \tag{38}
\end{aligned}$$

By (36),(37), the first term of the right-hand side of (38) is calculated as follows,

$$\begin{aligned}
& f_4(0) \left(\begin{array}{c} f_1(h) \quad f_2(h) \\ | \quad | \\ \hat{f}_6(h) \\ | \\ \hat{f}_5(h) \\ / \quad \backslash \\ f_3(h) \end{array} \right) \equiv_{\mathfrak{sl}_2} 2h f_4(0) \left(f_1(0) \left(\begin{array}{c} f_2(h)\hat{f}_6(-h) \quad f_3(h)\hat{f}_5(h) \\ \text{---} \\ \text{---} \end{array} \right) + \left(\begin{array}{c} f_2(h)\hat{f}_5(-h) \\ \text{---} \\ f_3(h)\hat{f}_6(h) \\ \text{---} \\ f_1(h) \end{array} \right) \right) \\
& \equiv_{\mathfrak{sl}_2} 2h f_4(0) \left(2h f_1(0) \left(\begin{array}{c} f_2(h)f_3(h)\hat{f}_5(h)\hat{f}_6(-h) \\ \text{---} \\ \text{---} \end{array} \right) - \begin{array}{c} f_2(h)f_3(-h)\hat{f}_5(-h)\hat{f}_6(-h) \\ \text{---} \\ \text{---} \end{array} \right) \\
& \quad + 2h f_1(0) \left(\begin{array}{c} f_2(h)f_3(-h)\hat{f}_5(-h)\hat{f}_6(-h) \\ \text{---} \\ \text{---} \end{array} \right) \Bigg) \\
& \equiv_{\mathfrak{sl}_2} 4h^2 f_1(0) f_4(0) \left(\begin{array}{c} f_2(h)f_3(h)\hat{f}_5(h)\hat{f}_6(-h) \\ \text{---} \\ \text{---} \end{array} \right) . \\
& \tag{39}
\end{aligned}$$

By (36),(37), the second term of the right-hand side of (38) is calculated as follows,

$$\begin{aligned}
& f_5(0) \left(\begin{array}{c} f_1(h) \quad f_2(h) \\ | \quad | \\ \hat{f}_6(h) \\ | \\ f_4(h) \\ / \quad \backslash \\ f_3(h) \end{array} \right) \equiv_{\mathfrak{sl}_2} 2h f_5(0) \left(f_2(0) \left(\begin{array}{c} f_1(h)\hat{f}_6(h) \quad f_3(h)f_4(-h) \\ \text{---} \\ \text{---} \end{array} \right) + \left(\begin{array}{c} f_1(h)f_4(h) \\ \text{---} \\ f_3(h)\hat{f}_6(-h) \\ \text{---} \\ f_2(h) \end{array} \right) \right)
\end{aligned}$$

$$\begin{aligned}
&\equiv_{\mathfrak{sl}_2} 2h f_5(0) \left(2h f_2(0) \left(\begin{array}{c} f_1(h)f_3(h)f_4(-h)\hat{f}_6(h) \\ \text{---} \\ \text{---} \end{array} - \begin{array}{c} f_1(h)f_3(-h)f_4(h)\hat{f}_6(h) \\ \text{---} \\ \text{---} \end{array} \right) \right. \\
&\quad \left. + 2h \left(f_2(0) \begin{array}{c} f_1(h)f_3(-h)f_4(h)\hat{f}_6(h) \\ \text{---} \\ \text{---} \end{array} + f_1(0)f_4(0) \begin{array}{c} f_2(h)f_3(h)\hat{f}_6(-h) \\ \text{---} \\ \text{---} \end{array} \right) \right) \\
&\equiv_{\mathfrak{sl}_2} 4h^2 \left(f_2(0)f_5(0) \begin{array}{c} f_1(h)f_3(h)f_4(-h)\hat{f}_6(h) \\ \text{---} \\ \text{---} \end{array} + f_1(0)f_4(0)f_5(0) \begin{array}{c} f_2(h)f_3(h)\hat{f}_6(-h) \\ \text{---} \\ \text{---} \end{array} \right). \tag{40}
\end{aligned}$$

