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A Note on Open Homomorphisms Between Global Solvably Closed Galois Groups

By

Yuichiro HOSHI

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京都大学　数理解析研究所

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES

KYOTO UNIVERSITY, Kyoto, Japan

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Abstract. — In the present paper, we study continuous open homomorphisms between the Galois groups of solvably closed Galois field extensions of number fields. In particular, we discuss Uchida's conjecture that asserts that an arbitrary continuous open homomorphism between the Galois groups of solvably closed Galois field extensions of number fields arises from a homomorphism between the given Galois field extensions. In the present paper, we prove that this conjecture is equivalent to the assertion that if the Galois group of a Galois field extension of a number field is isomorphic to an open subgroup of the maximal prosolvable quotient of the absolute Galois group of the field of rational numbers, then, for all prime numbers l and all but finitely many prime numbers p, the given Galois extension field contains l roots of the polynomial $t^l - p$. Moreover, we prove that this conjecture is also equivalent to the assertion that if the Galois group of a Galois field extension of an absolutely Galois number field is isomorphic to an open subgroup of the maximal prosolvable quotient of the absolute Galois group of the field of rational numbers, then the given Galois extension field is absolutely Galois.

Contents

Introduction

In the present paper, we study continuous open homomorphisms between the Galois groups of solvably closed Galois field extensions of number fields. We shall define

- a *number field* [cf. Definition 2.2, (i)] to be a field that is of characteristic zero and is finite over the minimal subfield of the field,

- a *solvably closed* field [cf. Definition 2.2, (ii)] to be a field that admits no nontrivial abelian field extension, and

- an *absolutely Galois* field [cf. Definition 3.3] to be a field that is [algebraic and] Galois over the minimal subfield of the field.

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In the present paper, we discuss the following conjecture posed by *K. Uchida* [cf. [8, Conjecture, p.595]]:

CONJECTURE A (Uchida). — *Let $F_\circ$, $F_\bullet$ be number fields, and let $\widetilde{F}_\circ$, $\widetilde{F}_\bullet$ be* ***Galois*** *extension fields of $F_\circ$, $F_\bullet$, respectively. Suppose that both $\widetilde{F}_\circ$ and $\widetilde{F}_\bullet$ are* ***solvably closed****. Let*

$$\alpha\colon \mathrm{Gal}(\widetilde{F}_\circ/F_\circ) \longrightarrow \mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$$

be a continuous ***open*** *homomorphism. Then there exists a homomorphism $\alpha_{\widetilde{F}}\colon \widetilde{F}_\bullet \hookrightarrow \widetilde{F}_\circ$ of rings* ***compatible*** *with the respective actions of $\mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$, $\mathrm{Gal}(\widetilde{F}_\circ/F_\circ)$, relative to the homomorphism α, i.e., such that, for each $\gamma \in \mathrm{Gal}(\widetilde{F}_\circ/F_\circ)$, the* ***equality*** *$\gamma \circ \alpha_{\widetilde{F}} = \alpha_{\widetilde{F}} \circ \alpha(\gamma)$ holds.*

Let us first recall that Uchida *solved affirmatively* the assertion obtained by replacing "a continuous *open* homomorphism" in the statement of Conjecture A by "a continuous *open injective* homomorphism" [cf. [7, Theorem]]. Moreover, Uchida also gave, in [8], some important results concerning Conjecture A. For instance, Uchida proved, in the situation of Conjecture A,

- the existence of a homomorphism "$\alpha_{\widetilde{F}}$" as in the statement of Conjecture A in the case where the number field $F_\circ$ is *isomorphic to the field of rational numbers* [cf. [8, Theorem 1]],
- the existence of a homomorphism "$\alpha_{\widetilde{F}}$" as in the statement of Conjecture A in the case where the homomorphism α *satisfies a certain condition concerning decomposition subgroups of nonarchimedean primes* [cf. [8, Theorem 2]], and
- the *uniqueness* of a homomorphism "$\alpha_{\widetilde{F}}$" as in the statement of Conjecture A [cf. [8, Proposition 2]].

Moreover, the author of the present paper

- studied Conjecture A from a "group-theoretic algorithmic" point of view [cf. [2], [4]] and
- proved the existence of a homomorphism "$\alpha_{\widetilde{F}}$" as in the statement of Conjecture A in the case where the homomorphism α is *compatible with the cyclotomic characters* [cf. [5, Theorem]].

The main result of the present paper may be stated as follows [cf. Corollary 2.7 and Corollary 3.5]:

THEOREM B. — *The following three assertions are equivalent:*

(1) *Let $F_\circ$, $F_\bullet$ be number fields, and let $\widetilde{F}_\circ$, $\widetilde{F}_\bullet$ be* ***Galois*** *extension fields of $F_\circ$, $F_\bullet$, respectively. Suppose that both $\widetilde{F}_\circ$ and $\widetilde{F}_\bullet$ are* ***solvably closed****. Let*

$$\alpha\colon \mathrm{Gal}(\widetilde{F}_\circ/F_\circ) \longrightarrow \mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$$

be a continuous ***open*** *homomorphism. Then there exists a homomorphism $\widetilde{F}_\bullet \hookrightarrow \widetilde{F}_\circ$ of rings* ***compatible*** *with the respective actions of $\mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$, $\mathrm{Gal}(\widetilde{F}_\circ/F_\circ)$, relative to the homomorphism α.*

(2) *Let $\overline{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$, and let $F \subseteq K \subseteq \overline{\mathbb{Q}}$ be subfields of $\overline{\mathbb{Q}}$ such that the field extension $F/\mathbb{Q}$ is* ***finite****, and, moreover, the field extension K/F is* ***Galois****. Suppose that the topological group* $\mathrm{Gal}(K/F)$ *is* ***isomorphic*** *to an open subgroup of the maximal prosolvable quotient of the absolute Galois group* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$*. Then, for all prime numbers l and all but finitely many prime numbers p,* ***every l-th power root of*** *p in $\overline{\mathbb{Q}}$ is* ***contained*** *in $K \subseteq \overline{\mathbb{Q}}$.*

(3) *Let $\overline{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$, and let $F \subseteq K \subseteq \overline{\mathbb{Q}}$ be subfields of $\overline{\mathbb{Q}}$ such that the field extension $F/\mathbb{Q}$ is* ***finite*** *and* ***Galois****, and, moreover, the field extension K/F is* ***Galois****. Suppose that the topological group* $\mathrm{Gal}(K/F)$ *is* ***isomorphic*** *to an open subgroup of the maximal prosolvable quotient of the absolute Galois group* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$*. Then the field extension $K/\mathbb{Q}$ is* ***Galois****.*

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1. Homomorphisms Between Topological Groups of MLF-type

In the present §1, we prove a technical lemma concerning continuous homomorphisms between topological groups *of MLF-type* [cf. Lemma 1.3 below]. This technical lemma may be regarded as a *partial generalization* of a result that was obtained in the study of the anabelian geometry of mixed-characteristic local fields [cf. Remark 1.3.1 below].

