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Hom-versions of the Combinatorial Grothendieck Conjecture I: Abelianizations and Graphically Full Actions

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Abstract

Semi-graphs of anabelioids of PSC-type and their PSC-fundamental groups (i.e., a combinatorial Galois-category-theoretic abstraction of pointed stable curves over algebraically closed fields of characteristic zero and their fundamental groups) are central objects in the study of combinatorial anabelian geometry. In the present series of papers, which consists of two successive works, we investigate combinatorial anabelian geometry of (not necessarily bijective) continuous homomorphisms between PSC-fundamental groups. This contrasts with previous researches, which focused only on continuous isomorphisms. More specifically, our main results of the present series of papers roughly state that, if a continuous homomorphism between PSC-fundamental groups is compatible with certain outer representations, then it satisfies a certain "group-theoretic compatibility property", i.e., the property that each of the images via the continuous homomorphism of certain VCN-subgroups of the domain are included in certain VCN-subgroups of the codomain. Such results may be considered as Hom-versions of the combinatorial version of the Grothendieck conjecture established in some previous works. As in the case of previous works (i.e., the Isom-versions), the proof requires different techniques depending on the types of outer representations under consideration. In the present paper, we will treat the case where the outer representations under consideration are assumed to be "l-graphically full", i.e., to satisfy a certain condition concerning "weights" considered with respect to the "l-adic cyclotomic character", where l is a certain

prime number. In addition, to prepare for this purpose, we include detailed expositions on "reduction techniques", namely, techniques of reduction to the maximal pro-Σ quotients and to the abelianizations of (various open subgroups of) the PSC-fundamental groups under consideration, where Σ is a certain set of prime numbers. Though the discussions of these "reduction techniques" are all essentially well-known to experts, we present the results in a highly unified/generalized fashion.

Contents

Introduction

Semi-graphs of anabelioids of PSC-type and their PSC-fundamental groups are central objects in the study of combinatorial anabelian geometry. In the present series of papers, which consists of two successive works, we investigate combinatorial anabelian geometry of (not necessarily bijective) continuous homomorphisms between PSC-fundamental groups. This contrasts with previous researches, which focused only on continuous isomorphisms. More specifically, our main results (cf. Theorem B below, [HmCbGCII], Theorem A, [HmCbGCII], Theorem B, and [HmCbGCII], Theorem C) roughly state that, if a continuous homomorphism between PSC-fundamental groups is compatible with certain outer representations, then it satisfies a certain "group-theoretic compatibility property". Such results may be considered as Hom-versions of the combinatorial version of the Grothendieck conjecture established in some previous works.

To begin with, we roughly explain our basic setting. The references are [SemiAn] and [CbGC]. We shall refer to as *a semi-graph* a triple $\mathbb{G} = (V, E, f)$, where V is a set, E is a set each of whose element is itself a set consisting of precisely two elements, and $f\colon \coprod_{e \in E} e \longrightarrow V \amalg \{\text{undefined}\}$, where $\{\text{undefined}\}$ is a singleton whose unique element is a formal symbol "undefined". Here, we think of V (resp. E; an element b of an element e of E; f) as the set of *vertices* of $\mathbb{G}$ (resp. the set of *edges* of $\mathbb{G}$; a *branch* of the edge e; determining the abutment relation). Note that it is allowed that a branch of an edge abuts (not to a vertex but) to "undefined", i.e., abuts nowhere. The vertices and the edges of a semi-graph $\mathbb{G}$ are collectively referred to as the *components* of $\mathbb{G}$. An edge of a semi-graph is said to be *closed* if both of the two branches abut to some vertex. An edge of a semi-graph is said to be *open* if it is not closed. Then a *semi-graph of anabelioids* is, by definition, a semi-graph equipped with a Galois category on each component, together with "an exact functor from the vertex to the edge" on each branch that abuts to a vertex (i.e., that does not abut to "undefined").