By (36),(37), the third term of the right-hand side of (38) is calculated as follows,

$$\begin{aligned}
&f_6(0) \begin{array}{c} f_1(h) \quad f_2(h) \\ \text{---} \\ \text{---} \\ f_5(h) \quad f_4(h) \\ \text{---} \\ f_3(h) \end{array} \equiv_{\mathfrak{sl}_2} 2h f_6(0) \left(f_3(0) \begin{array}{c} f_1(h)f_5(-h) \quad f_2(h)f_4(h) \\ \text{---} \quad \text{---} \end{array} + \begin{array}{c} f_1(h)f_4(-h) \\ \text{---} \\ f_2(h)f_5(h) \\ \text{---} \\ f_3(h) \end{array} \right) \\
&\equiv_{\mathfrak{sl}_2} 2h f_6(0) \left(2h f_3(0) \left(\begin{array}{c} f_1(h)f_2(h)f_4(h)f_5(-h) \\ \text{---} \\ \text{---} \end{array} - \begin{array}{c} f_1(h)f_2(-h)f_4(-h)f_5(-h) \\ \text{---} \\ \text{---} \end{array} \right) \right. \\
&\quad \left. + 2h \left(f_3(0) \begin{array}{c} f_1(h)f_2(-h)f_4(-h)f_5(-h) \\ \text{---} \\ \text{---} \end{array} + f_2(0)f_5(0) \begin{array}{c} f_1(h)f_3(h)f_4(-h) \\ \text{---} \\ \text{---} \end{array} + f_1(0)f_4(0) \begin{array}{c} f_2(h)f_3(h)f_5(h) \\ \text{---} \\ \text{---} \end{array} \right) \right) \\
&\equiv_{\mathfrak{sl}_2} 4h^2 \left(f_3(0)f_6(0) \begin{array}{c} f_1(h)f_2(h)f_4(h)f_5(-h) \\ \text{---} \\ \text{---} \end{array} + f_2(0)f_5(0)f_6(0) \begin{array}{c} f_1(h)f_3(h)f_4(-h) \\ \text{---} \\ \text{---} \end{array} \right. \\
&\quad \left. + f_1(0)f_4(0)f_6(0) \begin{array}{c} f_2(h)f_3(h)f_5(h) \\ \text{---} \\ \text{---} \end{array} \right). \tag{41}
\end{aligned}$$

Thus, by applying (39), (40), (41) to (38), we obtain

$$\begin{aligned}
& \equiv_{\text{sl}_2} 4h^2 \left(f_1(0)f_4(0) \text{ (oval)} + f_2(0)f_5(0) \text{ (oval)} \right. \\
& \quad + f_1(0)f_4(0)f_5(0) \text{ (oval)} + f_3(0)f_6(0) \text{ (oval)} \\
& \quad \left. + f_2(0)f_5(0)f_6(0) \text{ (oval)} + f_1(0)f_4(0)f_6(0) \text{ (oval)} \right) \\
& \equiv_{\text{sl}_2} 4h^2 \left(f_3(0)f_6(0) \text{ (oval)} + f_2(0)f_5(0) \text{ (oval)} \right. \\
& \quad \left. + f_1(0)f_4(0) \text{ (oval)} \right).
\end{aligned}$$

Therefore, we obtain the lemma. \square

Appendix

A The Duflo isomorphism and the proof of the connected sum formula for the 3-loop polynomial

In this section, we review the *Duflo isomorphism* ([4], [5]), and we prove Proposition 4.1 (the connected sum formula for the 3-loop polynomial) in Section 4.2. For the notation, see Section 4.2.

