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Abstract

The 3D-index is an invariant of a 3-manifold with cusps, which would be related to the volume conjecture, and it would be useful to study properties of this invariant. In this paper, we calculate the 3D-index of the nth cyclic covers of hyperbolic knot complements, and show that the dth coefficient of this 3D-index is equal to a polynomial in n of degree $\leq 2d$ for any sufficiently large n. In particular, we calculate these polynomials concretely for lower degrees for the 4_1, 5_2, 6_1 knots.

1 Introduction

The 3D-index of a 3-manifold M with cusps is introduced in [2] from the viewpoint of mathematical physics, which can be regarded as the partition function of $\mathrm{SL}(2,\mathbb{C})$ Chern–Simons theory as mentioned in [2]. They predict that the 3D-index is a topological invariant of M. Let $\mathcal{T}$ be an ideal triangulation of M. The 3D-index is defined as a power series in q with integer coefficients by using an infinite sum over integer labels of the edges of $\mathcal{T}$. In general, this sum does not necessarily converge. In order that this sum converges, we need the assumption that $\mathcal{T}$ has a strict angle structure. We assume that M is hyperbolic, and $\mathcal{T}$ is an ideal triangulation giving the hyperbolic structure of M; in this case, $\mathcal{T}$ has a strict angle structure, and the above mentioned sum converges. For details, see [3, 4, 5]. Further, from the mathematical viewpoint, another step is to show the topological invariance of the 3D-index. As an extension of the 3D-index, the meromorphic 3D-index of M is introduced in [6], which is a topological invariant of M. They show that, if $\mathcal{T}$ has a strict angle structure, the meromorphic 3D-index can be expanded into a power series which coincides with the 3D-index. Hence, if M is a hyperbolic 3-manifold with cusps, then the 3D-index of M is a topological invariant.

We expect that the 3D-index would be related to the volume conjecture. We recall that the volume conjecture for knots is proposed in [11, 13], and the volume conjecture for 3-manifolds is proposed in [1]. The volume conjecture states that the hyperbolic volume appears in the asymptotic expansion of quantum invariants of knots and 3-manifolds. The volume conjecture is proved for some knots *e.g.* in [16, 17, 19], and for some 3-manifolds in [18]. Further, "the volume conjecture for the meromorphic 3D-index" is proposed and numerically observed for some knots in [8]. From the viewpoint of mathematical physics, the volume conjecture is formally obtained by formally applying infinite dimensional saddle point method to the path integral of the partition function of $\mathrm{SL}(2,\mathbb{C})$ Chern–Simons theory, as mentioned in [18]. Further, it is conjectured in [12] that the hyperbolic volume appears in the asymptotic expansion of the meromorphic 3D-index. As we mention above, the 3D-index can be regarded as the partition function of $\mathrm{SL}(2,\mathbb{C})$ Chern–Simons theory. Hence, we expect that the 3D-index might be an approach to prove the volume conjecture. The volume conjecture is an important conjecture which relates quantum topology and hyperbolic geometry. Hence, we expect that it would be useful to study properties of the 3D-index, which might be related to quantum topology and hyperbolic geometry.

In this paper, we calculate the 3D-index of finite cyclic covers of some hyperbolic knot complements, and observe the behavior of the 3D-index of finite cyclic covers. Let K be a hyperbolic knot. We assume that S^3-K has an ideal triangulation which gives the hyperbolic structure of S^3-K. Let $M_n(K)$ be the n-fold cyclic cover of S^3-K. We put coefficients of its 3D-index as

$$I\big(M_n(K)\big) \;=\; 1 + c_1^{(n)}(K)\,q + c_2^{(n)}(K)\,q^2 + \cdots \;\in \mathbb{Z}[[q]].$$

In Theorem 3.1, we show that, for each $d>0$, $c_d^{(n)}(K)$ is equal to a polynomial in n of degree $\le 2d$ for any sufficiently large n. Since the hyperbolic volume of the n-fold cover of a hyperbolic manifold is equal to n times the hyperbolic volume of the hyperbolic manifold, this behavior of the 3D-index is an extension of the behavior of the hyperbolic volume. We calculate concrete examples of such polynomials in Theorems 3.2, 3.3, 3.4; for the 4_1 knot, we have that

$$c_1^{(n)}(4_1) \;=\; 0 \;\text{ for } n\ge 2, \quad c_2^{(n)}(4_1) \;=\; 0 \;\text{ for } n\ge 3, \quad c_3^{(n)}(4_1) \;=\; 0 \;\text{ for } n\ge 4,$$

and, for the 5_2 knot, we have that

$$c_1^{(n)}(5_2) \;=\; n(n-2) \;\text{ for } n\ge 2, \quad c_2^{(n)}(5_2) \;=\; \frac{1}{4}\,n\big(n^3-6n^2+n+36\big) \;\text{ for } n\ge 4,$$

and, for the 6_1 knot, we have that

$$c_1^{(n)}(6_1) \;=\; n(n-2) \;\text{ for } n\ge 2, \quad c_2^{(n)}(6_1) \;=\; \frac{1}{4}\,n\big(n^3-6n^2+n+32\big) \;\text{ for } n\ge 3.$$

We obtain these theorems in the following way. For example, for the 4_1 knot, $I\big(M_n(4_1)\big)$ is presented by

$$\begin{aligned} I\big(M_n(4_1)\big) \;=\; \sum_{\substack{a_0=0,\\ a_1,\cdots,a_{2n-1}\in\mathbb{Z}}} & \hat{J}_{4_1}(a_0+a_3, 2a_1, 2a_2)\, \hat{J}_{4_1}(a_1+a_4, 2a_2, 2a_3) \\ & \times \hat{J}_{4_1}(a_2+a_5, 2a_3, 2a_4)\, \hat{J}_{4_1}(a_3+a_6, 2a_4, 2a_5) \\ & \times \cdots \\ & \times \hat{J}_{4_1}(a_{2n-2}+a_{2n+1}, 2a_{2n-1}, 2a_{2n})\, \hat{J}_{4_1}(a_{2n-1}+a_{2n+2}, 2a_{2n}, 2a_{2n+1}), \end{aligned} \tag{1}$$

where we regard the subscript of a_i as modulo $2n$, as we show in Section B.1. Here, $\deg \hat{J}_{4_1}(\ell_1,\ell_2,\ell_3)\ge 0$ for $\ell_1,\ell_2,\ell_3\in\mathbb{Z}$, and the equality holds if and only if $\ell_1=\ell_2=\ell_3$. Further, there exists an constant $\delta>0$ such that,

$$\text{if } \deg \hat{J}_{4_1}(\ell_1,\ell_2,\ell_3) > 0, \text{ then } \deg \hat{J}_{4_1}(\ell_1,\ell_2,\ell_3) \ge \delta. \tag{2}$$

Hence, for any fixed $d > 0$, the degree $\le d$ part of $I\big(M_n(4_1)\big)$ depends on contributions from sequences $(a_0, a_1, \cdots, a_{n-1})$ for a sufficiently large n, only when a sequence is obtained as the union of constant sequences and particular sequences. We note that, by (2), there are finitely many such particular sequences of degree $\le d$. By classifying such particular sequences of degree $\le d$, we can calculate the degree $\le d$ part of $I\big(M_n(4_1)\big)$ as a polynomial in n for a sufficiently large n. In this approach, we obtain the degree $\le d$ part of $I\big(M_n(K)\big)$ as combinations of finitely many particular sequences, which can be classified for any given d. For details, see Section 5.

Another approach to calculate $I\big(M_n(K)\big)$ is calculation by using eigenvalues of a transfer matrix. For example, for the 4_1 knot, $I\big(M_n(4_1)\big)$ is presented by (1), where the range of the sum is given by

$$a_0 = 0, \qquad a_1, \cdots, a_{2n-1} \in \mathbb{Z}.$$

By putting

$$a'_k = a_k - a_{k-1},$$

the range of the sum is rewritten as

$$a'_0, a'_1, \cdots, a'_{2n-1} \in \mathbb{Z}, \qquad a'_0 + a'_1 + \cdots + a'_{2n-1} = 0.$$

Further, we put

$$\mathcal{M}^{(a'_1, a'_2)}_{(a'_3, a'_4)} = \hat{J}_{4_1}(a_0 + a_3, 2a_1, 2a_2)\, \hat{J}_{4_1}(a_1 + a_4, 2a_2, 2a_3)\, u^{a'_2 + a'_3},$$

where u is a variable whose power counts $a'_2 + a'_3 + \cdots$, and we put the transfer matrix by

$$\mathcal{M} = \Big(\mathcal{M}^{(a'_1, a'_2)}_{(a'_3, a'_4)}\Big).$$

Then, $I\big(M_n(4_1)\big)$ can be presented by

$$\begin{aligned} I\big(M_n(4_1)\big) &= \big(\text{the coefficient of } u^0 \text{ in } \ \operatorname{trace} \mathcal{M}^n\big) \\ &= \big(\text{the coefficient of } u^0 \text{ in } \ \lambda_1^n + \lambda_2^n + \cdots\big), \end{aligned}$$

where $\lambda_1, \lambda_2, \cdots$ are eigenvalues of $\mathcal{M}$. Hence, by calculating $\lambda_1, \lambda_2, \cdots$ concretely, we obtain concrete values of lower degree part of $I\big(M_n(K)\big)$. In this approach, the characteristic polynomial (22) of $\mathcal{M}^{-1}$ is itself an invariant of K, and it can be regarded as a universal invariant among $I\big(M_n(K)\big)$ for all n; see Remark 4.4. For details of this approach, see Section 4.

We comment on related works. We note that the "stability property" of the colored Jones polynomial $J_n(K) \in \mathbb{Z}[q^{\pm 1}]$ is discussed in [7], which means that there exists a relatively simple function of n and q such that it is equal to $J_n(K)$ for any sufficiently large n. From this viewpoint, Theorem 3.1 means that, for each d, $c_d^{(n)}$ has a "polynomial stability" for any sufficiently large n. We also note that the loop invariants and some kinds of quantum invariants of cyclic covers of hyperbolic knot complements are studied in [9, 10], where such invariants are expressed in terms of the twisted Neumann-Zagier matrix, which is a $\mathbb{Z}[t^{\pm 1}]$-lift of the Neumann-Zagier matrix of an ideal triangulation of the hyperbolic knot complement. The Neumann-Zagier matrix given in Section 5.5 is essentially equivalent to the twisted Neumann-Zagier matrix. Hence, the method of that section is partially similar to the method in [9, 10].

The paper is organized, as follows. In Section 2, we review the definition of the 3D-index. In Section 3, we give Theorems 3.2, 3.3, 3.4, which show concrete values of lower degree part of $I\big(M_n(K)\big)$ for the 4_1, 5_2, 6_1 knots. As a generalization of them, we give Theorem 3.1, which shows that the dth coefficient of $I\big(M_n(K)\big)$ is a polynomial in n of degree $\leq 2d$ for a sufficiently large n. In Section 4, we calculate lower degree part of $I\big(M_n(K)\big)$ by using eigenvalues of transfer matrices for the 4_1, 5_2, 6_1 knots. In Section 5, we give proofs of Theorems 3.2, 3.3, 3.4 by using particular sequences of parameters. Further, we give a proof of Theorem 3.1 as a generalization of them. In Appendix A, we classify particular sequences of parameters for lower degrees, which are used in Section 5. In Appendix B, we give concrete presentations of $I\big(M_n(K)\big)$ for the 4_1, 5_2, 6_1 knots.

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2 Preliminaries

In this section, we review the definition of the 3D-index of a hyperbolic 3-manifold M with a cusp. We also review a modification of the defining formula of the 3D-index such that the lowest degree of the summand is non-negative which is obtained by using the hyperbolic structure of M. For details, see [3, 4, 5].

We put

$$\tilde{I}(m,e) \;=\; \sum_{n=\max\{0,-e\}}^{\infty} (-1)^n \, \frac{q^{\frac{1}{2}n(n+1)-(n+\frac{1}{2}e)m}}{(q)_n\,(q)_{n+e}}$$

for $m, e \in \mathbb{Z}$, where

$$(q)_n \;=\; \begin{cases} 1 & \text{if } n=0, \\ (1-q)(1-q^2)\cdots(1-q^n) & \text{if } n>0. \end{cases}$$

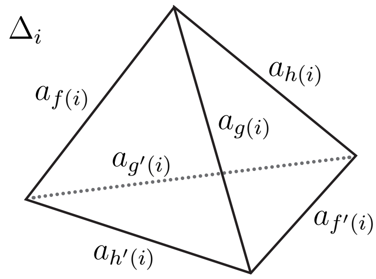
We put

$$\begin{aligned} I(\ell_1,\ell_2,\ell_3) \;&=\; (-q^{1/2})^{-\ell_1}\,\tilde{I}(\ell_1-\ell_2,\,\ell_3-\ell_1) \\ &=\; (-q^{1/2})^{-\ell_2}\,\tilde{I}(\ell_2-\ell_3,\,\ell_1-\ell_2) \\ &=\; (-q^{1/2})^{-\ell_3}\,\tilde{I}(\ell_3-\ell_1,\,\ell_2-\ell_3), \end{aligned}$$

noting that $I(\ell_1,\ell_2,\ell_3)$ is invariant under all permutations of ℓ_1, ℓ_2, ℓ_3, and satisfies that

$$I(\ell_1,\ell_2,\ell_3) \;=\; \big(-q^{1/2}\big)^{-c}\, I(\ell_1+c,\,\ell_2+c,\,\ell_3+c). \qquad (3)$$

Let M be a hyperbolic 3-manifold with a cusp. We assume that M has an ideal triangulation $\mathcal{T}$ which gives the hyperbolic structure of M, and $\mathcal{T}$ has m' tetrahedra and m edges. We assign an integer label a_j to the jth edge $\mathcal{E}_j$. Let Δ_i be the ith ideal tetrahedron of $\mathcal{T}$, whose edges are labeled as follows,



(4)

where f, f', g, g', h, h' are maps $\{1, \cdots, m'\} \to \{1, \cdots, m\}$ such that $\mathcal{E}_{f(i)}$ and $\mathcal{E}_{f'(i)}$, $\mathcal{E}_{g(i)}$ and $\mathcal{E}_{g'(i)}$, $\mathcal{E}_{h(i)}$ and $\mathcal{E}_{h'(i)}$ are opposite edges of Δ_i, and the edge $\mathcal{E}_j$ is labeled by an integer a_j. The *3D-index* of M is defined by

$$I(M) \;=\; \sum_{\substack{a_1,\cdots,a_m \in \mathbb{Z} \\ a_j=0}} q^{a_1+\cdots+a_m} \prod_i I\big(a_{f(i)}+a_{f'(i)},\, a_{g(i)}+a_{g'(i)},\, a_{h(i)}+a_{h'(i)}\big) \;\in \mathbb{Z}[[q]], \tag{5}$$

where we fix any j. We can show that the right-hand side of (5) does not depend on the choice of j, as follows. By counting the Euler number of the boundary torus of a neighborhood of the cusp of M using m and m', we can show that $m = m'$. Hence, by (3), the summand of (5) is invariant under replacing $(a_1, \cdots, a_m)$ with $(a_1+c, \cdots, a_m+c)$. Therefore, the right-hand side of (5) does not depend on the choice of j.

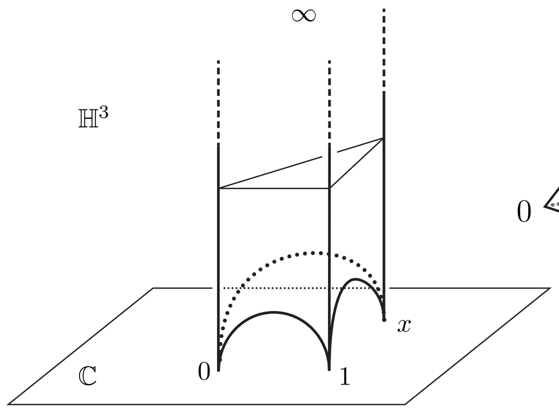
In general, the sum (5) does not necessarily converge. In order that the sum of (5) converges, we need the assumption that $\mathcal{T}$ has a strict angle structure. Actually, in this paper, we assume that M is hyperbolic, and $\mathcal{T}$ is an ideal triangulation giving the hyperbolic structure of M; in this case, $\mathcal{T}$ has a strict angle structure, and the sum of (5) converges. For details, see [3, 4, 5] and Section 5.4.

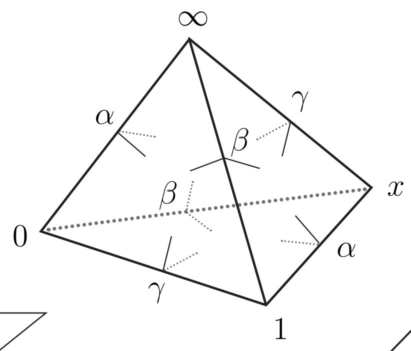
We define the degree of $I(\ell_1, \ell_2, \ell_3)$ to be the lowest degree of $I(\ell_1, \ell_2, \ell_3)$. Then, we have that

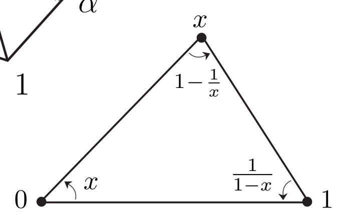
$$\deg I(\ell_1, \ell_2, \ell_3) \;=\; \begin{cases} \frac{1}{2}(\ell_2-\ell_1)(\ell_3-\ell_1) - \frac{1}{2}\ell_1 & \text{if } \ell_1 \le \ell_2 \text{ and } \ell_1 \le \ell_3, \\ \frac{1}{2}(\ell_1-\ell_2)(\ell_3-\ell_2) - \frac{1}{2}\ell_2 & \text{if } \ell_2 \le \ell_1 \text{ and } \ell_2 \le \ell_3, \\ \frac{1}{2}(\ell_1-\ell_3)(\ell_2-\ell_3) - \frac{1}{2}\ell_3 & \text{if } \ell_3 \le \ell_1 \text{ and } \ell_3 \le \ell_2. \end{cases} \tag{6}$$

We note that the degree of $I(\ell_1, \ell_2, \ell_3)$ is positive in many cases, but it can be negative in some cases.

We rewrite the sum (5) in such a way that the degree of the summand is non-negative. In order to this, we briefly review an ideal tetrahedron in the hyperbolic space $\mathbb{H}^3$. We assign labels to the four vertices of an ideal tetrahedron such that the labels are in $\mathbb{C} \cup \{\infty\} = \partial\mathbb{H}^3$. The shape of an ideal tetrahedron is determined by these labels, and, by the action of $\mathrm{PSL}_2\mathbb{C} = \mathrm{Isom}(\mathbb{H}^3)$, these labels are normalized in the form $\{\infty, 0, 1, x\}$ for some $x \in \mathbb{C} - \{0,1\}$.







Then, the angles of faces are given by

$$\operatorname{Arg} x, \qquad \operatorname{Arg} \frac{1}{1-x}, \qquad \operatorname{Arg} \big(1 - \frac{1}{x}\big),$$

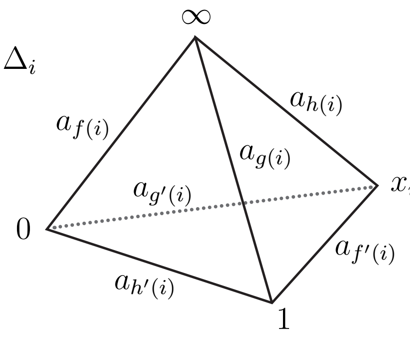
noting that the opposite edge has the same angle. We put

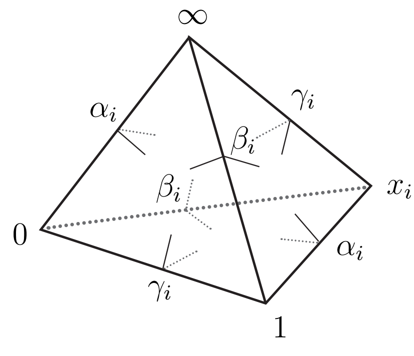
$$\alpha \;=\; \frac{1}{2\pi} \operatorname{Arg} x, \qquad \beta \;=\; \frac{1}{2\pi} \operatorname{Arg} \frac{1}{1-x}, \qquad \gamma \;=\; \frac{1}{2\pi} \operatorname{Arg} \big(1 - \frac{1}{x}\big).$$

We note that

$$\alpha + \beta + \gamma \;=\; \frac{1}{2}, \tag{7}$$

since $\operatorname{Arg} x + \operatorname{Arg} \frac{1}{1-x} + \operatorname{Arg} \big(1 - \frac{1}{x}\big) = \operatorname{Arg}(-1) = \pi$. We denote these angles for tetrahedra Δ_i by α_i, β_i, γ_i, as follows.





By (7), we have that

$$\alpha_i + \beta_i + \gamma_i \;=\; \frac{1}{2}. \tag{8}$$

Since the sum of angles around the edge $\mathcal{E}_j$ is equal to 2π, we have that

$$\sum_{f(i)=j} \alpha_i + \sum_{f'(i)=j} \alpha_i + \sum_{g(i)=j} \beta_i + \sum_{g'(i)=j} \beta_i + \sum_{h(i)=j} \gamma_i + \sum_{h'(i)=j} \gamma_i \;=\; 1$$

for each j. Hence,

$$\sum_i \alpha_i(a_{f(i)}+a_{f'(i)})+\beta_i(a_{g(i)}+a_{g'(i)})+\gamma_i(a_{h(i)}+a_{h'(i)})$$
$$= \sum_j \Big(\sum_{f(i)=j} \alpha_i + \sum_{f'(i)=j} \alpha_i + \sum_{g(i)=j} \beta_i + \sum_{g'(i)=j} \beta_i + \sum_{h(i)=j} \gamma_i + \sum_{h'(i)=j} \gamma_i \Big) a_j \;=\; \sum_j a_j. \tag{9}$$

We put

$$J_i(\ell_1,\ell_2,\ell_3) \;=\; q^{\alpha_i\ell_1+\beta_i\ell_2+\gamma_i\ell_3} I(\ell_1,\ell_2,\ell_3), \tag{10}$$

noting that

$$J_i(\ell_1,\ell_2,\ell_3) \;=\; J_i(\ell_1+2c,\ell_2+2c,\ell_3+2c)$$

by (3) and (8). Then, by (9), we have that

$$I(M) \;=\; \sum_{\substack{a_1,\cdots,a_m\in\mathbb{Z}\\ a_j=0}} \prod_i J_i\big(a_{f(i)}+a_{f'(i)},\, a_{g(i)}+a_{g'(i)},\, a_{h(i)}+a_{h'(i)}\big). \tag{11}$$

By (6) and (8), we have that

$$\deg J_i(\ell_1,\ell_2,\ell_3)$$
$$= \begin{cases} \frac12(\ell_2-\ell_1)(\ell_3-\ell_1)+\beta_i(\ell_2-\ell_1)+\gamma_i(\ell_3-\ell_1) & \text{if } \ell_1\le\ell_2 \text{ and } \ell_1\le\ell_3\,, \\ \frac12(\ell_1-\ell_2)(\ell_3-\ell_2)+\alpha_i(\ell_1-\ell_2)+\gamma_i(\ell_3-\ell_2) & \text{if } \ell_2\le\ell_1 \text{ and } \ell_2\le\ell_3\,, \\ \frac12(\ell_1-\ell_3)(\ell_2-\ell_3)+\alpha_i(\ell_1-\ell_3)+\beta_i(\ell_2-\ell_3) & \text{if } \ell_3\le\ell_1 \text{ and } \ell_3\le\ell_2\,. \end{cases} \tag{12}$$

In particular,

$$\deg J_i(\ell_1,\ell_2,\ell_3) \;\ge\; 0$$

for any $\ell_1,\ell_2,\ell_3\in\mathbb{Z}$. Further, the equality holds if and only if $\ell_1=\ell_2=\ell_3$.

Lemma 2.1. *There exists a constant $\delta>0$ such that, if*

$$\deg J_i\big(\ell_1,\ell_2,\ell_3\big) \;>\; 0,$$

then

$$\deg J_i\big(\ell_1,\ell_2,\ell_3\big) \;\ge\; \delta.$$

Proof. Let δ be the minimum of α_i, β_i, γ_i for all i. By (12), we have that

$$\deg J_i(\ell_1,\ell_2,\ell_3) \;\ge\; \begin{cases} \beta_i(\ell_2-\ell_1)+\gamma_i(\ell_3-\ell_1) & \text{if } \ell_1\le\ell_2 \text{ and } \ell_1\le\ell_3\,, \\ \alpha_i(\ell_1-\ell_2)+\gamma_i(\ell_3-\ell_2) & \text{if } \ell_2\le\ell_1 \text{ and } \ell_2\le\ell_3\,, \\ \alpha_i(\ell_1-\ell_3)+\beta_i(\ell_2-\ell_3) & \text{if } \ell_3\le\ell_1 \text{ and } \ell_3\le\ell_2\,. \end{cases}$$

Hence, unless $\ell_1=\ell_2=\ell_3$, we have that

$$\deg J_i\big(\ell_1,\ell_2,\ell_3\big) \;\ge\; \delta,$$

as required. □

3 Main results

In this section, we give Theorems 3.2, 3.3, 3.4, which show concrete values of lower degree part of $I\big(M_n(K)\big)$ for the 4_1, 5_2, 6_1 knots. For a general hyperbolic knot, we give Theorem 3.1, which states that, for any fixed d, the dth coefficient of $I\big(M_n(K)\big)$ is equal to a polynomial in n of degree $\leq 2d$ for any sufficiently large n.

Let K be a hyperbolic knot. We assume that there exists an ideal triangulation of $S^3 - K$ which gives the hyperbolic structure of $S^3 - K$, and we fix such a triangulation. Let $M_n(K)$ denote the n-fold cyclic cover of $S^3 - K$, which has an ideal triangulation as a lift of the ideal triangulation of $S^3 - K$. We regard $M_n(K)$ as a 3-manifold with a cusp. We put coefficients of $I\big(M_n(K)\big)$ as

$$I\big(M_n(K)\big) \;=\; 1 + c_1^{(n)}(K)\, q + c_2^{(n)}(K)\, q^2 + \cdots \;\in \mathbb{Z}[[q]].$$

As for behavior of values of lower degree part of $I\big(M_n(K)\big)$ for any sufficiently large n, we have the following theorem.

Theorem 3.1. *For any positive integer d, there exist a positive integer n_0 and a polynomial $p_d^K(n)$ in n of degree $\leq 2d$ such that $c_d^{(n)}(K) = p_d^K(n)$ for any $n \geq n_0$.*

We give a proof of the theorem in Section 5.5.

We put

$$\tilde{I}_K(n,q) \;=\; 1 + p_1^K(n)\, q + p_2^K(n)\, q^2 \cdots \;\in \mathbb{Z}[n][[q]].$$

As for concrete values of lower degree part of $I\big(M_n(4_1)\big)$ for any sufficiently large n, we have the following theorem.

Theorem 3.2. *We have that*

$$p_1^{4_1}(n) = 0, \qquad p_2^{4_1}(n) = 0, \qquad p_3^{4_1}(n) = 0.$$

Hence,

$$\tilde{I}_{4_1}(n,q) \;=\; 1 + O(q^4).$$

We give a proof of the theorem in Section 5.1. As for concrete values of degree ≤ 7 part of $I\big(M_n(4_1)\big)$ for $n \leq 8$, it is obtained by computer calculation that

$$\begin{aligned}
I\big(M_1(4_1)\big) &= 1 - 2q - 3q^2 + 2q^3 + 8q^4 + 18q^5 + 18q^6 + 14q^7 + O(q^8),\\
I\big(M_2(4_1)\big) &= 1 + 2q^2 + 8q^3 - 3q^4 - 32q^5 - 66q^6 - 56q^7 + O(q^8),\\
I\big(M_3(4_1)\big) &= 1 - 2q^3 - 18q^4 - 6q^5 + 138q^6 + 306q^7 + O(q^8),\\
I\big(M_4(4_1)\big) &= 1 + 2q^4 + 32q^5 + 48q^6 - 424q^7 + O(q^8),\\
I\big(M_5(4_1)\big) &= 1 - 2q^5 - 50q^6 - 160q^7 + O(q^8),\\
I\big(M_6(4_1)\big) &= 1 + 2q^6 + 72q^7 + O(q^8),\\
I\big(M_7(4_1)\big) &= 1 - 2q^7 + O(q^8),\\
I\big(M_8(4_1)\big) &= 1 + O(q^8).
\end{aligned}$$

We can observe that there is a particular property that coefficients of lower left part are 0. We explain a reason of this property in Theorem 4.5 later. By Theorem 3.2, we can verify that

$$c_1^{(n)}(4_1) = p_1^{4_1}(n) \quad \text{for any } n \geq 2, \tag{13}$$
$$c_2^{(n)}(4_1) = p_2^{4_1}(n) \quad \text{for any } n \geq 3, \tag{14}$$
$$c_3^{(n)}(4_1) = p_3^{4_1}(n) \quad \text{for any } n \geq 4, \tag{15}$$

which we prove for $n \geq 6$ in Section 5.1.

As for concrete values of lower degree part of $I\big(M_n(5_2)\big)$ for any sufficiently large n, we have the following theorem.

Theorem 3.3. *We have that*

$$p_1^{5_2}(n) = n(n-2), \qquad p_2^{5_2}(n) = \frac{1}{4}\, n\,(n^3 - 6n^2 + n + 36).$$

Hence,

$$\tilde{I}_{5_2}(n,q) \;=\; 1 + n(n-2)\,q + \frac{1}{4}\,n\,(n^3 - 6n^2 + n + 36)\,q^2 \;+ O(q^3).$$

We give a proof of the theorem in Section 5.2. As for concrete values of degree ≤ 3 part of $I\big(M_n(5_2)\big)$ for $n \leq 8$, it is obtained by computer calculation that

$$\begin{aligned}
I\big(M_1(5_2)\big) &= 1 - 4q - q^2 + 16q^3 + 26q^4 + O(q^5),\\
I\big(M_2(5_2)\big) &= 1 + 14q^2 + 6q^3 - 107q^4 + O(q^5),\\
I\big(M_3(5_2)\big) &= 1 + 3q + 15q^2 - 82q^3 - 24q^4 + O(q^5),\\
I\big(M_4(5_2)\big) &= 1 + 8q + 8q^2 - 72q^3 + O(q^4),\\
I\big(M_5(5_2)\big) &= 1 + 15q + 20q^2 + 45q^3 + O(q^4),\\
I\big(M_6(5_2)\big) &= 1 + 24q + 63q^2 + 216q^3 + O(q^3),\\
I\big(M_7(5_2)\big) &= 1 + 35q + 161q^2 + 546q^3 + O(q^3),\\
I\big(M_8(5_2)\big) &= 1 + 48q + 344q^2 + 1248q^3 + O(q^3).
\end{aligned}$$

Hence, we can observe for $n \leq 8$ that

$$c_1^{(n)}(5_2) = p_1^{5_2}(n) \quad \text{for any } n \geq 2, \tag{16}$$
$$c_2^{(n)}(5_2) = p_2^{5_2}(n) \quad \text{for any } n \geq 4, \tag{17}$$

which we prove for $n > 8$ in Section 5.2.

As for concrete values of lower degree part of $I\big(M_n(6_1)\big)$ for any sufficiently large n, we have the following theorem.

Theorem 3.4. *We have that*

$$p_1^{6_1}(n) = n(n-2), \qquad p_2^{6_1}(n) = \frac{1}{4}\, n\,(n^3 - 6n^2 + n + 32).$$

Hence,

$$\tilde{I}_{6_1}(n,q) \;=\; 1 + n(n-2)\,q + \frac{1}{4}\,n\,(n^3-6n^2+n+32)\,q^2 \; + O(q^3).$$

We give a proof of the theorem in Section 5.3. As for concrete values of degree ≤ 2 part of $I\big(M_n(6_1)\big)$ for $n \leq 8$, it is obtained by computer calculation that

$$\begin{aligned}
I\big(M_1(6_1)\big) &= 1 \; - 4q \; + q^2 + 18q^3 \; + O(q^4), \\
I\big(M_2(6_1)\big) &= 1 \qquad\quad + 14q^2 \; + O(q^3), \\
I\big(M_3(6_1)\big) &= 1 \; + 3q \; + 6q^2 \; + O(q^3), \\
I\big(M_4(6_1)\big) &= 1 \; + 8q \; + 4q^2 \; + O(q^3), \\
I\big(M_5(6_1)\big) &= 1 + 15q + 15q^2 \; + O(q^3), \\
I\big(M_6(6_1)\big) &= 1 + 24q + 57q^2 \; + O(q^3), \\
I\big(M_7(6_1)\big) &= 1 + 35q + 154q^2 \; + O(q^3), \\
I\big(M_8(6_1)\big) &= 1 + 48q + 336q^2 \; + O(q^3).
\end{aligned}$$

Hence, we can observe for $n \leq 8$ that

$$c_1^{(n)}(6_1) = p_1^{6_1}(n) \qquad \text{for any } n \geq 2, \tag{18}$$

$$c_2^{(n)}(6_1) = p_2^{6_1}(n) \qquad \text{for any } n \geq 3, \tag{19}$$

which we prove for $n \geq 5$ in Section 5.3.

4 Calculation of $I\big(M_n(K)\big)$ from eigenvalues of transfer matrices

The defining formula of $I\big(M_n(K)\big)$ can be rewritten by using the product of n copies of some matrix, which we call a *transfer matrix*. In this section, we calculate $I\big(M_n(K)\big)$ from eigenvalues of a transfer matrix.

We recall the finite dimensional case. Let $\mathcal{M}$ be an $m \times m$ matrix. We put

$$\tau_n \;=\; \operatorname{trace} \mathcal{M}^n.$$

We like to know the behavior of τ_n. We put

$$\begin{aligned}
\sigma_1 &= \tau_1\,, \\
\sigma_k &= \frac{1}{k}\big(\tau_k - \sigma_1\tau_{k-1} + \sigma_2\tau_{k-2} - \cdots + (-1)^{k-1}\sigma_{k-1}\tau_1\big) \qquad \text{for } k = 2, \cdots, m.
\end{aligned} \tag{20}$$

Then, the characteristic polynomial of $\mathcal{M}$ is given by

$$T^m - \sigma_1 T^{m-1} + \sigma_2 T^{m-2} - \sigma_3 T^{m-3} + \cdots + (-1)^m \sigma_m \;=\; 0.$$

The eigenvalues $\lambda_1, \cdots, \lambda_m$ are obtained as the solutions of this equation. Then, we have that

$$\tau_n \;=\; \lambda_1^n + \lambda_2^n + \cdots + \lambda_m^n.$$

The behavior of τ_n can be described by this equation.

Let K be a hyperbolic knot in S^3. We assume that there exists an ideal triangulation of $S^3 - K$ which gives the hyperbolic structure of $S^3 - K$, and we fix such a triangulation. We consider a fundamental domain C of the infinite cyclic cover of $S^3 - K$, which is a union of ideal tetrahedra. The boundary of the fundamental domain C consists of F_1 and F_2; they are unions of ideal triangles, and they are naturally identified in $S^3 - K$. F_1 and F_2 might have common edges. The n-fold cyclic cover of $S^3 - K$ is obtained by gluing n copies $C^{(k)}$ of C along n copies $F_1^{(k)}$, $F_2^{(k)}$ of F_1, F_2 by naturally identifying $F_1^{(k)}$ and $F_2^{(k-1)}$.

We consider a "transfer matrix" $\mathcal{M}$ which is an invariant of C. In fact, we can calculate an invariant of $\bigcup_{i\le k} C^{(i)}$ from an invariant of $\bigcup_{i<k} C^{(i)}$ by using an invariant of $C^{(k)}$, which can often been presented by a matrix $\mathcal{M}$, which we call a *transfer matrix*. In this case, $I\big(M_n(K)\big)$ is obtained from $\mathcal{M}^n$; for concrete formulas, see (23), (25), (27). As we can see in these formulas, we can define a matrix $\mathcal{M}$ whose entries are defined to be some factors in the defining formula of $I\big(M_n(K)\big)$ such that

$$I\big(M_n(K)\big) \;=\; \big(\text{the coefficient of } u^0 \text{ in } \;\operatorname{trace}\mathcal{M}^n\big). \tag{21}$$

For simplicity, we consider the following assumption.