Definition 1.1. — Let D be a topological group of MLF-type [cf. [2, Definition 1.1], [2, Proposition 1.2, (i), (ii)]], i.e., a topological group such that there exist

- a prime number p,
- a finite extension field k of $\mathbb{Q}_p$,
- an algebraic closure $\overline{k}$ of k, and
- an isomorphism $\alpha_D\colon \mathrm{Gal}(\overline{k}/k) \xrightarrow{\sim} D$ of topological groups.

(i) Let us recall the positive integers

$$p(D), \qquad f(D)$$

defined in [2, Theorem 1.4, (1), (2)]. In particular, it follows from [2, Theorem 1.4, (i)] that the existence of the above isomorphism α_D implies that

(i-a) the positive integer $p(D)$ coincides with the prime number p, and that

(i-b) the positive integer $f(D)$ coincides with the extension degree of the residue field of k over the minimal subfield of the residue field of k.

(ii) Let us recall the closed subgroups

$$P(D) \subseteq I(D) \subseteq D$$

of D defined in [2, Theorem 1.4, (3)]. In particular, it follows from [2, Theorem 1.4, (ii)] that

(ii-a) the above isomorphism α_D restricts to a continuous isomorphism of the inertia subgroup of $\mathrm{Gal}(\overline{k}/k)$ with the closed subgroup $I(D) \subseteq D$ of D, and that

(ii-b) the above isomorphism α_D restricts to a continuous isomorphism of the wild inertia subgroup of $\mathrm{Gal}(\overline{k}/k)$ with the closed subgroup $P(D) \subseteq D$ of D.

(iii) Let us recall the closed subgroup

$$\mathcal{O}^{\times}(D) \stackrel{\mathrm{def}}{=} \mathrm{Im}\big(I(D) \hookrightarrow D \twoheadrightarrow D^{\mathrm{ab}}\big) \subseteq D^{\mathrm{ab}}$$

of D^{ab} defined in [2, Theorem 1.4, (5)]. In particular, it follows from [2, Theorem 1.4, (iii)] that the existence of the above isomorphism α_D implies that

(iii-a) the topological group of units of the normalization of $\mathbb{Z}_p$ in k is isomorphic to the topological group $\mathcal{O}^{\times}(D)$.

Lemma 1.2. — *Let D be a topological group **of MLF-type**, and let l be a prime number **not equal** to $p(D)$. Then every pro-l-Sylow subgroup of $I(D)$ is **isomorphic** to the topological group $\mathbb{Z}_l$.*

Proof. — This assertion is well-known [cf., e.g., [3, Lemma 1.5, (ii)] and Definition 1.1, (i-a), (ii-a), (ii-b)]. □

Lemma 1.3. — *Let $D_\circ$, $D_\bullet$ be topological groups **of MLF-type**, and let $\alpha\colon D_\circ \to D_\bullet$ be a continuous homomorphism. Suppose that the following two conditions are satisfied:*

*(1) The **equality** $p(D_\circ) = p(D_\bullet)$ holds.*

*(2) Let l be a prime number **not equal** to $p(D_\circ) = p(D_\bullet)$ [cf. (1)]. Then there exist a pro-l-Sylow subgroup ${}_lI(D_\circ) \subseteq I(D_\circ)$ of $I(D_\circ)$ and a normal open subgroup $N \subseteq D_\bullet$ of $D_\bullet$ such that the image of the composite*

$${}_lI(D_\circ) \hookrightarrow I(D_\circ) \hookrightarrow D_\circ \xrightarrow{\alpha} D_\bullet \twoheadrightarrow D_\bullet/N$$

*— where the first and second arrows are the natural inclusions, and the fourth arrow is the natural continuous surjective homomorphism — is a **nontrivial** l**-Sylow** subgroup of the finite group $D_\bullet/N$.*

Then the following assertions hold:

*(i) Let l be a prime number **not equal** to $p(D_\circ) = p(D_\bullet)$ [cf. (1)], and let ${}_lI(D_\circ) \subseteq I(D_\circ)$ be a pro-l-Sylow subgroup of $I(D_\circ)$. Then the homomorphism α restricts to an **isomorphism** of ${}_lI(D_\circ)$ with a pro-l-Sylow subgroup of $I(D_\bullet)$.*

*(ii) The integer $f(D_\circ)$ is **divisible** by the integer $f(D_\bullet)$.*

Proof. — We begin the proof of Lemma 1.3 with the following claim:

> **Claim 1.3.A.** — Let l be a prime number *not equal* to $p(D_\circ) = p(D_\bullet)$ [cf. condition (1)], and let ${}_lI(D_\circ) \subseteq I(D_\circ)$ be a pro-l-Sylow subgroup of $I(D_\circ)$. Then the image of the composite
>
> $${}_lI(D_\circ) \hookrightarrow I(D_\circ) \hookrightarrow D_\circ \xrightarrow{\alpha} D_\bullet$$

— where the first and second arrows are the natural inclusions — is *contained* in the subgroup $I(D_\bullet) \subseteq D_\bullet$ of $D_\bullet$.

To this end, let us first observe that it is well-known [cf., e.g., [3, Lemma 1.5, (i)] and Definition 1.1, (ii-a)] that the quotient $D_\bullet/I(D_\bullet)$ is *abelian* and *torsion-free*. In particular, to verify Claim 1.3.A, it suffices to verify the *triviality* of the image of ${}_lI(D_\circ)$ in the maximal abelian torsion-free quotient of $D_\circ$. On the other hand, since [we have assumed that] $l \neq p(D_\circ)$, this *triviality* is well-known [cf., e.g., [3, Lemma 1.2, (i)], [3, Lemma 1.7, (i)], and Definition 1.1, (i-a), (ii-a)]. This completes the proof of Claim 1.3.A.

First, we verify assertion (i). Let $N \subseteq D_\bullet$ be as in condition (2). Let us first observe that it follows from Claim 1.3.A that there exists a pro-l-Sylow subgroup ${}_lI(D_\bullet) \subseteq I(D_\bullet)$ of $I(D_\bullet)$ that *contains* the image of the composite discussed in Claim 1.3.A. Let ${}_l(D_\bullet/N) \subseteq D_\bullet/N$ be an l-Sylow subgroup of $D_\bullet/N$ that *contains* the image of ${}_lI(D_\bullet) \subseteq I(D_\bullet)$ in $D_\bullet/N$. Then it follows from condition (2) that

- the group ${}_l(D_\bullet/N)$ is *nontrivial*, and that
- the composite

$$ {}_lI(D_\circ) \longrightarrow {}_lI(D_\bullet) \longrightarrow {}_l(D_\bullet/N) $$

— where the first arrow is the homomorphism induced by α, and the second arrow is the homomorphism induced by the natural continuous surjective homomorphism $D_\bullet \twoheadrightarrow D_\bullet/N$ — is *surjective*.

In particular, one concludes immediately from Lemma 1.2 that the homomorphism ${}_lI(D_\circ) \to {}_lI(D_\bullet)$ induced by α is an *isomorphism*, as desired. This completes the proof of assertion (i).