Let D' be a diagram which have at least one trivalent vertex on each component. Then we define $\partial_{D'} : \mathcal{B} \rightarrow \mathcal{B}$ by

$$\partial_{D'}(D) = \begin{cases} 0 & \text{if } D' \text{ has more legs than } D, \\ \text{the sum of all ways of gluing} & \\ \text{all the legs of } D' \text{ to some} & \text{otherwise} \\ \text{(or all) legs of } D & \end{cases}$$

Thus, we get

$$\begin{aligned}
\Upsilon^{-1}Z(K_1\#K_2) &\equiv \left(1 + \frac{1}{1152} \text{Diagram} \right) \sqcup \exp \left(\text{Diagram} \right) \\
&\quad + \text{Diagram} + \text{Diagram} + \gamma_1^{(2)} + \gamma_2^{(2)} + \gamma_1^{(3)} + \gamma_2^{(3)} \\
&\quad - \partial_C \left(\text{Diagram} \right) - \partial_C \left(\text{Diagram} \right) \\
&\quad - \frac{1}{48} \text{Diagram} - \frac{1}{48} \text{Diagram} \\
&\equiv \left(1 + \frac{1}{1152} \text{Diagram} \right) \sqcup \exp \left(\text{Diagram} \right) \\
&\quad + \gamma_1^{(2)} + \gamma_2^{(2)} + \gamma_1^{(3)} + \gamma_2^{(3)} - \partial_C \left(\text{Diagram} \right) \\
&\quad - \frac{1}{48} \text{Diagram} - \frac{1}{48} \text{Diagram} \Big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\chi^{-1}Z(K_1\#K_2) &= \partial_\Omega \Upsilon^{-1}Z(K_1\#K_2) \\
&\equiv \left(1 + \frac{1}{1152} \text{Diagram} \right) \sqcup \exp \left(\text{Diagram} + \gamma_1^{(2)} + \gamma_2^{(2)} + \gamma_1^{(3)} + \gamma_2^{(3)} \right) \\
&\quad - \partial_C \left(\text{Diagram} \right) - \frac{1}{48} \text{Diagram} \\
&\quad - \frac{1}{48} \text{Diagram} + \partial_C \exp \left(\text{Diagram} \right)
\end{aligned}$$

$$\begin{aligned}
&\equiv \left(1 + \frac{1}{1152} \text{diag} \right) \sqcup \exp \left(\text{diag}_{f(x)-g(x)} + \gamma_1^{(2)} + \gamma_2^{(2)} + \gamma_1^{(3)} + \gamma_2^{(3)} \right. \\
&\quad + \partial_C \left(\text{diag}_{-f(x)} \right) + \frac{1}{48} \text{diag}_{f'(x)} \\
&\quad - \frac{1}{24} \text{diag}_{f'(x)-g'_1(x)} - \frac{1}{24} \text{diag}_{f'(x)-g'_2(x)} \\
&\quad \left. + \frac{1}{24} \text{diag}_{f'(x)-g'_1(x)} + \frac{1}{24} \text{diag}_{f'(x)-g'_2(x)} \right).
\end{aligned}$$

Here, as in [4], [5], $\partial_\Omega(\Omega) = \langle \Omega, \Omega \rangle \Omega$, so we get

$$-\partial_C \left(\text{diag}_{-f(x)} \right) + \frac{1}{48} \text{diag}_{f'(x)} = \frac{1}{1152} \text{diag}.$$

This implies that

$$\exp \left(\partial_C \left(\text{diag}_{-f(x)} \right) \right) \equiv \left(1 - \frac{1}{1152} \text{diag} \right) \exp \left(\frac{1}{48} \text{diag}_{f'(x)} \right).$$