Assumption 4.1. We assume that the eigenvalues $\lambda_1, \lambda_2, \lambda_3, \cdots$ of $\mathcal{M}$ are in $(\mathbb{C}[u^{\pm1}])[[q^{1/2}]]$, and that the lowest degree of λ_k with respect to q goes to ∞ as $k \to \infty$.

We obtain lower degree part of λ_k, as follows.

$$\tau_k \;=\; \operatorname{trace}\mathcal{M}^k \;+ O(q^{m+\frac{1}{2}})$$

By (20), we obtain σ_k from τ_k. We consider the following equation,

$$1 - \sigma_1 \hat{T} + \sigma_2 \hat{T}^2 - \sigma_3 \hat{T}^3 + \cdots + (-1)^m \sigma_m \hat{T}^m \;=\; O(q^{m+\frac{1}{2}}), \tag{22}$$

and we assume that solutions of T are invertible elements in $q^{-m}\cdot\big(\mathbb{C}[u^{\pm1}]\big)[[q^{1/2}]]$. Then, the eigenvalues of $\mathcal{M}$ are obtained from its solutions, as follows,

$$\hat{T}^{-1} \;=\; \lambda_1, \lambda_2, \lambda_3, \cdots \;\in \big(\mathbb{C}[u^{\pm1}]\big)[[q^{1/2}]].$$

Recalling that

$$\operatorname{trace}\mathcal{M}^n \;=\; \tau_n \;=\; \lambda_1^n + \lambda_2^n + \lambda_3^n + \cdots \;\in \big(\mathbb{C}[u^{\pm1}]\big)[[q^{1/2}]],$$

the invariant $I\big(M_n(K)\big)$ can be presented in the following form,

$$I\big(M_n(K)\big) \;=\; \big(\text{the coefficient of } u^0 \text{ in } (\lambda_1^n + \lambda_2^n + \lambda_3^n + \cdots)\big) \;\in \mathbb{Z}[[q]].$$

In fact, the cases of the 4_1 knot and the 6_1 knot satisfy Assumption 4.1 in lower degrees as we see in Sections 4.1 and 4.3, but the case of the 5_2 knot does not satisfy Assumption 4.1 as we see in Section 4.2.

More generally, we need the following assumption instead of Assumption 4.1, which the case of the 5_2 knot satisfies in lower degrees.

Assumption 4.2. We assume that the eigenvalues $\lambda_1, \lambda_2, \lambda_3, \cdots$ of $\mathcal{M}$ are in $(\mathbb{C}[u^{\pm 1}])[[q^{1/2K_1}, q^{1/2K_2}, \cdots]]$, and that the lowest degree of λ_k with respect to q goes to ∞ as $k \to \infty$.

In general, the eigenvalues of $\mathcal{M}$ might belongs to $(\overline{\mathbb{C}[u^{\pm 1}]})[[q^{1/2K_1}, q^{1/2K_2}, \cdots]]$, where $\overline{\mathbb{C}[u^{\pm 1}]}$ denotes the algebraic closure of $\mathbb{C}[u^{\pm 1}]$.

Remark 4.3. As for other coefficients of (21), we have that

$$I\big(M_n(K)\big)\big(\frac{k}{2}(\text{longitude})\big) = \big(\text{the coefficient of } u^k \text{ in } \operatorname{trace} \mathcal{M}^n\big),$$

where the general 3D-index $I(M)(\gamma)$ is defined for $\gamma \in H_1(\partial M)$, regarding M as a 3-manifold with a torus boundary; see [5, 4].

Remark 4.4. The right-hand side of (22) is the characteristic polynomial of $\mathcal{M}^{-1}$, and it is itself an invariant of K, and it can be regarded as a universal invariant among $I\big(M_n(K)\big)$ for all n.

4.1 Calculation of $I\big(M_n(4_1)\big)$ from eigenvalues of a transfer matrix

In this section, we calculate $I\big(M_n(4_1)\big)$ from eigenvalues of a transfer matrix.

By (82), $I\big(M_n(4_1)\big)$ is presented by

$$\begin{aligned} I\big(M_n(4_1)\big) = \sum & \hat{J}_{4_1}(a_0+a_3, 2a_1, 2a_2)\, \hat{J}_{4_1}(a_1+a_4, 2a_2, 2a_3) \\ & \times \hat{J}_{4_1}(a_2+a_5, 2a_3, 2a_4)\, \hat{J}_{4_1}(a_3+a_6, 2a_4, 2a_5) \\ & \times \cdots \\ & \times \hat{J}_{4_1}(a_{2n-2}+a_{2n+1}, 2a_{2n-1}, 2a_{2n})\, \hat{J}_{4_1}(a_{2n-1}+a_{2n+2}, 2a_{2n}, 2a_{2n+1}), \end{aligned}$$

where we regard the subscript of a_i as modulo $2n$, and the range of the sum is given by

$$a_0 = 0, \qquad a_1, \cdots, a_{2n-1} \in \mathbb{Z}.$$

By putting

$$a'_k = a_k - a_{k-1},$$

the range of the sum is rewritten as

$$a'_0, a'_1, \cdots, a'_{2n-1} \in \mathbb{Z}, \qquad a'_0 + a'_1 + \cdots + a'_{2n-1} = 0.$$

Further, we put

$$\mathcal{M}^{(a'_1, a'_2)}_{(a'_3, a'_4)} = \hat{J}_{4_1}(a_0+a_3, 2a_1, 2a_2)\, \hat{J}_{4_1}(a_1+a_4, 2a_2, 2a_3)\, u^{a'_2+a'_3},$$

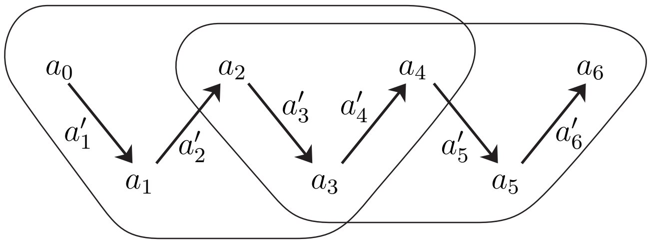
where u is a variable whose power counts $a'_2 + a'_3 + \cdots$. Since $\hat{J}_{4_1}(\ell_1+2, \ell_2+2, \ell_3+2) = \hat{J}_{4_1}(\ell_1, \ell_2, \ell_3)$, we note that the right-hand side depends only on a'_1, a'_2, a'_3, a'_4. Furthermore, we put

$$\mathcal{M} = \Big(\mathcal{M}^{(a'_1, a'_2)}_{(a'_3, a'_4)}\Big).$$

The product of copies of $\mathcal{M}$ is given by

$$\mathcal{M}^2 \;=\; \left(\sum_{a'_3, a'_4} \mathcal{M}^{(a'_1, a'_2)}_{(a'_3, a'_4)} \, \mathcal{M}^{(a'_3, a'_4)}_{(a'_5, a'_6)} \right),$$

where the parameters are related, as follows.



Hence, $I\big(M_n(4_1)\big)$ can be presented by

$$I\big(M_n(4_1)\big) \;=\; \big(\text{the coefficient of } u^0 \text{ in } \ \operatorname{trace} \mathcal{M}^n\big). \tag{23}$$

We calculate the first few eigenvalues of $\mathcal{M}$. We put

$$\tau_k \;=\; \operatorname{trace} \mathcal{M}^k.$$

Then, by computer calculation, we obtain that

$$\begin{aligned}
\tau_1 \;&= 1 - 2q + 2q^{3/2}(u + u^{-1}) - 3q^2 + q^3(2 + u^2 + u^{-2}) - 4q^{7/2}(u + u^{-1}) \\
&\quad + 2q^4(4 + u^2 + u^{-2}) \; + O(q^{9/2}), \\
\tau_2 \;&= 1 + 2q^2 - 4q^{5/2}(u + u^{-1}) + 2q^3(4 + u^2 + u^{-2}) - 4q^{7/2}(u + u^{-1}) \\
&\quad - q^4\big(3 + 4(u^2 + u^{-2})\big) + 4q^{9/2}\big(u^3 + u^{-3} + 4(u + u^{-1})\big) \\
&\quad - 16q^5(2 + u^2 + u^{-2}) \; + O(q^{11/2}), \\
\tau_3 \;&= 1 - 2q^3 + 6q^{7/2}(u + u^{-1}) - 6q^4(3 + u^2 + u^{-2}) \\
&\quad + 2q^{9/2}\big(u^3 + u^{-3} + 9(u + u^{-1})\big) + 3q^5(-2 + u^2 + u^{-2}) \\
&\quad - 6q^{11/2}\big(3(u^3 + u^{-3}) + 8(u + u^{-1})\big) \\
&\quad + 3q^6\big(46 + 28(u^2 + u^{-2}) + 3(u^4 + u^{-4})\big) \; + O(q^{13/2}), \\
\tau_4 \;&= 1 + 2q^4 - 8q^{9/2}(u + u^{-1}) + 4q^5\big(8 + 3(u^2 + u^{-2})\big) \\
&\quad - 8q^{11/2}\big(u^3 + u^{-3} + 6(u + u^{-1})\big) + 2q^6\big(24 + u^4 + u^{-4} + 8(u^2 + u^{-2})\big) \\
&\quad + 8q^{13/2}\big(5(u^3 + u^{-3}) + 11(u + u^{-1})\big) \\
&\quad - 4q^7\big(106 + 69(u^2 + u^{-2}) + 12(u^4 + u^{-4})\big) \; + O(q^{15/2}).
\end{aligned}$$

By observing the behavior of τ'_k, we can expect the first eigenvalue is given by

$$\lambda_1 \;=\; 1 \; + O(q^{15/2}).$$

In order to calculate the next eigenvalues concretely, we put

$$\tau'_n \;=\; (\tau_n - 1)/(-q)^n.$$

By (20), we obtain σ'_k from τ'_k, as follows,

$$\begin{aligned}
\sigma'_1 &= \tau'_1 \\
&= 2 - 2q^{1/2}(u+u^{-1}) + 3q - q^2(2-u^2-u^{-2}) + 4q^{5/2}(u+u^{-1}) \\
&\qquad - 2q^3(4+u^2+u^{-2}) \; + O(q^{7/2}), \\
\sigma'_2 &= -\frac{1}{2}(\tau'_2 - \sigma'_1\tau'_1) \\
&= 1 - 2q^{1/2}(u+u^{-1}) + q(6+u^2+u^{-2}) - 4q^{3/2}(u+u^{-1}) + 2q^2 \\
&\qquad + 6q^{5/2}(u+u^{-1}) - q^3\big(22+7(u+u^{-1})\big) \; + O(q^{7/2}), \\
\sigma'_3 &= \frac{1}{3}(\tau'_3 - \sigma'_1\tau'_2 + \sigma'_2\tau'_1) \\
&= q - 2q^{3/2}(u+u^{-1}) + q^2(4+u^2+u^{-2}) - q^3\big(11+4(u+u^{-1})\big) \; + O(q^{7/2}), \\
\sigma'_4 &= -\frac{1}{4}(\tau'_4 - \sigma'_1\tau'_3 + \sigma'_2\tau'_2 - \sigma'_3\tau'_1) \\
&= q^2 - 2q^{5/2}(u+u^{-1}) + q^3(4+u^2+u^{-2}) \; + O(q^{7/2}).
\end{aligned}$$

Further, we consider the following equation

$$1 - \sigma'_1\hat{T} + \sigma'_2\hat{T}^2 - \sigma'_3\hat{T}^3 + \sigma'_4\hat{T}^4 \;=\; O(q^{7/2}).$$

From solutions $\hat{T}$ of this equation, we obtain $\hat{T}^{-1} = \lambda'_2, \lambda'_3$, as follows,

$$\begin{aligned}
\lambda'_2 &= 1 \; - q^{1/2}(u+u^{-1}) \; - q\big(-1+\sqrt{-1}\,(u+u^{-1})\big) \\
&\qquad + q^{3/2}\sqrt{-1}\,\big(1+\frac{1}{2}(u^2+u^{-2})\big) \\
&\qquad - q^2\big(\frac{1}{2}(u^2+u^{-2}) + \frac{9\sqrt{-1}}{8}(u+u^{-1}) - \frac{\sqrt{-1}}{8}(u^3+u^{-3})\big) \\
&\qquad - q^{5/2}\big(-2(u+u^{-1}) - \frac{7\sqrt{-1}}{8} + \frac{\sqrt{-1}}{2}(u^2+u^{-2}) - \frac{\sqrt{-1}}{16}(u^4+u^{-4})\big) \\
&\qquad - q^3\sqrt{-1}\,\big(\frac{51}{64}(u+u^{-1}) + \frac{31}{128}(u^3+u^{-3}) - \frac{5}{128}(u^5+u^{-5})\big) \; + O(q^{7/2}), \\
\lambda'_3 &= \overline{\lambda'_2}.
\end{aligned}$$

Further, in order to calculate the next eigenvalues concretely, we put

$$\tau''_n \;=\; (\tau'_n - {\lambda'_2}^n - {\lambda'_3}^n)/q^n.$$

They are concretely presented by

$$\begin{aligned}
\tau''_1 &= 1 - 2q \; + O(q^{3/2}), \\
\tau''_2 &= -1 - 4q \; + O(q^{3/2}), \\
\tau''_3 &= -2 - 30q \; + O(q^{3/2}).
\end{aligned}$$

By (20), we obtain σ''_k from τ''_k, as follows,

$$\begin{aligned}
\sigma''_1 &= \tau''_1 = 1-2q + O(q^{3/2}),\\
\sigma''_2 &= -\frac{1}{2}(\tau''_2 - \sigma''_1\tau''_1) = 1-2q + O(q^{3/2}),\\
\sigma''_3 &= \frac{1}{3}(\tau''_3 - \sigma''_1\tau''_2 + \sigma''_2\tau''_1) = -10q + O(q^{3/2}).
\end{aligned}$$

In a similar way as above, we obtain the next eigenvalues, as follows.

$$\begin{aligned}
\lambda''_4 &= e^{-\pi\sqrt{-1}/3} + O(q^{1/2}),\\
\lambda''_5 &= e^{\pi\sqrt{-1}/3} + O(q^{1/2}).
\end{aligned}$$

Therefore, putting

$$\lambda_2 = -q\lambda'_2, \quad \lambda_3 = -q\lambda'_3, \quad \lambda_4 = -q^2\lambda''_4, \quad \lambda_5 = -q^2\lambda''_5,$$

we obtain lower degree parts of the first five eigenvalues, as follows,

$$\begin{aligned}
\lambda_1 &= 1 + O(q^{15/2}), \qquad (24)\\
\lambda_2 &= -q + q^{3/2}(u+u^{-1}) + q^2\big(-1+\sqrt{-1}\,(u+u^{-1})\big)\\
&\quad - q^{5/2}\sqrt{-1}\,\big(1+\frac{1}{2}(u^2+u^{-2})\big)\\
&\quad + q^3\big(\frac{1}{2}(u^2+u^{-2}) + \frac{9\sqrt{-1}}{8}(u+u^{-1}) - \frac{\sqrt{-1}}{8}(u^3+u^{-3})\big)\\
&\quad + q^{7/2}\big(-2(u+u^{-1}) - \frac{7\sqrt{-1}}{8} + \frac{\sqrt{-1}}{2}(u^2+u^{-2}) - \frac{\sqrt{-1}}{16}(u^4+u^{-4})\big)\\
&\quad + q^4\sqrt{-1}\,\big(\frac{51}{64}(u+u^{-1}) + \frac{31}{128}(u^3+u^{-3}) - \frac{5}{128}(u^5+u^{-5})\big) + O(q^{9/2}),\\
\lambda_3 &= \overline{\lambda_2},\\
\lambda_4 &= q^2 e^{2\pi\sqrt{-1}/3} + O(q^{5/2}),\\
\lambda_5 &= \overline{\lambda_4}.
\end{aligned}$$

Hence, we obtain the following theorem.

Theorem 4.5. *We can present $I\big(M_n(4_1)\big)$ in terms of the above eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_5$ by*

$$I\big(M_n(4_1)\big) = \big(\textit{the coefficient of } u^0 \textit{ in } (\lambda_1^n+\lambda_2^n+\lambda_3^n+\lambda_4^n+\lambda_5^n)\big) + O(q^{\min\{2n+1,8\}}).$$

We show concrete forms of $M_n(4_1)$ obtained from the theorem for $n \leq 8$, as follows,

$$\begin{aligned}
I\big(M_1(4_1)\big) &= 1 - 2q - 3q^2 \quad + O(q^3), \\
I\big(M_2(4_1)\big) &= 1 \qquad + 2q^2 + 8q^3 - 3q^4 \quad + O(q^5), \\
I\big(M_3(4_1)\big) &= 1 \qquad\qquad - 2q^3 - 18q^4 - 6q^5 + 138q^6 \quad + O(q^7), \\
I\big(M_4(4_1)\big) &= 1 \qquad\qquad\qquad + 2q^4 + 32q^5 + 48q^6 - 424q^7 \quad + O(q^8), \\
I\big(M_5(4_1)\big) &= 1 \qquad\qquad\qquad\qquad - 2q^5 - 50q^6 - 160q^7 \quad + O(q^8), \\
I\big(M_6(4_1)\big) &= 1 \qquad\qquad\qquad\qquad\qquad + 2q^6 + 72q^7 \quad + O(q^8), \\
I\big(M_7(4_1)\big) &= 1 \qquad\qquad\qquad\qquad\qquad\qquad - 2q^7 \quad + O(q^8), \\
I\big(M_8(4_1)\big) &= 1 \qquad\qquad\qquad\qquad\qquad\qquad\qquad + O(q^8).
\end{aligned}$$

We can verify these values by comparing to these values which we show after Theorem 3.2. In particular, by Theorem 4.5, we can observe that coefficients of lower left part are 0.

Remark 4.6. From Theorem 3.2 and (24), we obtain that

$$\tilde{I}_{4_1}(n, q) \;=\; \lambda_1^n \;+ O(q^4).$$

In fact, we expect that $\lambda_1 = 1$ and $\tilde{I}_{4_1}(n, q) \;=\; \lambda_1$, but, in order to show this in this way, we have technical difficulty that we must calculate λ_1 in (24) not only for lower degrees, but also for all degrees, and we need Assumption 4.1 to ignore contributions from other λ_n.

4.2 Calculation of $I\big(M_n(5_2)\big)$ from eigenvalues of a transfer matrix

In this section, we calculate $I\big(M_n(5_2)\big)$ from eigenvalues of a transfer matrix.

By (84), $I\big(M_n(5_2)\big)$ is presented by

$$\begin{aligned}
&I\big(M_n(5_2)\big) \;= \\
&\sum \hat{J}_{5_2}(a_0{+}c_0, c_0{+}b_0, a_1{+}b_1)\, \hat{J}_{5_2}(a_2{+}c_0, c_0{+}b_1, a_1{+}b_0)\, \hat{J}_{5_2}(b_0{+}b_1, a_1{+}c_0, a_0{+}a_2) \\
&\quad \times \hat{J}_{5_2}(a_1{+}c_1, c_1{+}b_1, a_2{+}b_2)\, \hat{J}_{5_2}(a_3{+}c_1, c_1{+}b_2, a_2{+}b_1)\, \hat{J}_{5_2}(b_1{+}b_2, a_2{+}c_1, a_1{+}a_3) \\
&\quad \times \cdots \\
&\quad \times \hat{J}_{5_2}(a_{n-1}{+}c_{n-1}, c_{n-1}{+}b_{n-1}, a_n{+}b_n)\, \hat{J}_{5_2}(a_{n+1}{+}c_{n-1}, c_{n-1}{+}b_n, a_n{+}b_{n-1}) \\
&\qquad \times \hat{J}_{5_2}(b_{n-1}{+}b_n, a_n{+}c_{n-1}, a_{n-1}{+}a_{n+1}),
\end{aligned}$$

where we regard the subscripts of a_i, b_i, c_i as modulo n, and the range of the sum is given by

$$a_0 = 0, \qquad a_1, \cdots, a_{n-1}, b_0, \cdots, b_{n-1}, c_0, \cdots, c_{n-1} \in \mathbb{Z}.$$

By putting

$$a'_k \;=\; a_k - a_{k-1}, \quad b'_k \;=\; b_k - a_k, \quad c'_k \;=\; c_k - b_k,$$

the range of the sum is rewritten as

$$a'_0, \cdots, a'_{n-1}, b'_0, \cdots, b'_{n-1}, c'_0, \cdots, c'_{n-1} \in \mathbb{Z}, \qquad a'_0 + a'_1 + \cdots + a'_{n-1} \;=\; 0.$$

Further, we put

$$\mathcal{M}^{(b_0',a_1')}_{(b_1',a_2')} \;=\; \sum_{c_0'\in\mathbb{Z}} \hat{J}_{5_2}(a_0{+}c_0, c_0{+}b_0, a_1{+}b_1)\, \hat{J}_{5_2}(a_2{+}c_0, c_0{+}b_1, a_1{+}b_0)\, \hat{J}_{5_2}(b_0{+}b_1, a_1{+}c_0, a_0{+}a_2)\, u^{a_1'},$$

where u is a variable whose power counts $a_1' + a_2' + \cdots$. Since $\hat{J}_{5_2}(\ell_1{+}2, \ell_2{+}2, \ell_3{+}2) = \hat{J}_{5_2}(\ell_1, \ell_2, \ell_3)$, we note that the right-hand side depends only on b_0', a_1', b_1', a_2'. Furthermore, we put

$$\mathcal{M} \;=\; \Big(\mathcal{M}^{(b_0',a_1')}_{(b_1',a_2')}\Big).$$

The product of copies of $\mathcal{M}$ is given by

$$\mathcal{M}^2 \;=\; \Bigg(\sum_{b_1',a_2'} \mathcal{M}^{(b_0',a_1')}_{(b_1',a_2')}\, \mathcal{M}^{(b_1',a_2')}_{(b_2',a_3')}\Bigg),$$

where the parameters are related, as follows.

$$\begin{tikzcd}
a_0 \arrow[r, "a_1'"] \arrow[d, "b_0'"'] & a_1 \arrow[r, "a_2'"] \arrow[d, "b_1'"'] & a_2 \arrow[r, "a_3'"] \arrow[d, "b_2'"] & a_3 \arrow[d, "b_3'"] \\
b_0 \arrow[d, "c_0'"'] & b_1 \arrow[d, "c_1'"] & b_2 \arrow[d, "c_2'"] & b_3 \arrow[d, "c_3'"] \\
c_0 & c_1 & c_2 & c_3
\end{tikzcd}$$

Hence, $I\big(M_n(5_2)\big)$ can be presented by

$$I\big(M_n(5_2)\big) \;=\; \big(\text{the coefficient of } u^0 \text{ in } \;\operatorname{trace}\mathcal{M}^n\big). \tag{25}$$

We calculate the first few eigenvalues of $\mathcal{M}$. We put

$$\tau_k \;=\; \operatorname{trace}\mathcal{M}^k.$$

Then, by computer calculation, we obtain that

$$\begin{aligned}
\tau_1 \;&= 1 + q^{1/2}(u+u^{-1}) + q(-4+u^2+u^{-2}) + q^{3/2}(u^3+u^{-3}) \\
&\quad + q^2(-1+u^4+u^{-4}) + q^{5/2}\big(u^5+u^{-5}-5(u+u^{-1})\big) \\
&\quad + q^3(16+u^6+u^{-6}) \;+ O(q^{7/2}), \\
\tau_2 \;&= 1 + 2q^{1/2}(u+u^{-1} + 3q(u^2+u^{-2}) + 4q^{3/2}(-u-u^{-1}+u^3+u^{-3}) \\
&\quad + q^2\big(14-4(u^2+u^{-2})+5(u^4+u^{-4})\big) \\
&\quad + 2q^{5/2}\big(-3(u+u^{-1})-2(u^3+u^{-3})+3(u^5+u^{-5})\big) \\
&\quad + q^3\big(6-12(u^2+u^{-2})-4(u^4+u^{-4})+7(u^6+u^{-6})\big) \;+ O(q^{7/2}),
\end{aligned}$$

$$
\begin{aligned}
\tau_3 \ &= 1+3q^{1/2}(u+u^{-1})+3q\big(1+2(u^2+u^{-2})\big)\\
&\quad+q^{3/2}\big(-3(u+u^{-1})+10(u^3+u^{-3})\big)\\
&\quad+3q^2\big(5-3(u^2+u^{-2})+5(u^4+u^{-4})\big)\\
&\quad+3q^{5/2}\big(9(u+u^{-1})-5(u^3+u^{-3})+7(u^5+u^{-5})\big)\\
&\quad+q^3\big(-82+9(u^2+u^{-2})-21(u^4+u^{-4})+28(u^6+u^{-6})\big)\ +O(q^{7/2}),\\
\tau_4 \ &= 1+4q^{1/2}(u+u^{-1})+2q(4+5(u^2+u^{-2}))+4q^{3/2}\big(u+u^{-1}+5(u^3+u^{-3})\big)\\
&\quad+q^2\big(8-8(u^2+u^{-2})+35(u^4+u^{-4})\big)\\
&\quad+4q^{5/2}\big(11(u+u^{-1})-7(u^3+u^{-3})+14(u^5+u^{-5})\big)\\
&\quad+4q^3\big(-18+16(u^2+u^{-2})-14(u^4+u^{-4})+21(u^6+u^{-6})\big)\ +O(q^{7/2}).
\end{aligned}
$$

By (20), we obtain σ_k from τ_k, as follows,

$$
\begin{aligned}
\sigma_1 \ &= \tau_1\\
&= 1+q^{1/2}(u+u^{-1})+q(-4+u^2+u^{-2})+q^{3/2}(u^3+u^{-3})\\
&\quad+q^2(-1+u^4+u^{-4})+q^{5/2}\big(u^5+u^{-5}-5(u+u^{-1})\big)\\
&\quad+q^3(16+u^6+u^{-6})\ +O(q^{7/2}),\\
\sigma_2 \ &= \ -\frac{1}{2}(\tau_2-\sigma_1\tau_1)\\
&= \ -3q-q^{3/2}(u+u^{-1})+q^2(1-u^2-u^{-2})+q^{5/2}\big(-u^3-u^{-3}-2(u+u^{-1})\big)\\
&\quad+q^3(8+u^2+u^{-2}-u^4-u^{-4})\ +O(q^{7/2}),\\
\sigma_3 \ &= \ \frac{1}{3}(\tau_3-\sigma_1\tau_2+\sigma_2\tau_1)\\
&= 5q^2+4q^{5/2}(u+u^{-1})+q^3\big(-5+4(u^2+u^{-2})\big)\ +O(q^{7/2}),\\
\sigma_4 \ &= \ -\frac{1}{4}(\tau_4-\sigma_1\tau_3+\sigma_2\tau_2-\sigma_3\tau_1)\\
&= 2q^2+2q^{5/2}(u+u^{-1})+2q^3(-1+u^2+u^{-2})\ +O(q^{7/2}).
\end{aligned}
$$

We consider the following equation,

$$1-\sigma_1\,\hat{T}+\sigma_2\,\hat{T}^2-\sigma_3\,\hat{T}^3+\sigma_4\,\hat{T}^4 \ = \ O(q^{7/2}).$$

From solutions $\hat{T}$ of this equation, we obtain $\hat{T}^{-1}=\lambda_1$, as follows,

$$
\begin{aligned}
\lambda_1 \ &= 1+q^{1/2}(u+u^{-1})+q(-1+u^2+u^{-2})+q^{3/2}\big(-2(u+u^{-1})+u^3+u^{-3}\big)\\
&\quad+q^2(8+u^4+u^{-4})+q^{5/2}\big(-5(u+u^{-1})+u^5+u^{-5}\big)\\
&\quad+q^3\big(-19-12(u^2+u^{-2})+u^6+u^{-6}\big)\ +O(q^{7/2}).
\end{aligned}
$$

Further, in order to calculate the next eigenvalues concretely, we put

$$\tau_n'=\tau_n-\lambda_1^n.$$

They are concretely presented by

$$\tau_1' \ = \ -3q+2q^{3/2}(u+u^{-1})-9q^2+q^3\big(35+12(u^2+u^{-2})\big)\ +O(q^{7/2}),$$

$$
\begin{aligned}
\tau_2' &= 3q^2 - 14q^{5/2}(u+u^{-1}) + q^3\big(70 + 4(u^2+u^{-2})\big) \ + O(q^{7/2}),\\
\tau_3' &= 6q^2 \ + O(q^{7/2})\\
\tau_4' &= -24q^3 \ + O(q^{7/2})\\
\tau_k' &= O(q^{7/2}) \quad \text{for } k = 5,6,7,8.
\end{aligned}
$$

By (20), we obtain σ_k' from τ_k', as follows,

$$
\begin{aligned}
\sigma_1' &= -3q + 2q^{3/2}(u+u^{-1}) - 9q^2 + q^3\big(35 + 12(u^2+u^{-2})\big) \ + O(q^{7/2}),\\
\sigma_2' &= 3q^2 + q^{5/2}(u+u^{-1}) - 4q^3 \ + O(q^{7/2}),\\
\sigma_3' &= 2q^2 \ + O(q^{7/2}),\\
\sigma_k' &= O(q^{7/2}) \quad \text{for } k = 4,5,\cdots,8.
\end{aligned}
$$

We put

$$
\sigma_k'' \ = \ \sigma_k'/(q^{2/3})^k.
$$

They are concretely presented by

$$
\begin{aligned}
\sigma_1'' &= -3q^{1/3} + 2q^{5/6}(u+u^{-1}) \ + O(q^{7/6}),\\
\sigma_2'' &= 3q^{2/3} \ + O(q^{7/6}),\\
\sigma_3'' &= 2 \ + O(q^{7/6}).
\end{aligned}
$$

We consider the following equation,

$$
1 - \sigma_1''\hat{T} + \sigma_2''\hat{T}^2 - \sigma_3''\hat{T}^3 \ = \ O(q^{7/6}).
$$

From solutions $\hat{T}$ of this equation, we obtain $\hat{T}^{-1} = \lambda_2'', \lambda_3'', \lambda_4''$, as follows,

$$
\begin{aligned}
\lambda_2'' &= 2^{1/3} - q^{1/3} + \frac{2}{3}\,q^{5/6}(u+u^{-1}) + \frac{1}{2^{1/3}}\,q \ + O(q^{7/6}),\\
\lambda_3'' &= 2^{1/3}e^{2\pi\sqrt{-1}/3} - q^{1/3} + \frac{2}{3}\,q^{5/6}(u+u^{-1}) + \frac{e^{-2\pi\sqrt{-1}/3}}{2^{1/3}}\,q \ + O(q^{7/6}),\\
\lambda_4'' &= 2^{1/3}e^{-2\pi\sqrt{-1}/3} - q^{1/3} + \frac{2}{3}\,q^{5/6}(u+u^{-1}) + \frac{e^{2\pi\sqrt{-1}/3}}{2^{1/3}}\,q \ + O(q^{7/6}).
\end{aligned}
$$

Therefore, putting

$$
\lambda_2 \ = \ q^{2/3}\lambda_2'', \quad \lambda_3 \ = \ q^{2/3}\lambda_3'', \quad \lambda_4 \ = \ q^{2/3}\lambda_4'',
$$

we obtain lower degree parts of the first four eigenvalues, as follows,

$$
\begin{aligned}
\lambda_1 &= 1 + q^{1/2}(u+u^{-1}) + q(-1+u^2+u^{-2}) + q^{3/2}\big(-2(u+u^{-1}) + u^3 + u^{-3}\big)\\
&\quad + q^2(8+u^4+u^{-4}) + q^{5/2}\big(-5(u+u^{-1}) + u^5 + u^{-5}\big)\\
&\quad + q^3\big(-19 - 12(u^2+u^{-2}) + u^6 + u^{-6}\big) \ + O(q^{7/2}), \qquad (26)\\
\lambda_2 &= 2^{1/3}q^{3/2} - 1 + \frac{2}{3}\,q^{3/2}(u+u^{-1}) + \frac{1}{2^{1/3}}\,q^{5/2} \ + O(q^{11/6}),
\end{aligned}
$$

$$
\begin{aligned}
\lambda_3 &= 2^{1/3} e^{2\pi\sqrt{-1}/3} q^{3/2} - 1 + \frac{2}{3}\, q^{3/2}(u+u^{-1}) + \frac{e^{-2\pi\sqrt{-1}/3}}{2^{1/3}}\, q^{5/3} \; + O(q^{11/6}), \\
\lambda_4 &= 2^{1/3} e^{-2\pi\sqrt{-1}/3} q^{3/2} - 1 + \frac{2}{3}\, q^{3/2}(u+u^{-1}) + \frac{e^{2\pi\sqrt{-1}/3}}{2^{1/3}}\, q^{5/3} \; + O(q^{11/6}).
\end{aligned}
$$

Hence, we obtain the following theorem.

Theorem 4.7. *We can present $I\big(M_n(5_2)\big)$ in terms of the above eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ by*

$$
I\big(M_n(5_2)\big) \;=\; \big(\textit{the coefficient of } u^0 \textit{ in } \lambda_1^n + \lambda_2^n + \lambda_3^n + \lambda_4^n\big) + \begin{cases} O(q^2) & \textit{if } n = 1, \\ O(q^3) & \textit{if } n = 2,3,4, \\ O(q^4) & \textit{if } n \geq 5. \end{cases}
$$

We show concrete forms of $M_n(5_2)$ obtained from the theorem for $n \leq 8$, as follows,

$$
\begin{aligned}
I\big(M_1(5_2)\big) &= 1 - 4q + O(q^2), \\
I\big(M_2(5_2)\big) &= 1 + 14q^2 + O(q^3), \\
I\big(M_3(5_2)\big) &= 1 + 3q + 15q^2 + O(q^3), \\
I\big(M_4(5_2)\big) &= 1 + 8q + 8q^2 + O(q^3), \\
I\big(M_5(5_2)\big) &= 1 + 15q + 20q^2 + 45q^3 + O(q^4), \\
I\big(M_6(5_2)\big) &= 1 + 24q + 63q^2 + 216q^3 + O(q^4), \\
I\big(M_7(5_2)\big) &= 1 + 35q + 161q^2 + 546q^3 + O(q^4), \\
I\big(M_8(5_2)\big) &= 1 + 48q + 344q^2 + 1248q^3 + O(q^4).
\end{aligned}
$$

We can verify these values by comparing to these values which we show after Theorem 3.3.

Remark 4.8. From Theorem 3.3 and (26), we obtain that

$$
\tilde{I}_{5_2}(n,q) \;=\; \big(\text{the coefficient of } u^0 \text{ in } \lambda_1^n\big) \; + O(q^3).
$$

In fact, we expect that

$$
\tilde{I}_{5_2}(n,q) \;=\; \big(\text{the coefficient of } u^0 \text{ in } \lambda_1^n\big),
$$

but, in order to show this in this way, we have technical difficulty that we must calculate λ_1 in (26) not only for lower degrees, but also for all degrees, and we need Assumption 4.2 to ignore contributions from other λ_n.

4.3 Calculation of $I\big(M_n(6_1)\big)$ from eigenvalues of a transfer matrix

In this section, we calculate $I\big(M_n(6_1)\big)$ from eigenvalues of a transfer matrix.