Next, we verify assertion (ii). It follows immediately from assertion (i) that, for each prime number l *not equal* to $p(D_\circ) = p(D_\bullet)$ [cf. condition (1)], the homomorphism α determines a *surjective* homomorphism from the [unique] pro-l Sylow subgroup of $\mathcal{O}^\times(D_\circ)$ to the [unique] pro-l Sylow subgroup of $\mathcal{O}^\times(D_\bullet)$. In particular, one concludes immediately from [3, Lemma 1.2, (i)] and Definition 1.1, (i-a), (i-b), (iii-a), that $p(D_\circ)^{f(D_\circ)} - 1$ is *divisible* by $p(D_\bullet)^{f(D_\bullet)} - 1$, which thus implies [cf. condition (1)] that $f(D_\circ)$ is *divisible* by $f(D_\bullet)$, as desired. This completes the proof of assertion (ii), hence also of Lemma 1.3. $\square$

Remark 1.3.1. — Let $D_\circ$, $D_\bullet$ be topological groups *of MLF-type*, and let $\alpha\colon D_\circ \to D_\bullet$ be a continuous homomorphism. Suppose that the homomorphism α is *surjective*. Then one verifies easily from [1, Proposition 3.4, (i), (iii)] [cf. also [3, Lemma 1.5, (ii)] and Definition 1.1, (i-a), (ii-a), (ii-b)] that conditions (1), (2) in the statement of Lemma 1.3 are satisfied. Moreover, it follows from the final assertion of [1, Proposition 3.4, (iii)] that the *equality* $f(D_\circ) = f(D_\bullet)$ holds. Thus, Lemma 1.3, (ii), may be regarded as a *partial generalization* of the final assertion of [1, Proposition 3.4, (iii)].

2. The First Equivalence

In the present §2, we give a proof of the first main result of the present paper [cf. Corollary 2.7 below].

Lemma 2.1. — *Let p, l be **distinct** prime numbers, $\overline{\mathbb{Q}}_p$ an algebraic closure of $\mathbb{Q}_p$, $\zeta_l \in \overline{\mathbb{Q}}_p$ a primitive l-th root of unity, $p^{1/l} \in \overline{\mathbb{Q}}_p$ an l-th power root of $p \in \overline{\mathbb{Q}}_p$, and $L \subseteq \mathbb{Q}_p$ a subfield of $\mathbb{Q}_p$. Write $D \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ for the absolute Galois group of $\mathbb{Q}_p$ determined by the algebraic closure $\overline{\mathbb{Q}}_p$ and $I \subseteq D$ for the inertia subgroup of D. Let ${}_lI \subseteq I$ be a pro-l-Sylow subgroup of I. Then the following assertions hold.*

(i) *The subgroup $\mathrm{Gal}(L(\zeta_l, p^{1/l})/L(\zeta_l)) \subseteq \mathrm{Gal}(L(\zeta_l, p^{1/l})/L)$ is a **unique nontrivial** l**-Sylow** subgroup of $\mathrm{Gal}(L(\zeta_l, p^{1/l})/L)$.*

(ii) *The continuous homomorphism $D \to \mathrm{Gal}(L(\zeta_l, p^{1/l})/L)$ induced by the natural inclusion $L(\zeta_l, p^{1/l}) \hookrightarrow \overline{\mathbb{Q}}_p$ restricts to a continuous **surjective** homomorphism*

$$ {}_lI \twoheadrightarrow \mathrm{Gal}\big(L(\zeta_l, p^{1/l})/L(\zeta_l)\big). $$

Proof. — These assertions are immediate. □

Definition 2.2.

(i) We shall say that a field is a *number field* if the field is of characteristic zero and finite over the minimal subfield of the field.

(ii) We shall say that a field is *solvably closed* if the field admits no nontrivial abelian field extension.

Lemma 2.3. — *Let F be a number field, $\widetilde{F}$ a **Galois** extension field of F that is **solvably closed**, D a topological group **of MLF-type**,*

$$ \alpha \colon D \longrightarrow \mathrm{Gal}(\widetilde{F}/F) $$

*a continuous homomorphism, and l a prime number **not equal** to $p(D)$. Suppose that the following two conditions are satisfied:*

(1) *The number field F is **totally imaginary**.*

(2) *The image of a pro-l-Sylow subgroup of $I(D)$ by α is **nontrivial**.*

*Then there exist a **unique** nonarchimedean prime $\mathfrak{p}$ of F and a **unique** decomposition subgroup $D_{\mathfrak{p}} \subseteq \mathrm{Gal}(\widetilde{F}/F)$ of $\mathrm{Gal}(\widetilde{F}/F)$ at $\mathfrak{p}$ such that the image of α is **contained** in $D_{\mathfrak{p}} \subseteq \mathrm{Gal}(\widetilde{F}/F)$. Moreover, in this situation, the **residue characteristic** of $\mathfrak{p}$ is **not equal** to l.*

Proof. — Let ${}_lI(D) \subseteq I(D)$ be a pro-l-Sylow subgroup of $I(D)$. Let us first observe that since [we have assumed that — cf. condition (1)] the number field F is *totally imaginary*, the group $\mathrm{Gal}(\widetilde{F}/F)$ has *no nontrivial torsion element* [cf., e.g., the argument given in [8, pp.596-597]]. Thus, since [we have assumed that — cf. condition (2)] the image of ${}_lI(D)$ by α is *nontrivial*, it follows from Lemma 1.2 that the restriction of α to ${}_lI(D)$ is *injective*. In particular, it follows immediately from the well-known structure of a pro-l-Sylow subgroup of D [cf., e.g., the classification of the topological quotients of "$G_{\mathfrak{p}_1,l}$" given in [8, p.596]; also Definition 1.1, (i-a), (ii-a)] that the restriction of α to a pro-l-Sylow subgroup of D is *injective*. Thus, it follows immediately from a similar argument to the argument given in [8, pp.595-596] [cf. also [6, Proposition 2.3, (iv)]] that there exist a *unique* nonarchimedean prime $\mathfrak{p}$ of F and a *unique* decomposition subgroup

$D_{\mathfrak{p}} \subseteq \mathrm{Gal}(\widetilde{F}/F)$ of $\mathrm{Gal}(\widetilde{F}/F)$ at $\mathfrak{p}$ that satisfy the desired conditions. This completes the proof of Lemma 2.3. □

Definition 2.4. — Let F be a number field, and let $\mathfrak{p}$ be a nonarchimedean prime of F.

(i) We shall say that $\mathfrak{p}$ is *of absolute degree one* if the completion of F at $\mathfrak{p}$ is isomorphic to $\mathbb{Q}_p$, where we write p for the residue characteristic of $\mathfrak{p}$.

(ii) We shall say that $\mathfrak{p}$ is *of absolute residue degree one* if the residue field of F at $\mathfrak{p}$ is isomorphic to $\mathbb{F}_p$, where we write p for the residue characteristic of $\mathfrak{p}$.