Let X be a pointed stable curve over an algebraically closed field of characteristic zero and Σ a non-empty set of prime numbers. Then X and Σ determine a semi-graph of anabelioids $\mathcal{G}^{\Sigma}_{X}$, which we call "a semi-graph of anabelioids of pro-Σ PSC-type associated to X", roughly as follows. The underlying semi-graph of $\mathcal{G}^{\Sigma}_{X}$ is the dual semi-graph of X, i.e., the semi-graph whose vertices (resp. closed edges; open edges) are the irreducible components (resp. the nodes, i.e., the singular points; the cusps, i.e., the marked points) of X, and the Galois category on each vertex v is the Galois category of finite étale coverings of the complement of the cusps in the regular locus of the irreducible component of X corresponding to v whose "Galois closure" has degree a number the prime factors of which belong to Σ. We omit the detail of the Galois categories on edges, but we mention that the construction is similar to the construction in the case of vertices and that their fundamental groups are all isomorphic to the maximal pro-Σ quotient $\hat{\mathbb{Z}}^{\Sigma}$ of $\hat{\mathbb{Z}}$.

Next, let $\mathcal{G}$ be a *semi-graph of anabelioids of pro-Σ PSC-type*, i.e., a semi-garph of anabelioids which is isomorphic, in the evident sense, to the "$\mathcal{G}^{\Sigma}_{X}$" for some X and Σ introduced above. Then we define the *PSC-fundamental group* $\Pi_{\mathcal{G}}$ of $\mathcal{G}$ as the profinite group $\mathrm{Aut}(\widetilde{\mathcal{G}}/\mathcal{G})^{\mathrm{op}}$, where $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ is a fixed "pro-Σ universal covering", which we omit the explanation here (cf. the discussion preceding [NodNon], Definition 1.1). We shall write $\mathrm{Vert}(\widetilde{\mathcal{G}})$ (resp. $\mathrm{Edge}(\widetilde{\mathcal{G}})$;

$\mathrm{VCN}(\widetilde{\mathcal{G}})$) for the set of vertices (resp. edges; components) of $\widetilde{\mathcal{G}}$. Then we have a natural right action of $\Pi_{\mathcal{G}}$ on $\mathrm{VCN}(\widetilde{\mathcal{G}})$. For every $\tilde{c} \in \mathrm{VCN}(\widetilde{\mathcal{G}})$, we shall write $\Pi_{\tilde{c}} \subset \Pi_{\mathcal{G}}$ for the stabilizer subgroup of $\tilde{c}$. If $\tilde{c}$ is a vertex (resp. a closed edge; an open edge; arbitrary), then $\Pi_{\tilde{c}}$ is said to be a verticial subgroup (resp. a nodal subgroup; a cuspidal subgroup; a VCN-subgroup) of $\Pi_{\mathcal{G}}$. An important fact is that the map from $\mathrm{VCN}(\widetilde{\mathcal{G}})$ to the set of VCN-subgroups of $\Pi_{\mathcal{G}}$ defined by $\tilde{c} \longmapsto \Pi_{\tilde{c}}$ is bijective (cf. Lemma 1.3, (2)).

Primitively, an anabelian geometer is interested in the following naive question.

> Question: to what extent the geometric information of an object may be recovered from the group-theoretic information of its fundamental group?

In the context of combinatorial anabelian geometry, one sometimes takes the "object" in the above question to be a semi-graph of anabelioids of pro-Σ PSC-type. Among the various results obtained in the previous research in this realm, the combinatorial version of the Grothendieck conjecture (or "the combinatorial Grothendieck conjecture" for short), which is an analogue of (a certain version of) the original arithmetic Grothendieck conjecture, is of particular interest and importance. Indeed, it is, on the one hand, a certain Galois-category-theoretic unified abstraction of some techniques in arithmetic anabelian geometry of hyperbolic curves (cf., e.g., the paragraph that starts with "The original motivation for..." in [CbGC], Introduction) and, on the other hand, a basic tool in the study of algebro-geometric anabelian geometry of configuration spaces of hyperbolic curves (cf., e.g., the proof of [NodNon], Theorem 6.1). The combinatorial Grothendieck conjecture states roughly that, if a continuous isomorphism between PSC-fundamental groups is compatible with certain outer representations, then the isomorphism under consideration preserves certain VCN-subgroups. To be more brief, it states that certain information of VCN-subgroups of a PSC-fundamental group is recovered from a certain outer representation on the PSC-fundamental group. Various versions of the combinatorial Grothendieck conjecture are obtained in [CbGC], [NodNon], and [CbTpII].