By (86), $I\bigl(M_n(6_1)\bigr)$ is presented by

$$\begin{aligned} I\bigl(M_n(6_1)\bigr) \;=\; \sum\; & \hat{J}_{6_1,3}(a_0+d_0, a_1+b_0, d_0+c_0)\, \hat{J}_{6_1,1}(d_0+b_0, a_1+c_0, a_0+b_1) \\ & \times \hat{J}_{6_1,2}(a_1+d_0, d_0+b_1, b_0+c_0)\, \hat{J}_{6_1,1}(2a_1, b_0+c_1, 2b_1) \\ & \times \hat{J}_{6_1,3}(a_1+d_1, a_2+b_1, d_1+c_1)\, \hat{J}_{6_1,1}(d_1+b_1, a_2+c_1, a_1+b_2) \\ & \times \hat{J}_{6_1,2}(a_2+d_1, d_1+b_2, b_1+c_1)\, \hat{J}_{6_1,1}(2a_2, b_1+c_2, 2b_2) \\ & \times \cdots \\ & \times \hat{J}_{6_1,3}(a_{n-1}+d_{n-1}, a_n+b_{n-1}, d_{n-1}+c_{n-1})\, \hat{J}_{6_1,1}(d_{n-1}+b_{n-1}, a_n+c_{n-1}, a_{n-1}+b_n) \\ & \times \hat{J}_{6_1,2}(a_n+d_{n-1}, d_{n-1}+b_n, b_{n-1}+c_{n-1})\, \hat{J}_{6_1,1}(2a_n, b_{n-1}+c_n, 2b_n), \end{aligned}$$

where we regard the subscripts of a_i, b_i, c_i, d_i as modulo n, and the range of the sum is given by

$$a_0 = 0, \quad a_1, \cdots, a_{n-1}, b_0, \cdots, b_{n-1}, c_0, \cdots, c_{n-1}, d_0, \cdots, d_{n-1} \in \mathbb{Z}.$$

By putting

$$a'_k \;=\; a_k - a_{k-1}, \quad b'_k \;=\; b_k - a_k, \quad c'_k \;=\; c_k - b_k, \quad d'_k \;=\; d_k - c_k,$$

the range of the sum is rewritten as

$$a'_0, \cdots, a'_{n-1}, b'_0, \cdots, b'_{n-1}, c'_0, \cdots, c'_{n-1}, d'_0, \cdots, d'_{n-1} \in \mathbb{Z}, \quad a'_0 + a'_1 + \cdots + a'_{n-1} \;=\; 0.$$

Further, we put

$$\begin{aligned} \mathcal{M}^{(b'_0,c'_0)}_{(b'_1,c'_1)} \;=\; \sum_{d'_0, a'_1 \in \mathbb{Z}} & \hat{J}_{6_1,3}(a_0+d_0, a_1+b_0, d_0+c_0)\, \hat{J}_{6_1,1}(d_0+b_0, a_1+c_0, a_0+b_1) \\ & \times \hat{J}_{6_1,2}(a_1+d_0, d_0+b_1, b_0+c_0)\, \hat{J}_{6_1,1}(2a_1, b_0+c_1, 2b_1)\, u^{a'_1}, \end{aligned}$$

where u is a variable whose power counts $a'_1 + a'_2 + \cdots$. Since $\hat{J}_{6_1,i}(\ell_1+2, \ell_2+2, \ell_3+2) = \hat{J}_{6_1,i}(\ell_1, \ell_2, \ell_3)$, we note that the right-hand side depends only on b'_1, c'_1, b'_2, c'_2. Furthermore, we put

$$\mathcal{M} \;=\; \Bigl(\mathcal{M}^{(b'_0,c'_0)}_{(b'_1,c'_1)}\Bigr).$$

The product of copies of $\mathcal{M}$ is given by

$$\mathcal{M}^2 \;=\; \left(\sum_{b'_1,c'_1} \mathcal{M}^{(b'_0,c'_0)}_{(b'_1,c'_1)}\, \mathcal{M}^{(b'_1,c'_1)}_{(b'_2,c'_2)}\right),$$

where the parameters are related, as follows.

$$\begin{tikzcd}
a_0 \arrow[r, "a_1'"] \arrow[d, "b_0'"'] & a_1 \arrow[r, "a_2'"] \arrow[d, "b_1'"'] & a_2 \arrow[d, "b_2'"'] \\
b_0 \arrow[d, "c_0'"'] & b_1 \arrow[d, "c_1'"'] & b_2 \arrow[d, "c_2'"'] \\
c_0 \arrow[d, "d_0'"'] & c_1 \arrow[d, "d_1'"] & c_2 \arrow[d, "d_2'"] \\
d_0 & d_1 & d_2
\end{tikzcd}$$

Hence, $I\bigl(M_n(6_1)\bigr)$ can be presented by

$$I\bigl(M_n(6_1)\bigr) \;=\; \bigl(\text{the coefficient of } u^0 \text{ in } \; \operatorname{trace}\mathcal{M}^n\bigr). \tag{27}$$

We calculate the first few eigenvalues of $\mathcal{M}$. We put

$$\tau_k \;=\; \operatorname{trace}\mathcal{M}^k.$$

Then, by computer calculation, we obtain that

$$\begin{aligned}
\tau_1 &= 1 + q^{1/2}(u+u^{-1}) + q(-4+u^2+u^{-2}) + q^{3/2}(u^3+u^{-3}) \\
&\quad + q^2(1+u^4+u^{-4}) \; + O(q^{5/2}), \\
\tau_2 &= 1 + 2q^{1/2}(u+u^{-1}) + 3q(u^2+u^{-2}) + 4q^{3/2}(-u-u^{-1}+u^3+u^{-3}) \\
&\quad + q^2(14+3u^4-4u^2-4u^{-2}+5u^{-4}) \; + O(q^{7/2}), \\
\tau_3 &= 1 + 3q^{1/2}(u+u^{-1}) + 3q\bigl(1+2(u^2+u^{-2})\bigr) \\
&\quad + q^{3/2}\bigl(-3(u+u^{-1}) + 10(u^3+u^{-3})\bigr) \\
&\quad + 3q^2(2-3(u^2+u^{-2})+4u^4+5u^{-4}) \; + O(q^{5/2}), \\
\tau_4 &= 1 + 4q^{1/2}(u+u^{-1}) + 2q\bigl(4+5(u^2+u^{-2})\bigr) \\
&\quad + 4q^{3/2}\bigl(u+u^{-1}+5(u^3+u^{-3})\bigr) \\
&\quad + q^2(4+31u^4-8u^2-8u^{-2}+35u^{-4}) \; + O(q^{5/2}),
\end{aligned}$$

By (20), we obtain σ_k from τ_k, as follows,

$$\begin{aligned}
\sigma_1 &= \tau_1 \\
&= 1 + q^{1/2}(u+u^{-1}) + q(-4+u^2+u^{-2}) + q^{3/2}(u^3+u^{-3}) + q^2(1+u^4+u^{-4}) \\
&\qquad + O(q^{5/2}), \\
\sigma_2 &= -\frac{1}{2}(\tau_2 - \sigma_1\tau_1) \\
&= -3q - q^{3/2}(u+u^{-1}) + q^2(3+u^4-u^2-u^{-2}) \; + O(q^{5/2}),
\end{aligned}$$

$$\begin{aligned}
\sigma_3 &= \frac{1}{3}(\tau_3 - \sigma_1\tau_2 + \sigma_2\tau_1) \\
&= 2q^2 \ + O(q^{5/2}), \\
\sigma_4 &= -\frac{1}{4}(\tau_4 - \sigma_1\tau_3 + \sigma_2\tau_2 - \sigma_3\tau_1) \\
&= O(q^{5/2}).
\end{aligned}$$

We consider the following equation,

$$1 - \sigma_1\hat{T} + \sigma_2\hat{T}^2 - \sigma_3\hat{T}^3 \ = \ O(q^{5/2}).$$

From solutions $\hat{T}$ of this equation, we obtain $\hat{T}^{-1} = \lambda_1$, as follows,

$$\begin{aligned}
\lambda_1 &= 1 + q^{1/2}(u + u^{-1}) + q(-1 + u^2 + u^{-2}) + q^{3/2}(-2(u + u^{-1}) + u^3 + u^{-3}) \\
&\quad + q^2(7 + u^{-4}) \ + O(q^{5/2}).
\end{aligned}$$

Further, in order to calculate the next eigenvalues concretely, we put

$$\tau'_n = (\tau_n - \lambda_1^n)/(-q)^n.$$

By (20), we obtain σ'_k from τ'_k, as follows,

$$\begin{aligned}
\sigma'_1 &= 3 - 2q^{1/2}(u + u^{-1}) + q(6 - u^4) \ + O(q^{3/2}), \\
\sigma'_2 &= 2 \ + O(q^{1/2}).
\end{aligned}$$

We consider the following equation,

$$1 - \sigma'_1\hat{T} + \sigma'_2\hat{T}^2 \ = \ O(q^{1/2}).$$

From solutions $\hat{T}$ of this equation, we obtain $\hat{T}^{-1} = \lambda'_2, \lambda'_3$, as follows,

$$\lambda'_2 \ = \ 2 \ + O(q^{1/2}), \qquad \lambda'_3 \ = \ 1 \ + O(q^{1/2})$$

Therefore, putting

$$\lambda_2 \ = \ -q\lambda'_2, \qquad \lambda_3 \ = \ -q\lambda'_3,$$

we obtain lower degree parts of the first three eigenvalues, as follows,

$$\begin{aligned}
\lambda_1 &= 1 + q^{1/2}(u + u^{-1}) + q(-1 + u^2 + u^{-2}) + q^{3/2}(-2(u + u^{-1}) + u^3 + u^{-3}) \\
&\quad + q^2(7 + u^{-4}) \ + O(q^{5/2}), \\
\lambda_2 &= -2q \ + O(q^{3/2}), \\
\lambda_3 &= -q \ + O(q^{3/2}).
\end{aligned} \tag{28}$$

Hence, we obtain the following theorem.

Theorem 4.9. *We can present* $I\big(M_n(6_1)\big)$ *in terms of the above eigenvalues* $\lambda_1, \lambda_2, \lambda_3$ *by*

$$I\big(M_n(6_1)\big) \ = \ \big(\textit{the coefficient of } u^0 \textit{ in } (\lambda_1^n + \lambda_2^n + \lambda_3^n)\big) \ + \begin{cases} O(q^2) & \textit{if } n = 1, \\ O(q^3) & \textit{if } n \geq 2. \end{cases}$$

We show concrete forms of $M_n(6_1)$ obtained from the theorem for $n \leq 8$, as follows,

$$\begin{aligned}
I\big(M_1(6_1)\big) &= 1 - 4q + O(q^2),\\
I\big(M_2(6_1)\big) &= 1 + 14q^2 + O(q^3),\\
I\big(M_3(6_1)\big) &= 1 + 3q + 6q^2 + O(q^3),\\
I\big(M_4(6_1)\big) &= 1 + 8q + 4q^2 + O(q^3),\\
I\big(M_5(6_1)\big) &= 1 + 15q + 15q^2 + O(q^3),\\
I\big(M_6(6_1)\big) &= 1 + 24q + 57q^2 + O(q^3),\\
I\big(M_7(6_1)\big) &= 1 + 35q + 154q^2 + O(q^3),\\
I\big(M_8(6_1)\big) &= 1 + 48q + 336q^2 + O(q^3).
\end{aligned}$$

We can verify these values by comparing to these values which we show after Theorem 3.4.

Remark 4.10. From Theorem 3.4 and (28), we obtain that

$$\tilde{I}_{6_1}(n,q) = \big(\text{the coefficient of } u^0 \text{ in } \lambda_1^n\big) + O(q^3).$$

In fact, we expect that

$$\tilde{I}_{6_1}(n,q) = \big(\text{the coefficient of } u^0 \text{ in } \lambda_1^n\big),$$

but, in order to show this in this way, we have technical difficulty that we must calculate λ_1 in (28) not only for lower degrees, but also for all degrees, and we need Assumption 4.1 to ignore contributions from other λ_n.

5 Calculation of $I\big(M_n(K)\big)$ from contributions from subsequences of parameters

In this section, we show Theorems 3.2, 3.3, 3.4 in Sections 5.1, 5.2, 5.3, respectively. We also show Theorem 3.1 in Section 5.5.

5.1 Proof of Theorem 3.2 for the 4_1 knot

In this section, we give a proof of Theorem 3.2. In fact, we prove (13), (14) and (15) for $n \geq 6$, noting that (13), (14) and (15) are verified for $n < 6$ by concrete computational results shown after Theorem 3.2, *i.e.*, we calculate the degree ≤ 3 part of $I\big(M_n(4_1)\big)$ for $n \geq 6$.

As we mention in Section 2, we put

$$\hat{J}_{4_1}(\ell_1,\ell_2,\ell_3) = q^{\alpha\ell_1+\beta\ell_2+\gamma\ell_3} I(\ell_1,\ell_2,\ell_3)$$

where $\alpha = \beta = \gamma = \frac{1}{6}$, and this value is obtained in Section B.1.

Proof of Theorem 3.2. As mentioned above, it is sufficient to calculate the degree ≤ 3 part of $I\big(M_n(4_1)\big)$ for $n \geq 6$. By (82), $I\big(M_n(4_1)\big)$ is presented by

$$\begin{aligned} I\big(M_n(4_1)\big) \;=\; \sum_{\substack{a_0=0 \\ a_1,\cdots,a_{2n-1}\in\mathbb{Z}}} & \hat{J}_{4_1}(a_0{+}a_3, 2a_1, 2a_2)\, \hat{J}_{4_1}(a_1{+}a_4, 2a_2, 2a_3) \\ & \times \hat{J}_{4_1}(a_2{+}a_5, 2a_3, 2a_4)\, \hat{J}_{4_1}(a_3{+}a_6, 2a_4, 2a_5) \\ & \times \cdots \\ & \times \hat{J}_{4_1}(a_{2n-2}{+}a_{2n+1}, 2a_{2n-1}, 2a_{2n})\, \hat{J}_{4_1}(a_{2n-1}{+}a_{2n+2}, 2a_{2n}, 2a_{2n+1}), \end{aligned}$$

where we regard the subscript of a_i as modulo $2n$, *i.e.*, $a_{2n} = a_0,\ a_{2n+1} = a_1,\ a_{2n+2} = a_2$.

We consider a sequence of the form

$$U \;=\; (a, a, a,\ a_3, a_4, \cdots, a_{\ell-1},\ a', a', a'). \tag{29}$$

We define the *length* of U to be ℓ, and define the *height* of U to be $a' - a$. We define $J_{4_1}(U)$ by

$$\begin{aligned} J_{4_1}(U) \;=\; & \hat{J}_{4_1}(a{+}a_3, 2a, 2a)\, \hat{J}_{4_1}(a{+}a_4, 2a, 2a_3) \\ & \times \hat{J}_{4_1}(a{+}a_5, 2a_3, 2a_4)\, \hat{J}_{4_1}(a_3{+}a_6, 2a_4, 2a_5) \\ & \times \cdots \\ & \times \hat{J}_{4_1}(a_{2\ell-1}{+}a', 2a', 2a'), \end{aligned}$$

and we define the *degree* of U to be the lowest degree of $J_{4_1}(U)$, which is a non-negative half integer. We note that, for any fixed d, there are a finite number of such sequences U of degree d; we show a classification of such sequences of degree 1, 2, 3 in Section A.1.

For another sequence of the form (29)

$$U' \;=\; (a', a', a',\ a'_3, a'_4, \cdots, a'_{\ell'-1},\ a'', a'', a'').$$

we define the union of U and U' by

$$U \cdot U' \;=\; (a, a, a,\ a_3, a_4, \cdots, a_{\ell-1},\ a', a', a'\ a'_3, a'_4, \cdots, a'_{\ell'-1},\ a'', a'', a'').$$

We note that the degree of $U \cdot U'$ is equal to the sum of degrees of U and U'. Further, we consider a constant sequence

$$U^{\ell}_{\mathrm{const}} \;=\; (\underbrace{a, a, a, \cdots, a}_{\ell},\ a, a, a),$$

noting that $J_{4_1}(U^{\ell}_{\mathrm{const}}) = I(0,0,0)^{\ell}$, and its degree is 0.

We calculate the degree ≤ 3 part of $I\big(M_n(4_1)\big)$. For a sufficiently large n, there are contributions to $I\big(M_n(4_1)\big)$ only from a union of finite number of sequences of the form (29) and constant sequences,

$$U^{\ell_0}_{\mathrm{const}} \cdot U_1 \cdot U^{\ell_1}_{\mathrm{const}} \cdot U_2 \cdot\ \cdots\ \cdot U_m \cdot U^{\ell_m}_{\mathrm{const}}.$$

It is sufficient to consider the following cases,

$$\begin{aligned}(1,0) &= \sum_i \big(\text{degree of } U_i,\ \text{height of } U_i\big),\\ (2,0) &= \sum_i \big(\text{degree of } U_i,\ \text{height of } U_i\big),\\ (3,0) &= \sum_i \big(\text{degree of } U_i,\ \text{height of } U_i\big).\end{aligned}$$

It follows from the classification in Section A.1 that the range of $(d,h) = (\text{degree},\ \text{height})$ satisfies that

$$\big|\,h\,\big| \ \le\ 2d, \qquad \big|\,h\,\big| \ \le\ 6-2d. \tag{30}$$

Hence, as concrete sums of $(d,h) = (\text{degree},\ \text{height})$, it is sufficient to consider the following cases,

$$\begin{aligned}&(0,0),\\ &(1,0),\\ &(2,0),\\ &(2,0) = (1,0)+(1,0),\\ &(3,0),\\ &(3,0) = (2,0)+(1,0),\\ &(3,0) = (1,0)+(1,0)+(1,0).\end{aligned}$$

We consider a sequence

$$U \ =\ (a_0, a_1, \cdots, a_{2n-1}),$$

which we regard as a cyclic sequence, *i.e.*, $a_{2n}=a_0$, $a_{2n+1}=a_1$, $a_{2n+2}=a_2$. In this proof, we write $f\equiv g$ if $f = g+O(q^4)$.

Case $(\mathbf{0},\mathbf{0})$ In this case, we have the following sequence,

$$U \ =\ (a_0,\cdots,a_{2n-1}) \ =\ U_{\mathrm{const}}^{2n}\,,$$

and hence,

$$\begin{aligned}&J_{4_1}(U) \ =\ I(0,0,0)^{2n} \ \equiv\ (1-q-2q^2-2q^3)^{2n}\\ &\equiv\ 1-2n\,q + \Big(2n\,(-q^2)+\frac{1}{2}\cdot 2n(2n-1)\,(-q)^2\Big)\\ &\quad + \Big(2n(-2q^3)+2n(2n-1)\,(-q)(-2q^2)+\frac{1}{6}\cdot 2n(2n-1)(2n-2)\,(-q)^3\Big)\\ &\equiv\ 1-2n\,q+\frac{1}{2}\cdot 2n(2n-5)\,q^2-\frac{1}{3}\cdot 2n\,(2n^2-15n+13)\,q^3.\end{aligned}\tag{31}$$

Case $(\mathbf{1},\mathbf{0})$ In this case, we have the following sequence,

$$U \ =\ U_{\mathrm{const}}^{\ell_0}\cdot U_1^{1,0,4}\cdot U_{\mathrm{const}}^{\ell_1}\,,$$

and hence,

$$\begin{aligned}
\sum_{\ell_0} J_{4_1}(U) &= 2n\, J_{4_1}(U_1^{1,0,4})\, I(0,0,0)^{2n-4} \equiv 2n\,(q-2q^3)(1-q-2q^2)^{2n-4} \\
&\equiv 2n\, q(1-2q^2)\Big(1-(2n-4)q+\big((2n-4)(-2q^2)+\frac{1}{2}\,(2n-4)(2n-5)\,(-q)^2\big)\Big) \\
&\equiv 2n\, q(1-2q^2)\big(1-(2n-4)\,q+(2n^2-13n+18)\,q^2\big) \\
&\equiv 2n\, q - 2n\,(2n-4)\,q^2 + 2n\,(2n^2-13n+16)\,q^3. \qquad (32)
\end{aligned}$$

The sum of (31) and (32) is given by

$$1 + \frac{1}{2}\cdot 2n(2n-3)\,q^2 + \frac{1}{3}\cdot 2n(4n^2-24n+35)\,q^3, \qquad (33)$$

whose degree ≤ 1 part gives the degree ≤ 1 part of the required formula.

Case $(\mathbf{2},\mathbf{0})$ In this case, we have the following sequences,

$$\begin{aligned}
U_{(1)} &= U_{\mathrm{const}}^{\ell_0}\cdot U_1^{2,0,4}\cdot U_{\mathrm{const}}^{\ell_1}, \\
U_{(2)} &= U_{\mathrm{const}}^{\ell_0}\cdot U_1^{2,0,7}\cdot U_{\mathrm{const}}^{\ell_1},
\end{aligned}$$

and hence,

$$\begin{aligned}
\sum_{\ell_0} J_{4_1}(U_{(1)}) &= 2n\, J_{4_1}(U_1^{2,0,4})\, I(0,0,0)^{2n-4} \equiv 2n\, q^2\,(1-q)^{2n-4} \\
&\equiv 2n\, q^2\big(1-(2n-4)q\big) \equiv 2n\, q^2 - 2n(2n-4)\,q^3, \\
\sum_{\ell_0} J_{4_1}(U_{(2)}) &= 2n\, J_{4_1}(U_1^{2,0,7})\, I(0,0,0)^{2n-7} \equiv 2n\, q^2\,(1-q)^{2n-7} \\
&\equiv 2n\, q^2\big(1-(2n-7)q\big) \equiv 2n\, q^2 - 2n(2n-7)\,q^3.
\end{aligned}$$

Their sum is given by

$$4n\, q^2 - 2n\,(4n-11)\,q^3. \qquad (34)$$

Case $(\mathbf{2},\mathbf{0}) = (\mathbf{1},\mathbf{0}) + (\mathbf{1},\mathbf{0})$ In this case, we have the following sequence,

$$U = U_{\mathrm{const}}^{\ell_0}\cdot U_1^{1,0,4}\cdot U_{\mathrm{const}}^{\ell_1}\cdot U_1^{1,0,4}\cdot U_{\mathrm{const}}^{\ell_2}, \qquad (35)$$

and hence,

$$\begin{aligned}
\sum_{\ell_0,\ell_1} J_{4_1}(U) &= \frac{2n(2n-7)}{2}\cdot J_{4_1}(U_1^{1,0,4})^2\, I(0,0,0)^{2n-8} \\
&\equiv \frac{1}{2}\cdot 2n(2n-7)\,(q-2q^3)^2(1-q)^{2n-8} \\
&\equiv \frac{1}{2}\cdot 2n(2n-7)\,q^2(1-4q^2)\big(1-(2n-8)q\big) \\
&\equiv \frac{1}{2}\cdot 2n(2n-7)\,q^2 - \frac{1}{2}\cdot 2n(2n-7)(2n-8)\,q^3. \qquad (36)
\end{aligned}$$

The sum of (33), (34) and (36) is given by

$$1 + \frac{1}{3} \cdot 2n(4n^2 - 24n + 35)\, q^3, \tag{37}$$

whose degree ≤ 2 part gives the degree ≤ 2 part of the required formula.

Case $(\mathbf{3}, \mathbf{0})$ In this case, we have the following sequences,

$$\begin{aligned} U_{(1)} &= U_{\text{const}}^{\ell_0} \cdot U_1^{3,0,4} \cdot U_{\text{const}}^{\ell_1}\,, \\ U_{(2),i} &= U_{\text{const}}^{\ell_0} \cdot U_i^{3,0,7} \cdot U_{\text{const}}^{\ell_1} \qquad \text{for } i = 1, 2, \\ U_{(3)} &= U_{\text{const}}^{\ell_0} \cdot U_1^{3,0,10} \cdot U_{\text{const}}^{\ell_1}\,, \end{aligned}$$

and hence,

$$\begin{aligned} \sum_{\ell_0} J_{4_1}(U_{(1)}) &= 2n\, J_{4_1}(U_1^{3,0,4}) \equiv 2n\, q^3, \\ \sum_{\ell_0} \sum_{1 \leq i \leq 2} J_{4_1}(U_{(2),i}) &= \sum_{1 \leq i \leq 2} 2n\, J_{4_1}(U_i^{3,0,7}) \equiv 2n \cdot 2q^3, \\ \sum_{\ell_0} J_{4_1}(U_{(3)}) &= 2n\, J_{4_1}(U_1^{3,0,10}) \equiv 2n\, q^3. \end{aligned}$$

Their sum is given by

$$2n \cdot 4q^3. \tag{38}$$

Case $(\mathbf{3}, \mathbf{0}) = (\mathbf{2}, \mathbf{0}) + (\mathbf{1}, \mathbf{0})$ In this case, we have the following sequences,

$$\begin{aligned} U_{(1)} &= U_{\text{const}}^{\ell_0} \cdot U_1^{2,0,4} \cdot U_{\text{const}}^{\ell_1} \cdot U_1^{1,0,4} \cdot U_{\text{const}}^{\ell_2}\,, \\ U_{(2)} &= U_{\text{const}}^{\ell_0} \cdot U_1^{2,0,7} \cdot U_{\text{const}}^{\ell_1} \cdot U_1^{1,0,4} \cdot U_{\text{const}}^{\ell_2}\,, \end{aligned}$$

and hence,

$$\begin{aligned} \sum_{\ell_0, \ell_1} J_{4_1}(U_{(1)}) &= 2n(2n-7)\, J_{4_1}(U_1^{2,0,4})\, J_{4_1}(U_1^{1,0,4}) \\ &\equiv 2n(2n-7)\, q \cdot q^2 \equiv 2n(2n-7)\, q^3, \\ \sum_{\ell_0, \ell_1} J_{4_1}(U_{(2)}) &= 2n(2n-10)\, J_{4_1}(U_1^{2,0,7})\, J_{4_1}(U_1^{1,0,4}) \\ &\equiv 2n(2n-10)\, q \cdot q^2 \equiv 2n(2n-10)\, q^3. \end{aligned}$$

Their sum is given by

$$2n(4n-17)\, q^3. \tag{39}$$

Case $(\mathbf{3}, \mathbf{0}) = (\mathbf{1}, \mathbf{0}) + (\mathbf{1}, \mathbf{0}) + (\mathbf{1}, \mathbf{0})$ In this case, we have the following sequence,

$$U = U_{\text{const}}^{\ell_0} \cdot U_1^{1,0,4} \cdot U_{\text{const}}^{\ell_1} \cdot U_1^{1,0,4} \cdot U_{\text{const}}^{\ell_2} \cdot U_1^{1,0,4} \cdot U_{\text{const}}^{\ell_3}\,,$$

and hence,

$$\sum_{\ell_0,\ell_1,\ell_3} J_{4_1}(U) \;=\; \frac{2n(2n-10)(2n-11)}{6}\, J_{4_1}(U_1^{1,0,4})^3 \;\equiv\; \frac{1}{6}\cdot 2n(2n-10)(2n-11)\, q^3. \tag{40}$$

The sum of (37), (38), (39) and (40) is given by

$$1.$$

This is the degree ≤ 3 part of $I\big(M_n(4_1)\big)$ for a sufficiently large n. Therefore, we obtain the theorem. $\square$

5.2 Proof of Theorem 3.3 for the 5_2 knot

In this section, we give a proof of Theorem 3.3. In fact, we prove (16) and (17) for $n \ge 8$, noting that (16) and (17) are verified for $n \le 8$ by concrete computational results shown after Theorem 3.3, *i.e.*, we calculate the degree ≤ 2 part of $I\big(M_n(4_1)\big)$ for $n \ge 8$.

As we mention in Section 2, we put

$$\hat{J}_{5_2}(\ell_1,\ell_2,\ell_3) \;=\; q^{\alpha\ell_1+\beta\ell_2+\gamma\ell_3} I(\ell_1,\ell_2,\ell_3)$$

where $\alpha = 0.164$, $\beta = 0.224$, $\gamma = 0.112$, and these values are obtained in Section B.2.

Proof of Theorem 3.3. As mentioned above, it is sufficient to calculate the degree ≤ 2 part of $I\big(M_n(5_2)\big)$ for $n \ge 8$. By (84), $I\big(M_n(5_2)\big)$ is presented by

$$\begin{aligned} I\big(M_n(5_2)\big) \;=\; & \\ \sum_{\substack{a_0=0\\ a_1,\cdots,a_{n-1}\in\mathbb{Z}\\ b_0,\cdots,b_{n-1}\in\mathbb{Z}\\ c_0,\cdots,c_{n-1}\in\mathbb{Z}}} & \hat{J}_{5_2}(a_0{+}c_0, c_0{+}b_0, a_1{+}b_1)\, \hat{J}_{5_2}(a_2{+}c_0, c_0{+}b_1, a_1{+}b_0)\, \hat{J}_{5_2}(b_0{+}b_1, a_1{+}c_0, a_0{+}a_2) \\ & \times \hat{J}_{5_2}(a_1{+}c_1, c_1{+}b_1, a_2{+}b_2)\, \hat{J}_{5_2}(a_3{+}c_1, c_1{+}b_2, a_2{+}b_1)\, \hat{J}_{5_2}(b_1{+}b_2, a_2{+}c_1, a_1{+}a_3) \\ & \times \cdots \\ & \times \hat{J}_{5_2}(a_{n-1}{+}c_{n-1}, c_{n-1}{+}b_{n-1}, a_n{+}b_n)\, \hat{J}_{5_2}(a_{n+1}{+}c_{n-1}, c_{n-1}{+}b_n, a_n{+}b_{n-1}) \\ & \quad \times \hat{J}_{5_2}(b_{n-1}{+}b_n, a_n{+}c_{n-1}, a_{n-1}{+}a_{n+1}), \end{aligned}$$

where we regard the subscripts of a_i, b_i, c_i as modulo n.

We consider a sequence of the form

$$V \;=\; (a, a, c_0,\; a, b_1, c_1,\; a_2, b_2, c_2,\; \cdots,\; a_{\ell-1}, b_{\ell-1}, c_{\ell-1},\; a', a', *,\; a'). \tag{41}$$

We define the *length* of V to be ℓ, and define the *height* of V to be $a'-a$. We define $J_{5_2}(V)$ by

$$\begin{aligned} J_{5_2}(V) \;=\; & \hat{J}_{5_2}(a{+}c_0, c_0{+}a, a{+}b_1)\, \hat{J}_{5_2}(a_2{+}c_0, c_0{+}b_1, a{+}a)\, \hat{J}_{5_2}(a{+}b_1, a{+}c_0, a{+}a_2) \\ & \times \hat{J}_{5_2}(a{+}c_1, c_1{+}b_1, a_2{+}b_2)\, \hat{J}_{5_2}(a_3{+}c_1, c_1{+}b_2, a_2{+}b_1)\, \hat{J}_{5_2}(b_1{+}b_2, a_2{+}c_1, a{+}a_3) \\ & \times \cdots \\ & \times \hat{J}_{5_2}(a_{\ell-1}{+}c_{\ell-1}, c_{\ell-1}{+}b_{\ell-1}, a'{+}a')\, \hat{J}_{5_2}(a'{+}c_{\ell-1}, c_{\ell-1}{+}a', a'{+}b_{\ell-1}) \\ & \quad \times \hat{J}_{5_2}(b_{\ell-1}{+}a', a'{+}c_{\ell-1}, a_{\ell-1}{+}a'), \end{aligned}$$

and we define the *degree* of V to be the lowest degree of $J_{5_2}(V)$, which is a non-negative half integer. We note that, for any fixed d, there are a finite number of such sequences V of degree d; we show a classification of such sequences of degree 1, 2 in Section A.2.

For another sequence of the form (41)

$$V' \;=\; (a', a', c_0',\; a', b_1', c_1',\; a_2', b_2', c_2',\; \cdots,\; a'_{\ell'-1}, b'_{\ell'-1}, c'_{\ell'-1},\; a'', a'', *,\; a''),$$

we define the union of V and V' by

$$\begin{aligned} V\cdot V' \;=\; & (a, a, c_0,\; a, b_1, c_1,\; a_2, b_2, c_2,\; \cdots,\; a_{\ell-1}, b_{\ell-1}, c_{\ell-1}, \\ & a', a', c_0',\; a', b_1', c_1',\; a_2', b_2', c_2',\; \cdots,\; a'_{\ell'-1}, b'_{\ell'-1}, c'_{\ell'-1},\; a'', a'', *,\; a''). \end{aligned}$$

We note that the degree of $V\cdot V'$ is equal to the sum of degrees of V and V'. Further, we consider a constant sequence

$$V^{\ell}_{\text{const}} \;=\; (\underbrace{a, a, a,\; a, a, a,\; \cdots,\; a, a, a}_{3\ell},\; a, a, *,\; a),$$

noting that $J_{5_2}(V^{\ell}_{\text{const}}) = I(0,0,0)^{3\ell}$, and its degree is 0.

We calculate the degree ≤ 2 part of $I\big(M_n(5_2)\big)$. For a sufficiently large n, there are contributions to $I\big(M_n(5_2)\big)$ only from a union of finite number of sequences of the form (41) and constant sequences,

$$V^{\ell_0}_{\text{const}}\cdot V_1\cdot V^{\ell_1}_{\text{const}}\cdot V_2\cdot\;\cdots\;\cdot V_m\cdot V^{\ell_m}_{\text{const}}.$$

It is sufficient to consider the following cases,

$$\begin{aligned} (1,0) \;&=\; \sum_i \big(\text{degree of } V_i,\; \text{height of } V_i\big), \\ (2,0) \;&=\; \sum_i \big(\text{degree of } V_i,\; \text{height of } V_i\big). \end{aligned}$$

It follows from the classification in Section A.2 that the range of $(d,h) = (\text{degree},\ \text{height})$ satisfies that

$$\big|\, h\,\big| \;\le\; 2d, \qquad \big|\, h\,\big| \;\le\; 4-2d. \tag{42}$$

Hence, concrete values of $(d,h) = (\text{degree},\ \text{height})$ are

$$(d,h) \;=\; (\frac{1}{2}, \pm 1),\; (1,0),\; (1,\pm 2),\; (\frac{3}{2}, \pm 1).$$

Therefore, as concrete sums of $(d,h) = (\text{degree, height})$, it is sufficient to consider the following cases,

$$\begin{aligned}
&(0,0),\\
&(1,0),\\
&(1,0) = (\frac{1}{2},1) + (\frac{1}{2},-1),\\
&(2,0),\\
&(2,0) = (1,0) + (1,0),\\
&(2,0) = (1,2) + (1,-2),\\
&(2,0) = (\frac{3}{2},1) + (\frac{1}{2},-1),\\
&(2,0) = (\frac{3}{2},-1) + (\frac{1}{2},1),\\
&(2,0) = (1,0) + (\frac{1}{2},1) + (\frac{1}{2},-1),\\
&(2,0) = (1,2) + (\frac{1}{2},-1) + (\frac{1}{2},-1),\\
&(2,0) = (1,-2) + (\frac{1}{2},1) + (\frac{1}{2},1),\\
&(2,0) = (\frac{1}{2},1) + (\frac{1}{2},1) + (\frac{1}{2},-1) + (\frac{1}{2},-1).
\end{aligned}$$

We consider a sequence

$$V \;=\; (a_0, b_0, c_0, \cdots, a_{n-1}, b_{n-1}, c_{n-1}),$$

where we regard the subscripts of a_i, b_i, c_i as modulo n. In this proof, we write $f \equiv g$ if $f = g + O(q^3)$.