Lemma 2.5. — *Let $F_\circ$, $F_\bullet$ be number fields, and let $\widetilde{F}_\circ$, $\widetilde{F}_\bullet$ be **Galois** extension fields of $F_\circ$, $F_\bullet$, respectively. Suppose that both $\widetilde{F}_\circ$ and $\widetilde{F}_\bullet$ are **solvably closed**. Let p be a prime number, $\mathfrak{p}_\circ$ a nonarchimedean prime of $F_\circ$ **of residue characteristic** p, $D_\circ \subseteq \mathrm{Gal}(\widetilde{F}_\circ/F_\circ)$ a decomposition subgroup of $\mathrm{Gal}(\widetilde{F}_\circ/F_\circ)$ at $\mathfrak{p}_\circ$, and*

$$\alpha\colon \mathrm{Gal}(\widetilde{F}_\circ/F_\circ) \twoheadrightarrow \mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$$

*a continuous **surjective** homomorphism. Suppose, moreover, that the following three conditions are satisfied:*

(1) *The number field $F_\bullet$ is **totally imaginary**.*

(2) *The nonarchimedean prime $\mathfrak{p}_\circ$ is **of absolute degree one**.*

(3) *Write ${}^{\alpha}F_\circ \subseteq \widetilde{F}_\circ$ for the subfield of $\widetilde{F}_\circ$ that corresponds to the kernel of the continuous homomorphism α. Then, for all prime numbers l **not equal** to p, **every l-th power root of** p in $\widetilde{F}_\circ$ is **contained** in ${}^{\alpha}F_\circ \subseteq \widetilde{F}_\circ$.*

Then the following assertions hold:

(i) *There exist a **unique** nonarchimedean prime $\mathfrak{p}_\bullet$ of $F_\bullet$ and a **unique** decomposition subgroup $D_\bullet \subseteq \mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$ of $\mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$ at $\mathfrak{p}_\bullet$ that satisfy the following four conditions:*

(a) *The image of $D_\circ \subseteq \mathrm{Gal}(\widetilde{F}_\circ/F_\circ)$ by α is **contained** in $D_\bullet \subseteq \mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$.*

(b) *The nonarchimedean prime $\mathfrak{p}_\bullet$ is **of residue characteristic** p.*

(c) *Let l be a prime number **not equal** to p. Then the homomorphism α restricts to an **isomorphism** of a pro-l-Sylow subgroup of the inertia subgroup of $D_\circ \subseteq \mathrm{Gal}(\widetilde{F}_\circ/F_\circ)$ with a pro-l-Sylow subgroup of the inertia subgroup of $D_\bullet \subseteq \mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$.*

(d) *The nonarchimedean prime $\mathfrak{p}_\bullet$ is **of absolute residue degree one**.*

(ii) *Suppose, moreover, that the following two conditions are satisfied:*

(4) *The prime number p is **odd**.*

(5) *Recall the subfield ${}^{\alpha}F_\circ \subseteq \widetilde{F}_\circ$ of $\widetilde{F}_\circ$ defined in* (3). *Then there exists a finite set S of prime numbers such that if q is a prime number **not contained** in S, then **every square root of** q in $\widetilde{F}_\circ$ is **contained** in ${}^{\alpha}F_\circ \subseteq \widetilde{F}_\circ$.*

Then the homomorphism $D_\circ^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z} \to D_\bullet^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z}$ induced by α [cf. (a)] *is an **isomorphism**.*

Proof. — We begin the proof of Lemma 2.5 with the following claim:

Claim 2.5.A. — Let l be a prime number *not equal* to p, ${}_lI_\circ \subseteq D_\circ$ a pro-l-Sylow subgroup of the inertia subgroup of $D_\circ$, $\zeta_l \in \widetilde{F}_\circ$ a primitive l-th root of unity, and $p^{1/l} \in \widetilde{F}_\circ$ an l-th power root of $p \in \widetilde{F}_\circ$. Then the image of the composite

$${}_lI_\circ \hookrightarrow \mathrm{Gal}(\widetilde{F}_\circ/F_\circ) \twoheadrightarrow \mathrm{Gal}({}^\alpha F_\circ/F_\circ) \twoheadrightarrow \mathrm{Gal}\big(F_\circ(\zeta_l, p^{1/l})/F_\circ\big)$$

— where the first arrow is the natural inclusion, and the second, third arrows are the continuous surjective homomorphisms determined by the natural inclusions ${}^\alpha F_\circ \hookrightarrow \widetilde{F}_\circ$, $F_\circ(\zeta_l, p^{1/l}) \hookrightarrow {}^\alpha F_\circ$ [cf. condition (3)], respectively — is a *unique nontrivial l-Sylow* subgroup of $\mathrm{Gal}(F(\zeta_l, p^{1/l})/F_\circ)$.

To this end, let us first recall that [we have assumed that — cf. condition (2)] the nonarchimedean prime $\mathfrak{p}_\circ$ is *of absolute degree one*. Thus, Claim 2.5.A follows immediately form Lemma 2.1, (i), (ii). This completes the proof of Claim 2.5.A.

First, we verify assertion (i). Observe that it is immediate from Claim 2.5.A that, for each prime number l *not equal* to p and each pro-l-Sylow subgroup ${}_lI_\circ \subseteq D_\circ$ of the inertia subgroup of $D_\circ$, the image of the composite

$${}_lI_\circ \hookrightarrow \mathrm{Gal}(\widetilde{F}_\circ/F_\circ) \xrightarrow{\alpha} \mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$$

— where the first arrow is the natural inclusion — is *nontrivial*. Thus, one concludes immediately from Lemma 2.3 [cf. also condition (1)] that there exist a *unique* nonarchimedean prime $\mathfrak{p}_\bullet$ of $F_\bullet$ and a *unique* decomposition subgroup $D_\bullet \subseteq \mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$ of $\mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$ at $\mathfrak{p}_\bullet$ that satisfy conditions (a), (b). Moreover, since [we have assumed that — cf. condition (2)] the nonarchimedean prime $\mathfrak{p}_\circ$ is *of absolute degree one*, hence also *of absolute residue degree one*, by applying Lemma 1.3, (i), (ii) [cf. also Definition 1.1, (i-a), (i-b), (ii-a)], to the homomorphism $D_\circ \to D_\bullet$ induced by α [cf. condition (a)], one also concludes immediately from condition (b) and Claim 2.5.A that conditions (c), (d) are satisfied. This completes the proof of assertion (i).

Next, we verify assertion (ii). Write $I_\circ \subseteq D_\circ$, $I_\bullet \subseteq D_\bullet$ for the respective inertia subgroups of $D_\circ$, $D_\bullet$. Now recall that [we have assumed that — cf. condition (4)] the prime number p is *odd*. Thus, it is well-known [cf., e.g., [3, Lemma 1.5, (ii)]] that if ${}_2I_\circ \subseteq I_\circ$, ${}_2I_\bullet \subseteq I_\bullet$ are pro-2-Sylow subgroups of $I_\circ$, $I_\bullet$, respectively, then the natural homomorphisms ${}_2I_\circ^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z} \to I_\circ^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z}$, ${}_2I_\bullet^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z} \to I_\bullet^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z}$ are *isomorphisms* [cf. also condition (b)]. In particular, it follows from conditions (a), (c) that the homomorphism α induces a homomorphism

$$(D_\circ/I_\circ)^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z} \longrightarrow (D_\bullet/I_\bullet)^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z},$$

and, moreover, to verify assertion (ii), it suffices to verify that this homomorphism is an *isomorphism*. Thus, since [it is well-known — cf., e.g., [3, Lemma 1.5, (i)] — that] both $(D_\circ/I_\circ)^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z}$ and $(D_\bullet/I_\bullet)^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z}$ are *of order two*, one concludes [cf. also condition (c)] that, to verify assertion (ii), it suffices to verify that there exists a homomorphism $\mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet) \to \mathbb{Z}/2\mathbb{Z}$ such that the composite

$$D_\circ \hookrightarrow \mathrm{Gal}(\widetilde{F}_\circ/F_\circ) \xrightarrow{\alpha} \mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet) \longrightarrow \mathbb{Z}/2\mathbb{Z}$$