Considering the intriguing nature and significance of the combinatorial Grothendieck conjecture, it is natural to contemplate its new variants and generalizations. This is precisely the aim of the present series of papers.

More specifically, in the present series of papers, we will examine the extent to which one may generalize the combinatorial Grothendieck conjecture to (possibly non-bijective) continuous homomorphisms between PSC-fundamental groups, i.e., the extent to which one may establish certain "Hom-versions" of the combinatorial Grothendieck conjecture. Here, we should mention the general context that the anabelian geometers are interested also in Hom-versions of the original arithmetic Grothendieck conjecture.

Let $\mathcal{G}$ and $\mathcal{H}$ be semi-graphs of anabelioids of PSC-type, $\alpha\colon \Pi_{\mathcal{G}} \longrightarrow \Pi_{\mathcal{H}}$ a continuous homomorphism, $S \subset \mathrm{Vert}(\widetilde{\mathcal{G}})$, and $T \subset \mathrm{Vert}(\widetilde{\mathcal{H}})$. Then we shall say that α is (S, T)-*compatible* if, for every $\tilde{c} \in S$, either $\alpha(\Pi_{\tilde{c}}) = 1$ or there exists $\tilde{d} \in T$ such that $\alpha(\Pi_{\tilde{c}}) \subset \Pi_{\tilde{d}}$. The properties of being (S, T)-compatible for some S, T are collectively referred to as the *group-theoretic compatibility properties*. The Hom-versions of the combinatorial Grothendieck conjecture established in the present series of papers provide us with various sufficient conditions for certain group-theoretic compatibility properties of continuous homomorphisms between PSC-fundamental groups.

In §1, we review some basic knowledge of semi-graphs of anabelioids of PSC-type, and we define several new terminologies some of which play central roles in the present series of papers. In particular, we define the notion of group-theoretic compatibility properties introduced above.

In §2, we give a detailed exposition of a well-known "reduction technique", namely, reduction to the maximal pro-Σ quotients of (various open subgroups of) the PSC-fundamental groups under consideration, where Σ is a certain set of prime numbers. We also examine the extent to which these reduction techniques may be applied to our Hom-version situation. Though the discussions in §2 are all essentially well-known to experts, the results are formulated in a highly unified fashion.

In §3, we apply the technique established in §2 to verify another "reduction technique" which is also well-known to experts, namely, reduction to the abelianizations of (various open subgroups of) the PSC-fundamental groups under consideration. As a result, we verify the following highly unified generalization of this technique to our Hom-version situation.

Theorem A (Theorem 3.2)**.** *Let $\mathcal{G}$ be a semi-graph of anabelioids of pro-$\Sigma_{\mathcal{G}}$ PSC-type, $\mathcal{H}$ a semi-graph of anabelioids of pro-$\Sigma_{\mathcal{H}}$ PSC-type, $\alpha\colon \Pi_{\mathcal{G}} \longrightarrow \Pi_{\mathcal{H}}$ a continuous homomorphism between their PSC-fundamental groups, $S \subset \mathrm{VCN}(\widetilde{\mathcal{G}})$, and $T \subset \mathrm{VCN}(\widetilde{\mathcal{H}})$. Let us consider the following three conditions,* *where the closures* $\overline{S}$ *and* $\overline{T}$ *are taken with respect to the natural profinite topology of* $\mathrm{VCN}(\widetilde{\mathcal{G}})$ *and* $\mathrm{VCN}(\widetilde{\mathcal{H}})$*:*

(i) α *is* (S,T)*-compatible.*

(ii) α *is* (S,T)*-filtration-preserving (cf. Definition 1.7, (2)). That is to say, roughly speaking, for every open subgroup* $U \subset \Pi_{\mathcal{H}}$*, the image of the "*$S$*-like" submodule of the abelianization* $\alpha^{-1}(U)^{\mathrm{ab}}$ *via the induced homomorphism* $\alpha^{-1}(U)^{\mathrm{ab}} \longrightarrow U^{\mathrm{ab}}$ *is included in the "*T*-like" submodule of the abelianization* U^{ab}*.*

(iii) α *is* $(\overline{S},\overline{T})$*-compatible.*

Then the implications (i) $\Longrightarrow$ *(ii)* $\Longrightarrow$ *(iii) holds. In particular, if we suppose further that* $S = \overline{S}$ *and* $T = \overline{T}$*, then the conditions above are all equivalent.*

We observe that Theorem A may be considered as a unified generalized Hom-version of [CbGC], Theorem 1.6, (ii).