Case $(\mathbf{0},\mathbf{0})$ In this case, we have the following sequence,

$$V \;=\; V_{\text{const}}^{n}\,,$$

and hence,

$$J_{5_2}(V) \;=\; I(0,0,0)^{3n} \;\equiv\; (1-q-2q^2)^{3n} \;\equiv\; 1 - 3n\,q + \frac{1}{2}\cdot 3n(3n-5)\,q^2. \tag{43}$$

Case $(\mathbf{1},\mathbf{0})$ In this case, we have the following sequences,

$$\begin{aligned}
V_{(1)} \;&=\; V_{\text{const}}^{\ell_0}\cdot V_1^{1,0,1}\cdot V_{\text{const}}^{\ell_1}\,,\\
V_{(2)} \;&=\; V_{\text{const}}^{\ell_0}\cdot V_1^{1,0,2}\cdot V_{\text{const}}^{\ell_1}\,,\\
V_{(3),i} \;&=\; V_{\text{const}}^{\ell_0}\cdot V_i^{1,0,3}\cdot V_{\text{const}}^{\ell_1} \qquad \text{for } i=1,2,3,4,
\end{aligned}$$

and hence,

$$\begin{aligned}
\sum_{\ell_0} J_{5_2}(V_{(1)}) &= n\, J_{5_2}(V_1^{1,0,1})\, I(0,0,0)^{3(n-1)} \\
&\equiv n\,(-q-q^2)(1-q)^{3n-3} \;\equiv\; -n\,q + n\,(3n-4)\,q^2, \\
\sum_{\ell_0} J_{5_2}(V_{(2)}) &= n\, J_{5_2}(V_1^{1,0,2})\, I(0,0,0)^{3(n-2)} \\
&\equiv n\,q(1-q)^{3n-6} \;\equiv\; n\,q - n\,(3n-6)\,q^2. \\
\sum_{\ell_0} J_{5_2}(V_{(3),1}) &= n\, J_{5_2}(V_1^{1,0,3})\, I(0,0,0)^{3(n-3)} \\
&\equiv n\,(q-3q^2)(1-q)^{3n-9} \;\equiv\; n\,q - n\,(3n-6)\,q^2, \\
\sum_{\ell_0} J_{5_2}(V_{(3),2}) &= n\, J_{5_2}(V_2^{1,0,3})\, I(0,0,0)^{3(n-3)} \\
&\equiv n\,(q-2q^2)(1-q)^{3n-9} \;\equiv\; n\,q - n\,(3n-7)\,q^2, \\
\sum_{\ell_0} J_{5_2}(V_{(3),3}) &= n\, J_{5_2}(V_3^{1,0,3})\, I(0,0,0)^{3(n-3)} \\
&\equiv n\,(q-4q^2)(1-q)^{3n-9} \;\equiv\; n\,q - n\,(3n-5)\,q^2, \\
\sum_{\ell_0} J_{5_2}(V_{(3),4}) &= n\, J_{5_2}(V_4^{1,0,3})\, I(0,0,0)^{3(n-3)} \\
&\equiv n\,(q-4q^2)(1-q)^{3n-9} \;\equiv\; n\,q - n\,(3n-5)\,q^2.
\end{aligned}$$

Their sum is given by

$$4n\,q \;-n\,(12n-25)\,q^2. \tag{44}$$

Case $(\mathbf{1},\mathbf{0}) = (\frac{\mathbf{1}}{\mathbf{2}},\mathbf{1}) + (\frac{\mathbf{1}}{\mathbf{2}},-\mathbf{1})$ In this case, we have the following sequence,

$$V \;=\; V_{\mathrm{const}}^{\ell_0}\cdot V_1^{1/2,1,2}\cdot V_{\mathrm{const}}^{\ell_1}\cdot V_1^{1/2,-1,2}\cdot V_{\mathrm{const}}^{\ell_2}\,,$$

and hence,

$$\begin{aligned}
\sum_{\ell_0,\ell_1} J_{5_2}(V) &= n(n-3)\, J_{5_2}(V_1^{1/2,1,2})\, J_{5_2}(V_1^{1/2,-1,2})\, I(0,0,0)^{3(n-4)} \\
&\equiv n(n-3)\,(q^{1/2}-3q^{3/2})\,(q^{1/2}-3q^{3/2})\,(1-q)^{3n-12} \\
&\equiv n(n-3)\,q(1-6q)\big(1-(3n-12)q\big) \\
&\equiv n(n-3)q - n(n-3)(3n-6)q^2.
\end{aligned} \tag{45}$$

The sum of (43), (44) and (45)

$$1\,+n(n-2)\,q\;-\frac{1}{2}\,n(6n^2-15n+1)\,q^2\;+O(q^3), \tag{46}$$

whose degree ≤ 1 part gives the degree ≤ 1 part of the required formula. The above 3 cases determine the coefficient of q. Hence, $c_1^{(n)}(5_2) = n(n-2)$ for any $n \geq 4$. Therefore, $p_1^{5_2}(n) = n(n-2)$.

In the remaining cases, we calculate the coefficient of q^2.

Case $(\mathbf{2},\mathbf{0})$ In this case, we have the following sequences,

$$
\begin{aligned}
V_{(1)} &= V_{\text{const}}^{\ell_0} \cdot V_1^{2,0,1} \cdot V_{\text{const}}^{\ell_1}\,, \\
V_{(2)} &= V_{\text{const}}^{\ell_0} \cdot V_1^{2,0,2} \cdot V_{\text{const}}^{\ell_1}\,, \\
V_{(3),i} &= V_{\text{const}}^{\ell_0} \cdot V_i^{2,0,3} \cdot V_{\text{const}}^{\ell_1}\,, \\
V_{(4),i} &= V_{\text{const}}^{\ell_0} \cdot V_i^{2,0,4} \cdot V_{\text{const}}^{\ell_1}\,, \\
V_{(5),i} &= V_{\text{const}}^{\ell_0} \cdot V_i^{2,0,5} \cdot V_{\text{const}}^{\ell_1}\,,
\end{aligned}
$$

and hence,

$$
\begin{aligned}
\sum_{\ell_0} J_{5_2}(V_{(1)}) &= n\, J_{5_2}(V_1^{2,0,1}) \equiv n\, q^2, \\
\sum_{\ell_0} J_{5_2}(V_{(2)}) &= n\, J_{5_2}(V_1^{2,0,2}) \equiv n\, q^2, \\
\sum_{1\leq i\leq 11} \sum_{\ell_0} J_{5_2}(V_{(3),i}) &= \sum_{1\leq i\leq 11} n\, J_{5_2}(V_i^{2,0,3}) \equiv n\,(6q^2 - 5q^2) \equiv n\, q^2, \\
\sum_{1\leq i\leq 17} \sum_{\ell_0} J_{5_2}(V_{(4),i}) &= \sum_{1\leq i\leq 17} n\, J_{5_2}(V_i^{2,0,4}) \equiv 17\, n\, q^2, \\
\sum_{1\leq i\leq 22} \sum_{\ell_0} J_{5_2}(V_{(5),i}) &= \sum_{1\leq i\leq 22} n\, J_{5_2}(V_i^{2,0,5}) \equiv 22\, n\, q^2.
\end{aligned}
$$

Their sum is given by

$$
42\, n\, q^2. \tag{47}
$$

Case $(\mathbf{2},\mathbf{0}) = (\mathbf{1},\mathbf{0}) + (\mathbf{1},\mathbf{0})$ In this case, we have the following sequences,

$$
\begin{aligned}
V_{(1)} &= V_{\text{const}}^{\ell_0} \cdot V_1^{1,0,1} \cdot V_{\text{const}}^{\ell_1} \cdot V_1^{1,0,1} \cdot V_{\text{const}}^{\ell_2}\,, \\
V_{(2)} &= V_{\text{const}}^{\ell_0} \cdot V_1^{1,0,1} \cdot V_{\text{const}}^{\ell_1} \cdot V_1^{1,0,2} \cdot V_{\text{const}}^{\ell_2}\,, \\
V_{(3),i} &= V_{\text{const}}^{\ell_0} \cdot V_1^{1,0,1} \cdot V_{\text{const}}^{\ell_1} \cdot V_i^{1,0,3} \cdot V_{\text{const}}^{\ell_2}\,, \\
V_{(4)} &= V_{\text{const}}^{\ell_0} \cdot V_1^{1,0,2} \cdot V_{\text{const}}^{\ell_1} \cdot V_1^{1,0,2} \cdot V_{\text{const}}^{\ell_2}\,, \\
V_{(5),i} &= V_{\text{const}}^{\ell_0} \cdot V_1^{1,0,2} \cdot V_{\text{const}}^{\ell_1} \cdot V_i^{1,0,3} \cdot V_{\text{const}}^{\ell_2}\,, \\
V_{(6),i} &= V_{\text{const}}^{\ell_0} \cdot V_i^{1,0,3} \cdot V_{\text{const}}^{\ell_1} \cdot V_i^{1,0,3} \cdot V_{\text{const}}^{\ell_2}\,, \\
V_{(7),i,j} &= V_{\text{const}}^{\ell_0} \cdot V_i^{1,0,3} \cdot V_{\text{const}}^{\ell_1} \cdot V_j^{1,0,3} \cdot V_{\text{const}}^{\ell_2}\,,
\end{aligned}
$$

and hence,

$$\begin{aligned}
\sum_{\ell_0,\ell_1} J_{5_2}(V_{(1)}) &= \frac{n(n-1)}{2}\cdot J_{5_2}(V_1^{1,0,1})^2 \\
&\equiv \frac{1}{2}\,n(n-1)\,(-q)^2 \;\equiv\; \frac{1}{2}\,n(n-1)\,q^2, \\
\sum_{\ell_0,\ell_1} J_{5_2}(V_{(2)}) &= n(n-2)\,J_{5_2}(V_1^{1,0,1})\,J_{5_2}(V_1^{1,0,2}) \\
&\equiv n(n-2)\,(-q)\cdot q \;\equiv\; -n(n-2)\,q^2, \\
\sum_{1\le i\le 4}\sum_{\ell_0,\ell_1} J_{5_2}(V_{(3),i}) &= \sum_{1\le i\le 4} n(n-3)\,J_{5_2}(V_1^{1,0,1})\,J_{5_2}(V_i^{1,0,3}) \\
&\equiv 4n(n-3)\,(-q)\cdot q \;\equiv\; -4n(n-3)\,q^2, \\
\sum_{\ell_0,\ell_1} J_{5_2}(V_{(4)}) &= \frac{n(n-3)}{2}\,J_{5_2}(V_1^{1,0,2})^2 \;\equiv\; \frac{1}{2}\,n(n-3)\,q^2, \\
\sum_{1\le i\le 4}\sum_{\ell_0,\ell_1} J_{5_2}(V_{(5),i}) &= \sum_{1\le i\le 4} n(n-4)\,J_{5_2}(V_1^{1,0,2})\,J_{5_2}(V_i^{1,0,3}) \\
&\equiv 4n(n-4)\,q\cdot q \;\equiv\; 4n(n-4)\,q^2, \\
\sum_{1\le i\le 4}\sum_{\ell_0,\ell_1} J_{5_2}(V_{(6),i}) &= \sum_{1\le i\le 4} \frac{n(n-5)}{2}\,J_{5_2}(V_i^{1,0,3})^2 \;\equiv\; 2n(n-5)\,q^2, \\
\sum_{1\le i<j\le 4}\sum_{\ell_0,\ell_1} J_{5_2}(V_{(7),i,j}) &= \sum_{1\le i<j\le 4} n(n-5)\,J_{5_2}(V_i^{1,0,3})\,J_{5_2}(V_j^{1,0,3}) \;\equiv\; 6n(n-5)\,q^2.
\end{aligned}$$

Their sum is given by

$$n\,(8n-44)\,q^2. \tag{48}$$

Case $(\mathbf{2},\mathbf{0}) = (\mathbf{1},\mathbf{2}) + (\mathbf{1},-\mathbf{2})$ In this case, we have the following sequences,

$$\begin{aligned}
V_{(1)} &= V_{\mathrm{const}}^{\ell_0}\cdot V_1^{1,2,2}\cdot V_{\mathrm{const}}^{\ell_1}\cdot V_1^{1,-2,2}\cdot V_{\mathrm{const}}^{\ell_2}\,, \\
V_{(2)} &= V_{\mathrm{const}}^{\ell_0}\cdot V_1^{1,2,2}\cdot V_{\mathrm{const}}^{\ell_1}\cdot V_1^{1,-2,3}\cdot V_{\mathrm{const}}^{\ell_2}\,, \\
V_{(3)} &= V_{\mathrm{const}}^{\ell_0}\cdot V_1^{1,2,3}\cdot V_{\mathrm{const}}^{\ell_1}\cdot V_1^{1,-2,2}\cdot V_{\mathrm{const}}^{\ell_2}\,, \\
V_{(4)} &= V_{\mathrm{const}}^{\ell_0}\cdot V_1^{1,2,3}\cdot V_{\mathrm{const}}^{\ell_1}\cdot V_1^{1,-2,3}\cdot V_{\mathrm{const}}^{\ell_2}\,,
\end{aligned}$$

and hence,

$$\begin{aligned}
\sum_{\ell_0,\ell_1} J_{5_2}(V_{(1)}) &= n(n-3)\, J_{5_2}(V_1^{1,2,2})\, J_{5_2}(V_1^{1,-2,2}) \equiv n(n-3)\, q^2, \\
\sum_{\ell_0,\ell_1} J_{5_2}(V_{(2)}) &= n(n-4)\, J_{5_2}(V_1^{1,2,2})\, J_{5_2}(V_1^{1,-2,3}) \equiv n(n-4)\, q^2, \\
\sum_{\ell_0,\ell_1} J_{5_2}(V_{(3)}) &= n(n-4)\, J_{5_2}(V_1^{1,2,3})\, J_{5_2}(V_1^{1,-2,2}) \equiv n(n-4)\, q^2, \\
\sum_{\ell_0,\ell_1} J_{5_2}(V_{(4)}) &= n(n-5)\, J_{5_2}(V_1^{1,2,3})\, J_{5_2}(V_1^{1,-2,3}) \equiv n(n-5)\, q^2.
\end{aligned}$$

Their sum is given by

$$4n(n-4)\, q^2. \tag{49}$$

Case $(\mathbf{2},\mathbf{0}) = (\frac{\mathbf{3}}{\mathbf{2}},\mathbf{1}) + (\frac{\mathbf{1}}{\mathbf{2}},-\mathbf{1})$ In this case, we have the following sequences,

$$\begin{aligned}
V_{(1),i} &= V_{\mathrm{const}}^{\ell_0} \cdot V_i^{3/2,1,2} \cdot V_{\mathrm{const}}^{\ell_1} \cdot V_1^{1/2,-1,2} \cdot V_{\mathrm{const}}^{\ell_2}\,, \\
V_{(2),i} &= V_{\mathrm{const}}^{\ell_0} \cdot V_i^{3/2,1,3} \cdot V_{\mathrm{const}}^{\ell_1} \cdot V_1^{1/2,-1,2} \cdot V_{\mathrm{const}}^{\ell_2}\,, \\
V_{(3),i} &= V_{\mathrm{const}}^{\ell_0} \cdot V_i^{3/2,1,4} \cdot V_{\mathrm{const}}^{\ell_1} \cdot V_1^{1/2,-1,2} \cdot V_{\mathrm{const}}^{\ell_2}\,,
\end{aligned}$$

and hence,

$$\begin{aligned}
\sum_{1\le i\le 2}\sum_{\ell_0,\ell_1} J_{5_2}(V_{(1),i}) &= \sum_{1\le i\le 2} n(n-3)\, J_{5_2}(V_i^{3/2,1,2})\, J_{5_2}(V_1^{1/2,-1,2}) \\
&\equiv 2n(n-3)\,(-q^{3/2}) \cdot q^{1/2} \equiv -2n(n-3)\, q^2, \\
\sum_{1\le i\le 5}\sum_{\ell_0,\ell_1} J_{5_2}(V_{(2),i}) &= \sum_{1\le i\le 5} n(n-4)\, J_{5_2}(V_i^{3/2,1,3})\, J_{5_2}(V_1^{1/2,-1,2}) \\
&\equiv 5n(n-4)\, q^{3/2} \cdot q^{1/2} \equiv 5n(n-4)\, q^2, \\
\sum_{1\le i\le 7}\sum_{\ell_0,\ell_1} J_{5_2}(V_{(3),i}) &= \sum_{1\le i\le 7} n(n-5)\, J_{5_2}(V_i^{3/2,1,4})\, J_{5_2}(V_1^{1/2,-1,2}) \\
&\equiv 7n(n-5)\, q^{3/2} \cdot q^{1/2} \equiv 7n(n-5)\, q^2.
\end{aligned}$$

Their sum is given by

$$n(10n-49)\, q^2. \tag{50}$$

Case $(\mathbf{2},\mathbf{0}) = (\frac{\mathbf{3}}{\mathbf{2}},-\mathbf{1}) + (\frac{\mathbf{1}}{\mathbf{2}},\mathbf{1})$ In this case, we have the following sequences,

$$\begin{aligned}
V_{(1),i} &= V_{\mathrm{const}}^{\ell_0} \cdot V_i^{3/2,-1,2} \cdot V_{\mathrm{const}}^{\ell_1} \cdot V_1^{1/2,1,2} \cdot V_{\mathrm{const}}^{\ell_2}\,, \\
V_{(2),i} &= V_{\mathrm{const}}^{\ell_0} \cdot V_i^{3/2,-1,3} \cdot V_{\mathrm{const}}^{\ell_1} \cdot V_1^{1/2,1,2} \cdot V_{\mathrm{const}}^{\ell_2}\,, \\
V_{(3),i} &= V_{\mathrm{const}}^{\ell_0} \cdot V_i^{3/2,-1,4} \cdot V_{\mathrm{const}}^{\ell_1} \cdot V_1^{1/2,1,2} \cdot V_{\mathrm{const}}^{\ell_2}\,,
\end{aligned}$$

and hence,

$$\begin{aligned}
\sum_{1\le i\le 2}\sum_{\ell_0,\ell_1} J_{5_2}(V_{(1),i}) &= \sum_{1\le i\le 2} n(n-3)\, J_{5_2}(V_i^{3/2,-1,2})\, J_{5_2}(V_1^{1/2,1,2}) \\
&\equiv 2n(n-3)\,(-q^{3/2})\cdot q^{1/2} \equiv -2n(n-3)\,q^2, \\
\sum_{1\le i\le 5}\sum_{\ell_0,\ell_1} J_{5_2}(V_{(2),i}) &= \sum_{1\le i\le 5} n(n-4)\, J_{5_2}(V_i^{3/2,-1,3})\, J_{5_2}(V_1^{1/2,1,2}) \\
&\equiv 5n(n-4)\,q^{3/2}\cdot q^{1/2} \equiv 5n(n-4)\,q^2, \\
\sum_{1\le i\le 7}\sum_{\ell_0,\ell_1} J_{5_2}(V_{(3),i}) &= \sum_{1\le i\le 7} n(n-5)\, J_{5_2}(V_i^{3/2,-1,4})\, J_{5_2}(V_1^{1/2,1,2}) \\
&\equiv 7n(n-5)\,q^{3/2}\cdot q^{1/2} \equiv 7n(n-5)\,q^2.
\end{aligned}$$

Their sum is given by

$$n(10n-49)\,q^2. \tag{51}$$

Case $\mathbf{(2,0)=(1,0)+(\frac{1}{2},1)+(\frac{1}{2},-1)}$ In this case, we have the following sequences,

$$\begin{aligned}
V_{(1)} &= V_{\text{const}}^{\ell_0}\cdot V_1^{1,0,1}\cdot V_{\text{const}}^{\ell_1}\cdot V_1^{1/2,1,2}\cdot V_{\text{const}}^{\ell_2}\cdot V_1^{1/2,-1,2}\cdot V_{\text{const}}^{\ell_3}, \\
V_{(2)} &= V_{\text{const}}^{\ell_0}\cdot V_1^{1,0,2}\cdot V_{\text{const}}^{\ell_1}\cdot V_1^{1/2,1,2}\cdot V_{\text{const}}^{\ell_2}\cdot V_1^{1/2,-1,2}\cdot V_{\text{const}}^{\ell_3}, \\
V_{(3),i} &= V_{\text{const}}^{\ell_0}\cdot V_i^{1,0,3}\cdot V_{\text{const}}^{\ell_1}\cdot V_1^{1/2,1,2}\cdot V_{\text{const}}^{\ell_2}\cdot V_1^{1/2,-1,2}\cdot V_{\text{const}}^{\ell_3},
\end{aligned}$$

and hence,

$$\begin{aligned}
\sum_{\ell_0,\ell_1,\ell_2} J_{5_2}(V_{(1)}) &= n(n-3)(n-4)\, J_{5_2}(V_1^{1,0,1})\, J_{5_2}(V_1^{1/2,1,2})\, J_{5_2}(V_1^{1/2,-1,2}) \\
&\equiv n(n-3)(n-4)\,(-q)\cdot q^{1/2}\cdot q^{1/2} \equiv -n(n-3)(n-4)\,q^2, \\
\sum_{\ell_0,\ell_1,\ell_2} J_{5_2}(V_{(2)}) &= n(n-4)(n-5)\, J_{5_2}(V_1^{1,0,2})\, J_{5_2}(V_1^{1/2,1,2})\, J_{5_2}(V_1^{1/2,-1,2}) \\
&\equiv n(n-4)(n-5)\,q\cdot q^{1/2}\cdot q^{1/2} \equiv n(n-4)(n-5)\,q^2, \\
\sum_{1\le i\le 4}\sum_{\ell_0,\ell_1,\ell_2} J_{5_2}(V_{(3),i}) &= \sum_{1\le i\le 4} n(n-5)(n-6)\, J_{5_2}(V_i^{1,0,3})\, J_{5_2}(V_1^{1/2,1,2})\, J_{5_2}(V_1^{1/2,-1,2}) \\
&\equiv 4n(n-5)(n-6)\,q\cdot q^{1/2}\cdot q^{1/2} \equiv 4n(n-5)(n-6)\,q^2.
\end{aligned}$$

Their sum is given by

$$n(4n^2-46n+128). \tag{52}$$

Case $\mathbf{(2,0)=(1,2)+(\frac{1}{2},-1)+(\frac{1}{2},-1)}$ In this case, we have the following sequences,

$$\begin{aligned}
V_{(1)} &= V_{\text{const}}^{\ell_0}\cdot V_1^{1,2,2}\cdot V_{\text{const}}^{\ell_1}\cdot V_1^{1/2,-1,2}\cdot V_{\text{const}}^{\ell_2}\cdot V_1^{1/2,-1,2}\cdot V_{\text{const}}^{\ell_3}, \\
V_{(2)} &= V_{\text{const}}^{\ell_0}\cdot V_1^{1,2,3}\cdot V_{\text{const}}^{\ell_1}\cdot V_1^{1/2,-1,2}\cdot V_{\text{const}}^{\ell_2}\cdot V_1^{1/2,-1,2}\cdot V_{\text{const}}^{\ell_3},
\end{aligned}$$

and hence,

$$\begin{aligned}
\sum_{\ell_0,\ell_1,\ell_2} J_{5_2}(V_{(1)}) &= \frac{1}{2}\,n(n-4)(n-5)\,J_{5_2}(V_1^{1,2,2})\,J_{5_2}(V_1^{1/2,-1,2})^2 \\
&\equiv \frac{1}{2}\,n(n-4)(n-5)\,q\cdot(q^{1/2})^2 \;\equiv\; \frac{1}{2}\,n(n-4)(n-5)\,q^2, \\
\sum_{\ell_0,\ell_1,\ell_2} J_{5_2}(V_{(2)}) &= \frac{1}{2}\,n(n-5)(n-6)\,J_{5_2}(V_1^{1,2,3})\,J_{5_2}(V_1^{1/2,-1,2})^2 \\
&\equiv \frac{1}{2}\,n(n-5)(n-6)\,q\cdot(q^{1/2})^2 \;\equiv\; \frac{1}{2}\,n(n-5)(n-6)\,q^2.
\end{aligned}$$

Their sum is given by

$$n(n-5)^2. \tag{53}$$

Case $(\mathbf{2},\mathbf{0}) = (\mathbf{1},-\mathbf{2}) + (\frac{\mathbf{1}}{\mathbf{2}},\mathbf{1}) + (\frac{\mathbf{1}}{\mathbf{2}},\mathbf{1})$ In this case, we have the following sequences,

$$\begin{aligned}
V_{(1)} &= V_{\mathrm{const}}^{\ell_0}\cdot V_1^{1,-2,2}\cdot V_{\mathrm{const}}^{\ell_1}\cdot V_1^{1/2,1,2}\cdot V_{\mathrm{const}}^{\ell_2}\cdot V_1^{1/2,1,2}\cdot V_{\mathrm{const}}^{\ell_3}\,, \\
V_{(2)} &= V_{\mathrm{const}}^{\ell_0}\cdot V_1^{1,-2,3}\cdot V_{\mathrm{const}}^{\ell_1}\cdot V_1^{1/2,1,2}\cdot V_{\mathrm{const}}^{\ell_2}\cdot V_1^{1/2,1,2}\cdot V_{\mathrm{const}}^{\ell_3}\,,
\end{aligned}$$

and hence,

$$\begin{aligned}
\sum_{\ell_0,\ell_1,\ell_2} J_{5_2}(V_{(1)}) &= \frac{1}{2}\,n(n-4)(n-5)\,J_{5_2}(V_1^{1,-2,2})\,J_{5_2}(V_1^{1/2,1,2})^2 \\
&\equiv \frac{1}{2}\,n(n-4)(n-5)\,q\cdot(q^{1/2})^2 \;\equiv\; \frac{1}{2}\,n(n-4)(n-5)\,q^2, \\
\sum_{\ell_0,\ell_1,\ell_2} J_{5_2}(V_{(2)}) &= \frac{1}{2}\,n(n-5)(n-6)\,J_{5_2}(V_1^{1,-2,3})\,J_{5_2}(V_1^{1/2,1,2})^2 \\
&\equiv \frac{1}{2}\,n(n-5)(n-6)\,q\cdot(q^{1/2})^2 \;\equiv\; \frac{1}{2}\,n(n-5)(n-6)\,q^2.
\end{aligned}$$

Their sum is given by

$$n(n-5)^2. \tag{54}$$

Case $(\mathbf{2},\mathbf{0}) = (\frac{\mathbf{1}}{\mathbf{2}},\mathbf{1}) + (\frac{\mathbf{1}}{\mathbf{2}},\mathbf{1}) + (\frac{\mathbf{1}}{\mathbf{2}},-\mathbf{1}) + (\frac{\mathbf{1}}{\mathbf{2}},-\mathbf{1})$ In this case, we have the following sequence,

$$V = V_{\mathrm{const}}^{\ell_0}\cdot V_1^{1/2,1,2}\cdot V_{\mathrm{const}}^{\ell_1}\cdot V_1^{1/2,1,2}\cdot V_{\mathrm{const}}^{\ell_2}\cdot V_1^{1/2,1,2}\cdot V_{\mathrm{const}}^{\ell_3}\cdot V_1^{1/2,-1,2}\cdot V_{\mathrm{const}}^{\ell_3}\cdot V_1^{1/2,-1,2}\cdot V_{\mathrm{const}}^{\ell_4}\,,$$

and hence,

$$\begin{aligned}
\sum_{\ell_0,\ell_1,\ell_2,\ell_3} J_{5_2}(V) &= \frac{1}{4}\,n(n-5)(n-6)(n-7)\,J_{5_2}(V_1^{1/2,1,2})^2\,J_{5_2}(V_1^{1/2,-1,2})^2 \\
&\equiv \frac{1}{4}\,n(n-5)(n-6)(n-7)\,(q^{1/2})^2\cdot(q^{1/2})^2 \\
&\equiv \frac{1}{4}\,n(n-5)(n-6)(n-7)\,q^2. \qquad (55)
\end{aligned}$$

The sum of (46), (47), (48), $\cdots$, (55) is given by

$$1 + n(n-2)\,q + \frac{1}{4}\,n\,(n^3 - 6n^2 + n + 36)\,q^2.$$

This is the degree ≤ 2 part of $I\bigl(M_n(5_2)\bigr)$ for a sufficiently large n. Therefore, we obtain the theorem. $\square$

5.3 Proof of Theorem 3.4 for the 6_1 knot

In this section, we give a proof of Theorem 3.4. In fact, we prove (18) and (19) for $n \ge 6$, noting that (18) and (19) are verified for $n < 6$ by concrete computational results shown after Theorem 3.3, *i.e.*, we calculate the degree ≤ 2 part of $I\bigl(M_n(6_1)\bigr)$ for $n \ge 6$.

As we mention in Section 2, we put

$$\begin{aligned}
\hat{J}_{6_1,1}(\ell_1,\ell_2,\ell_3) &= q^{\alpha_1\ell_1+\beta_1\ell_2+\gamma_1\ell_3} I(\ell_1,\ell_2,\ell_3),\\
\hat{J}_{6_1,2}(\ell_1,\ell_2,\ell_3) &= q^{\alpha_2\ell_1+\beta_2\ell_2+\gamma_2\ell_3} I(\ell_1,\ell_2,\ell_3),\\
\hat{J}_{6_1,3}(\ell_1,\ell_2,\ell_3) &= q^{\alpha_3\ell_1+\beta_3\ell_2+\gamma_3\ell_3} I(\ell_1,\ell_2,\ell_3),
\end{aligned}$$

where

$$\begin{aligned}
&\alpha_1 = 0.166, \quad \beta_1 = 0.24, \quad \gamma_1 = 0.094,\\
&\alpha_2 = 0.224, \quad \beta_2 = 0.146, \quad \gamma_2 = 0.13,\\
&\alpha_3 = 0.074, \quad \beta_3 = 0.036, \quad \gamma_3 = 0.39,
\end{aligned}$$

and these values are obtained in Section B.3.

Proof of Theorem 3.4. As mentioned above, it is sufficient to calculate the degree ≤ 2 part of $I\bigl(M_n(6_1)\bigr)$ for $n \ge 6$. By (86), $I\bigl(M_n(6_1)\bigr)$ is presented by

$$\begin{aligned}
I\bigl(M_n(6_1)\bigr) \;=\; \sum\; & \hat{J}_{6_1,3}(a_0{+}d_0, a_1{+}b_0, d_0{+}c_0)\,\hat{J}_{6_1,1}(d_0{+}b_0, a_1{+}c_0, a_0{+}b_1)\\
&\times \hat{J}_{6_1,2}(a_1{+}d_0, d_0{+}b_1, b_0{+}c_0)\,\hat{J}_{6_1,1}(2a_1, b_0{+}c_1, 2b_1)\\
&\times \hat{J}_{6_1,3}(a_1{+}d_1, a_2{+}b_1, d_1{+}c_1)\,\hat{J}_{6_1,1}(d_1{+}b_1, a_2{+}c_1, a_1{+}b_2)\\
&\times \hat{J}_{6_1,2}(a_2{+}d_1, d_1{+}b_2, b_1{+}c_1)\,\hat{J}_{6_1,1}(2a_2, b_1{+}c_2, 2b_2)\\
&\times \cdots\\
&\times \hat{J}_{6_1,3}(a_{n-1}{+}d_{n-1}, a_n{+}b_{n-1}, d_{n-1}{+}c_{n-1})\,\hat{J}_{6_1,1}(d_{n-1}{+}b_{n-1}, a_n{+}c_{n-1}, a_{n-1}{+}b_n)\\
&\times \hat{J}_{6_1,2}(a_n{+}d_{n-1}, d_{n-1}{+}b_n, b_{n-1}{+}c_{n-1})\,\hat{J}_{6_1,1}(2a_n, b_{n-1}{+}c_n, 2b_n),
\end{aligned}$$

where we regard the subscripts of a_i, b_i, c_i, d_i as modulo n.

We consider a sequence of the form

$$W \;=\; (a,a,a,d_0,\; a_1,b_1,c_1,d_1,\; \cdots,\; a_{\ell-1},b_{\ell-1},c_{\ell-1},d_{\ell-1},\; a',a',a'). \tag{56}$$

We define the *length* of W to be ℓ, and define the *height* of W to be $a'-a$. We define $J_{6_1}(W)$ by

$$\begin{aligned}
J_{6_1}(W) \;=\; & \hat{J}_{6_1,3}(a{+}d_0, a_1{+}a, d_0{+}a)\, \hat{J}_{6_1,1}(d_0{+}a, a_1{+}a, a{+}b_1) \\
& \times \hat{J}_{6_1,2}(a_1{+}d_0, d_0{+}b_1, a{+}a)\, \hat{J}_{6_1,1}(2a_1, a{+}c_1, 2b_1) \\
& \times \hat{J}_{6_1,3}(a_1{+}d_1, a_2{+}b_1, d_1{+}c_1)\, \hat{J}_{6_1,1}(d_1{+}b_1, a_2{+}c_1, a_1{+}b_2) \\
& \times \hat{J}_{6_1,2}(a_2{+}d_1, d_1{+}b_2, b_1{+}c_1)\, \hat{J}_{6_1,1}(2a_2, b_1{+}c_2, 2b_2) \\
& \times \cdots \\
& \times \hat{J}_{6_1,3}(a_{\ell-1}{+}d_{\ell-1}, a'{+}b_{\ell-1}, d_{\ell-1}{+}c_{\ell-1})\, \hat{J}_{6_1,1}(d_{\ell-1}{+}b_{\ell-1}, a'{+}c_{\ell-1}, a_{\ell-1}{+}a') \\
& \times \hat{J}_{6_1,2}(a'{+}d_{\ell-1}, d_{\ell-1}{+}a', b_{\ell-1}{+}c_{\ell-1})\, \hat{J}_{6_1,1}(2a', b_{\ell-1}{+}a', 2a'),
\end{aligned}$$

and we define the *degree* of W to be the lowest degree of $J_{6_1}(W)$, which is a non-negative half integer. We note that, for any fixed d, there are a finite number of such sequences W of degree d; we show a classification of such sequences of degree 1, 2 in Section A.3.

For another sequence of the form (56)

$$W' \;=\; (a', a', a', d'_0,\; a'_1, b'_1, c'_1, d'_1,\; \cdots,\; a'_{\ell'-1}, b'_{\ell'-1}, c'_{\ell'-1}, d'_{\ell'-1},\; a'', a'', a''),$$

we define the union of W and W' by

$$\begin{aligned}
W\cdot W' \;=\; & (a, a, a, d_0,\; a_1, b_1, c_1, d_1,\; \cdots,\; a_{\ell-1}, b_{\ell-1}, c_{\ell-1}, d_{\ell-1}, \\
& a', a', a', d'_0,\; a'_1, b'_1, c'_1, d'_1,\; \cdots,\; a'_{\ell'-1}, b'_{\ell'-1}, c'_{\ell'-1}, d'_{\ell'-1},\; a'', a'', a'').
\end{aligned}$$

We note that the degree of $W\cdot W'$ is equal to the sum of degrees of W and W'. Further, we consider a constant sequence

$$W^{\ell}_{\mathrm{const}} \;=\; (\underbrace{a, a, a, a,\; \cdots, a, a, a, a}_{4\ell},\; a, a, a),$$

noting that $J_{6_1}(W^{\ell}_{\mathrm{const}}) = I(0,0,0)^{4\ell}$, and its degree is 0.

We calculate the degree ≤ 2 part of $I\big(M_n(6_1)\big)$. For a sufficiently large n, there are contributions to $I\big(M_n(6_1)\big)$ only from a union of finite number of sequences of the form (56) and constant sequences,

$$W^{\ell_0}_{\mathrm{const}}\cdot W_1\cdot W^{\ell_1}_{\mathrm{const}}\cdot W_2\cdot\;\cdots\;\cdot W_m\cdot QW^{\ell_m}_{\mathrm{const}}.$$

It is sufficient to consider the following cases,

$$\begin{aligned}
(1,0) \;&=\; \sum_i \big(\text{degree of } W_i,\;\text{height of } W_i\big), \\
(2,0) \;&=\; \sum_i \big(\text{degree of } W_i,\;\text{height of } W_i\big).
\end{aligned}$$

It follows from the classification in Section A.3 that the range of $(d,h) = (\text{degree},\;\text{height})$ satisfies that

$$\big|\,h\,\big| \;\le\; 2d, \qquad \big|\,h\,\big| \;\le\; 4-2d. \tag{57}$$

Hence, concrete values of $(d,h) = (\text{degree}, \text{ height})$ are

$$(d,h) \;=\; (\frac{1}{2}, \pm 1),\ (1,0),\ (1,\pm 2),\ (\frac{3}{2},\pm 1).$$

Therefore, as concrete sums of $(d,h) = (\text{degree}, \text{ height})$, it is sufficient to consider the following cases,

$$\begin{aligned}
&(0,0),\\
&(1,0),\\
&(1,0) = (\frac{1}{2},1) + (\frac{1}{2},-1),\\
&(2,0),\\
&(2,0) = (1,0) + (1,0),\\
&(2,0) = (1,2) + (1,-2),\\
&(2,0) = (\frac{3}{2},1) + (\frac{1}{2},-1),\\
&(2,0) = (\frac{3}{2},-1) + (\frac{1}{2},1),\\
&(2,0) = (1,0) + (\frac{1}{2},1) + (\frac{1}{2},-1),\\
&(2,0) = (1,2) + (\frac{1}{2},-1) + (\frac{1}{2},-1),\\
&(2,0) = (1,-2) + (\frac{1}{2},1) + (\frac{1}{2},1),\\
&(2,0) = (\frac{1}{2},1) + (\frac{1}{2},1) + (\frac{1}{2},-1) + (\frac{1}{2},-1).
\end{aligned}$$

We consider a sequence

$$W \;=\; (a_0, b_0, c_0, d_0, \cdots, a_{n-1}, b_{n-1}, c_{n-1}, d_{n-1}),$$

where we regard the subscripts of a_i, b_i, c_i, d_i as modulo n. In this proof, we write $f \equiv g$ if $f = g + O(q^3)$.