— where the first arrow is the natural inclusion, and the third arrow is the homomorphism under consideration — determines an *isomorphism* $(D_\circ/I_\circ)^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z} \xrightarrow{\sim} \mathbb{Z}/2\mathbb{Z}$. On the other hand, such a homomorphism $\mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet) \to \mathbb{Z}/2\mathbb{Z}$ may be obtained by pulling back, by the inverse of the isomorphism $\mathrm{Gal}({}^{\alpha}F_\circ/F_\circ) \xrightarrow{\sim} \mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$ determined by α, the continuous surjective homomorphism $\mathrm{Gal}({}^{\alpha}F_\circ/F_\circ) \twoheadrightarrow \mathrm{Gal}(F_\circ(q^{1/2})/F_\circ)$ determined by the natural inclusion $F_\circ(q^{1/2}) \hookrightarrow {}^{\alpha}F_\circ$, where q is a prime number *not contained* in the finite set S of condition (5) such that the image of q in $\mathbb{F}_p$ is *not contained* in $\mathbb{F}_p^2$ ($\stackrel{\mathrm{def}}{=} \{ a^2 \in \mathbb{F}_p \mid a \in \mathbb{F}_p \}$), and $q^{1/2} \in {}^{\alpha}F_\circ$ is a *square root* of q [cf. condition (5)]. [Note that it follows from *Dirichlet's theorem on primes in arithmetic progressions* that the set consisting of prime numbers whose images in $\mathbb{F}_p$ are *not contained* in $\mathbb{F}_p^2$ is *infinite.*] This completes the proof of assertion (ii), hence also of Lemma 2.5. □

Theorem 2.6. — *Let $F_\circ$, $F_\bullet$ be number fields, and let $\widetilde{F}_\circ$, $\widetilde{F}_\bullet$ be* ***Galois*** *extension fields of $F_\circ$, $F_\bullet$, respectively. Suppose that both $\widetilde{F}_\circ$ and $\widetilde{F}_\bullet$ are* ***solvably closed***. *Let*

$$\alpha\colon \mathrm{Gal}(\widetilde{F}_\circ/F_\circ) \longrightarrow \mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$$

be a continuous ***open*** *homomorphism. Then the following two conditions are equivalent:*

(1) *There exists a homomorphism $\widetilde{F}_\bullet \hookrightarrow \widetilde{F}_\circ$ of rings* ***compatible*** *with the respective actions of $\mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$, $\mathrm{Gal}(\widetilde{F}_\circ/F_\circ)$, relative to the homomorphism α.*

(2) *Write ${}^{\alpha}F_\circ \subseteq \widetilde{F}_\circ$ for the subfield of $\widetilde{F}_\circ$ that corresponds to the kernel of the continuous homomorphism α. Then, for all prime numbers l and all but finitely many prime numbers p,* ***every l-th power root of*** *p in $\widetilde{F}_\circ$ is* ***contained*** *in ${}^{\alpha}F_\circ \subseteq \widetilde{F}_\circ$.*

Proof. — First, we verify the implication (1) ⇒ (2). Suppose that condition (1) is satisfied. Then it is immediate that the field ${}^{\alpha}F_\circ$ contains the field *isomorphic* to $\widetilde{F}_\bullet$. Thus, since [we have assumed that] the field $\widetilde{F}_\bullet$ is *solvably closed*, it is immediate that, for all prime numbers l, p, *every l-th power root of* p in $\widetilde{F}_\circ$ is *contained* in ${}^{\alpha}F_\circ \subseteq \widetilde{F}_\circ$, as desired. This completes the proof of the implication (1) ⇒ (2).

Next, we verify the implication (2) ⇒ (1). Suppose that condition (2) is satisfied. Let us first observe that since [we have assumed that] the continuous homomorphism α is *open*, to verify condition (1), we may assume without loss of generality, by replacing $\mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$ by the image of α, that α is *surjective*.

Let $K_\bullet \subseteq \widetilde{F}_\bullet$ be a finite Galois extension field of $F_\bullet$ contained in $\widetilde{F}_\bullet$ that is *totally imaginary*. Write $K_\circ \subseteq \widetilde{F}_\circ$ for the finite Galois extension field of $F_\circ$ contained in $\widetilde{F}_\circ$ that corresponds to the normal open subgroup of $\mathrm{Gal}(\widetilde{F}_\circ/F_\circ)$ obtained by forming the inverse image by the continuous surjective homomorphism α of $\mathrm{Gal}(\widetilde{F}_\bullet/K_\bullet) \subseteq \mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$. Thus, we have a commutative diagram of topological groups

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{Gal}(\widetilde{F}_\circ/K_\circ) & \longrightarrow & \mathrm{Gal}(\widetilde{F}_\circ/F_\circ) & \longrightarrow & \mathrm{Gal}(K_\circ/F_\circ) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow \alpha & & \wr\downarrow \alpha_K & & \\
1 & \longrightarrow & \mathrm{Gal}(\widetilde{F}_\bullet/K_\bullet) & \longrightarrow & \mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet) & \longrightarrow & \mathrm{Gal}(K_\bullet/F_\bullet) & \longrightarrow & 1
\end{array}$$

— where the horizontal sequences are *exact*, the vertical arrows are *surjective*, and the right-hand vertical arrow is an *isomorphism.* Then one concludes immediately from [5, Corollary 2.8] and Lemma 2.5, (i), (ii) [cf. also condition (2)], that the continuous surjective homomorphism

$$\mathrm{Gal}(\widetilde{F}_\circ/K_\circ)^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z} \longrightarrow\!\!\!\!\!\rightarrow \mathrm{Gal}(\widetilde{F}_\bullet/K_\bullet)^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z}$$

determined by the left-hand vertical arrow $\mathrm{Gal}(\widetilde{F}_\circ/K_\circ) \to \mathrm{Gal}(\widetilde{F}_\bullet/K_\bullet)$ of the above diagram *arises* from a *uniquely determined* homomorphism of rings

$$K_\bullet \hookrightarrow K_\circ.$$

Next, write

$$\mathrm{Aut}^*\big(\mathrm{Gal}(\widetilde{F}_\circ/K_\circ)^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z}\big) \subseteq \mathrm{Aut}\big(\mathrm{Gal}(\widetilde{F}_\circ/K_\circ)^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z}\big)$$

for the subgroup consisting of the continuous automorphisms of the topological group $\mathrm{Gal}(\widetilde{F}_\circ/K_\circ)^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z}$ that *preserve* the kernel of the continuous surjective homomorphism $\mathrm{Gal}(\widetilde{F}_\circ/K_\circ)^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z} \twoheadrightarrow \mathrm{Gal}(\widetilde{F}_\bullet/K_\bullet)^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z}$ determined by the left-hand vertical arrow of the above diagram. Then the above diagram determines a commutative diagram of groups

$$\begin{array}{ccc} \mathrm{Gal}(K_\circ/F_\circ) & \longrightarrow & \mathrm{Aut}^*\big(\mathrm{Gal}(\widetilde{F}_\circ/K_\circ)^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z}\big) \\ {\scriptstyle \alpha_K}\downarrow{\scriptstyle \wr} & & \downarrow \\ \mathrm{Gal}(K_\bullet/F_\bullet) & \longrightarrow & \mathrm{Aut}\big(\mathrm{Gal}(\widetilde{F}_\bullet/K_\bullet)^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Z}/2\mathbb{Z}\big) \end{array}$$