Finally, in §4, we apply the techniques established in §2 and §3 to obtain a Hom-version of the combinatorial version of the Grothendieck conjecture, where the outer representations under consideration are assumed to be "l-graphically full", i.e., to satisfy a certain condition concerning "weights" considered with respect to the "l-adic cyclotomic character", where l is a certain prime number (cf. [CbGC], Definition 2.3, (iii)). In general, an *outer representation* of a profinite group I on another profinite group G is defined to be a continuous homomorphism $I \longrightarrow \mathrm{Out}(G)$, where $\mathrm{Out}(G)$ denotes the group of continuous outer automorphisms of G equipped with the topology induced by the compact-open topology on $\mathrm{Aut}(G)$, the group of continuous automorphisms of G. It is well-known (and not difficult to see) that, if G is topologically finitely generated, then $\mathrm{Aut}(G)$, hence also $\mathrm{Out}(G)$, is profinite. If $\rho\colon I \longrightarrow \mathrm{Out}(G)$ is an outer representation of a profinite group I on a profinite topologically finitely generated group G which is center-free, then, by pulling back the natural exact sequence of profinite groups and continuous homomorphisms

$$1 \longrightarrow G \longrightarrow \mathrm{Aut}(G) \longrightarrow \mathrm{Out}(G) \longrightarrow 1,$$

where the injection $G \longrightarrow \mathrm{Aut}(G)$ is given by $g \longmapsto (h \longmapsto ghg^{-1})$ (cf. the assumption that G is center-free), via the continuous homomorphism

$\rho\colon I \longrightarrow \mathrm{Out}(G)$, we obtain an exact sequence of profinite groups and continuous homomorphisms

$$1 \longrightarrow G \longrightarrow G \overset{\text{out}}{\rtimes} I \longrightarrow I \longrightarrow 1.$$

In light of these observations, our main result is formulated as follows.

Theorem B (Theorem 4.4)**.** *Let I and J be profinite groups, $\mathcal{G}$ a semi-graph of anabelioids of pro-$\Sigma_{\mathcal{G}}$ PSC-type, $\mathcal{H}$ a semi-graph of anabelioids of pro-$\Sigma_{\mathcal{H}}$ PSC-type, and $\rho_I\colon I \longrightarrow \mathrm{Out}(\Pi_{\mathcal{G}})$ and $\rho_J\colon J \longrightarrow \mathrm{Out}(\Pi_{\mathcal{H}})$ outer representations such that the images of ρ_I and ρ_J are respectively included in $\mathrm{Aut}(\mathcal{G}) \subset \mathrm{Out}(\Pi_{\mathcal{G}})$ and $\mathrm{Aut}(\mathcal{H}) \subset \mathrm{Out}(\Pi_{\mathcal{H}})$, where the usage of the symbol "$\subset$" is a slight abuse of notation (cf. the discussion preceding [CbGC], Lemma 2.1). Let us consider the following commutative diagram of profinite groups and continuous homomorphisms:*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Pi_{\mathcal{G}} & \longrightarrow & \Pi_{\mathcal{G}} \overset{\text{out}}{\rtimes} I & \longrightarrow & I & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \tilde{\alpha}} & & \downarrow{\scriptstyle \beta} & & \\ 1 & \longrightarrow & \Pi_{\mathcal{H}} & \longrightarrow & \Pi_{\mathcal{H}} \overset{\text{out}}{\rtimes} J & \longrightarrow & J & \longrightarrow & 1, \end{array}$$

where the two horizontal sequences are the respective exact sequences associated to ρ_I and to ρ_J (cf. the discussion preceding the present theorem; also [CbGC], Remark 1.1.3, for the fact that $\Pi_{\mathcal{G}}$ and $\Pi_{\mathcal{H}}$ are topologically finitely generated and center-free). Let $\Box \in \{\mathrm{Vert}, \mathrm{Edge}\}$ and $S \subset \Box(\widetilde{\mathcal{G}})$. Suppose further that there exists $l \in \Sigma_{\mathcal{G}} \cap \Sigma_{\mathcal{H}}$ for which the following four conditions hold.