Case $(\mathbf{0},\mathbf{0})$ In this case, we have the following sequence,

$$W \;=\; W_{\text{const}}^{n}\,,$$

and hence,

$$J_{6_1}(W) \;=\; I(0,0,0)^{4n} \;\equiv\; (1-q-2q^2)^{4n} \;\equiv\; 1 - 4n\,q + \frac{1}{2}\cdot 4n(4n-5)\,q^2. \tag{58}$$

Case $(\mathbf{1},\mathbf{0})$ In this case, we have the following sequences,

$$\begin{aligned}
W_{(1)} &= W_{\text{const}}^{\ell_0}\cdot W_1^{1,0,1}\cdot W_{\text{const}}^{\ell_1}\,,\\
W_{(2)} &= W_{\text{const}}^{\ell_0}\cdot W_1^{1,0,2}\cdot W_{\text{const}}^{\ell_1}\,,\\
W_{(3)} &= W_{\text{const}}^{\ell_0}\cdot W_2^{1,0,2}\cdot W_{\text{const}}^{\ell_1}\,,\\
W_{(4)} &= W_{\text{const}}^{\ell_0}\cdot W_3^{1,0,2}\cdot W_{\text{const}}^{\ell_1}\,,\\
W_{(5)} &= W_{\text{const}}^{\ell_0}\cdot W_4^{1,0,2}\cdot W_{\text{const}}^{\ell_1}\,,\\
W_{(6)} &= W_{\text{const}}^{\ell_0}\cdot W_5^{1,0,2}\cdot W_{\text{const}}^{\ell_1}\,,
\end{aligned}$$

and hence,

$$\begin{aligned}
\sum_{\ell_0} J_{6_1}(W_{(1)}) &= n\, J_{6_1}(W_1^{1,0,1})\, I(0,0,0)^{4(n-1)} \\
&\equiv n\,(-q)(1-q)^{4n-4} \;\equiv\; -n\,q + n\,(4n-4)\,q^2, \\
\sum_{\ell_0} J_{6_1}(W_{(2)}) &= n\, J_{6_1}(W_1^{1,0,2})\, I(0,0,0)^{4(n-2)} \\
&\equiv n\,(q-4q^2)(1-q)^{4n-8} \;\equiv\; n\,q - n\,(4n-4)\,q^2, \\
\sum_{\ell_0} J_{6_1}(W_{(3)}) &= n\, J_{6_1}(W_2^{1,0,2})\, I(0,0,0)^{4(n-2)} \\
&\equiv n\,(q-2q^2)(1-q)^{4n-8} \;\equiv\; n\,q - n\,(4n-6)\,q^2, \\
\sum_{\ell_0} J_{6_1}(W_{(4)}) &= n\, J_{6_1}(W_3^{1,0,2})\, I(0,0,0)^{4(n-2)} \\
&\equiv n\,(q-q^2)(1-q)^{4n-8} \;\equiv\; n\,q - n\,(4n-7)\,q^2, \\
\sum_{\ell_0} J_{6_1}(W_{(5)}) &= n\, J_{6_1}(W_4^{1,0,2})\, I(0,0,0)^{4(n-2)} \\
&\equiv n\,(q-3q^2)(1-q)^{4n-8} \;\equiv\; n\,q - n\,(4n-5)\,q^2, \\
\sum_{\ell_0} J_{6_1}(W_{(6)}) &= n\, J_{6_1}(W_5^{1,0,2})\, I(0,0,0)^{4(n-2)} \\
&\equiv n\,(q-2q^2)(1-q)^{4n-8} \;\equiv\; n\,q - n\,(4n-6)\,q^2.
\end{aligned}$$

Their sum is given by

$$4n\,q \;-n\,(16n-24)\,q^2. \tag{59}$$

Case $(\mathbf{1},\mathbf{0}) = (\tfrac{\mathbf{1}}{\mathbf{2}},\mathbf{1}) + (\tfrac{\mathbf{1}}{\mathbf{2}},-\mathbf{1})$ In this case, we have the following sequence,

$$W \;=\; W_{\rm const}^{\ell_0}\cdot W_1^{1/2,1,2}\cdot W_{\rm const}^{\ell_1}\cdot W_1^{1/2,-1,1}\cdot W_{\rm const}^{\ell_2}\,,$$

and hence,

$$\begin{aligned}
\sum_{\ell_0,\ell_1} J_{6_1}(W) &= n(n-2)\, J_{6_1}(W_1^{1/2,1,2})\, J_{6_1}(W_1^{1/2,-1,1})\, I(0,0,0)^{4(n-3)} \\
&\equiv n(n-2)\,(q^{1/2}-4q^{3/2})\,(q^{1/2}-2q^{3/2})\,(1-q)^{4n-12} \\
&\equiv n(n-2)\,q(1-6q)\big(1-(4n-12)q\big) \\
&\equiv n(n-2)\,q \;-n(n-2)(4n-6)\,q^2.
\end{aligned} \tag{60}$$

The sum of (58), (59) and (60) is given by

$$1\;+n(n-2)\,q\;+n(-4n^2+6n+2)\,q^2. \tag{61}$$

whose degree ≤ 1 part gives the degree ≤ 1 part of the required formula. Hence, $c_1^{(n)}(5_2) = n(n-2)$ for any $n \geq 4$. Therefore, $p_1^{5_2}(n) = n(n-2)$.

In the remaining cases, we calculate the coefficient of q^2.

Case $(\mathbf{2}, \mathbf{0})$ In this case, we have the following sequences,

$$\begin{aligned} W_{(1)} &= W_{\mathrm{const}}^{\ell_0} \cdot W_1^{2,0,1} \cdot W_{\mathrm{const}}^{\ell_1}, \\ W_{(2),i} &= W_{\mathrm{const}}^{\ell_0} \cdot W_i^{2,0,2} \cdot W_{\mathrm{const}}^{\ell_1}, \\ W_{(3),i} &= W_{\mathrm{const}}^{\ell_0} \cdot W_i^{2,0,3} \cdot W_{\mathrm{const}}^{\ell_1}, \end{aligned}$$

and hence,

$$\begin{aligned} \sum_{\ell_0} J_{6_1}(W_{(1)}) &= n\, J_{6_1}(W_1^{2,0,1}) \equiv n\, q^2, \\ \sum_{1\leq i\leq 10} \sum_{\ell_0} J_{6_1}(W_{(2),i}) &= \sum_{1\leq i\leq 10} n\, J_{6_1}(W_i^{2,0,2}) \equiv n(7q^2 - 3q^2) \equiv 4n\, q^2, \\ \sum_{1\leq i\leq 17} \sum_{\ell_0} J_{6_1}(W_{(3),i}) &= \sum_{1\leq i\leq 17} n\, J_{6_1}(W_i^{2,0,3}) \equiv 17n\, q^2. \end{aligned}$$

Their sum is given by

$$22\, n\, q^2. \tag{62}$$

Case $(\mathbf{2}, \mathbf{0}) = (\mathbf{1}, \mathbf{0}) + (\mathbf{1}, \mathbf{0})$ In this case, we have the following sequences,

$$\begin{aligned} W_{(1)} &= W_{\mathrm{const}}^{\ell_0} \cdot W_1^{1,0,1} \cdot W_{\mathrm{const}}^{\ell_1} \cdot W_1^{1,0,1} \cdot W_{\mathrm{const}}^{\ell_2}, \\ W_{(2),i} &= W_{\mathrm{const}}^{\ell_0} \cdot W_1^{1,0,1} \cdot W_{\mathrm{const}}^{\ell_1} \cdot W_i^{1,0,2} \cdot W_{\mathrm{const}}^{\ell_2}, \\ W_{(3),i} &= W_{\mathrm{const}}^{\ell_0} \cdot W_i^{1,0,2} \cdot W_{\mathrm{const}}^{\ell_1} \cdot W_i^{1,0,2} \cdot W_{\mathrm{const}}^{\ell_2}, \\ W_{(4),i,j} &= W_{\mathrm{const}}^{\ell_0} \cdot W_i^{1,0,2} \cdot W_{\mathrm{const}}^{\ell_1} \cdot W_j^{1,0,2} \cdot W_{\mathrm{const}}^{\ell_2}, \end{aligned}$$

and hence,

$$\begin{aligned} \sum_{\ell_0,\ell_1} J_{6_1}(W_{(1)}) &= \frac{n(n-1)}{2} \cdot J_{6_1}(W_1^{1,0,1})^2 \\ &\equiv \frac{1}{2} n(n-1)\,(-q)^2 \equiv \frac{1}{2} n(n-1)\, q^2, \\ \sum_{1\leq i\leq 5} \sum_{\ell_0,\ell_1} J_{6_1}(W_{(2),i}) &= \sum_{1\leq i\leq 5} n(n-2)\, J_{6_1}(W_1^{1,0,1})\, J_{6_1}(W_i^{1,0,2}) \\ &\equiv 5n(n-2)\,(-q)\cdot q \equiv -5n(n-2)\, q^2, \\ \sum_{1\leq i\leq 5} \sum_{\ell_0,\ell_1} J_{6_1}(W_{(3),i}) &= \sum_{1\leq i\leq 5} \frac{n(n-3)}{2} J_{6_1}(W_i^{1,0,2})^2 \equiv \frac{5}{2} n(n-3)\, q^2, \\ \sum_{1\leq i<j\leq 5} \sum_{\ell_0,\ell_1} J_{6_1}(W_{(4),i,j}) &= \sum_{1\leq i<j\leq 5} n(n-3)\, J_{6_1}(W_i^{1,0,2})\, J_{6_1}(W_j^{1,0,2}) \equiv 10n(n-3)\, q^2. \end{aligned}$$

Their sum is given by

$$n\,(8n-28)\,q^2. \tag{63}$$

Case $(\mathbf{2},\mathbf{0})=(\mathbf{1},\mathbf{2})+(\mathbf{1},-\mathbf{2})$ In this case, we have the following sequences,

$$\begin{aligned} W_{(1)} &= W_{\text{const}}^{\ell_0}\cdot W_1^{1,2,2}\cdot W_{\text{const}}^{\ell_1}\cdot W_1^{1,-2,1}\cdot W_{\text{const}}^{\ell_2}\,,\\ W_{(2)} &= W_{\text{const}}^{\ell_0}\cdot W_1^{1,2,3}\cdot W_{\text{const}}^{\ell_1}\cdot W_1^{1,-2,1}\cdot W_{\text{const}}^{\ell_2}\,, \end{aligned}$$

and hence,

$$\begin{aligned} \sum_{\ell_0,\ell_1} J_{6_1}(W_{(1)}) &= n(n-2)\,J_{6_1}(W_1^{1,2,2})\,J_{6_1}(W_1^{1,-2,1}) \;\equiv\; n(n-2)\,q^2,\\ \sum_{\ell_0,\ell_1} J_{6_1}(W_{(2)}) &= n(n-3)\,J_{6_1}(W_1^{1,2,3})\,J_{6_1}(W_1^{1,-2,1}) \;\equiv\; n(n-3)\,q^2. \end{aligned}$$

Their sum is given by

$$n(2n-5)\,q^2. \tag{64}$$

Case $(\mathbf{2},\mathbf{0})=(\frac{\mathbf{3}}{\mathbf{2}},\mathbf{1})+(\frac{\mathbf{1}}{\mathbf{2}},-\mathbf{1})$ In this case, we have the following sequences,

$$\begin{aligned} W_{(1),i} &= W_{\text{const}}^{\ell_0}\cdot W_i^{3/2,1,2}\cdot W_{\text{const}}^{\ell_1}\cdot W_1^{1/2,-1,1}\cdot W_{\text{const}}^{\ell_2}\,,\\ W_{(2),i} &= W_{\text{const}}^{\ell_0}\cdot W_i^{3/2,1,3}\cdot W_{\text{const}}^{\ell_1}\cdot W_1^{1/2,-1,1}\cdot W_{\text{const}}^{\ell_2}\,, \end{aligned}$$

and hence,

$$\begin{aligned} \sum_{1\le i\le 5}\sum_{\ell_0,\ell_1} J_{6_1}(W_{(1),i}) &= \sum_{1\le i\le 5} n(n-2)\,J_{6_1}(W_i^{3/2,1,2})\,J_{6_1}(W_1^{1/2,-1,1})\\ &\equiv n(n-2)\,(3q^{3/2}-2q^{3/2})\cdot q^{1/2} \;\equiv\; n(n-2)\,q^2,\\ \sum_{1\le i\le 6}\sum_{\ell_0,\ell_1} J_{6_1}(W_{(2),i}) &= \sum_{1\le i\le 6} n(n-3)\,J_{6_1}(W_i^{3/2,1,3})\,J_{6_1}(W_1^{1/2,-1,1})\\ &\equiv 6n(n-3)\,q^{3/2}\cdot q^{1/2} \;\equiv\; 6n(n-3)\,q^2, \end{aligned}$$

Their sum is given by

$$n(7n-20)\,q^2. \tag{65}$$

Case $(\mathbf{2},\mathbf{0})=(\frac{\mathbf{3}}{\mathbf{2}},-\mathbf{1})+(\frac{\mathbf{1}}{\mathbf{2}},\mathbf{1})$ In this case, we have the following sequences,

$$\begin{aligned} W_{(1),i} &= W_{\text{const}}^{\ell_0}\cdot W_i^{3/2,-1,1}\cdot W_{\text{const}}^{\ell_1}\cdot W_1^{1/2,1,2}\cdot W_{\text{const}}^{\ell_2}\,,\\ W_{(2),i} &= W_{\text{const}}^{\ell_0}\cdot W_i^{3/2,-1,2}\cdot W_{\text{const}}^{\ell_1}\cdot W_1^{1/2,1,2}\cdot W_{\text{const}}^{\ell_2}\,, \end{aligned}$$

and hence,

$$
\begin{aligned}
\sum_{1\le i\le 2}\sum_{\ell_0,\ell_1} J_{6_1}(W_{(1),i}) &= \sum_{1\le i\le 2} n(n-2)\, J_{6_1}(W_i^{3/2,-1,1})\, J_{6_1}(W_1^{1/2,1,2}) \\
&\equiv 2n(n-2)\,(-q^{3/2})\cdot q^{1/2} \;\equiv\; -2n(n-2)\,q^2, \\
\sum_{1\le i\le 7}\sum_{\ell_0,\ell_1} J_{6_1}(W_{(2),i}) &= \sum_{1\le i\le 7} n(n-3)\, J_{6_1}(W_i^{3/2,-1,2})\, J_{6_1}(W_1^{1/2,1,2}) \\
&\equiv 7n(n-3)\,q^{3/2}\cdot q^{1/2} \;\equiv\; 7n(n-3)\,q^2.
\end{aligned}
$$

Their sum is given by

$$
n(5n-17)\,q^2. \tag{66}
$$

Case $(\mathbf{2},\mathbf{0}) = (\mathbf{1},\mathbf{0}) + (\frac{\mathbf{1}}{\mathbf{2}},\mathbf{1}) + (\frac{\mathbf{1}}{\mathbf{2}},-\mathbf{1})$ In this case, we have the following sequences,

$$
\begin{aligned}
W_{(1)} &= W_{\rm const}^{\ell_0}\cdot W_1^{1,0,1}\cdot W_{\rm const}^{\ell_1}\cdot W_1^{1/2,1,2}\cdot W_{\rm const}^{\ell_2}\cdot W_1^{1/2,-1,2}\cdot W_{\rm const}^{\ell_3}\,, \\
W_{(2),i} &= W_{\rm const}^{\ell_0}\cdot W_i^{1,0,2}\cdot W_{\rm const}^{\ell_1}\cdot W_1^{1/2,1,2}\cdot W_{\rm const}^{\ell_2}\cdot W_1^{1/2,-1,2}\cdot W_{\rm const}^{\ell_3}\,,
\end{aligned}
$$

and hence,

$$
\begin{aligned}
\sum_{\ell_0,\ell_1,\ell_2} J_{6_1}(W_{(1)}) &= n(n-2)(n-3)\, J_{6_1}(W_1^{1,0,1})\, J_{6_1}(W_1^{1/2,1,2})\, J_{6_1}(W_1^{1/2,-1,1}) \\
&\equiv n(n-2)(n-3)\,(-q)\cdot q^{1/2}\cdot q^{1/2} \\
&\equiv -n(n-2)(n-3)\,q^2, \\
\sum_{1\le i\le 5}\sum_{\ell_0,\ell_1,\ell_2} J_{6_1}(W_{(2),i}) &= \sum_{1\le i\le 5} n(n-3)(n-4)\, J_{6_1}(W_i^{1,0,2})\, J_{6_1}(W_1^{1/2,1,2})\, J_{6_1}(W_1^{1/2,-1,1}) \\
&\equiv 5n(n-3)(n-4)\,q\cdot q^{1/2}\cdot q^{1/2} \\
&\equiv 5n(n-3)(n-4)\,q^2.
\end{aligned}
$$

Their sum is given by

$$
n(n-3)(4n-18)\,q^2. \tag{67}
$$

Case $(\mathbf{2},\mathbf{0}) = (\mathbf{1},\mathbf{2}) + (\frac{\mathbf{1}}{\mathbf{2}},-\mathbf{1}) + (\frac{\mathbf{1}}{\mathbf{2}},-\mathbf{1})$ In this case, we have the following sequences,

$$
\begin{aligned}
W_{(1)} &= W_{\rm const}^{\ell_0}\cdot W_1^{1,2,2}\cdot W_{\rm const}^{\ell_1}\cdot W_1^{1/2,-1,2}\cdot W_{\rm const}^{\ell_2}\cdot W_1^{1/2,-1,2}\cdot W_{\rm const}^{\ell_3}\,, \\
W_{(2)} &= W_{\rm const}^{\ell_0}\cdot W_1^{1,2,3}\cdot W_{\rm const}^{\ell_1}\cdot W_1^{1/2,-1,2}\cdot W_{\rm const}^{\ell_2}\cdot W_1^{1/2,-1,2}\cdot W_{\rm const}^{\ell_3}\,,
\end{aligned}
$$

and hence,

$$\begin{aligned}
\sum_{\ell_0,\ell_1,\ell_2} J_{6_1}(W_{(1)}) &= \frac{1}{2}\,n(n-2)(n-3)\,J_{6_1}(W_1^{1,2,2})\,J_{6_1}(W_1^{1/2,-1,1})^2 \\
&\equiv \frac{1}{2}\,n(n-2)(n-3)\,q\cdot(q^{1/2})^2 \;\equiv\; \frac{1}{2}\,n(n-2)(n-3)\,q^2, \\
\sum_{\ell_0,\ell_1,\ell_2} J_{6_1}(W_{(2)}) &= \frac{1}{2}\,n(n-3)(n-4)\,J_{6_1}(W_1^{1,2,3})\,J_{6_1}(W_1^{1/2,-1,1})^2 \\
&\equiv \frac{1}{2}\,n(n-3)(n-4)\,q\cdot(q^{1/2})^2 \;\equiv\; \frac{1}{2}\,n(n-3)(n-4)\,q^2.
\end{aligned}$$

Their sum is given by

$$n(n-3)^2\,q^2. \tag{68}$$

Case $(\mathbf{2},\mathbf{0}) = (\mathbf{1},-\mathbf{2}) + (\frac{\mathbf{1}}{\mathbf{2}},\mathbf{1}) + (\frac{\mathbf{1}}{\mathbf{2}},\mathbf{1})$ In this case, we have the following sequence,

$$W \;=\; W_{\rm const}^{\ell_0}\cdot W_1^{1,-2,1}\cdot W_{\rm const}^{\ell_1}\cdot W_1^{1/2,1,2}\cdot W_{\rm const}^{\ell_2}\cdot W_1^{1/2,1,2}\cdot W_{\rm const}^{\ell_3}\,,$$

and hence,

$$\begin{aligned}
\sum_{\ell_0,\ell_1,\ell_2} J_{6_1}(W) &= \frac{1}{2}\,n(n-3)(n-4)\,J_{6_1}(W_1^{1,-2,1})\,J_{6_1}(W_1^{1/2,1,2})^2 \\
&\equiv \frac{1}{2}\,n(n-3)(n-4)\,q\cdot(q^{1/2})^2 \;\equiv\; \frac{1}{2}\,n(n-3)(n-4)\,q^2.
\end{aligned} \tag{69}$$

Case $(\mathbf{2},\mathbf{0}) = (\frac{\mathbf{1}}{\mathbf{2}},\mathbf{1}) + (\frac{\mathbf{1}}{\mathbf{2}},\mathbf{1}) + (\frac{\mathbf{1}}{\mathbf{2}},-\mathbf{1}) + (\frac{\mathbf{1}}{\mathbf{2}},-\mathbf{1})$ In this case, we have the following sequence,

$$W \;=\; W_{\rm const}^{\ell_0}\cdot W_1^{1/2,1,2}\cdot W_{\rm const}^{\ell_1}\cdot W_1^{1/2,1,2}\cdot W_{\rm const}^{\ell_2}\cdot W_1^{1/2,-1,2}\cdot W_{\rm const}^{\ell_3}\cdot W_1^{1/2,-1,2}\cdot W_{\rm const}^{\ell_3}\,,$$

and hence,

$$\begin{aligned}
\sum_{\ell_0,\ell_1,\ell_2,\ell_3} J_{6_1}(W) &= \frac{1}{4}\,n(n-3)(n-4)(n-5)\,J_{6_1}(W_1^{1/2,1,2})^2\,J_{6_1}(W_1^{1/2,-1,1})^2 \\
&\equiv \frac{1}{4}\,n(n-3)(n-4)(n-5)\,(q^{1/2})^2\cdot(q^{1/2})^2 \\
&\equiv \frac{1}{4}\,n(n-3)(n-4)(n-5)\,q^2.
\end{aligned} \tag{70}$$

The sum of (61), (62), (63), $\cdots$, (70) is given by

$$1\,+n(n-2)\,q\,+\frac{1}{4}\,n\,(n^3-6n^2+n+32)\,q^2.$$

This is the degree ≤ 2 part of $I\big(M_n(6_1)\big)$ for a sufficiently large n. Therefore, we obtain the theorem. $\square$

5.4 Convergence of the sum of the defining formula of the 3D-index

It is shown in [3] (see also [4]) that the infinite sum of the defining formula of the 3D-index converges. In this section, we review the proof of the convergence. We generalize the method of this section to the case of cyclic covers of a hyperbolic knot complement in the next section.

Let M be a hyperbolic 3-manifold with a cusp. We assume that there exists an ideal triangulation $\mathcal{T}$ which gives the hyperbolic structure of M, and $\mathcal{T}$ has m tetrahedra and m edges. We denote by a^j the label of the jth edge $\mathcal{E}^j$, and, as in (4), we denote labels of edges of the ith tetrahedron Δ^i by $a^{f(i)}$, $a^{f'(i)}$, $a^{g(i)}$, $a^{g'(i)}$, $a^{h(i)}$, $a^{h'(i)}$ in this section. We recall that the defining formula (11) of the 3D-index is given by

$$I\big(M\big) \;=\; \sum_{\substack{a^2,\cdots,a^m\in\mathbb{Z}\\ a^1=0}} \prod_i J_i\big(a^{f(i)}+a^{f'(i)}, a^{g(i)}+a^{g'(i)}, a^{h(i)}+a^{h'(i)}\big). \tag{71}$$

We review the Neumann-Zagier matrix, as follows. We put three $m\times m$ matrices by

$$\overline{\mathbf{A}} \;=\; \big(A_{ji}\big), \qquad A_{ji} \;=\; \begin{cases} 0 & \text{if } f(i)\neq j \text{ and } f'(i)\neq j,\\ 1 & \text{if } f(i)= j \text{ and } f'(i)\neq j,\\ 1 & \text{if } f(i)\neq j \text{ and } f'(i)= j,\\ 2 & \text{if } f(i)= j \text{ and } f'(i)= j,\end{cases}$$

$$\overline{\mathbf{B}} \;=\; \big(B_{ji}\big), \qquad B_{ji} \;=\; \begin{cases} 0 & \text{if } g(i)\neq j \text{ and } g'(i)\neq j,\\ 1 & \text{if } g(i)= j \text{ and } g'(i)\neq j,\\ 1 & \text{if } g(i)\neq j \text{ and } g'(i)= j,\\ 2 & \text{if } g(i)= j \text{ and } g'(i)= j,\end{cases}$$

$$\overline{\mathbf{C}} \;=\; \big(C_{ji}\big), \qquad C_{ji} \;=\; \begin{cases} 0 & \text{if } h(i)\neq j \text{ and } h'(i)\neq j,\\ 1 & \text{if } h(i)= j \text{ and } h'(i)\neq j,\\ 1 & \text{if } h(i)\neq j \text{ and } h'(i)= j,\\ 2 & \text{if } h(i)= j \text{ and } h'(i)= j.\end{cases}$$

Further, we put

$$\mathbf{A} \;=\; \overline{\mathbf{A}}-\overline{\mathbf{C}}, \qquad \mathbf{B} \;=\; \overline{\mathbf{B}}-\overline{\mathbf{C}}.$$

The *Neumann-Zagier matrix* is defined to be $\big(\mathbf{A}\big|\mathbf{B}\big)$. It is shown in [15, Theorem 2.2 – Proposition 2.5] and [14, Theorem 4.1] that the rank of the Neumann-Zagier matrix $\big(\mathbf{A}\big|\mathbf{B}\big)$ is $m-1$.

Example 5.1. For the 6_1 knot K_{6_1}, its complement $S^3-K_{6_1}$ is $M_1(6_1)$. Hence, by (86), its 3D-index is presented by

$$\begin{aligned} I\big(S^3-K_{6_1}\big) \;=\; \sum_{\substack{a=0\\ b,c,d\in\mathbb{Z}}} q^{a+b+c+d}\, I(a+d,\, a+b,\, d+c)\, I(b+d,\, a+c,\, a+b) \\ \times\, I(a+d,\, b+d,\, b+c)\, I(2a,\, b+c,\, 2b). \end{aligned} \tag{72}$$

We have that

$$\overline{\mathbf{A}} = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \qquad \overline{\mathbf{B}} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad \overline{\mathbf{C}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

These matrices are obtained in such a way that, for example, the first entries of four factors in the right-hand side of (72) is given by

$$\begin{pmatrix} a & b & c & d \end{pmatrix} \overline{\mathbf{A}} = (a{+}d,\, b{+}d,\, a{+}d,\, 2a).$$

Proposition 5.2 ([3], see also [4])**.** *The 3D-index $I(M)$ is well-defined, i.e., the infinite sum of the defining formula* (11) *of the 3D-index converges as a power series.*

Proof. We put

$$\begin{aligned} \mathbb{E} &= \big\{ (a^1, a^2, \cdots, a^m) \in \mathbb{Z}^m \big\} = \mathbb{Z}^m, \\ Q^i(\mathbb{Z}) &= \big\{ (\ell_1, \ell_2, \ell_3) \in \mathbb{Z}^3 \big\} = \mathbb{Z}^3. \end{aligned}$$

We define

$$\mathcal{F}^i : \mathbb{E} \longrightarrow Q^i(\mathbb{Z})$$

by

$$\mathcal{F}^i(a^1, a^2, \cdots, a^m) = \big(a^{f(i)} + a^{f'(i)},\, a^{g(i)} + a^{g'(i)},\, a^{h(i)} + a^{h'(i)} \big).$$

Further, we define

$$\mathcal{D}^i : Q^i(\mathbb{Z}) \longrightarrow \mathbb{R}_{\geq 0}$$

by

$$\begin{aligned} \mathcal{D}^i(\ell_1, \ell_2, \ell_3) &= \deg J_i(\ell_1, \ell_2, \ell_3) \\ &= \begin{cases} \frac{1}{2}(\ell_2{-}\ell_1)(\ell_3{-}\ell_1) + \beta_i(\ell_2{-}\ell_1) + \gamma_i(\ell_3{-}\ell_1) & \text{if } \ell_1 \leq \ell_2 \text{ and } \ell_1 \leq \ell_3, \\ \frac{1}{2}(\ell_1{-}\ell_2)(\ell_3{-}\ell_2) + \alpha_i(\ell_1{-}\ell_2) + \gamma_i(\ell_3{-}\ell_2) & \text{if } \ell_2 \leq \ell_1 \text{ and } \ell_2 \leq \ell_3, \\ \frac{1}{2}(\ell_1{-}\ell_3)(\ell_2{-}\ell_3) + \alpha_i(\ell_1{-}\ell_3) + \beta_i(\ell_2{-}\ell_3) & \text{if } \ell_3 \leq \ell_1 \text{ and } \ell_3 \leq \ell_2, \end{cases} \end{aligned} \tag{73}$$

where the second equality is obtained by (12).

We put $\mathbb{E}_{\mathbb{R}} = \mathbb{E} \otimes \mathbb{R} = \mathbb{R}^m$ and $Q^i(\mathbb{R}) = Q^i(\mathbb{Z}) \otimes \mathbb{R} = \mathbb{R}^3$, and we give standard metrics to them. The above maps $\mathcal{F}^i$ and $\mathcal{D}^i$ are naturally extended to the following maps, which we also denote by $\mathcal{F}^i$ and $\mathcal{D}^i$,

$$\mathbb{E}_{\mathbb{R}} \xrightarrow{\mathcal{F}^i} Q^i(\mathbb{R}) \xrightarrow{\mathcal{D}^i} \mathbb{R}_{\geq 0}. \tag{74}$$

By (73), $\mathcal{D}^i$ naturally induces $\hat{\mathcal{D}}^i : Q^i(\mathbb{R})/\mathbb{R} \to \mathbb{R}_{\geq 0}$, where the denominator $\mathbb{R}$ of $Q^i(\mathbb{R})/\mathbb{R}$ is $\mathrm{span}_{\mathbb{R}}\{(1,1,1)\} \subset Q^i(\mathbb{R})$. Further, $\mathcal{F}^i : \mathbb{E}_{\mathbb{R}} \to Q^i(\mathbb{R})/\mathbb{R}$ naturally induces $\hat{\mathcal{F}}^i : \mathbb{E}_{\mathbb{R}}/\mathbb{R} \to Q^i(\mathbb{R})/\mathbb{R}$, where the denominator $\mathbb{R}$ of $\mathbb{E}_{\mathbb{R}}/\mathbb{R}$ is $\mathrm{span}_{\mathbb{R}}\{(1,1,\cdots,1)\} \subset \mathbb{E}_{\mathbb{R}}$. Then, from (74), we obtain the following maps,

$$\mathbb{E}_{\mathbb{R}}/\mathbb{R} \xrightarrow{\hat{\mathcal{F}}^i} Q^i(\mathbb{R})/\mathbb{R} \xrightarrow{\hat{\mathcal{D}}^i} \mathbb{R}_{\geq 0}\,.$$

By making the direct sum of the middle vector space with respect to i, we obtain the following maps,

$$\mathbb{E}_{\mathbb{R}}/\mathbb{R} \xrightarrow{\hat{\mathcal{F}}} \bigoplus_i \big(Q^i(\mathbb{R})/\mathbb{R}\big) \xrightarrow{\hat{\mathcal{D}}} \mathbb{R}_{\geq 0}\,,$$

where we put $\hat{\mathcal{F}} = \oplus \hat{\mathcal{F}}^i$ and $\hat{\mathcal{D}} = \sum \hat{\mathcal{D}}^i$. We consider the dual map of $\hat{\mathcal{F}}$, as follows,

$$\hat{\mathcal{F}}^* : \bigoplus_i \big(Q^i(\mathbb{R})/\mathbb{R}\big)^* \longrightarrow \big(\mathbb{E}_{\mathbb{R}}/\mathbb{R}\big)^*.$$

This map is surjective by Lemma 5.3 below. Hence, the map $\hat{\mathcal{F}}$ is injective.

Let D be any positive integer. From the definition (73) of $\mathcal{D}^i$, we obtain that ${\hat{\mathcal{D}}^i}{}^{-1}\big([0,D]\big)$ is bounded. Hence, $\hat{\mathcal{D}}^{-1}\big([0,D]\big)$ is bounded. Therefore, $\hat{\mathcal{F}}^{-1}\hat{\mathcal{D}}^{-1}\big([0,D]\big)$ is bounded, since $\hat{\mathcal{F}}$ is injective. When we restrict the sum (71) to the part of degree $\leq D$, the sum can be regarded as a sum over integer points of $\hat{\mathcal{F}}^{-1}\hat{\mathcal{D}}^{-1}\big([0,D]\big)$. Hence, this restricted sum is a finite sum.

Therefore, we obtain the proposition. $\square$

Lemma 5.3. *The map $\hat{\mathcal{F}}^*$ is surjective.*

Proof. The matrix $\big(\overline{\mathbf{A}}\,\big|\,\overline{\mathbf{B}}\,\big|\,\overline{\mathbf{C}}\big)^T$ is a presentation matrix of the linear map,

$$\oplus \mathcal{F}^i : \mathbb{E}_{\mathbb{R}} \longrightarrow \bigoplus_i Q^i(\mathbb{R}).$$

Hence, the matrix $\big(\mathbf{A}\big|\mathbf{B}\big)^T$ is a presentation matrix of the linear map,

$$\oplus \hat{\mathcal{F}}^i : \mathbb{E}_{\mathbb{R}} \longrightarrow \bigoplus_i \big(Q^i(\mathbb{R})/\mathbb{R}\big).$$

Therefore, the matrix $\big(\mathbf{A}\big|\mathbf{B}\big)$ is a presentation matrix of the dual linear map,

$$\bigoplus_i \big(Q^i(\mathbb{R})/\mathbb{R}\big)^* \longrightarrow \mathbb{E}_{\mathbb{R}}^*\,.$$

As mentioned before, the rank of the Neumann-Zagier matrix $\big(\mathbf{A}\big|\mathbf{B}\big)$ is $m-1$. Hence, the image of the above linear map is the $(m-1)$-dimensional subspace $(\mathbb{E}_{\mathbb{R}}/\mathbb{R})^*$ of $\mathbb{E}_{\mathbb{R}}^*$,

$$\bigoplus_i \big(Q^i(\mathbb{R})/\mathbb{R}\big)^* \xrightarrow{\hat{\mathcal{F}}^*} \big(\mathbb{E}_{\mathbb{R}}/\mathbb{R}\big)^* \subset \mathbb{E}_{\mathbb{R}}^*\,.$$

Therefore, $\hat{\mathcal{F}}^*$ is surjective, as required. $\square$

5.5 Proof of Theorem 3.1

In this section, we give a proof of Theorem 3.1. We generalize the method of the previous section to the case of cyclic covers of a hyperbolic knot complement. This proof is also a generalization of the proofs in Sections 5.1, 5.2, 5.3.

As in Section 2, let K be a hyperbolic knot. We assume that there exists an ideal triangulation $\mathcal{T}$ which gives the hyperbolic structure of the complement of K, and $\mathcal{T}$ has m tetrahedra and m edges. We denote by a^j the label of the jth edge $\mathcal{E}^j$, and, as in (4), we denote labels of edges of the ith tetrahedron Δ^i by $a^{f(i)}$, $a^{f'(i)}$, $a^{g(i)}$, $a^{g'(i)}$, $a^{h(i)}$, $a^{h'(i)}$ in this section.