— where the horizontal arrows are the respective natural continuous actions, i.e., determined by the horizontal sequences of the above diagram, and the right-hand vertical arrow is the homomorphism induced by the left-hand vertical arrow of the above diagram. In particular, one concludes immediately from the *commutativity* of this diagram that the homomorphism $K_\bullet \hookrightarrow K_\circ$ of rings that appears in the preceding paragraph is *compatible* with the respective actions of $\mathrm{Gal}(K_\bullet/F_\bullet)$, $\mathrm{Gal}(K_\circ/F_\circ)$, relative to the isomorphism $\alpha_K\colon \mathrm{Gal}(K_\circ/F_\circ) \xrightarrow{\sim} \mathrm{Gal}(K_\bullet/F_\bullet)$. Thus, by allowing "$K_\bullet$" to vary, one also concludes that there exists a homomorphism $\widetilde{F}_\bullet \hookrightarrow \widetilde{F}_\circ$ of rings *compatible* with the respective actions of $\mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$, $\mathrm{Gal}(\widetilde{F}_\circ/F_\circ)$, relative to the homomorphism α, as desired. This completes the proof of the implication (2) ⇒ (1), hence also of Theorem 2.6. □

Corollary 2.7. — *The following two assertions are equivalent:*

(1) *Let $F_\circ$, $F_\bullet$ be number fields, and let $\widetilde{F}_\circ$, $\widetilde{F}_\bullet$ be **Galois** extension fields of $F_\circ$, $F_\bullet$, respectively. Suppose that both $\widetilde{F}_\circ$ and $\widetilde{F}_\bullet$ are **solvably closed**. Let*

$$\alpha\colon \mathrm{Gal}(\widetilde{F}_\circ/F_\circ) \longrightarrow \mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$$

*be a continuous **open** homomorphism. Then there exists a homomorphism $\widetilde{F}_\bullet \hookrightarrow \widetilde{F}_\circ$ of rings **compatible** with the respective actions of $\mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$, $\mathrm{Gal}(\widetilde{F}_\circ/F_\circ)$, relative to the homomorphism α.*

(2) *Let $\overline{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$, and let $F \subseteq K \subseteq \overline{\mathbb{Q}}$ be subfields of $\overline{\mathbb{Q}}$ such that the field extension $F/\mathbb{Q}$ is **finite**, and, moreover, the field extension K/F is **Galois**. Suppose that the topological group $\mathrm{Gal}(K/F)$ is **isomorphic** to an open subgroup of the maximal prosolvable quotient of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Then, for all prime numbers l and all but finitely many prime numbers p, **every l-th power root of** p in $\overline{\mathbb{Q}}$ is **contained** in $K \subseteq \overline{\mathbb{Q}}$.*

Proof. — First, we verify the implication (1) ⇒ (2). Suppose that assertion (1) is satisfied. Let $\overline{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$. Write $\mathbb{Q}^{\mathrm{slv}} \subseteq \overline{\mathbb{Q}}$ for the maximal prosolvable extension field of $\mathbb{Q}$ in $\overline{\mathbb{Q}}$. Let $F \subseteq K \subseteq \overline{\mathbb{Q}}$ be subfields of $\overline{\mathbb{Q}}$ as in assertion (2), which thus implies that there exists a continuous *open injective* homomorphism $\mathrm{Gal}(K/F) \hookrightarrow \mathrm{Gal}(\mathbb{Q}^{\mathrm{slv}}/\mathbb{Q})$. Then, by applying assertion (1) to the composite

$$\mathrm{Gal}(\overline{\mathbb{Q}}/F) \twoheadrightarrow \mathrm{Gal}(K/F) \hookrightarrow \mathrm{Gal}(\mathbb{Q}^{\mathrm{slv}}/\mathbb{Q})$$

— where the first arrow is the continuous surjective homomorphism determined by the natural inclusion $K \hookrightarrow \overline{\mathbb{Q}}$, and the second arrow is a continuous *open injective* homomorphism — one concludes immediately that the field K *contains* $\mathbb{Q}^{\mathrm{slv}}$. In particular, for all prime numbers l, p, *every l-th power root of p* in $\overline{\mathbb{Q}}$ is *contained* in $K \subseteq \overline{\mathbb{Q}}$, as desired. This completes the proof of the implication (1) $\Rightarrow$ (2).

Next, we verify the implication (2) $\Rightarrow$ (1). Suppose that assertion (2) is satisfied. Let $F_\circ$, $F_\bullet$, $\widetilde{F}_\circ$, $\widetilde{F}_\bullet$, $\alpha\colon \mathrm{Gal}(\widetilde{F}_\circ/F_\circ) \to \mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$ be as in assertion (1). Write $(F_\bullet)_0 \subseteq F_\bullet$ for the minimal subfield of $F_\bullet$, $(\widetilde{F}_\bullet)_0 \subseteq \widetilde{F}_\bullet$ for the maximal prosolvable extension field of $(F_\bullet)_0$ in $\widetilde{F}_\bullet$, and

$$\pi\colon \mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet) \longrightarrow \mathrm{Gal}\big((\widetilde{F}_\bullet)_0/(F_\bullet)_0\big)$$

for the continuous open homomorphism determined by the natural inclusion $(\widetilde{F}_\bullet)_0 \hookrightarrow \widetilde{F}_\bullet$. Write, moreover, ${}^{\pi\circ\alpha}F_\circ \subseteq {}^{\alpha}F_\circ \subseteq \widetilde{F}_\circ$ for the respective subfields of $\widetilde{F}_\circ$ that correspond to the kernels of the continuous homomorphisms $\pi \circ \alpha$, α. Then since [it is immediate that] the topological group $\mathrm{Gal}({}^{\pi\circ\alpha}F_\circ/F_\circ)$ is *isomorphic* to an open subgroup of $\mathrm{Gal}((\widetilde{F}_\bullet)_0/(F_\bullet)_0)$, it follows immediately from assertion (2) that, for all prime numbers l and all but finitely many prime numbers p, *every l-th power root of p* in $\widetilde{F}_\circ$ is *contained* in ${}^{\pi\circ\alpha}F_\circ \subseteq \widetilde{F}_\circ$, hence also in ${}^{\alpha}F_\circ \subseteq \widetilde{F}_\circ$. In particular, it follows from the implication (2) $\Rightarrow$ (1) of Theorem 2.6 that there exists a homomorphism $\widetilde{F}_\bullet \hookrightarrow \widetilde{F}_\circ$ of rings *compatible* with the respective actions of $\mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$, $\mathrm{Gal}(\widetilde{F}_\circ/F_\circ)$, relative to the homomorphism α, as desired. This completes the proof of the implication (2) $\Rightarrow$ (1), hence also of Corollary 2.7. $\square$

3. The Second Equivalence

In the present §3, we give a proof of the second main result of the present paper [cf. Corollary 3.5 below].