(i) For every $\tilde{c} \in S$, there exists an open subgroup $U \subset \Pi_{\mathcal{H}}$ such that the image of $\alpha(\Pi_{\tilde{c}}) \cap U$ via the natural surjection $U \twoheadrightarrow U^{(l)} \cong \Pi_{(\mathcal{H}')^{(l)}}$ is non-trivial, where the supersprict "(l)" denotes the maximal pro-l quotient.

(ii) I is l-graphically full with respect to the outer representation $\rho_I\colon I \longrightarrow \mathrm{Aut}(\mathcal{G})$ (cf. [CbGC], Definition 2.3, (iii)).

(iii) I is l-graphically full with respect to the outer representation $\rho_J \circ \beta\colon I \longrightarrow \mathrm{Aut}(\mathcal{H})$ (cf. [CbGC], Definition 2.3, (iii)).

(iv) *If, moreover,* $\square =$ Edge, *then the two cyclotomic characters* $I \longrightarrow \mathbb{Z}_l^\times$ *associated to* $\rho_I \colon I \longrightarrow \mathrm{Aut}(\mathcal{G})$ *and to* $\rho_J \circ \beta \colon I \longrightarrow \mathrm{Aut}(\mathcal{H})$ *coincide.*

Then α *is* $\big(\overline{S}, \square(\widetilde{\mathcal{H}})\big)$*-compatible.*

Though the appearance of Theorem B is slightly different from that of Theorem 4.4, it is immediate that the these are essentially the same. Theorem B, along with its proof, can be considered as a Hom-version of [CbGC], Corollary 2.7, (ii). Also, it is possible to verify [CbGC], Corollary 2.7, (ii), utilizing Theorem B, albeit not straightforwardly (cf. the assumption (iv) of Theorem B; Remark 4.4.1).

0 Notations and Conventions

Numbers

$\mathbb{N}_{>0}$ denotes the set or the (multiplicative) monoid of positive rational integers. $\mathbb{Z}$ denotes the set or the (additive) group of rational integers. $\mathbb{Q}$ denotes the set of rational numbers. $\hat{\mathbb{Z}}$ denotes the profinite completion of the group $\mathbb{Z}$.

Topological Groups

Let K be a topological group and $G, H \subset K$ closed subgroups of K. Then we shall write $C_G(H)$ for the *commensurator subgroup* of H in G, i.e.,

$$C_G(H) \stackrel{\mathrm{def}}{=} \{g \in G \mid H \cap g^{-1}Hg \text{ is of finte index in } H \text{ and } g^{-1}Hg.\}.$$

We shall say that the subgroup H is *commensurably terminal* in K if $H = C_K(H)$.

Let Σ be a set of prime numbers, l a prime number, and G a profinite group. Then we shall write G^Σ for the *maximal pro-*Σ *quotient* of G and $G^{(l)} \stackrel{\mathrm{def}}{=} G^{\{l\}}$. We shall write G^{ab} for the abelianization of the profinite group G, i.e., the quotient of G by the closure of the commutator subgroup of G. Moreover, if $\alpha \colon G \longrightarrow H$ is a continuous homomorphism of profinite groups, then we shall write α^Σ (resp. $\alpha^{(l)}$; α^{ab}) for the homomorphism $G^\Sigma \longrightarrow H^\Sigma$ (resp. $G^{(l)} \longrightarrow H^{(l)}$; $G^{\mathrm{ab}} \longrightarrow H^{\mathrm{ab}}$) induced by α. We shall write $\mathrm{Aut}(G)$