We consider the n-fold cyclic cover $\hat{\mathcal{T}}$ of $\mathcal{T}$, which is a triangulation of the n-fold cyclic cover $M_n(K)$ of the complement of K. We denote by Δ^i_k and $\mathcal{E}^j_k$ (for $0 \le k < n$) lifts of Δ^i and $\mathcal{E}^j$ in $\hat{\mathcal{T}}$ such that the deck transformation takes Δ^i_k and $\mathcal{E}^j_k$ to Δ^i_{k+1} and $\mathcal{E}^j_{k+1}$. Then, the labels of edges of Δ^i_k are given by $a^{f(i)}_{k+\varepsilon_{i,1}}$, $a^{f'(i)}_{k+\varepsilon'_{i,1}}$, $a^{g(i)}_{k+\varepsilon_{i,2}}$, $a^{g'(i)}_{k+\varepsilon'_{i,2}}$, $a^{h(i)}_{k+\varepsilon_{i,3}}$, $a^{h'(i)}_{k+\varepsilon'_{i,3}}$ for some $\varepsilon_{i,1}$, $\varepsilon'_{i,1}$, $\varepsilon_{i,2}$, $\varepsilon'_{i,2}$, $\varepsilon_{i,3}$, $\varepsilon'_{i,3}$ which are constants independent of k. Without loss of generality, we assume that $\min\bigl\{\varepsilon_{i,1}, \varepsilon'_{i,1}, \varepsilon_{i,2}, \varepsilon'_{i,2}, \varepsilon_{i,3}, \varepsilon'_{i,3} \bigm| 1 \le i \le m\bigr\} = 0$. The 3D-index of $M_n(K)$ is given by

$$I\bigl(M_n(K)\bigr) \;=\; \sum \prod_k \prod_i \; J_i\bigl(a^{f(i)}_{k+\varepsilon_{i,1}} + a^{f'(i)}_{k+\varepsilon'_{i,1}},\, a^{g(i)}_{k+\varepsilon_{i,2}} + a^{g'(i)}_{k+\varepsilon'_{i,2}},\, a^{h(i)}_{k+\varepsilon_{i,3}} + a^{h'(i)}_{k+\varepsilon'_{i,3}}\bigr), \tag{75}$$

where the sum is taken over

$$a^1_0 = 0, \qquad a^2_0, \cdots, a^m_0 \in \mathbb{Z}, \quad a^1_1, \cdots, a^m_1 \in \mathbb{Z}, \quad \cdots, \quad a^1_{n-1}, \cdots, a^m_{n-1} \in \mathbb{Z}.$$

We put

$$\overline{\mathbf{A}}_k \;=\; \bigl(A_{ji,k}\bigr), \quad A_{ji,k} \;=\; \begin{cases} 0 & \text{if } \bigl(f(i), \varepsilon_{i,1}\bigr) \ne (j,k) \text{ and } \bigl(f'(i), \varepsilon'_{i,1}\bigr) \ne (j,k), \\ 1 & \text{if } \bigl(f(i), \varepsilon_{i,1}\bigr) = (j,k) \text{ and } \bigl(f'(i), \varepsilon'_{i,1}\bigr) \ne (j,k), \\ 1 & \text{if } \bigl(f(i), \varepsilon_{i,1}\bigr) \ne (j,k) \text{ and } \bigl(f'(i), \varepsilon'_{i,1}\bigr) = (j,k), \\ 2 & \text{if } \bigl(f(i), \varepsilon_{i,1}\bigr) = (j,k) \text{ and } \bigl(f'(i), \varepsilon'_{i,1}\bigr) = (j,k), \end{cases}$$

$$\overline{\mathbf{B}}_k \;=\; \bigl(B_{ji,k}\bigr), \quad B_{ji,k} \;=\; \begin{cases} 0 & \text{if } \bigl(g(i), \varepsilon_{i,2}\bigr) \ne (j,k) \text{ and } \bigl(g'(i), \varepsilon'_{i,2}\bigr) \ne (j,k), \\ 1 & \text{if } \bigl(g(i), \varepsilon_{i,2}\bigr) = (j,k) \text{ and } \bigl(g'(i), \varepsilon'_{i,2}\bigr) \ne (j,k), \\ 1 & \text{if } \bigl(g(i), \varepsilon_{i,2}\bigr) \ne (j,k) \text{ and } \bigl(g'(i), \varepsilon'_{i,2}\bigr) = (j,k), \\ 2 & \text{if } \bigl(g(i), \varepsilon_{i,2}\bigr) = (j,k) \text{ and } \bigl(g'(i), \varepsilon'_{i,2}\bigr) = (j,k), \end{cases}$$

$$\overline{\mathbf{C}}_k \;=\; \bigl(C_{ji,k}\bigr), \quad C_{ji,k} \;=\; \begin{cases} 0 & \text{if } \bigl(h(i), \varepsilon_{i,3}\bigr) \ne (j,k) \text{ and } \bigl(h'(i), \varepsilon'_{i,3}\bigr) \ne (j,k), \\ 1 & \text{if } \bigl(h(i), \varepsilon_{i,3}\bigr) = (j,k) \text{ and } \bigl(h'(i), \varepsilon'_{i,3}\bigr) \ne (j,k), \\ 1 & \text{if } \bigl(h(i), \varepsilon_{i,3}\bigr) \ne (j,k) \text{ and } \bigl(h'(i), \varepsilon'_{i,3}\bigr) = (j,k), \\ 2 & \text{if } \bigl(h(i), \varepsilon_{i,3}\bigr) = (j,k) \text{ and } \bigl(h'(i), \varepsilon'_{i,3}\bigr) = (j,k). \end{cases}$$

For simplicity, when $\max\{\varepsilon_{i,1}, \varepsilon'_{i,1}, \varepsilon_{i,2}, \varepsilon'_{i,2}, \varepsilon_{i,3}, \varepsilon'_{i,3} \mid 1 \le i \le m\} = 1$, we put

$$\overline{\mathbf{A}} = \begin{pmatrix} \overline{\mathbf{A}}_0 & & & & \overline{\mathbf{A}}_1 \\ \overline{\mathbf{A}}_1 & \overline{\mathbf{A}}_0 & & & \\ & \overline{\mathbf{A}}_1 & \ddots & & \\ & & \ddots & \overline{\mathbf{A}}_0 & \\ & & & \overline{\mathbf{A}}_1 & \overline{\mathbf{A}}_0 \end{pmatrix},$$

$$\overline{\mathbf{B}} = \begin{pmatrix} \overline{\mathbf{B}}_0 & & & & \overline{\mathbf{B}}_1 \\ \overline{\mathbf{B}}_1 & \overline{\mathbf{B}}_0 & & & \\ & \overline{\mathbf{B}}_1 & \ddots & & \\ & & \ddots & \overline{\mathbf{B}}_0 & \\ & & & \overline{\mathbf{B}}_1 & \overline{\mathbf{B}}_0 \end{pmatrix},$$

$$\overline{\mathbf{C}} = \begin{pmatrix} \overline{\mathbf{C}}_0 & & & & \overline{\mathbf{C}}_1 \\ \overline{\mathbf{C}}_1 & \overline{\mathbf{C}}_0 & & & \\ & \overline{\mathbf{C}}_1 & \ddots & & \\ & & \ddots & \overline{\mathbf{C}}_0 & \\ & & & \overline{\mathbf{C}}_1 & \overline{\mathbf{C}}_0 \end{pmatrix}.$$

Then, the Neumann-Zagier matrix $\big(\mathbf{A}\big|\mathbf{B}\big)$ is given by

$$\mathbf{A} = \overline{\mathbf{A}} - \overline{\mathbf{C}}, \qquad \mathbf{B} = \overline{\mathbf{B}} - \overline{\mathbf{C}}.$$

Example 5.4. For the 6_1 knot, by (86), we have that

$$\begin{aligned} I\big(M_n(6_1)\big) &= \sum_{\substack{a_0=0,\ a_1,\cdots,a_{n-1}\in\mathbb{Z} \\ b_0,\cdots,b_{n-1}\in\mathbb{Z} \\ c_0,\cdots,c_{n-1}\in\mathbb{Z} \\ d_0,\cdots,d_{n-1}\in\mathbb{Z}}} q^{a_0+\cdots+a_{n-1}+b_0+\cdots+b_{n-1}+c_0+\cdots+c_{n-1}+d_0+\cdots+d_{n-1}} \\ &\qquad \times I(a_0+d_0, a_1+b_0, d_0+c_0)\, I(b_0+d_0, a_1+c_0, a_0+b_1) \\ &\qquad\quad \times I(a_1+d_0, b_1+d_0, b_0+c_0)\, I(2a_1, b_0+c_1, 2b_1) \\ &\qquad \times I(a_1+d_1, a_2+b_1, d_1+c_1)\, I(b_1+d_1, a_2+c_1, a_1+b_2) \\ &\qquad\quad \times I(a_2+d_1, b_2+d_1, b_1+c_1)\, I(2a_2, b_1+c_2, 2b_2) \\ &\qquad \times \cdots \\ &\times I(a_{n-1}+d_{n-1}, a_n+b_{n-1}, d_{n-1}+c_{n-1})\, I(b_{n-1}+d_{n-1}, a_n+c_{n-1}, a_{n-1}+b_n) \\ &\quad \times I(a_n+d_{n-1}, b_n+d_{n-1}, b_{n-1}+c_{n-1})\, I(2a_n, b_{n-1}+c_n, 2b_n), \end{aligned} \tag{76}$$

where we regard the subscripts of a_i, b_i, c_i, d_i as modulo n. Hence, for example, $\overline{\mathbf{A}}_0$ and

$\overline{\mathbf{A}}_1$ are given by

$$\overline{\mathbf{A}}_0 \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \qquad \overline{\mathbf{A}}_1 \;=\; \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

These matrices are obtained in such a way that the first entries of the second and third lines of (76) is given by

$$\begin{pmatrix} a_0 & b_0 & c_0 & d_0 \end{pmatrix} \overline{\mathbf{A}}_0 \;+\; \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \end{pmatrix} \overline{\mathbf{A}}_1 \;=\; \big(a_0{+}d_0,\, b_0{+}d_0,\, a_1{+}d_0,\, 2a_1\big).$$

As mentioned before, we assume that $\min\big\{\varepsilon_{i,1}, \varepsilon'_{i,1}, \varepsilon_{i,2}, \varepsilon'_{i,2}, \varepsilon_{i,3}, \varepsilon'_{i,3} \;\big|\; 1 \le i \le m\big\} = 0$. Further, we put $\max\big\{\varepsilon_{i,1}, \varepsilon'_{i,1}, \varepsilon_{i,2}, \varepsilon'_{i,2}, \varepsilon_{i,3}, \varepsilon'_{i,3} \;\big|\; 1 \le i \le m\big\} = \overline{\varepsilon}$.

Proof of Theorem 3.1. We put

$$\begin{aligned} &\mathbb{E} \;=\; \big\{(a_0^1, a_0^2, \cdots, a_0^m,\;\; a_1^1, a_1^2, \cdots, a_1^m,\;\; \cdots,\;\; a_{n-1}^1, a_{n-1}^2, \cdots, a_{n-1}^m) \in \mathbb{Z}^{mn}\big\} \;=\; \mathbb{Z}^{mn}, \\ &Q_k^i(\mathbb{Z}) \;=\; \big\{(\ell_1, \ell_2, \ell_3) \in \mathbb{Z}^3\big\} \;=\; \mathbb{Z}^3. \end{aligned}$$

We define

$$\mathcal{F}_k^i :\; \mathbb{E} \;\longrightarrow\; Q_k^i(\mathbb{Z})$$

by

$$\begin{aligned} &\mathcal{F}_k^i(a_0^1, a_0^2, \cdots, a_0^m,\;\; a_1^1, a_1^2, \cdots, a_1^m,\;\; \cdots,\;\; a_{n-1}^1, a_{n-1}^2, \cdots, a_{n-1}^m) \\ &\;=\; \big(a_{k+\varepsilon_{i,1}}^{f(i)} + a_{k+\varepsilon'_{i,1}}^{f'(i)},\; a_{k+\varepsilon_{i,2}}^{g(i)} + a_{k+\varepsilon'_{i,2}}^{g'(i)},\; a_{k+\varepsilon_{i,3}}^{h(i)} + a_{k+\varepsilon'_{i,3}}^{h'(i)}\big), \end{aligned}$$

where we regard the subscripts in the right-hand side as modulo n. Further, we define

$$\mathcal{D}_k^i :\; Q_k^i(\mathbb{Z}) \;\longrightarrow\; \mathbb{R}_{\ge 0}$$

by

$$\begin{aligned} \mathcal{D}_k^i(\ell_1, \ell_2, \ell_3) \;&=\; \deg J_i(\ell_1, \ell_2, \ell_3) \\ &=\; \begin{cases} \frac{1}{2}(\ell_2{-}\ell_1)(\ell_3{-}\ell_1) + \beta_i(\ell_2{-}\ell_1) + \gamma_i(\ell_3{-}\ell_1) & \text{if } \ell_1 \le \ell_2 \text{ and } \ell_1 \le \ell_3, \\ \frac{1}{2}(\ell_1{-}\ell_2)(\ell_3{-}\ell_2) + \alpha_i(\ell_1{-}\ell_2) + \gamma_i(\ell_3{-}\ell_2) & \text{if } \ell_2 \le \ell_1 \text{ and } \ell_2 \le \ell_3, \\ \frac{1}{2}(\ell_1{-}\ell_3)(\ell_2{-}\ell_3) + \alpha_i(\ell_1{-}\ell_3) + \beta_i(\ell_2{-}\ell_3) & \text{if } \ell_3 \le \ell_1 \text{ and } \ell_3 \le \ell_2, \end{cases} \end{aligned} \tag{77}$$

where the second equality is obtained by (12). By Lemma 2.1, there exists a constant $\delta > 0$ such that,

$$\text{if } \;\mathcal{D}_k^i(\ell_1, \ell_2, \ell_3) > 0, \;\text{ then }\; \mathcal{D}_k^i(\ell_1, \ell_2, \ell_3) \ge \delta. \tag{78}$$

We put $\mathbb{E}_{\mathbb{R}} = \mathbb{E} \otimes \mathbb{R}$ and $Q_k^i(\mathbb{R}) = Q_k^i(\mathbb{Z}) \otimes \mathbb{R}$. The above maps $\mathcal{F}_k^i$ and $\mathcal{D}_k^i$ are naturally extended to the following maps,

$$\mathbb{E}_{\mathbb{R}} \;\xrightarrow{\mathcal{F}_i^i}\; Q_k^i(\mathbb{R}) \;\xrightarrow{\mathcal{D}_i^i}\; \mathbb{R}_{\ge 0}\,.$$

Similarly as in the previous section, these maps induces the following maps,

$$\mathbb{E}_{\mathbb{R}}/\mathbb{R} \ \xrightarrow{\hat{\mathcal{F}}^i_k}\ Q^i_k(\mathbb{R})/\mathbb{R} \ \xrightarrow{\hat{\mathcal{D}}^i_k}\ \mathbb{R}_{\geq 0}\,,$$

and we obtain the following maps,

$$\mathbb{E}_{\mathbb{R}}/\mathbb{R} \ \xrightarrow{\hat{\mathcal{F}}}\ \bigoplus_{i,k} \big(Q^i_k(\mathbb{R})/\mathbb{R}\big) \ \xrightarrow{\hat{\mathcal{D}}}\ \mathbb{R}_{\geq 0}\,,$$

As shown in the previous section, the map $\hat{\mathcal{F}}$ is injective.

We fix any positive integer D. Similarly as in the previous section, we show that the degree $\leq D$ part of the sum (75) can be reduced to a finite sum, as follows. By (78), the number of (i,k) with $\mathcal{D}^i_k(\cdots) > 0$ is bounded by D/δ. We consider a sequence of such (i,k),

$$\begin{aligned} &S \ = \ \big((i_1,k_1),\,(i_2,k_2),\,\cdots,\,(i_h,k_h)\big),\\ &\text{where}\ \ k_j \leq k_{j+1}\ \text{ for each } j,\ \text{ and } i_j < i_{j+1} \text{ if } k_j = k_{j+1},\ \text{ and }\ h \leq D/\delta, \end{aligned} \tag{79}$$

noting that there are finitely many such sequences. We put

$$Q_S \ = \ \bigoplus_{i,k} \begin{cases} \big(Q^i_k(\mathbb{R})/\mathbb{R}\big)^{\times} & \text{if } (i,k) \in S, \\ \{0\} & \text{if } (i,k) \notin S, \end{cases}$$

where $\big(Q^i_k(\mathbb{R})/\mathbb{R}\big)^{\times} \ = \ \big(Q^i_k(\mathbb{R})/\mathbb{R}\big) - \{0\}$. Then, $\bigoplus_{i,k} \big(Q^i_k(\mathbb{R})/\mathbb{R}\big)$ is presented as the disjoint union of finitely many Q_S,

$$\bigoplus_{i,k} \big(Q^i_k(\mathbb{R})/\mathbb{R}\big) \ = \ \bigsqcup_{S} Q_S \ \sqcup \ Q', \tag{80}$$

where Q' is the complement of $\sqcup_S Q_S$, noting that the degree obtained from a sequence of Q' is greater than D, which we can ignore. Similarly as in the previous section,

$$\prod_{(i,k)\,\in\, S} {\hat{\mathcal{D}}^i_k}^{\,-1}\big([0,D]\big)$$

is bounded by a constant independent of n. Hence, $\hat{\mathcal{D}}^{-1}\big([0,D]\big)$ is bounded. Therefore, similarly as in the previous section, we obtain that the degree $\leq D$ part of the sum (75) can be reduced to a finite sum, and hence, the sum converges. However, we note that the bounding constant of the range of this sum depends on n.

We consider to reduce the degree D part of the sum (75) further, in such a way that the bounding constant of the range of the sum is independent of n, as follows. Let $\mathcal{S}$ be the set of sequence of the form (79). When $1 \leq j < j' < h$ and $k_{j-1} + \overline{\varepsilon} < k_j$ and $k_{j'-1} + \overline{\varepsilon} + 1 < k_{j'}$, we consider a sequence,

$$S' \ = \ \big((i_1,k_1),\cdots,(i_{j-1},k_{j-1}),(i_j,k_j{+}1),\cdots,(i_{j'-1},k_{j'-1}{+}1),(i_{j'},k_{j'}),\cdots,(i_h,k_h)\big).$$

We consider the equivalence relation of $\mathcal{S}$ generated by the equivalence between such S and S'. In each equivalence class, we say that S of the form (79) is *minimal* if $k_1+\cdots+k_h$ is minimal. We choose a representative sequence of such equivalence class from minimal sequences. Let $\hat{\mathcal{S}}$ be the set of representative sequences. We note that the cardinality of $\hat{\mathcal{S}}$ is bounded by $(\overline{\varepsilon}+1)\cdot D/\delta$, which is independent of n. We consider to rewrite $\sqcup_S Q_S$ of (80) by

$$\bigsqcup_{S\in\mathcal{S}} Q_S \;=\; \bigsqcup_{\hat{S}\in\hat{\mathcal{S}}}\;\bigsqcup_{S\sim\hat{S}} Q_S.$$

In Sections 5.1, 5.2, 5.3, we consider "case" when we calculate the 3D-index. For example, in (35), we consider the sequence,

$$U \;=\; U_{\mathrm{const}}^{\ell_0}\cdot U_1^{1,0,4}\cdot U_{\mathrm{const}}^{\ell_1}\cdot U_1^{1,0,4}\cdot U_{\mathrm{const}}^{\ell_2}\,,$$

where we can change ℓ_0,ℓ_1,ℓ_2 in the same case, and $J_{4_1}(U)$ does not change independently of ℓ_0,ℓ_1,ℓ_2. Hence, the contribution from this case can be written in terms of a power series with coefficients of polynomials in n for a sufficiently large n. Here, this polynomial is obtained from the number of the ways of putting two copies of $U_1^{1,0,4}$ in the whole sequence of length $2n$. Hence, this polynomial is a polynomial in n of degree 2 in this case. More generally, we consider a sequence,

$$U \;=\; U_{\mathrm{const}}^{\ell_0}\cdot U_1\cdot U_{\mathrm{const}}^{\ell_1}\cdot U_2\cdot\cdots\cdot U_d\cdot U_{\mathrm{const}}^{\ell_d}.$$

Then, the number of ways of putting $U_1,\cdots,U_d$ in the whole sequence is a polynomial in n of degree d. Since the degree of $J_{4_1}(U_i)$ is at least $\frac{1}{2}$, we have that $\frac{1}{2}d\le D$. Hence, the polynomial is a polynomial of degree $\le 2D$. Further, we note that, if two sequences $\mathbf{a},\mathbf{a}'\in\mathbb{E}$ are in the same case, $\hat{\mathcal{F}}(\mathbf{a})$ and $\hat{\mathcal{F}}(\mathbf{a}')$ belong to Q_S and $Q_{S'}$ of equivalent S and S'. Similarly, when S and S' are equivalent,

$$\prod_{(i,k)\in S}{\hat{\mathcal{D}}_k^{i}}^{-1}\big([0,D]\big)\qquad\text{and}\qquad\prod_{(i',k')\in S'}{\hat{\mathcal{D}}_{k'}^{i'}}^{-1}\big([0,D]\big)$$

can be naturally identified, and identified sequences give the same contribution to the sum of the 3D-index, because of the same reason as above. Therefore, the sum of the defining formula (75) of $I\big(M_n(K)\big)$ can be written as the sum of partial sums over $\hat{\mathcal{S}}$ such that the contribution from each partial sum is a power series with coefficients of polynomials in n of degree $\le 2D$ for a sufficiently large n. Since the cardinality of $\hat{\mathcal{S}}$ is bounded by a constant independent of n, this sum is a power series with coefficients of polynomials in n of degree $\le 2D$.

Hence, we obtain the theorem. □

A Classification of particular sequences of parameters to calculate $I\big(M_n(K)\big)$

As we mention in Section 5, in order to calculate the lower degree part of $I\big(M_n(K)\big)$, it is sufficient to calculate contributions only from a union of finite number of particular sequences and constant sequences of parameters in the defining formula of $I\big(M_n(K)\big)$.

In this section, we classify such particular sequences for the 4_1, 5_2, 6_1 knots in Sections A.1, A.2, A.3 respectively.

A.1 Sequences for the 4_1 knot

In this section, in order to calculate the degree ≤ 3 part of $I\big(M_n(4_1)\big)$, we classify particular sequences of parameters which contribute to this part, by computer search.

We denote by $U_*^{d,h,\ell}$ a sequence of the form (29) of degree d, height h and length ℓ, where we define the degree of $U_*^{d,h,\ell}$ to be the lowest degree of $J_{4_1}(U_*^{d,h,\ell})$. It is sufficient to classify such sequences of degree ≤ 3.

Degree 1: The sequence of the form $U_*^{1,0,*}$ of degree 1 is given by

$$U_1^{1,0,4} = (0,0,0,1,0,0,0), \quad J_{4_1}(U_1^{1,0,4}) \equiv q - 2q^3.$$

Degree 2: The sequences of the form $U_*^{2,0,*}$ of degree 2 are given by

$$\begin{aligned} U_1^{2,0,4} &= (0,0,0,2,0,0,0), & J_{4_1}(U_1^{1,0,4}) &\equiv q^2, \\ U_1^{2,0,7} &= (0,0,0,1,0,0,1,0,0,0), & J_{4_1}(U_1^{2,0,7}) &\equiv q^2 \end{aligned}$$

Degree 3: The sequences of the form $U_*^{3,0,*}$ of degree 3 are given by

$$\begin{aligned} U_1^{3,0,4} &= (0,0,0,3,0,0,0), & J_{4_1}(U_1^{3,0,4}) &\equiv q^3, \\ U_1^{3,0,7} &= (0,0,0,1,0,0,2,0,0,0), & J_{4_1}(U_1^{3,0,7}) &\equiv q^3, \\ U_2^{3,0,7} &= (0,0,0,2,0,0,1,0,0,0), & J_{4_1}(U_1^{3,0,7}) &\equiv q^3, \\ U_2^{3,0,10} &= (0,0,0,1,0,0,1,0,0,1,0,0,0), & J_{4_1}(U_1^{3,0,10}) &\equiv q^3. \end{aligned}$$

A.2 Sequences for the 5_2 knot

In this section, in order to calculate the degree ≤ 2 part of $I\big(M_n(5_2)\big)$, we classify particular sequences of parameters which contribute to this part, by computer search.

We denote by $V_*^{d,h,\ell}$ a sequence of the form (41) of degree d, height h and length ℓ, where we define the degree of $V_*^{d,h,\ell}$ to be the lowest degree of $J_{5_2}(V_*^{d,h,\ell})$. It is sufficient to classify such sequences of degree ≤ 2.

Degree $\frac{1}{2}$: The sequences of the form $V_*^{1/2,*,*}$ of degree $\frac{1}{2}$ are given by

$$\begin{aligned} V_1^{1/2,-1,2} &= (0,0,-1,\ 0,-1,-1,\ -1,-1,*,\ -1), & J_{5_2}(V_1^{1/2,-1,2}) &\equiv q^{1/2} - 3q^{3/2}, \\ V_1^{1/2,1,2} &= (0,0,0,\ 0,0,0,\ 1,1,*,\ 1), & J_{5_2}(V_1^{1/2,1,2}) &\equiv q^{1/2} - 3q^{3/2}. \end{aligned}$$

Degree 1:

- *Height 0*: The sequences of the form $V_*^{1,0,*}$ of degree 1 and height 0 are given by

$$\begin{aligned}
&V_1^{1,0,1} = (0,0,-1,\ 0,0,*,\ 0), && J_{5_2}(V_1^{1,0,1}) \equiv -q-q^2,\\
&V_1^{1,0,2} = (0,0,0,\ 0,1,0,\ 0,0,*,\ 0), && J_{5_2}(V_1^{1,0,2}) \equiv q,\\
&V_1^{1,0,3} = (0,0,0,\ 0,0,-1,\ 1,0,0,\ 0,0,*,\ 0), && J_{5_2}(V_1^{1,0,3}) \equiv q-3q^2,\\
&V_2^{1,0,3} = (0,0,0,\ 0,0,0,\ 1,0,0,\ 0,0,*,\ 0), && J_{5_2}(V_2^{1,0,3}) \equiv q-2q^2,\\
&V_3^{1,0,3} = (0,0,0,\ 0,0,1,\ 1,0,0,\ 0,0,*,\ 0), && J_{5_2}(V_3^{1,0,3}) \equiv q-4q^2,\\
&V_4^{1,0,3} = (0,0,-1,\ 0,-1,-1,\ -1,-1,-1,\ 0,0,*,\ 0), && J_{5_2}(V_4^{1,0,3}) \equiv q-4q^2.
\end{aligned}$$

- *Height 2*: The sequences of the form $V_*^{1,2,*}$ of degree 1 and height 2 are given by

$$\begin{aligned}
&V_1^{1,2,2} = (0,0,0,\ 0,0,0,\ 2,2,*,\ 2), && J_{5_2}(V_1^{1,2,2}) = q-3q^2,\\
&V_1^{1,2,3} = (0,0,0,\ 0,0,0,\ 1,1,1,\ 2,2,*,\ 2), && J_{5_2}(V_1^{1,2,3}) = q-3q^2.
\end{aligned}$$

- *Height* -2: The sequences of the form $V_*^{1,-2,*}$ of degree 1 and height -2 are given by

$$\begin{aligned}
&V_1^{1,-2,2} = (0,0,-2,\ 0,-2,-2,\ -2,-2,*,\ -2), && J_{5_2}(V_1^{1,-2,2}) = q-3q^2,\\
&V_1^{1,-2,3} = (0,0,-1,\ 0,-1,-2,\ -1,-2,-2,\ -2,-2,*,\ -2), && J_{5_2}(V_1^{1,-2,3}) = q-3q^2
\end{aligned}$$

Degree 2:

- *Length* ≤ 2: The sequences of the form $V_*^{2,0,\ell}$ $(\ell=1,2)$ of degree 2 and length ≤ 2 are given by

$$\begin{aligned}
&V_1^{2,0,1} = (0,0,1,\ 0,0,*,\ 0), && J_{5_2}(V_1^{2,0,1}) \equiv q^2,\\
&V_1^{2,0,2} = (0,0,0,\ 0,2,0,\ 0,0,*,\ 0), && J_{5_2}(V_1^{2,0,2}) \equiv q^2.
\end{aligned}$$

- *Length 3*: The sequences of the form $V_*^{2,0,3}$ of degree 2 and length 3 are given by

$$\begin{aligned}
V_1^{2,0,3} &= (0,0,0,\ 0,-1,-1,\ -1,-1,-1,\ 0,0,*,\ 0),\\
V_2^{2,0,3} &= (0,0,0,\ 0,0,-2,\ 1,0,0,\ 0,0,*,\ 0),\\
V_3^{2,0,3} &= (0,0,0,\ 0,0,-2,\ 2,0,0,\ 0,0,*,\ 0),\\
V_4^{2,0,3} &= (0,0,0,\ 0,0,-1,\ 2,0,0,\ 0,0,*,\ 0),\\
V_5^{2,0,3} &= (0,0,0,\ 0,0,0,\ 2,0,0,\ 0,0,*,\ 0),\\
V_6^{2,0,3} &= (0,0,0,\ 0,0,1,\ 2,0,0,\ 0,0,*,\ 0),\\
V_7^{2,0,3} &= (0,0,0,\ 0,0,2,\ 2,0,0,\ 0,0,*,\ 0),\\
V_8^{2,0,3} &= (0,0,-2,\ 0,-2,-2,\ -2,-2,-2,\ 0,0,*,\ 0),\\
V_9^{2,0,3} &= (0,0,-2,\ 0,-1,-1,\ -1,-1,-1,\ 0,0,*,\ 0),\\
V_{10}^{2,0,3} &= (0,0,-1,\ 0,-1,-1,\ -1,-1,-2,\ 0,0,*,\ 0),\\
V_{11}^{2,0,3} &= (0,0,-1,\ 0,-1,-1,\ -1,-1,0,\ 0,0,*,\ 0).
\end{aligned}$$

Their invariants are given by

$$J_{5_2}(V_i^{2,0,3}) \equiv \begin{cases} -q^2 & \text{if } i = 1,2,9,10,11, \\ q^2 & \text{otherwise.} \end{cases}$$

- *Length 4*: The sequences of the form $V_*^{2,0,4}$ of degree 2 and length 4 are given by

$$\begin{aligned}
V_1^{2,0,4} &= (0,0,0,\ 0,0,-2,\ 1,-1,-1,\ -1,-1,-1,\ 0,0,*,\ 0),\\
V_2^{2,0,4} &= (0,0,0,\ 0,0,-1,\ 1,-1,-1,\ -1,-1,-1,\ 0,0,*,\ 0),\\
V_3^{2,0,4} &= (0,0,0,\ 0,0,0,\ 1,-1,-1,\ -1,-1,-1,\ 0,0,*,\ 0),\\
V_4^{2,0,4} &= (0,0,0,\ 0,0,0,\ 1,1,-1,\ 2,0,0,\ 0,0,*,\ 0),\\
V_5^{2,0,4} &= (0,0,0,\ 0,0,0,\ 1,1,0,\ 2,0,0,\ 0,0,*,\ 0),\\
V_6^{2,0,4} &= (0,0,0,\ 0,0,0,\ 1,1,1,\ 2,0,0,\ 0,0,*,\ 0),\\
V_7^{2,0,4} &= (0,0,0,\ 0,0,0,\ 1,2,0,\ 1,0,0,\ 0,0,*,\ 0),\\
V_8^{2,0,4} &= (0,0,0,\ 0,0,-1,\ 2,1,0,\ 1,0,0,\ 0,0,*,\ 0),\\
V_9^{2,0,4} &= (0,0,0,\ 0,0,0,\ 2,1,0,\ 1,0,0,\ 0,0,*,\ 0),\\
V_{10}^{2,0,4} &= (0,0,0,\ 0,0,1,\ 2,1,0,\ 1,0,0,\ 0,0,*,\ 0),\\
V_{11}^{2,0,4} &= (0,0,0,\ 0,1,-1,\ 0,-1,-1,\ -1,-1,-1,\ 0,0,*,\ 0),\\
V_{12}^{2,0,4} &= (0,0,-2,\ 0,-2,-2,\ -2,-2,-2,\ -1,-1,-1,\ 0,0,*,\ 0),\\
V_{13}^{2,0,4} &= (0,0,-1,\ 0,-1,-2,\ -1,-2,-2,\ -2,-2,-2,\ 0,0,*,\ 0),\\
V_{14}^{2,0,4} &= (0,0,-1,\ 0,-1,-1,\ -1,-1,-1,\ 0,1,0,\ 0,0,*,\ 0),\\
V_{15}^{2,0,4} &= (0,0,-1,\ 0,-1,-1,\ -1,-1,-2,\ 1,0,0,\ 0,0,*,\ 0),\\
V_{16}^{2,0,4} &= (0,0,-1,\ 0,-1,-1,\ -1,-1,-1,\ 1,0,0,\ 0,0,*,\ 0),\\
V_{17}^{2,0,4} &= (0,0,-1,\ 0,-1,-1,\ -1,-1,0,\ 1,0,0,\ 0,0,*,\ 0).
\end{aligned}$$

Their invariants are given by

$$J_{5_2}(V_i^{2,0,4}) \equiv q^2 \quad \text{for } i = 1,2,\cdots,17.$$

- *Length 5*: The sequences of the form $V_*^{2,0,5}$ of degree 2 and length 5 are given by

$$\begin{aligned}
V_1^{2,0,5} &= (0,0,0,\ 0,0,-1,\ 1,0,-1,\ 0,-1,-1,\ -1,-1,-1,\ 0,0,*,\ 0),\\
V_2^{2,0,5} &= (0,0,0,\ 0,0,-1,\ 1,0,0,\ 0,0,-1,\ 1,0,0,\ 0,0,*,\ 0),\\
V_3^{2,0,5} &= (0,0,0,\ 0,0,-1,\ 1,0,0,\ 0,0,0,\ 1,0,0,\ 0,0,*,\ 0),\\
V_4^{2,0,5} &= (0,0,0,\ 0,0,-1,\ 1,0,0,\ 0,0,1,\ 1,0,0,\ 0,0,*,\ 0),\\
V_5^{2,0,5} &= (0,0,0,\ 0,0,0,\ 1,0,-1,\ 0,-1,-1,\ -1,-1,-1,\ 0,0,*,\ 0),\\
V_6^{2,0,5} &= (0,0,0,\ 0,0,0,\ 1,0,0,\ 0,0,-1,\ 1,0,0,\ 0,0,*,\ 0),\\
V_7^{2,0,5} &= (0,0,0,\ 0,0,0,\ 1,0,0,\ 0,0,0,\ 1,0,0,\ 0,0,*,\ 0),\\
V_8^{2,0,5} &= (0,0,0,\ 0,0,0,\ 1,0,0,\ 0,0,1,\ 1,0,0,\ 0,0,*,\ 0),\\
V_9^{2,0,5} &= (0,0,0,\ 0,0,0,\ 1,1,0,\ 2,1,0,\ 1,0,0,\ 0,0,*,\ 0),
\end{aligned}$$

$$
\begin{aligned}
V_{10}^{2,0,5} &= (0,0,0,\ 0,0,0,\ 1,1,1,\ 2,1,0,\ 1,0,0,\ 0,0,*,\ 0),\\
V_{11}^{2,0,5} &= (0,0,0,\ 0,0,0,\ 1,1,2,\ 2,1,0,\ 1,0,0,\ 0,0,*,\ 0),\\
V_{12}^{2,0,5} &= (0,0,0,\ 0,0,1,\ 1,0,-1,\ 0,-1,-1,\ -1,-1,-1,\ 0,0,*,\ 0),\\
V_{13}^{2,0,5} &= (0,0,0,\ 0,0,1,\ 1,0,0,\ 0,0,-1,\ 1,0,0,\ 0,0,*,\ 0),\\
V_{14}^{2,0,5} &= (0,0,0,\ 0,0,1,\ 1,0,0,\ 0,0,0,\ 1,0,0,\ 0,0,*,\ 0),\\
V_{15}^{2,0,5} &= (0,0,0,\ 0,0,1,\ 1,0,0,\ 0,0,1,\ 1,0,0,\ 0,0,*,\ 0),\\
V_{16}^{2,0,5} &= (0,0,-1,\ 0,-1,-2,\ -1,-2,-2,\ -2,-2,-2,\ -1,-1,-1,\ 0,0,*,\ 0),\\
V_{17}^{2,0,5} &= (0,0,-1,\ 0,-1,-1,\ -1,-1,-2,\ 0,-1,-1,\ -1,-1,-1,\ 0,0,*,\ 0),\\
V_{18}^{2,0,5} &= (0,0,-1,\ 0,-1,-1,\ -1,-1,-1,\ 0,-1,-1,\ -1,-1,-1,\ 0,0,*,\ 0),\\
V_{19}^{2,0,5} &= (0,0,-1,\ 0,-1,-1,\ -1,-1,-1,\ 0,0,-1,\ 1,0,0,\ 0,0,*,\ 0),\\
V_{20}^{2,0,5} &= (0,0,-1,\ 0,-1,-1,\ -1,-1,-1,\ 0,0,0,\ 1,0,0,\ 0,0,*,\ 0),\\
V_{21}^{2,0,5} &= (0,0,-1,\ 0,-1,-1,\ -1,-1,-1,\ 0,0,1,\ 1,0,0,\ 0,0,*,\ 0),\\
V_{22}^{2,0,5} &= (0,0,-1,\ 0,-1,-1,\ -1,-1,0,\ 0,-1,-1,\ -1,-1,-1,\ 0,0,*,\ 0).
\end{aligned}
$$