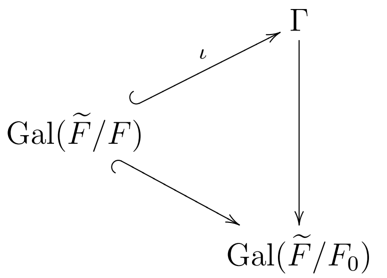
Lemma 3.1. — *Let F be a number field, and let $\widetilde{F}$ be a* ***Galois*** *extension field of F that is* ***solvably closed****. Write $\mathrm{Aut}(\mathrm{Gal}(\widetilde{F}/F))$ for the group of continuous automorphisms of the topological group $\mathrm{Gal}(\widetilde{F}/F)$. Then there exist a subfield $F_0 \subseteq F$ of F over which $\widetilde{F}$ is* ***Galois*** *and a commutative diagram of groups*

$$\begin{array}{ccc} & & \mathrm{Aut}\big(\mathrm{Gal}(\widetilde{F}/F)\big) \\ & \nearrow & \downarrow \wr \\ \mathrm{Gal}(\widetilde{F}/F) & & \\ & \searrow & \\ & & \mathrm{Gal}(\widetilde{F}/F_0) \end{array}$$

— *where the upper diagonal arrow is the continuous action by conjugation, the lower diagonal arrow is the continuous open injective homomorphism determined by the natural inclusion $F_0 \hookrightarrow F$, and the right-hand vertical arrow is an* ***isomorphism****.*

PROOF. — This assertion is a formal consequence of [7, Theorem]. $\square$

LEMMA 3.2. — *Let F be a number field, $\widetilde{F}$ a* ***Galois*** *extension field of F that is* ***solvably closed****, Γ a topological group, and $\iota\colon \mathrm{Gal}(\widetilde{F}/F) \hookrightarrow \Gamma$ a continuous* ***injective*** *homomorphism. Suppose that the image of ι is* ***normal*** *in Γ. Then there exist a subfield $F_0 \subseteq F$ of F over which $\widetilde{F}$ is* ***Galois*** *and a commutative diagram of groups*



— *where the lower diagonal arrow is the continuous open injective homomorphism determined by the natural inclusion $F_0 \hookrightarrow F$, and the image of the right-hand vertical arrow is* ***open****.*

PROOF. — One concludes immediately this assertion from Lemma 3.1 by considering the continuous action of Γ on $\mathrm{Gal}(\widetilde{F}/F)$ by conjugation. $\square$

DEFINITION 3.3. — We shall say that a field is *absolutely Galois* if the field is [algebraic and] Galois over the minimal subfield of the field.

THEOREM 3.4. — *Let $F_\circ$, $F_\bullet$ be number fields, and let $\widetilde{F}_\circ$, $\widetilde{F}_\bullet$ be* ***Galois*** *extension fields of $F_\circ$, $F_\bullet$, respectively. Suppose that both $\widetilde{F}_\circ$ and $\widetilde{F}_\bullet$ are* ***solvably closed****. Let*

$$\alpha\colon \mathrm{Gal}(\widetilde{F}_\circ/F_\circ) \longrightarrow \mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$$

be a continuous ***open*** *homomorphism. Suppose, moreover, that the subfield of $\widetilde{F}_\circ$ that corresponds to the kernel of the continuous homomorphism α is* ***absolutely Galois****. Then there exists a homomorphism $\widetilde{F}_\bullet \hookrightarrow \widetilde{F}_\circ$ of rings* ***compatible*** *with the respective actions of $\mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$, $\mathrm{Gal}(\widetilde{F}_\circ/F_\circ)$, relative to the homomorphism α.*

PROOF. — Write $(F_\circ)_0 \subseteq F_\circ$ for the minimal subfield of $F_\circ$ and ${}^{\alpha}F_\circ \subseteq \widetilde{F}_\circ$ for the subfield of $\widetilde{F}_\circ$ that corresponds to the kernel of the continuous homomorphism α. Let us first observe that since [we have assumed that] the continuous homomorphism α is *open*, to verify Theorem 3.4, we may assume without loss of generality, by replacing $\mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$ by the image of α, that α is *surjective*. Moreover, observe that, to verify Theorem 3.4, we may assume without loss of generality, by replacing $\widetilde{F}_\circ$ by an algebraic closure of $\widetilde{F}_\circ$, that $\widetilde{F}_\circ$ is *algebraically closed*. Thus, to verify Theorem 3.4, we may assume without loss of generality, by replacing $F_\circ$ by a suitable finite extension field of $F_\circ$ in $\widetilde{F}_\circ$, that $F_\circ$ is *Galois* over $(F_\circ)_0$.

Next, observe that it follows from our assumption that ${}^{\alpha}F_{\circ}$ is *Galois* over $(F_{\circ})_0$ that the natural inclusions $(F_{\circ})_0 \hookrightarrow F_{\circ} \hookrightarrow {}^{\alpha}F_{\circ} \hookrightarrow \widetilde{F}_{\circ}$ determine a commutative diagram of topological groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{Gal}(\widetilde{F}_{\circ}/F_{\circ}) & \longrightarrow & \mathrm{Gal}\big(\widetilde{F}_{\circ}/(F_{\circ})_0\big) & \longrightarrow & \mathrm{Gal}\big(F_{\circ}/(F_{\circ})_0\big) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \mathrm{Gal}({}^{\alpha}F_{\circ}/F_{\circ}) & \longrightarrow & \mathrm{Gal}\big({}^{\alpha}F_{\circ}/(F_{\circ})_0\big) & \longrightarrow & \mathrm{Gal}\big(F_{\circ}/(F_{\circ})_0\big) & \longrightarrow & 1 \end{array}$$

— where the horizontal sequences are *exact*, and the vertical arrows are *surjective*. Thus, since $\mathrm{Gal}({}^{\alpha}F_{\circ}/F_{\circ})$ is *isomorphic* to $\mathrm{Gal}(\widetilde{F}_{\bullet}/F_{\bullet})$, it follows immediately from Lemma 3.2 that there exist a subfield $(F_{\bullet})_0 \subseteq F_{\bullet}$ of $F_{\bullet}$ over which $\widetilde{F}_{\bullet}$ is *Galois* and a commutative diagram of topological groups

$$\begin{array}{ccc} \mathrm{Gal}(\widetilde{F}_{\circ}/F_{\circ}) & \hookrightarrow & \mathrm{Gal}\big(\widetilde{F}_{\circ}/(F_{\circ})_0\big) \\ {\scriptstyle\alpha}\downarrow & & \downarrow \\ \mathrm{Gal}(\widetilde{F}_{\bullet}/F_{\bullet}) & \hookrightarrow & \mathrm{Gal}\big(\widetilde{F}_{\bullet}/(F_{\bullet})_0\big) \end{array}$$