Therefore,

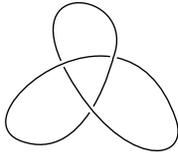
$$\begin{aligned}
\chi^{-1} Z(K_1 \# K_2) &\equiv \exp \left(\text{diag}_{f(x)-g(x)} + \gamma_1^{(2)} + \gamma_2^{(2)} + \gamma_1^{(3)} + \gamma_2^{(3)} \right. \\
&\quad + \frac{1}{24} \text{diag}_{f'(x)-g'_1(x)} - \frac{1}{24} \text{diag}_{f'(x)-g'_2(x)} \\
&\quad \left. - \frac{1}{24} \text{diag}_{f'(x)-g'_2(x)} + \frac{1}{24} \text{diag}_{f'(x)-g'_1(x)} \right) \\
&= \exp \left(\text{diag}_{f(x)-g(x)} + \gamma_1^{(2)} + \gamma_2^{(2)} + \gamma_1^{(3)} + \gamma_2^{(3)} \right. \\
&\quad \left. + \frac{1}{24} \text{diag}_{g'_1(x)} + \frac{1}{24} \text{diag}_{g'_2(x)} \right)
\end{aligned}$$

$$\begin{aligned}
&= \exp \left(\begin{array}{c} f(x) - g(x) \\ \text{[Diagram: a rounded rectangle]} \\ + \gamma_1^{(2)} + \gamma_2^{(2)} + \gamma_1^{(3)} + \gamma_2^{(3)} \\ + \frac{1}{6} \begin{array}{c} g_1'(x) \quad g_2'(x) \\ \text{[Diagram: a circle divided into three sectors]} \\ + \frac{1}{6} \begin{array}{c} g_1'(x) \\ \text{[Diagram: a circle divided into three sectors]} \\ g_2'(x) \end{array} \end{array} \right).
\end{aligned}$$

Hence, we obtain the proposition. □

B A Vassiliev invariant of degree 4 of the untwisted Whitehead double of the trefoil knot

In this section, we calculate a Vassiliev invariant of degree 4 of the untwisted Whitehead double of the trefoil knot concretely. We calculate it in two ways; one is by using a calculator and the other is by using our main theorem. Thereby, we verify the result of the main theorem.

The trefoil knot is $T(2, 3) =$  , and we give it 0-framing.

By using a calculator, we can get its degree 4 part of the Kontsevich invariant,

$$\log \left(Z(Wh^\pm(T(2, 3))) \# \nu^{-1} \right)^{(\text{degree } 4)} = \left(\frac{1}{8} \mp \frac{1}{4} \right) \left[\begin{array}{c} \text{[Diagram: a vertical line with three circles]} \end{array} \right]. \tag{42}$$

On the other hand, a straightforward calculation shows that (see Example 3.6)

$$Z(T(2, 3)) \# \nu^{-1} \equiv \exp \left(\begin{array}{c} -\frac{1}{2} \left[\begin{array}{c} \text{[Diagram: a vertical line with a circle]} \end{array} \right] \\ + \frac{1}{4} \left[\begin{array}{c} \text{[Diagram: a vertical line with two circles]} \end{array} \right] \\ + \frac{5}{24} \left[\begin{array}{c} \text{[Diagram: a vertical line with a circle and a loop]} \end{array} \right] \end{array} \right).$$

Note that $\Delta_{Wh^\pm(T(2,3))}(t) = 1$. So by Theorem 3.1, we get

$$\begin{aligned} & \log \left(\chi^{-1} \left(Z(Wh^\pm(T(2,3))) \# \nu^{-1} \right) \right)^{(\text{degree } 4)} \\ &= \left(-4 \cdot \left(-\frac{1}{2}\right) \cdot 0 - \frac{(\pm 1)^2 \cdot (-1/2)}{4} - 2 \cdot (\pm 1) \cdot \frac{1}{4} \right) \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} - \frac{(\pm 1)^2 \cdot (-1/2)}{8} \cdot 2 \cdot 2 \begin{array}{c} \text{---} \quad \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \\ &= \left(\frac{1}{8} \mp \frac{1}{4} \right) \begin{array}{c} \circ \\ | \\ \circ \\ | \\ \circ \end{array} . \end{aligned}$$

This matches to (42).

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