Their invariants are given by

$$J_{5_2}(V_i^{2,0,5}) \ \equiv \ q^2 \quad \text{for } \ i = 1, 2, \cdots, 22.$$

Degree $\frac{3}{2}$:

- *Height 1, length 2*: The sequences of the form $V_*^{3/2,1,2}$ of degree $\frac{3}{2}$, height 1, length 2 are given by

$$
\begin{aligned}
V_1^{3/2,1,2} &= (0,0,0,\ 0,0,-1,\ 1,1,*,\ 1),\\
V_2^{3/2,1,2} &= (0,0,0,\ 0,0,1,\ 1,1,*,\ 1).
\end{aligned}
$$

Their invariants are given by

$$J_{5_2}(V_i^{3/2,1,2}) = -q^{3/2} \quad \text{for } i = 1, 2.$$

- *Height 1, length 3*: The sequences of the form $V_*^{3/2,1,3}$ of degree $\frac{3}{2}$, height 1, length 3 are given by

$$
\begin{aligned}
V_1^{3/2,1,3} &= (0,0,0,\ 0,0,0,\ 1,2,1,\ 1,1,*,\ 1),\\
V_2^{3/2,1,3} &= (0,0,0,\ 0,0,-1,\ 2,1,1,\ 1,1,*,\ 1),\\
V_3^{3/2,1,3} &= (0,0,0,\ 0,0,0,\ 2,1,1,\ 1,1,*,\ 1),\\
V_4^{3/2,1,3} &= (0,0,0,\ 0,0,1,\ 2,1,1,\ 1,1,*,\ 1),\\
V_5^{3/2,1,3} &= (0,0,-1,\ 0,-1,-1,\ -1,-1,-1,\ 1,1,*,\ 1).
\end{aligned}
$$

Their invariants are given by

$$J_{5_2}(V_i^{3/2,1,3}) = q^{3/2} \quad \text{for } i = 1, 2, \cdots, 5.$$

- *Height 1, length 4*: The sequences of the form $V_*^{3/2,1,4}$ of degree $\frac{3}{2}$, height 1, length 4 are given by

$$\begin{aligned}
V_1^{3/2,1,4} &= (0,0,0,\ 0,0,-1,\ 1,0,0,\ 0,0,0,\ 1,1,*,\ 1),\\
V_2^{3/2,1,4} &= (0,0,0,\ 0,0,0,\ 1,0,0,\ 0,0,0,\ 1,1,*,\ 1),\\
V_3^{3/2,1,4} &= (0,0,0,\ 0,0,0,\ 1,1,0,\ 2,1,1,\ 1,1,*,\ 1),\\
V_4^{3/2,1,4} &= (0,0,0,\ 0,0,0,\ 1,1,1,\ 2,1,1,\ 1,1,*,\ 1),\\
V_5^{3/2,1,4} &= (0,0,0,\ 0,0,0,\ 1,1,2,\ 2,1,1,\ 1,1,*,\ 1),\\
V_6^{3/2,1,4} &= (0,0,0,\ 0,0,1,\ 1,0,0,\ 0,0,0,\ 1,1,*,\ 1),\\
V_7^{3/2,1,4} &= (0,0,-1,\ 0,-1,-1,\ -1,-1,-1,\ 0,0,0,\ 1,1,*,\ 1).
\end{aligned}$$

Their invariants are given by

$$J_{5_2}(V_i^{3/2,1,4}) = q^{3/2} \quad \text{for } i = 1,2,\cdots,7.$$

- *Height -1, length 2*: The sequences of the form $V_*^{3/2,-1,2}$ of degree $\frac{3}{2}$, height -1, length 2 are given by

$$\begin{aligned}
V_1^{3/2,-1,2} &= (0,0,0,\ 0,-1,-1,\ -1,-1,*,\ -1),\\
V_2^{3/2,-1,2} &= (0,0,-2,\ 0,-1,-1,\ -1,-1,*,\ -1).
\end{aligned}$$

Their invariants are given by

$$J_{5_2}(V_i^{3/2,-1,2}) = -q^{3/2} \quad \text{for } i = 1,2.$$

- *Height -1, length 3*: The sequences of the form $V_*^{3/2,-1,3}$ of degree $\frac{3}{2}$, height -1, length 3 are given by

$$\begin{aligned}
V_1^{3/2,-1,3} &= (0,0,0,\ 0,0,-2,\ 1,-1,-1,\ -1,-1,*,\ -1),\\
V_2^{3/2,-1,3} &= (0,0,0,\ 0,0,-1,\ 1,-1,-1,\ -1,-1,*,\ -1),\\
V_3^{3/2,-1,3} &= (0,0,0,\ 0,0,0,\ 1,-1,-1,\ -1,-1,*,\ -1),\\
V_4^{3/2,-1,3} &= (0,0,0,\ 0,1,-1,\ 0,-1,-1,\ -1,-1,*,\ -1),\\
V_5^{3/2,-1,3} &= (0,0,-2,\ 0,-2,-2,\ -2,-2,-2,\ -1,-1,*,\ -1).
\end{aligned}$$

Their invariants are given by

$$J_{5_2}(V_i^{3/2,-1,3}) = q^{3/2} \quad \text{for } i = 1,2,\cdots,5.$$

- *Height -1, length 4*: The sequences of the form $V_*^{3/2,-1,4}$ of degree $\frac{3}{2}$, height -1, length 4 are given by

$$\begin{aligned}
V_1^{3/2,-1,4} &= (0,0,0,\ 0,0,-1,\ 1,0,-1,\ 0,-1,-1,\ -1,-1,*,\ -1),\\
V_2^{3/2,-1,4} &= (0,0,0,\ 0,0,0,\ 1,0,-1,\ 0,-1,-1,\ -1,-1,*,\ -1),
\end{aligned}$$

$$
\begin{aligned}
V_3^{3/2,-1,4} &= (0,0,0,\ 0,0,1,\ 1,0,-1,\ 0,-1,-1,\ -1,-1,*,\ -1),\\
V_4^{3/2,-1,4} &= (0,0,-1,\ 0,-1,-2,\ -1,-2,-2,\ -2,-2,-2,\ -1,-1,*,\ -1),\\
V_5^{3/2,-1,4} &= (0,0,-1,\ 0,-1,-1,\ -1,-1,-2,\ 0,-1,-1,\ -1,-1,*,\ -1),\\
V_6^{3/2,-1,4} &= (0,0,-1,\ 0,-1,-1,\ -1,-1,-1,\ 0,-1,-1,\ -1,-1,*,\ -1),\\
V_7^{3/2,-1,4} &= (0,0,-1,\ 0,-1,-1,\ -1,-1,0,\ 0,-1,-1,\ -1,-1,*,\ -1),
\end{aligned}
$$

Their invariants are given by

$$
J_{5_2}(V_i^{3/2,-1,4}) = q^{3/2} \quad \text{for } i = 1,2,\cdots,7.
$$

A.3 Sequences for the 6_1 knot

In this section, in order to calculate the degree ≤ 2 part of $I\big(M_n(6_1)\big)$, we classify particular sequences of parameters which contribute to this part, by computer search.

We denote by $W_*^{d,h,\ell}$ a sequence of the form (56) of degree d, height h and length ℓ, where we define the degree of $W_*^{d,h,\ell}$ to be the lowest degree of $J_{6_1}(W_*^{d,h,\ell})$. It is sufficient to classify such sequences of degree ≤ 2.

Degree $\frac{1}{2}$: The sequences of the form $W_*^{1/2,*,*}$ and their invariants which have the lowest degree ≤ 2 are given by

$$
\begin{aligned}
W_1^{1/2,-1,1} &= (0,0,0,-1,\ -1,-1,-1), & J_{6_1}(W_1^{1/2,-1,1}) &\equiv q^{1/2} - 2q^{3/2},\\
W_1^{1/2,1,2} &= (0,0,0,0,\ 0,1,0,0,\ 1,1,1), & J_{6_1}(W_1^{1/2,1,2}) &\equiv q^{1/2} - 4q^{3/2}.
\end{aligned}
$$

Degree 1:

- *Height 0*: The sequences of the form $W_*^{1,0,*}$ of degree 1, height 0 are given by

$$
\begin{aligned}
W_1^{1,0,1} &= (0,0,0,-1,0,0,0), & J_{6_1}(W_1^{1,0,1}) &\equiv -q,\\
W_1^{1,0,2} &= (0,0,0,0,\ 0,0,1,0,\ 0,0,0), & J_{6_1}(W_1^{1,0,2}) &\equiv q - 4q^2,\\
W_2^{1,0,2} &= (0,0,0,0,\ 0,1,0,-1,\ 0,0,0), & J_{6_1}(W_2^{1,0,2}) &\equiv q - 2q^2,\\
W_3^{1,0,2} &= (0,0,0,0,\ 0,1,0,0,\ 0,0,0), & J_{6_1}(W_3^{1,0,2}) &\equiv q - q^2,\\
W_4^{1,0,2} &= (0,0,0,0,\ 0,1,0,1,\ 0,0,0), & J_{6_1}(W_4^{1,0,2}) &\equiv q - 3q^2,\\
W_5^{1,0,2} &= (0,0,0,0,\ 1,0,0,0,\ 0,0,0), & J_{6_1}(W_5^{1,0,2}) &\equiv q - 2q^2.
\end{aligned}
$$

- *Height* -2: The sequence of the form $W_*^{1,-2,*}$ of degree 1, height -2 is given by

$$
W_1^{1,-2,1} = (0,0,0,-2,\ -2,-2,-2), \quad J_{6_1}(W_1^{1,-2,1}) \equiv q - 2q^2.
$$

- *Height 2*: The sequences of the form $W_*^{1,2,*}$ of degree 1, height 2 are given by

$$
\begin{aligned}
W_1^{1,2,2} &= (0,0,0,0,\ 0,2,0,0,\ 2,2,2), & J_{6_1}(W_1^{1,2,2}) &\equiv q - 4q^2,\\
W_1^{1,2,3} &= (0,0,0,0,\ 0,1,0,0,\ 1,2,1,1,\ 2,2,2), & J_{6_1}(W_1^{1,2,3}) &\equiv q - 4q^2.
\end{aligned}
$$

Degree 2:

- *Length 1*: The sequence of the form $W_*^{2,0,1}$ of degree2, length 1 is given by

$$W_1^{2,0,1} = (0,0,0,1,0,0,0), \quad J_{6_1}(W_1^{2,0,1}) \equiv q^2.$$

- *Length 2*: The sequences of the form $W_*^{2,0,2}$ of degree2, length 2 are given by

$$\begin{aligned}
W_1^{2,0,2} &= (0,0,0,-1,\ 0,0,1,0,\ 0,0,0),\\
W_2^{2,0,2} &= (0,0,0,0,\ 0,0,-1,-1,\ 0,0,0),\\
W_3^{2,0,2} &= (0,0,0,0,\ 0,0,2,0,\ 0,0,0),\\
W_4^{2,0,2} &= (0,0,0,0,\ 0,1,0,-2,\ 0,0,0),\\
W_5^{2,0,2} &= (0,0,0,0,\ 0,2,0,-2,\ 0,0,0),\\
W_6^{2,0,2} &= (0,0,0,0,\ 0,2,0,-1,\ 0,0,0),\\
W_7^{2,0,2} &= (0,0,0,0,\ 0,2,0,0,\ 0,0,0),\\
W_8^{2,0,2} &= (0,0,0,0,\ 0,2,0,1,\ 0,0,0),\\
W_9^{2,0,2} &= (0,0,0,0,\ 0,2,0,2,\ 0,0,0),\\
W_{10}^{2,0,2} &= (0,0,0,0,\ 2,0,0,0,\ 0,0,0).
\end{aligned}$$

Their invariants are given by

$$J_{6_1}(W_i^{2,0,2}) \equiv \begin{cases} -q^2 & \text{for } i = 1,2,4, \\ q^2 & \text{otherwise.} \end{cases}$$

- *Length 3*: The sequences of the form $W_*^{2,0,3}$ of degree2, length 3 are given by

$$\begin{aligned}
W_1^{2,0,3} &= (0,0,0,-1,\ -1,-1,-2,-2,\ -1,0,-1,-1,\ 0,0,0),\\
W_2^{2,0,3} &= (0,0,0,0,\ 0,0,1,0,\ 0,0,1,0,\ 0,0,0),\\
W_3^{2,0,3} &= (0,0,0,0,\ 0,1,0,-1,\ 0,0,-1,-1,\ 0,0,0),\\
W_4^{2,0,3} &= (0,0,0,0,\ 0,1,0,-1,\ 0,0,1,0,\ 0,0,0),\\
W_5^{2,0,3} &= (0,0,0,0,\ 0,1,0,0,\ 0,0,-1,-1,\ 0,0,0),\\
W_6^{2,0,3} &= (0,0,0,0,\ 0,1,0,0,\ 0,0,1,0,\ 0,0,0),\\
W_7^{2,0,3} &= (0,0,0,0,\ 0,1,0,0,\ 1,1,2,0,\ 0,0,0),\\
W_8^{2,0,3} &= (0,0,0,0,\ 0,1,0,0,\ 1,2,1,-1,\ 0,0,0),\\
W_9^{2,0,3} &= (0,0,0,0,\ 0,1,0,0,\ 1,2,1,0,\ 0,0,0),\\
W_{10}^{2,0,3} &= (0,0,0,0,\ 0,1,0,0,\ 1,2,1,1,\ 0,0,0),\\
W_{11}^{2,0,3} &= (0,0,0,0,\ 0,1,0,0,\ 2,1,1,0,\ 0,0,0),\\
W_{12}^{2,0,3} &= (0,0,0,0,\ 0,1,0,1,\ 0,0,-1,-1,\ 0,0,0),\\
W_{13}^{2,0,3} &= (0,0,0,0,\ 0,1,0,1,\ 0,0,1,0,\ 0,0,0),\\
W_{14}^{2,0,3} &= (0,0,0,0,\ 1,0,0,0,\ 0,0,1,0,\ 0,0,0),
\end{aligned}$$

$$
\begin{aligned}
W_{15}^{2,0,3} &= (0,0,0,0,\ 1,0,0,0,\ 0,1,0,-1,\ 0,0,0),\\
W_{16}^{2,0,3} &= (0,0,0,0,\ 1,0,0,0,\ 0,1,0,0,\ 0,0,0),\\
W_{17}^{2,0,3} &= (0,0,0,0,\ 1,0,0,0,\ 0,1,0,1,\ 0,0,0).
\end{aligned}
$$

Their invariants are given by

$$J_{6_1}(W_i^{2,0,3}) \equiv q^2 \quad \text{for } i = 1,2,\cdots,17.$$

Degree $\frac{3}{2}$:

- *Height -1, length 1*: The sequences of the form $W_*^{3/2,-1,1}$ of degree $\frac{3}{2}$, height -1, length 1 are given by

$$
\begin{aligned}
W_1^{3/2,-1,1} &= (0,0,0,-2,\ -1,-1,-1),\\
W_2^{3/2,-1,1} &= (0,0,0,0,\ -1,-1,-1).
\end{aligned}
$$

Their invariants are given by

$$J_{6_1}(W_i^{3/2,-1,1}) \equiv -q^{3/2} \quad \text{for } i = 1,2.$$

- *Height -1, length 2*: The sequences of the form $W_*^{3/2,-1,2}$ of degree $\frac{3}{2}$, height -1, length 2 are given by

$$
\begin{aligned}
W_1^{3/2,-1,2} &= (0,0,0,-1,\ -1,-1,-2,-2,\ -1,-1,-1),\\
W_2^{3/2,-1,2} &= (0,0,0,-1,\ -1,-1,0,-1,\ -1,-1,-1),\\
W_3^{3/2,-1,2} &= (0,0,0,0,\ 0,0,1,-1,\ -1,-1,-1),\\
W_4^{3/2,-1,2} &= (0,0,0,0,\ 0,1,0,-2,\ -1,-1,-1),\\
W_5^{3/2,-1,2} &= (0,0,0,0,\ 0,1,0,-1,\ -1,-1,-1),\\
W_6^{3/2,-1,2} &= (0,0,0,0,\ 0,1,0,0,\ -1,-1,-1),\\
W_7^{3/2,-1,2} &= (0,0,0,0,\ 1,0,0,-1,\ -1,-1,-1).
\end{aligned}
$$

Their invariants are given by

$$J_{6_1}(W_i^{3/2,-1,2}) \equiv q^{3/2} \quad \text{for } i = 1,2,\cdots,7.$$

- *Height 1, length 2*: The sequences of the form $W_*^{3/2,1,2}$ of degree $\frac{3}{2}$, height 1, length 2 are given by

$$
\begin{aligned}
W_1^{3/2,1,2} &= (0,0,0,0,\ 0,1,0,-1,\ 1,1,1),\\
W_2^{3/2,1,2} &= (0,0,0,0,\ 0,1,0,1,\ 1,1,1),\\
W_3^{3/2,1,2} &= (0,0,0,0,\ 0,2,0,-1,\ 1,1,1),\\
W_4^{3/2,1,2} &= (0,0,0,0,\ 0,2,0,0,\ 1,1,1),\\
W_5^{3/2,1,2} &= (0,0,0,0,\ 0,2,0,1,\ 1,1,1).
\end{aligned}
$$

Their invariants are given by

$$J_{6_1}(W_i^{3/2,1,2}) \equiv \begin{cases} -q^{3/2} & \text{for } i = 1,2, \\ q^{3/2} & \text{otherwise.} \end{cases}$$

- *Height 1, length 3*: The sequences of the form $W_*^{3/2,1,3}$ of degree $\frac{3}{2}$, height 1, length 3 are given by

$$\begin{aligned} W_1^{3/2,1,3} &= (0,0,0,0,\ 0,1,0,0,\ 1,1,2,1,\ 1,1,1), \\ W_2^{3/2,1,3} &= (0,0,0,0,\ 0,1,0,0,\ 1,2,1,0,\ 1,1,1), \\ W_3^{3/2,1,3} &= (0,0,0,0,\ 0,1,0,0,\ 1,2,1,1,\ 1,1,1), \\ W_4^{3/2,1,3} &= (0,0,0,0,\ 0,1,0,0,\ 1,2,1,2,\ 1,1,1), \\ W_5^{3/2,1,3} &= (0,0,0,0,\ 0,1,0,0,\ 2,1,1,1,\ 1,1,1), \\ W_6^{3/2,1,3} &= (0,0,0,0,\ 1,0,0,0,\ 0,1,0,0,\ 1,1,1). \end{aligned}$$

Their invariants are given by

$$J_{6_1}(W_i^{3/2,1,3}) \equiv q^{3/2} \quad \text{for } i = 1,2,\cdots,6.$$

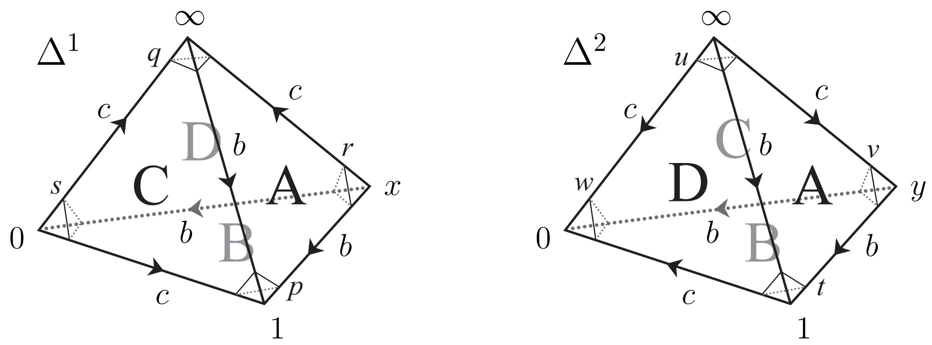
B Presentation of $I\big(M_n(K)\big)$

In this section, we review an ideal triangulation of a hyperbolic knot complement; for details of this topic, see [20] for the 4_1 knot, and [21, 22, 23] for other hyperbolic knots. Further, we consider the nth cyclic cover of this ideal triangulation, and we obtain a presentation of $I\big(M_n(K)\big)$. Furthermore, by using the hyperbolic structure of the knot complement, we modify the presentation of $I\big(M_n(K)\big)$ in such a way that the lowest degree of each summand of the presentation is positive. This modified presentation is used in Sections 4 and 5. We calculate such presentations for the 4_1, 5_2, 6_1 knots in Sections B.1, B.2, B.3 respectively.

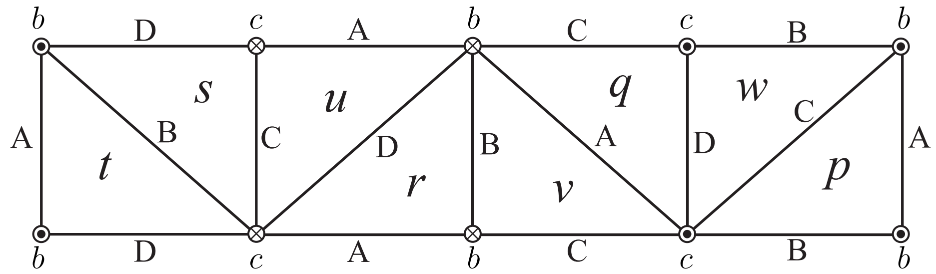
B.1 Presentation of $I\big(M_n(4_1)\big)$

In this section, we review an ideal triangulation of the 4_1 knot complement; see [20] for details of this topic. Further, we consider the nth cyclic cover of this ideal triangulation, and obtain a presentation of $I\big(M_n(4_1)\big)$.

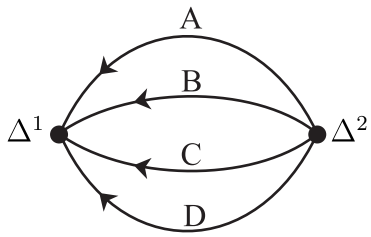
It is known [20] that the 4_1 knot complement can expressed as the union of the following two ideal tetrahedra.



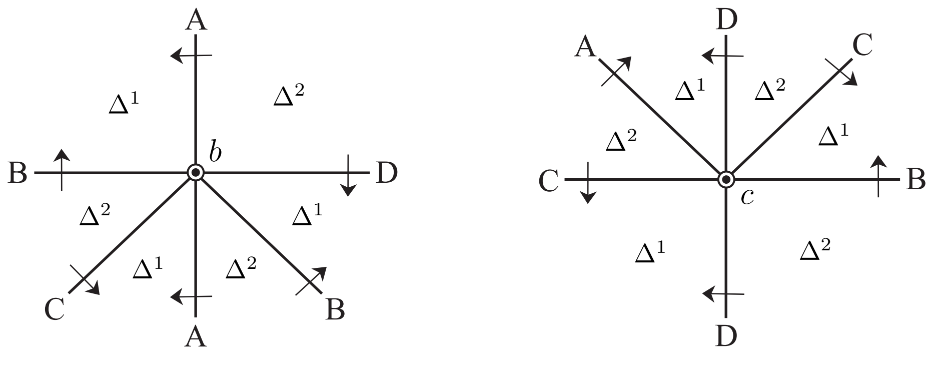
Here, the 4 faces "A", "B", "C", "D" are glued respectively, where the glay characters are on the back side of tetrahedra. The labels of vertices of a tetrahedron are regarded in $\mathbb{C} \cup \{\infty\} = \partial\mathbb{H}^3$, where $\mathbb{H}^3$ denotes the hyperbolic 3-space. The boundary torus of a tubular neighbourhood of the 4_1 knot is expressed as the union of 8 triangles "p", "q", $\cdots$ "w", which appear in neighbourhoods of the vertices of the ideal tetrahedra. A fundamental domain of the torus is depicted as follows.



We consider the dual decomposition of the above ideal triangulation. Its 1-skelton is depicted as follows.



Since the 2-cells of the dual decomposition are given by



the boundary cycles of the 2-cells are given by

$$C - D, \qquad B - A.$$

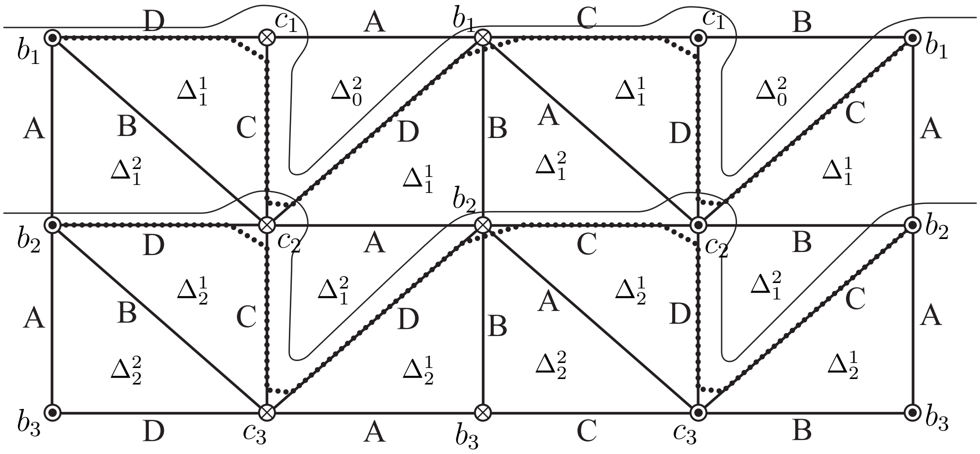
Hence, the first homology of the 4_1 knot complement is presented by

$$H_1(S^3 - K_{4_1}) \;\cong\; \mathrm{kernel}\big(\mathrm{span}_{\mathbb{Z}}\{A, B, C, D\} \longrightarrow \mathrm{span}_{\mathbb{Z}}\{\Delta^1, \Delta^2\}\big)/(A = B,\ C = D).$$

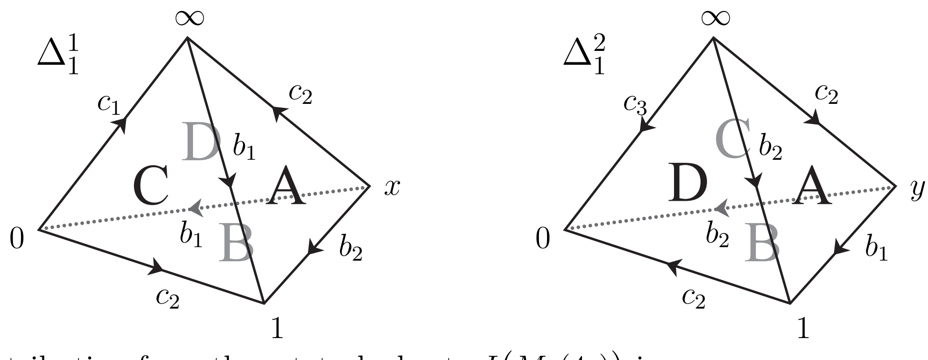
By homotopy equivalence collapsing the edge A, we have that

$$H_1(S^3 - K_{4_1}) \;\cong\; \mathrm{span}_{\mathbb{Z}}\{C, D\}/(C = D), \qquad A = B = 0.$$

We consider the infinite cyclic ocver of the torus in the infinite cyclic cover of the 4_1 knot complement. We can choose a fundamental domain as the domain between two dotted lines in the following figure.



Here, we denote by Δ_1^1 and Δ_1^2 the lifts of Δ^1 and Δ^2 in this fundamental domain. The deck transformation of the cyclic cover takes Δ_k^1 and Δ_k^2 to Δ_{k+1}^1 and Δ_{k+1}^2. Further, we obtain thin lines from dotted lines by pushing them to the direction from Δ_{k+1}^* to Δ_k^*. We denote by b_1 and c_1 the lifts of b and c in the domain between two thin lines. Further, the deck transformation of the cyclic cover takes b_k and c_k to b_{k+1} and c_{k+1}. Hence, the edges of Δ_1^1 and Δ_1^2 are labeled, as follows.



The contribution from these tetrahedra to $I\big(M_n(4_1)\big)$ is

$$I(c_1{+}b_2,\ 2b_1,\ 2c_2)\, I(b_1{+}c_3,\ 2b_2,\ 2c_2).$$

The labels of other tetrahedra Δ^1_{k+1} and Δ^2_{k+1} in the cyclic cover are obtained by replacing b_i and c_i with b_{i+k} and c_{i+k}. Hence, $I\big(M_n(4_1)\big)$ is presented by

$$\begin{aligned} I\big(M_n(4_1)\big) \;=\; & \sum_{\substack{c_0=0,\\ c_1,\cdots,c_{n-1}\in\mathbb{Z},\\ b_0,\cdots,b_{n-1}\in\mathbb{Z}}} q^{c_0+\cdots+c_{n-1}+b_0+\cdots+b_{n-1}} I(c_0+b_1,2b_0,2c_1)\, I(b_0+c_2,2b_1,2c_1) \\ & \times I(c_1+b_2,2b_1,2c_2)\, I(b_1+c_3,2b_2,2c_2) \\ & \times \cdots \\ & \times I(c_{n-1}+b_n,2b_{n-1},2c_n)\, I(b_{n-1}+c_{n+1},2b_n,2c_n), \end{aligned} \tag{81}$$

where we regard the subscripts of b_i and c_i as modulo n.

The hyperbolicity equations are given by

$$\frac{(1-x)(1-y)}{x} \;=\; -1 \;=\; \frac{(1-x)(1-y)}{y}\,.$$

This is rewritten as

$$x \;=\; y, \qquad x^2-x+1 \;=\; 0.$$

The hyperbolic structure of the 4_1 knot complement is given by the solution

$$x \;=\; y \;=\; e^{\pi\sqrt{-1}/3}.$$

Hence, we have that

$$\begin{aligned} \frac{1}{2\pi}\operatorname{Arg} x \;&=\; \frac{1}{2\pi}\operatorname{Arg}\frac{1}{1-x} \;=\; \frac{1}{2\pi}\operatorname{Arg}\big(1-\frac{1}{x}\big) \;=\; \frac{1}{6}\,,\\ \frac{1}{2\pi}\operatorname{Arg} y \;&=\; \frac{1}{2\pi}\operatorname{Arg}\frac{1}{1-y} \;=\; \frac{1}{2\pi}\operatorname{Arg}\big(1-\frac{1}{y}\big) \;=\; \frac{1}{6}\,. \end{aligned}$$

We put

$$\alpha \;=\; \beta = \gamma \;=\; \frac{1}{6}\,.$$

As mentioned in Section 2, we put

$$\hat{J}_{4_1}(\ell_1,\ell_2,\ell_3) \;=\; q^{\alpha\ell_1+\beta\ell_2+\gamma\ell_3} I(\ell_1,\ell_2,\ell_3).$$

Then, (81) is rewritten as

$$\begin{aligned} I\big(M_n(4_1)\big) \;=\; & \sum_{\substack{c_0=0,\\ c_1,\cdots,c_{n-1}\in\mathbb{Z},\\ b_0,\cdots,b_{n-1}\in\mathbb{Z}}} \hat{J}_{4_1}(c_0+b_1,2b_0,2c_1)\, \hat{J}_{4_1}(b_0+c_2,2b_1,2c_1) \\ & \times \hat{J}_{4_1}(c_1+b_2,2b_1,2c_2)\, \hat{J}_{4_1}(b_1+c_3,2b_2,2c_2) \\ & \times \cdots \\ & \times \hat{J}_{4_1}(c_{n-1}+b_n,2b_{n-1},2c_n)\, \hat{J}_{4_1}(b_{n-1}+c_{n+1},2b_n,2c_n), \end{aligned}$$

where we regard the subscripts of b_i and c_i as modulo n. Further, by putting $c_k = a_{2k}$ and $b_k = a_{2k+1}$, the above formula is rewritten as

$$
\begin{aligned}
I\big(M_n(4_1)\big) \;=\; & \sum_{\substack{a_0=0,\\ a_1,\cdots,a_{2n-1}\in\mathbb{Z}}} \hat{J}_{4_1}(a_0{+}a_3, 2a_1, 2a_2)\, \hat{J}_{4_1}(a_1{+}a_4, 2a_2, 2a_3) \\
& \times \hat{J}_{4_1}(a_2{+}a_5, 2a_3, 2a_4)\, \hat{J}_{4_1}(a_3{+}a_6, 2a_4, 2a_5) \\
& \times \cdots \\
& \times \hat{J}_{4_1}(a_{2n-2}{+}a_{2n+1}, 2a_{2n-1}, 2a_{2n})\, \hat{J}_{4_1}(a_{2n-1}{+}a_{2n+2}, 2a_{2n}, 2a_{2n+1}), \qquad (82)
\end{aligned}
$$

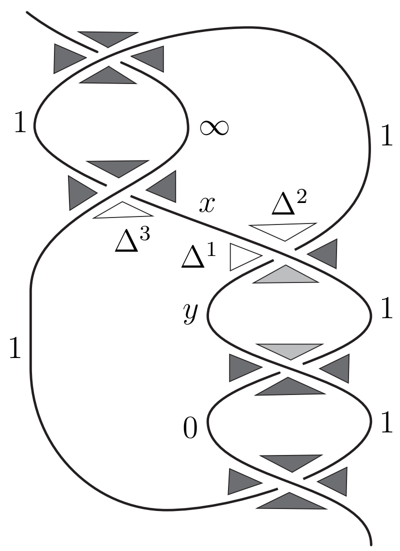
where we regard the subscript of a_i as modulo $2n$.

We use the formula (82) in Sections 4.1 and 5.1.

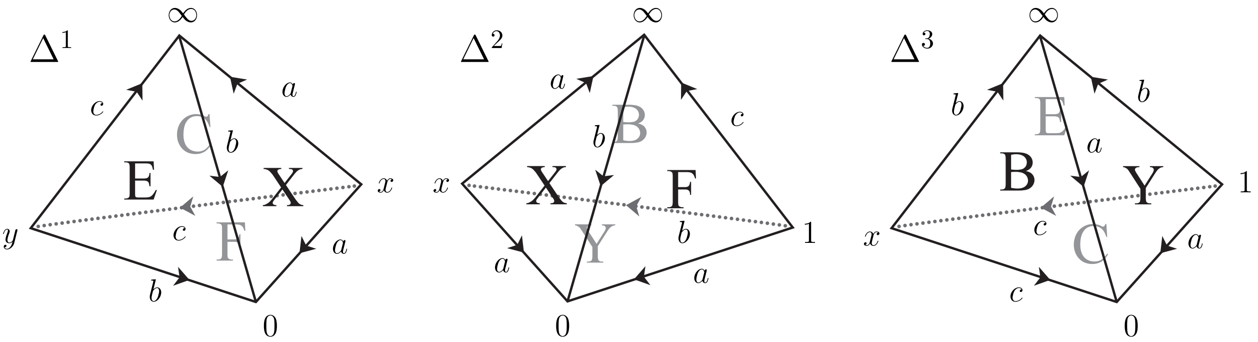
B.2 Presentation of $I\big(M_n(5_2)\big)$

In this section, we review an ideal triangulation of the 5_2 knot complement; see [21, 22, 23] for details of this topic. Further, we consider the nth cyclic cover of this ideal triangulation, and obtain a presentation of $I\big(M_n(5_2)\big)$.