for the group of automorphisms of G (as a profinite group) and $\mathrm{Inn}(G)$ for the group of inner automorphisms of G. Thus we have a homomorphism of groups $G \longrightarrow \mathrm{Aut}(G)$ determind by conjugation whose image is $\mathrm{Inn}(G) \subset G$. We shall write $\mathrm{Out}(G)$ for the quotient of $\mathrm{Aut}(G)$ by the normal subgroup $\mathrm{Inn}(G) \subset \mathrm{Aut}(G)$. In particular, if G is center-free, then the natural homomorphism $G \longrightarrow \mathrm{Inn}(G)$ is an isomorphism; thus we have a natural exact sequence of groups

$$1 \longrightarrow G \longrightarrow \mathrm{Aut}(G) \longrightarrow \mathrm{Out}(G) \longrightarrow 1.$$

If, moreover, G is topologically finitely generated, then one verifies easily that the topology of G admits a basis of characteristic open subgroups, which thus induces a profinite topology on the groups $\mathrm{Aut}(G)$ and $\mathrm{Out}(G)$, with respect to which the above exact sequence forms an exact sequence of profinite groups and continuous homomorphisms. In this situation, if, moreover, $\rho\colon J \longrightarrow \mathrm{Out}(G)$ is a continuous homomorphism, then we shall write $G \overset{\mathrm{out}}{\rtimes} J$ for the profinite group obtained by pulling back the above exact sequence of profinite groups via ρ. Thus we have a natural exact sequence of profinite groups and continuous homomorphisms

$$1 \longrightarrow G \longrightarrow G \overset{\mathrm{out}}{\rtimes} J \longrightarrow J \longrightarrow 1.$$

Group Actions

Let G be a group, X a set, and $\phi\colon X \times G \longrightarrow X$ a right action of G on X. Then we shall simply write "$X \curvearrowleft G$" and omit "ϕ" to express this situation. Also, for $x \in X$ and $g \in G$, we shall simply write x^g instead of $\phi(x, g)$.

1 Preliminaries

In this section, we prepare some definitions, notational conventions, and basic lemmas concerning the PSC-fundamental groups of semi-graphs of anabelioids of PSC-type and continuous homomorphisms between them (which are not necessarily isomorphisms). In particular, we define the *group-theoretic compatibility properties* of such homomorphisms (cf. Definition 1.7 and Remark 1.7.4). The study of these properties is the central issue of the present series of papers.

A basic reference for the theory of semi-graphs of anabelioids of PSC-type is [CbGC]. We shall use the terminologies "*semi-graph of anabelioids of PSC-type*", "*PSC-fundamental group of a semi-graph of anabelioids of PSC-type*", "*finite étale covering of semi-graphs of anabelioids of PSC-type*", "*vertex*", "*edge*", "*node*", "*cusp*", "*verticial subgroup*", "*edge-like subgroup*", "*nodal subgroup*", and "*cuspidal subgroup*", as they are defined in [CbGC], Definition 1.1. Moreover, if $\mathcal{G}$ is a semi-graph of anabelioids of PSC-type, then we shall write $\Sigma_{\mathcal{G}}$ for the (necessarily unique — cf. [CbGC], Remark 1.1.2) set of prime numbers such that $\mathcal{G}$ is a semi-graph of anabelioids of pro-$\Sigma_{\mathcal{G}}$ PSC-type. Also, we shall apply the various notational conventions established in [NodNon], Definition 1.1; in particular, if $\mathcal{G}$ is a semi-graph of anabelioids of PSC-type, then the pro-$\Sigma_{\mathcal{G}}$ universal covering $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ is fixed throughout the discussion, and the PSC-fundamental group $\Pi_{\mathcal{G}}$ is always considered to be associated to the fixed pro-$\Sigma_{\mathcal{G}}$ universal covering $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$. Thus there is a natural action $\widetilde{\mathcal{G}} \curvearrowleft \Pi_{\mathcal{G}}$, which induces a natural bijection

$$\begin{array}{ccc} \big(\text{the set of open subgroups of } \Pi_{\mathcal{G}}\big) & \xrightarrow{\sim} & \begin{pmatrix}\text{the set of the isomorphism classes} \\ \text{of the connected finite étale} \\ \text{subcoverings of } \widetilde{\mathcal{G}} \longrightarrow \mathcal{G}\end{pmatrix}; \\ \big(U \subset \Pi_{\mathcal{G}}\big) & \longmapsto & \big(\widetilde{\mathcal{G}}/U \longrightarrow \mathcal{G}\big). \end{array}$$