— where the upper, lower horizontal arrows are the respective continuous open injective homomorphisms determined by the natural inclusions $(F_{\circ})_0 \hookrightarrow F_{\circ}$, $(F_{\bullet})_0 \hookrightarrow F_{\bullet}$, and the vertical arrows are *surjective*. In particular, to verify Theorem 3.4, we may assume without loss of generality, by replacing α by the right-hand vertical arrow of this diagram, that $F_{\circ}$ is *isomorphic to the field of rational numbers.* On the other hand, it follows from [8, Theorem 1] that there exists a homomorphism $\widetilde{F}_{\bullet} \hookrightarrow \widetilde{F}_{\circ}$ of rings *compatible* with the respective actions of $\mathrm{Gal}(\widetilde{F}_{\bullet}/F_{\bullet})$, $\mathrm{Gal}(\widetilde{F}_{\circ}/F_{\circ})$, relative to the homomorphism α whenever $F_{\circ}$ is *isomorphic to the field of rational numbers.* This completes the proof of Theorem 3.4. □

Corollary 3.5. — *The following two assertions are equivalent:*

(1) *Let $F_{\circ}$, $F_{\bullet}$ be number fields, and let $\widetilde{F}_{\circ}$, $\widetilde{F}_{\bullet}$ be* ***Galois*** *extension fields of $F_{\circ}$, $F_{\bullet}$, respectively. Suppose that both $\widetilde{F}_{\circ}$ and $\widetilde{F}_{\bullet}$ are* ***solvably closed****. Let*

$$\alpha\colon \mathrm{Gal}(\widetilde{F}_{\circ}/F_{\circ}) \longrightarrow \mathrm{Gal}(\widetilde{F}_{\bullet}/F_{\bullet})$$

be a continuous ***open*** *homomorphism. Then there exists a homomorphism $\widetilde{F}_{\bullet} \hookrightarrow \widetilde{F}_{\circ}$ of rings* ***compatible*** *with the respective actions of* $\mathrm{Gal}(\widetilde{F}_{\bullet}/F_{\bullet})$, $\mathrm{Gal}(\widetilde{F}_{\circ}/F_{\circ})$*, relative to the homomorphism α.*

(2) *Let $\overline{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$, and let $F \subseteq K \subseteq \overline{\mathbb{Q}}$ be subfields of $\overline{\mathbb{Q}}$ such that the field extension $F/\mathbb{Q}$ is* ***finite*** *and* ***Galois****, and, moreover, the field extension K/F is* ***Galois****. Suppose that the topological group* $\mathrm{Gal}(K/F)$ *is* ***isomorphic*** *to an open subgroup of the maximal prosolvable quotient of the absolute Galois group* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$*. Then the field extension $K/\mathbb{Q}$ is* ***Galois****.*

Proof. — First, we verify the implication (1) ⇒ (2). Suppose that assertion (1) is satisfied. Let $\overline{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$. Write $\mathbb{Q}^{\mathrm{slv}} \subseteq \overline{\mathbb{Q}}$ for the maximal prosolvable extension field of $\mathbb{Q}$ in $\overline{\mathbb{Q}}$. Let $F \subseteq K \subseteq \overline{\mathbb{Q}}$ be subfields of $\overline{\mathbb{Q}}$ as in assertion (2), which thus implies that there exists a continuous *open injective* homomorphism $\mathrm{Gal}(K/F) \hookrightarrow \mathrm{Gal}(\mathbb{Q}^{\mathrm{slv}}/\mathbb{Q})$. Then, by applying assertion (1) to the composite

$$\mathrm{Gal}(\overline{\mathbb{Q}}/F) \twoheadrightarrow \mathrm{Gal}(K/F) \hookrightarrow \mathrm{Gal}(\mathbb{Q}^{\mathrm{slv}}/\mathbb{Q})$$

— where the first arrow is the continuous surjective homomorphism determined by the natural inclusion $K \hookrightarrow \overline{\mathbb{Q}}$, and the second arrow is a continuous *open injective* homomorphism — one concludes immediately that the *equality* $K = F \cdot \mathbb{Q}^{\mathrm{slv}}$ in $\overline{\mathbb{Q}}$ holds. In particular, the field extension $K/\mathbb{Q}$ is *Galois*, as desired. This completes the proof of the implication (1) ⇒ (2).

Next, we verify the implication (2) ⇒ (1). Suppose that assertion (2) is satisfied. Let $F_\circ$, $F_\bullet$, $\widetilde{F}_\circ$, $\widetilde{F}_\bullet$, $\alpha\colon \mathrm{Gal}(\widetilde{F}_\circ/F_\circ) \to \mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$ be as in assertion (1). Let us first observe that, to verify the existence of a homomorphism "$\widetilde{F}_\bullet \hookrightarrow \widetilde{F}_\circ$" as in assertion (1), we may assume without loss of generality, by replacing $\widetilde{F}_\circ$ by an algebraic closure of $\widetilde{F}_\circ$, that $\widetilde{F}_\circ$ is *algebraically closed*. Thus, to verify the existence of a homomorphism "$\widetilde{F}_\bullet \hookrightarrow \widetilde{F}_\circ$" as in assertion (1), we may assume without loss of generality, by replacing $F_\circ$ by a suitable finite extension field of $F_\circ$ in $\widetilde{F}_\circ$, that $F_\circ$ is *absolutely Galois*. Write $(F_\bullet)_0 \subseteq F_\bullet$ for the minimal subfield of $F_\bullet$, $(\widetilde{F}_\bullet)_0 \subseteq \widetilde{F}_\bullet$ for the maximal prosolvable extension field of $(F_\bullet)_0$ in $\widetilde{F}_\bullet$, and

$$\pi\colon \mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet) \longrightarrow \mathrm{Gal}\big((\widetilde{F}_\bullet)_0/(F_\bullet)_0\big)$$

for the continuous open homomorphism determined by the natural inclusion $(\widetilde{F}_\bullet)_0 \hookrightarrow \widetilde{F}_\bullet$. Then it follows immediately from Theorem 2.6 that, to verify the existence of a homomorphism "$\widetilde{F}_\bullet \hookrightarrow \widetilde{F}_\circ$" as in assertion (1), we may assume without loss of generality, by replacing α by $\pi \circ \alpha$, that $F_\bullet$ is *isomorphic to the field of rational numbers*, and $\widetilde{F}_\bullet$ is *isomorphic to a maximal prosolvable extension field of the field of rational numbers*. Write ${}^{\alpha}F_\circ \subseteq \widetilde{F}_\circ$ for the subfield of $\widetilde{F}_\circ$ that corresponds to the kernel of the continuous homomorphism α. Then since [it is immediate that] the topological group $\mathrm{Gal}({}^{\alpha}F_\circ/F_\circ)$ is *isomorphic* to an open subgroup of $\mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$, it follows from assertion (2) that the field ${}^{\alpha}F_\circ$ is *absolutely Galois*. In particular, it follows from Theorem 3.4 that there exists a homomorphism $\widetilde{F}_\bullet \hookrightarrow \widetilde{F}_\circ$ of rings *compatible* with the respective actions of $\mathrm{Gal}(\widetilde{F}_\bullet/F_\bullet)$, $\mathrm{Gal}(\widetilde{F}_\circ/F_\circ)$, relative to the homomorphism α, as desired. This completes the proof of the implication (2) ⇒ (1), hence also of Corollary 3.5. □

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(Yuichiro Hoshi) Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, JAPAN

Email address: yuichiro@kurims.kyoto-u.ac.jp