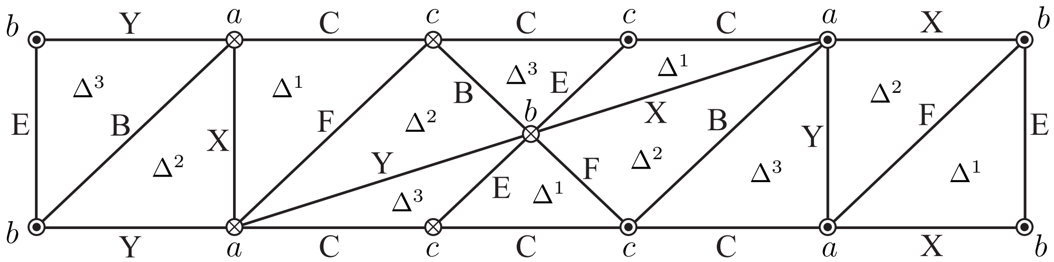
We review an ideal triangulation of the 5_2 knot complement. We consider the following 1-tangle diagram whose closure is the 5_2 knot.



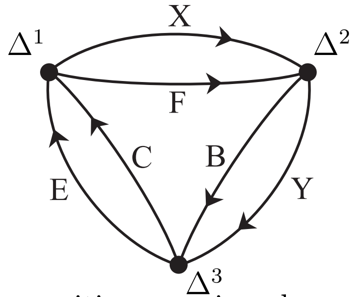
The edges of the diagram are labeled by parameters, which give the hyperbolic structure of the 5_2 knot complement later. We consider 4 tetrahedra at each crossing of the diagram. We glue them, and collapse dark gray tetrahedra near the end points of the 1-tangle diagram, and collapse dark gray tetrahedra adjacent to the unbounded regions the 1-tangle diagram, in the way shown in [21, 22, 23]. Then, we obtain an ideal triangulation of the 5_2 knot complement. Further, we cancel two light gray tetrahedra by the 0-2 Pachner move. Then, we obtain the ideal triangulation of the 5_2 knot complement, which consists of the following ideal three tetrahedra.



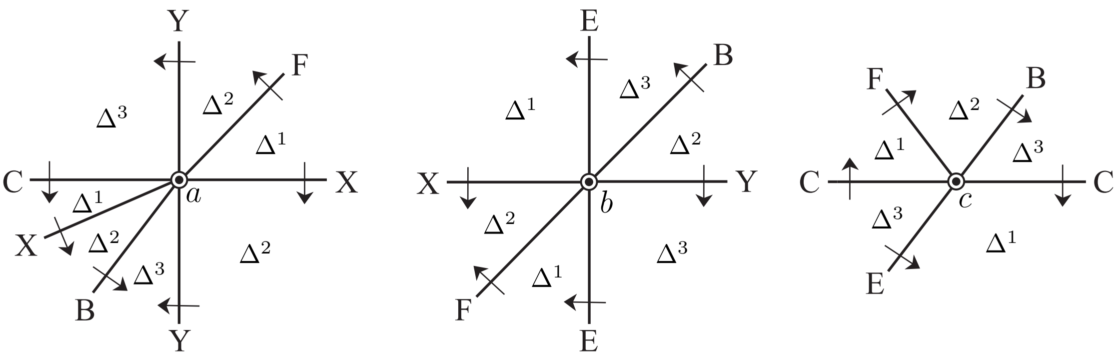
Similarly as in Section B.1, the boundary torus of a tubular neighbourhood of the 5_2 knot is expressed as the union of 12 triangles, which appear in neighbourhoods of the vertices of the ideal tetrahedra. A fundamental domain of the torus is depicted as follows.



We consider the dual decomposition of the above ideal triangulation. Its 1-skelton is depicted as follows.



Since the 2-cells of the dual decomposition are given by



the boundary cycles of the 2-cells are given by

$$B + C + F, \qquad B - F + X - Y, \qquad -B - 2C + E - F.$$

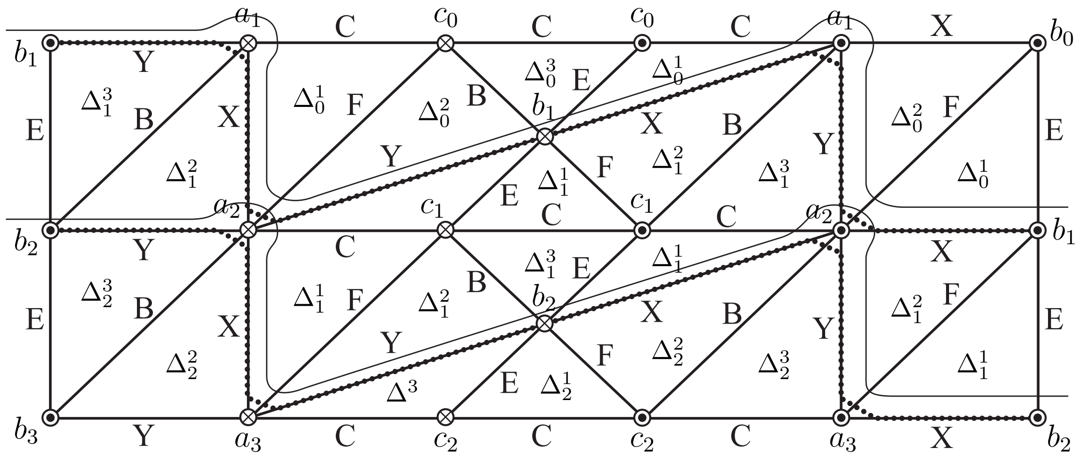
Hence, the first homology of the 5_2 knot complement is presented by

$$H_1(S^3 - K_{5_2}) \;\cong\; \frac{\operatorname{kernel}\big(\operatorname{span}_{\mathbb{Z}}\{B, C, E, F, X, Y\} \longrightarrow \operatorname{span}_{\mathbb{Z}}\{\Delta^1, \Delta^2, \Delta^3\}\big)}{(B+C+F = 0,\ B-F+X-Y = 0,\ -B-2C+E-F = 0)}.$$

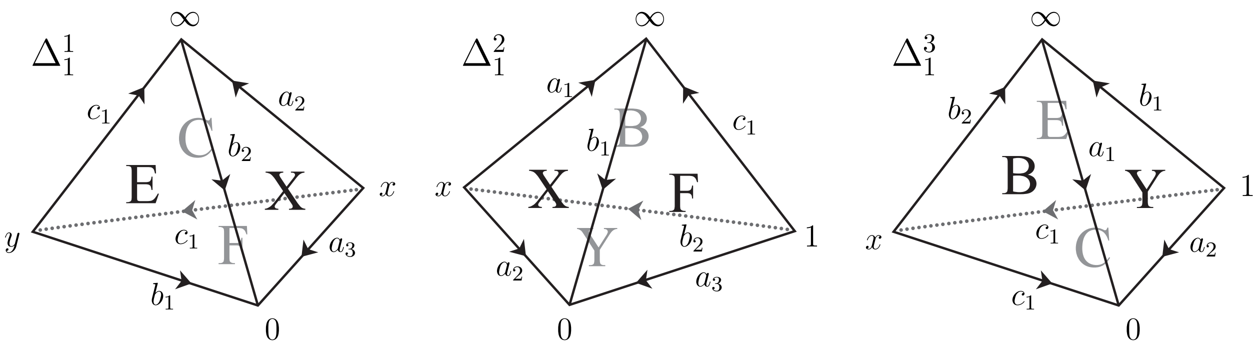
By homotopy equivalence collapsing the edges B and C, we have that

$$H_1(S^3 - K_{5_2}) \;\cong\; \operatorname{span}_{\mathbb{Z}}\{X, Y\}/(X = Y), \qquad B = C = E = F = 0.$$

We consider the infinite cyclic ocver of the torus in the infinite cyclic cover of the 5_2 knot complement. We can choose a fundamental domain as the domain between two dotted lines in the following figure.



Hence, in a similar way as in Section B.1, the edges of Δ_1^1, Δ_1^2, Δ_1^3 are labeled, as follows.



The contribution from these tetrahedra to $I\big(M_n(5_2)\big)$ is

$$I(a_3+c_1,\ c_1+b_2,\ a_2+b_1)\, I(b_1+b_2,\ a_2+c_1,\ a_1+a_3)\, I(a_1+c_1,\ c_1+b_1,\ a_2+b_2).$$

The labels of other tetrahedra Δ_{k+1}^1, Δ_{k+1}^2 and Δ_{k+1}^3 in the cyclic cover are obtained by replacing a_i, b_i and c_i with a_{i+k}, b_{i+k} and c_{i+k}. Hence, $I\big(M_n(5_2)\big)$ is presented by

$$
\begin{aligned}
I\big(M_n(5_2)\big) \;=\; & \\
\sum_{\substack{a_0=0,\quad a_1,\cdots,a_{n-1}\in\mathbb{Z}\\ b_0,\cdots,b_{n-1}\in\mathbb{Z}\\ c_0,\cdots,c_{n-1}\in\mathbb{Z}}} & q^{a_0+\cdots+a_{n-1}+b_0+\cdots+b_{n-1}c_0+\cdots+c_{n-1}} \\
& \times I(a_0+c_0, c_0+b_0, a_1+b_1)\, I(a_2+c_0, c_0+b_1, a_1+b_0)\, I(b_0+b_1, a_1+c_0, a_0+a_2) \\
& \times I(a_1+c_1, c_1+b_1, a_2+b_2)\, I(a_3+c_1, c_1+b_2, a_2+b_1)\, I(b_1+b_2, a_2+c_1, a_1+a_3) \\
& \times \cdots \\
& \times I(a_{n-1}+c_{n-1}, c_{n-1}+b_{n-1}, a_n+b_n)\, I(a_{n+1}+c_{n-1}, c_{n-1}+b_n, a_n+b_{n-1}) \\
& \quad \times I(b_{n-1}+b_n, a_n+c_{n-1}, a_{n-1}+a_{n+1}),
\end{aligned}
\tag{83}
$$

where we regard the subscripts of a_i, b_i, c_i as modulo n.

The hyperbolicity equations are given by

$$
(1-x)\big(1-\frac{1}{x}\big) \;=\; 1-\frac{y}{x}, \qquad \big(1-\frac{y}{x}\big)\big(1-\frac{1}{y}\big) \;=\; 1-y.
$$

They are rewritten

$$
y \;=\; x^2-x+1, \qquad y+1-\frac{y}{x} \;=\; 0.
$$

Further, they are rewritten

$$
x^3-2x^2+3x-1 \;=\; 0, \qquad y=\frac{x}{1-x}\,.
$$

Putting $y'=y/x$, we have that

$$
y' \;=\; \frac{1}{1-x}\,, \qquad \frac{1}{1-y'} \;=\; 1-\frac{1}{x}\,, \qquad 1-\frac{1}{y'} \;=\; x,
$$

noting that we can replace the labels x and y of Δ_1^1 with 1 and y'. The hyperbolic structure of the 5_2 knot complement is given by the solution

$$
x \;=\; 0.7849201454.....+\sqrt{-1}\;1.3071412786.....\,.
$$

Then, we have that

$$
\begin{aligned}
\frac{1}{2\pi}\mathrm{Arg}\,x \;&=\; \frac{1}{2\pi}\mathrm{Arg}\,\big(1-\frac{1}{y'}\big) \;=\; 0.1639326.....\,, \\
\frac{1}{2\pi}\mathrm{Arg}\,\frac{1}{1-x} \;&=\; \frac{1}{2\pi}\mathrm{Arg}\,y' \;=\; 0.2240448.....\,, \\
\frac{1}{2\pi}\mathrm{Arg}\,\big(1-\frac{1}{x}\big) \;&=\; \frac{1}{2\pi}\mathrm{Arg}\,\frac{1}{1-y'} \;=\; 0.1120224.....\,.
\end{aligned}
$$

As approximations of these values, we put

$$
\alpha \;=\; 0.164, \qquad \beta \;=\; 0.224, \qquad \gamma \;=\; 0.112.
$$

As mentioned in Section 2, we put

$$\hat{J}_{5_2}(\ell_1,\ell_2,\ell_3) \;=\; q^{\alpha\ell_1+\beta\ell_2+\gamma\ell_3} I(\ell_1,\ell_2,\ell_3).$$

Then, (83) is rewritten as

$$\begin{aligned}
&I\big(M_n(5_2)\big) \;= \\
&\sum_{\substack{a_0=0,\quad a_1,\cdots,a_{n-1}\in\mathbb{Z}\\ b_0,\cdots,b_{n-1}\in\mathbb{Z}\\ c_0,\cdots,c_{n-1}\in\mathbb{Z}}} \hat{J}_{5_2}(a_0+c_0,c_0+b_0,a_1+b_1)\,\hat{J}_{5_2}(a_2+c_0,c_0+b_1,a_1+b_0)\,\hat{J}_{5_2}(b_0+b_1,a_1+c_0,a_0+a_2) \\
&\quad\times \hat{J}_{5_2}(a_1+c_1,c_1+b_1,a_2+b_2)\,\hat{J}_{5_2}(a_3+c_1,c_1+b_2,a_2+b_1)\,\hat{J}_{5_2}(b_1+b_2,a_2+c_1,a_1+a_3) \\
&\quad\times\cdots \\
&\quad\times \hat{J}_{5_2}(a_{n-1}+c_{n-1},c_{n-1}+b_{n-1},a_n+b_n)\,\hat{J}_{5_2}(a_{n+1}+c_{n-1},c_{n-1}+b_n,a_n+b_{n-1}) \\
&\qquad\times \hat{J}_{5_2}(b_{n-1}+b_n,a_n+c_{n-1},a_{n-1}+a_{n+1}),
\end{aligned} \tag{84}$$

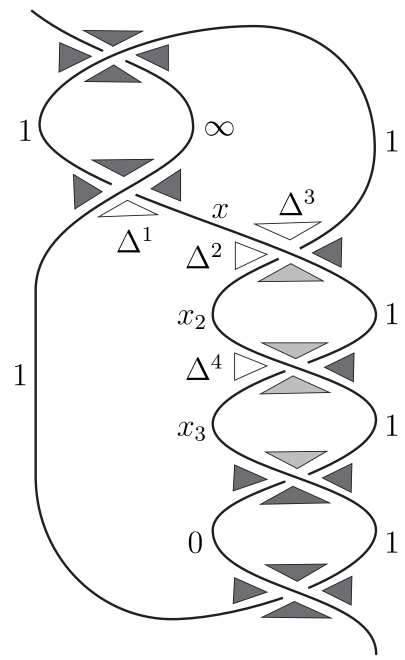
where we regard the subscripts of a_i, b_i, c_i as modulo n.

We use the formula (84) in Sections 4.2 and 5.2.

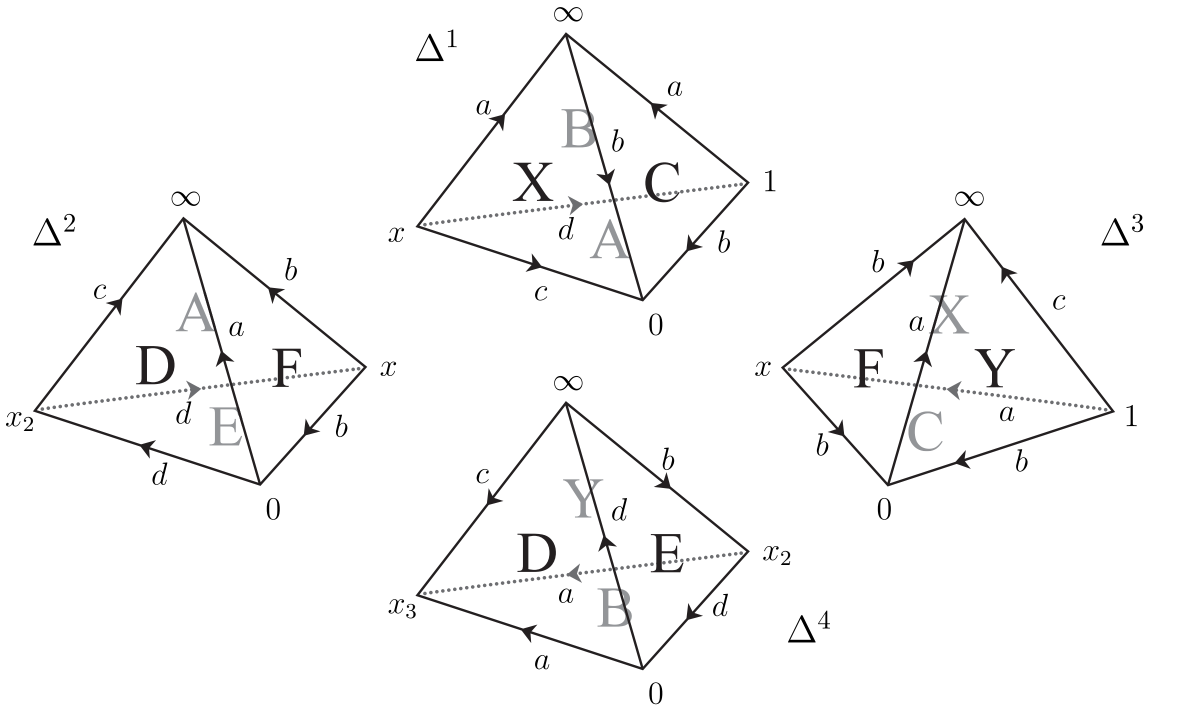
B.3 Presentation of $I\big(M_n(6_1)\big)$

In this section, we review an ideal triangulation of the 6_1 knot complement; see [21, 22, 23] for details of this topic. Further, we consider the nth cyclic cover of this ideal triangulation, and obtain a presentation of $I\big(M_n(6_1)\big)$.

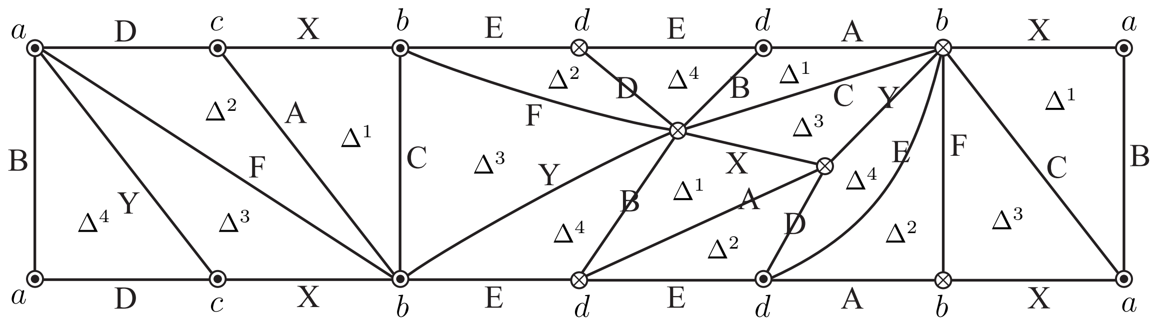
We review an ideal triangulation of the 6_1 knot complement. We consider the following 1-tangle diagram whose closure is the 6_1 knot.



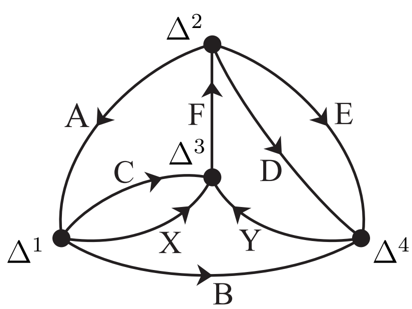
In a similar way as in Section B.2, we obtain the ideal triangulation of the 6_1 knot complement, which consists of the following ideal four tetrahedra.



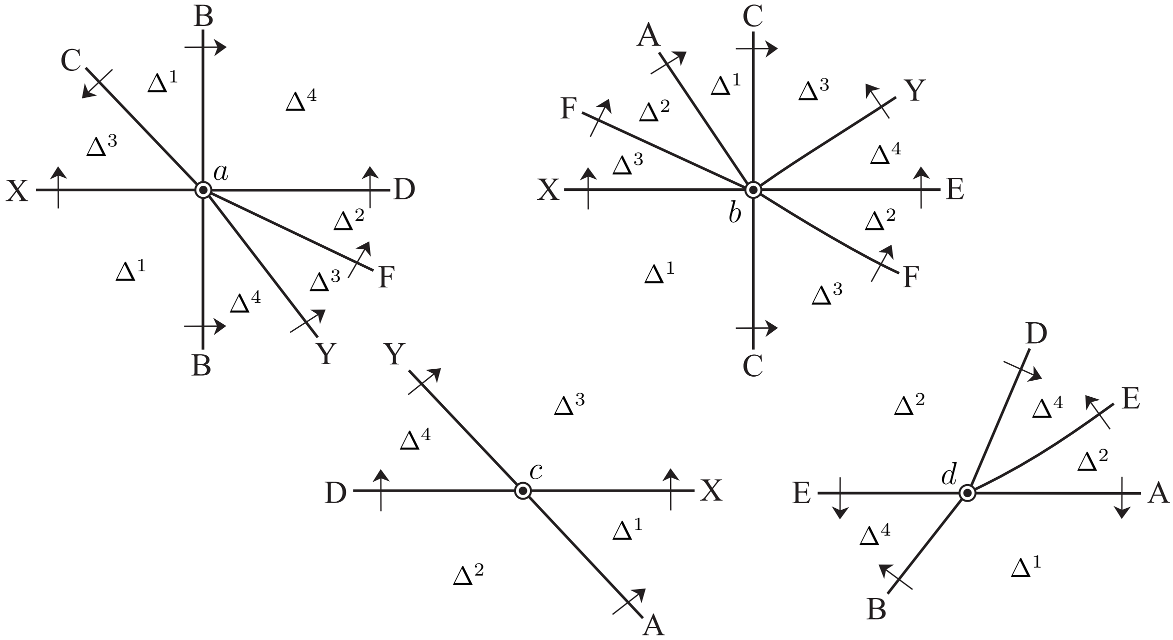
Similarly as in Section B.1, the boundary torus of a tubular neighborhood of the 6_1 knot is expressed as the union of 16 triangles, which appear in neighborhoods of the vertices of the ideal tetrahedra. A fundamental domain of the torus is depicted as follows.



We consider the dual decomposition of the above ideal triangulation. Its 1-skelton is depicted as follows.



Since the 2-cells of the dual decomposition are given by



the boundary cycles of the 2-cells are given by

$$-B+C+D+F-X+Y, \quad -A+E-X+Y, \quad A-D+X-Y, \quad -A-B-D+2E,$$

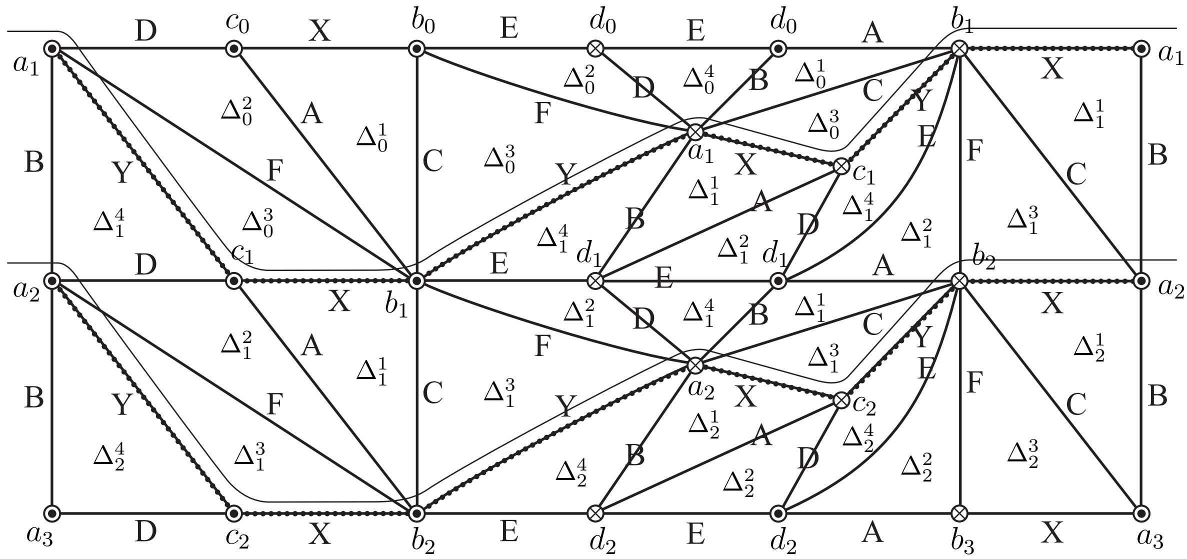
Hence, the first homology of the 6_1 knot complement is presented by

$$H_1(S^3 - K_{6_1}) \;\cong\; \frac{\operatorname{kernel}\big(\operatorname{span}_{\mathbb{Z}}\{A,B,C,D,E,F,X,Y\} \longrightarrow \operatorname{span}_{\mathbb{Z}}\{\Delta^1,\Delta^2,\Delta^3,\Delta^4\}\big)}{\left(\begin{array}{l} -B+C+D+F-X+Y=0, \quad -A+E-X+Y=0, \\ A-D+X-Y=0, \quad -A-B-D+2E=0 \end{array}\right)}.$$

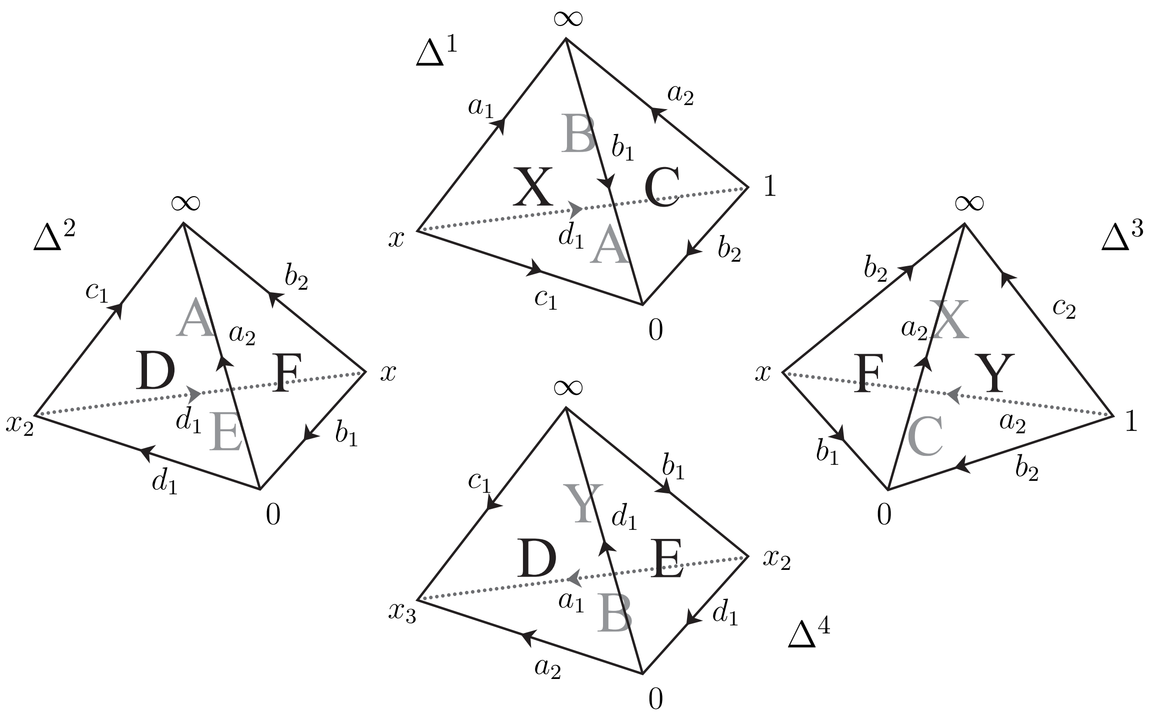
By homotopy equivalence collapsing the edges A, E and F, we have that

$$H_1(S^3 - K_{6_1}) \;\cong\; \operatorname{span}_{\mathbb{Z}}\{X,Y\}/(X=Y), \qquad A=B=C=D=E=F=0.$$

We consider the infinite cyclic cover of the torus in the infinite cyclic cover of the 6_1 knot complement. We can choose a fundamental domain as the domain between two dotted lines in the following figure.



Hence, in a similar way as in Section B.1, the edges of Δ_1^1, Δ_1^2, Δ_1^3, Δ_1^4 are labeled, as follows.



The contribution from these tetrahedra to $I\big(M_n(6_1)\big)$ is

$$I(d_1{+}b_1,\ a_2{+}c_1,\ a_1{+}b_2)\, I(a_2{+}d_1,\ d_1{+}b_2,\ b_1{+}c_1)$$
$$\times\, I(2a_2,\ b_1{+}c_2,\ 2b_2)\, I(a_1{+}d_1,\ a_2{+}b_1,\ d_1{+}c_1).$$

The labels of other tetrahedra Δ^1_{k+1}, Δ^2_{k+1}, Δ^3_{k+1} and Δ^4_{k+1} in the cyclic cover are obtained by replacing a_i, b_i, c_i and d_i with a_{i+k}, b_{i+k}, c_{i+k} and d_{i+k}. Hence, $I\big(M_n(6_1)\big)$ is presented by

$$\begin{aligned}
I\big(M_n(6_1)\big) \;=\; \sum_{\substack{a_0=0,\ a_1,\cdots,a_{n-1}\in\mathbb{Z}\\ b_0,\cdots,b_{n-1}\in\mathbb{Z}\\ c_0,\cdots,c_{n-1}\in\mathbb{Z}\\ d_0,\cdots,d_{n-1}\in\mathbb{Z}}} & q^{a_0+\cdots+a_{n-1}+b_0+\cdots+b_{n-1}+c_0+\cdots+c_{n-1}+d_0+\cdots+d_{n-1}} \\
& \times I(a_0+d_0, a_1+b_0, d_0+c_0)\, I(d_0+b_0, a_1+c_0, a_0+b_1) \\
& \quad \times I(a_1+d_0, d_0+b_1, b_0+c_0)\, I(2a_1, b_0+c_1, 2b_1) \\
& \times I(a_1+d_1, a_2+b_1, d_1+c_1)\, I(d_1+b_1, a_2+c_1, a_1+b_2) \\
& \quad \times I(a_2+d_1, d_1+b_2, b_1+c_1)\, I(2a_2, b_1+c_2, 2b_2) \\
& \times \cdots \\
\times I(a_{n-1}+d_{n-1}, a_n+b_{n-1}, & d_{n-1}+c_{n-1})\, I(d_{n-1}+b_{n-1}, a_n+c_{n-1}, a_{n-1}+b_n) \\
\times I(a_n+d_{n-1}, d_{n-1}+b_n, & b_{n-1}+c_{n-1})\, I(2a_n, b_{n-1}+c_n, 2b_n), \qquad (85)
\end{aligned}$$

where we regard the subscripts of a_i, b_i, c_i, d_i as modulo n.

The hyperbolicity equations are given by

$$\begin{aligned}
1-\frac{x_2}{x} &= (1-x)\big(1-\frac{1}{x}\big), \\
\big(1-\frac{x_2}{x}\big)\big(1-\frac{1}{x_2}\big) &= (1-x_2)\big(1-\frac{x_3}{x_2}\big), \\
\big(1-\frac{x_3}{x_2}\big)\big(1-\frac{1}{x_3}\big) &= 1-x_3.
\end{aligned}$$

They are rewritten

$$x_2 \;=\; x^2-x+1, \qquad x_3 \;=\; x_2+1-\frac{x_2}{x}, \qquad x_3+1-\frac{x_3}{x_2} \;=\; 0.$$

Further, they are rewritten

$$x^4-3x^3+6x^2-5x+2 \;=\; 0, \quad x_2 \;=\; x^2-x+1, \quad x_3 \;=\; \frac{x^3-2x^2+3x-1}{x}.$$

The hyperbolic structure of the 6_1 knot complement is given by the solution

$$x \;=\; 0.8951233822.....+\sqrt{-1}\;1.5524918200..... .$$

Putting $x_2' = x_2/x$ and $x_3' = x_3/x_2$, we have that

$$\begin{aligned}
&\frac{1}{2\pi}\operatorname{Arg} x \;=\; 0.16675... , \quad \frac{1}{2\pi}\operatorname{Arg}\frac{1}{1-x} \;=\; 0.23926... , \quad \frac{1}{2\pi}\operatorname{Arg}\big(1-\frac{1}{x}\big) \;=\; 0.09397... , \\
&\frac{1}{2\pi}\operatorname{Arg} x_2' \;=\; 0.22434... , \quad \frac{1}{2\pi}\operatorname{Arg}\frac{1}{1-x_2'} \;=\; 0.14528... , \quad \frac{1}{2\pi}\operatorname{Arg}\big(1-\frac{1}{x_2'}\big) \;=\; 0.13036... , \\
&\frac{1}{2\pi}\operatorname{Arg} x_3' \;=\; 0.07250... , \quad \frac{1}{2\pi}\operatorname{Arg}\frac{1}{1-x_3'} \;=\; 0.03639... , \quad \frac{1}{2\pi}\operatorname{Arg}\big(1-\frac{1}{x_3'}\big) \;=\; 0.39110... .
\end{aligned}$$

As approximations of these values, we put

$$\begin{aligned} \alpha_1 &= 0.166, & \beta_1 &= 0.24, & \gamma_1 &= 0.094, \\ \alpha_2 &= 0.224, & \beta_2 &= 0.146, & \gamma_2 &= 0.13, \\ \alpha_3 &= 0.074, & \beta_3 &= 0.036, & \gamma_3 &= 0.39. \end{aligned}$$

As mentioned in Section 2, we put

$$\begin{aligned} \hat{J}_{6_1,1}(\ell_1,\ell_2,\ell_3) &= q^{\alpha_1\ell_1+\beta_1\ell_2+\gamma_1\ell_3} I(\ell_1,\ell_2,\ell_3), \\ \hat{J}_{6_1,2}(\ell_1,\ell_2,\ell_3) &= q^{\alpha_2\ell_1+\beta_2\ell_2+\gamma_2\ell_3} I(\ell_1,\ell_2,\ell_3), \\ \hat{J}_{6_1,3}(\ell_1,\ell_2,\ell_3) &= q^{\alpha_3\ell_1+\beta_3\ell_2+\gamma_3\ell_3} I(\ell_1,\ell_2,\ell_3). \end{aligned}$$

Then, (85) is rewritten as

$$\begin{aligned} I\big(M_n(6_1)\big) = \sum_{\substack{a_0=0,\ a_1,\cdots,a_{n-1}\in\mathbb{Z} \\ b_0,\cdots,b_{n-1}\in\mathbb{Z} \\ c_0,\cdots,c_{n-1}\in\mathbb{Z} \\ d_0,\cdots,d_{n-1}\in\mathbb{Z}}} & \hat{J}_{6_1,3}(a_0+d_0, a_1+b_0, d_0+c_0)\, \hat{J}_{6_1,1}(d_0+b_0, a_1+c_0, a_0+b_1) \\ & \times \hat{J}_{6_1,2}(a_1+d_0, d_0+b_1, b_0+c_0)\, \hat{J}_{6_1,1}(2a_1, b_0+c_1, 2b_1) \\ & \times \hat{J}_{6_1,3}(a_1+d_1, a_2+b_1, d_1+c_1)\, \hat{J}_{6_1,1}(d_1+b_1, a_2+c_1, a_1+b_2) \\ & \times \hat{J}_{6_1,2}(a_2+d_1, d_1+b_2, b_1+c_1)\, \hat{J}_{6_1,1}(2a_2, b_1+c_2, 2b_2) \\ & \times \cdots \\ & \times \hat{J}_{6_1,3}(a_{n-1}+d_{n-1}, a_n+b_{n-1}, d_{n-1}+c_{n-1})\, \hat{J}_{6_1,1}(d_{n-1}+b_{n-1}, a_n+c_{n-1}, a_{n-1}+b_n) \\ & \times \hat{J}_{6_1,2}(a_n+d_{n-1}, d_{n-1}+b_n, b_{n-1}+c_{n-1})\, \hat{J}_{6_1,1}(2a_n, b_{n-1}+c_n, 2b_n), \end{aligned} \tag{86}$$

where we regard the subscripts of a_i, b_i, c_i, d_i as modulo n.

We use the formula (86) in Sections 4.3 and 5.3.

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