If $\mathcal{G}' \longrightarrow \mathcal{G}$ is a connected finite étale subcovering of the pro-$\Sigma_{\mathcal{G}}$ universal covering $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$, then we always choose the implicit structure morphism $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}'$ as the pro-$\Sigma_{\mathcal{G}'}$ universal covering of $\mathcal{G}'$. (Here, we note that $\Sigma_{\mathcal{G}} = \Sigma_{\mathcal{G}'}$.) Under this convention, the inverse of the natural bijection above is given by $\big(\mathcal{G}' \longrightarrow \mathcal{G}\big) \longmapsto \big(\Pi_{\mathcal{G}'} \subset \Pi_{\mathcal{G}}\big)$. Finally, we shall refer to the "PSC-fundamental group of a semi-graph of anabelioids of PSC-type" simply as the "fundamental group" (of the semi-graph of anabelioids of PSC-type).

In the following, the letters "$\mathcal{G}$" and "$\mathcal{H}$" always denote semi-graphs of anabelioids of PSC-type. Note that we do not assume that $\Sigma_{\mathcal{G}} = \Sigma_{\mathcal{H}}$.

Definition 1.1. We shall refer to any element of $\mathrm{VCN}(\mathcal{G})$ (resp. $\mathrm{VCN}(\widetilde{\mathcal{G}})$) as a *component* of $\mathcal{G}$ (resp. $\widetilde{\mathcal{G}}$). Moreover, if $\tilde{c} \in \mathrm{VCN}(\widetilde{\mathcal{G}})$, then we shall write $\Pi_{\tilde{c}}$ for the VCN-subgroup of $\Pi_{\mathcal{G}}$ associated to $\tilde{c}$ (cf. also Remark 1.3.2 for the notation "Π_c" for $c \in \mathrm{VCN}(\mathcal{G})$).

Let us recall here the following fundamental results, which will be of frequent use in the present series of papers.

Lemma 1.2. *The following assertions hold.*

(1) Let $\tilde{c}, \tilde{d} \in \mathrm{Edge}(\widetilde{\mathcal{G}})$. Then one of the following two (mutually exclusive) conditions holds.

- *It holds that $\tilde{c} = \tilde{d}$. In particular, the equality $\Pi_{\tilde{c}} = \Pi_{\tilde{d}}$ holds.*
- *It holds that $\tilde{c} \neq \tilde{d}$, and the equality $\Pi_{\tilde{c}} \cap \Pi_{\tilde{d}} = 1$ holds.*

(2) Let $\tilde{c} \in \mathrm{Edge}(\widetilde{\mathcal{G}})$ and $\tilde{d} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$. Then one of the following two (mutually exclusive) conditions holds.

- *$\tilde{c}$ abuts to $\tilde{d}$. In particular, the inclusion $\Pi_{\tilde{c}} \subsetneq \Pi_{\tilde{d}}$ holds.*
- *$\tilde{c}$ does not abut to $\tilde{d}$, and the equality $\Pi_{\tilde{c}} \cap \Pi_{\tilde{d}} = 1$ holds.*

(3) Let $\tilde{c}, \tilde{d} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$. Then one of the following three (mutually exclusive) conditions holds.

- *It holds that $\tilde{c} = \tilde{d}$. In particular, the equality $\Pi_{\tilde{c}} = \Pi_{\tilde{d}}$ holds.*
- *It holds that $\tilde{c} \neq \tilde{d}$, and there exists a unique node $\tilde{e} \in \mathrm{Node}(\widetilde{\mathcal{G}})$ which abuts both to $\tilde{c}$ and to $\tilde{d}$. Moreover, the equality $\Pi_{\tilde{c}} \cap \Pi_{\tilde{d}} = \Pi_{\tilde{e}}$ holds.*
- *It holds that $\tilde{c} \neq \tilde{d}$, and the equality $\Pi_{\tilde{c}} \cap \Pi_{\tilde{d}} = 1$ holds. In particular (cf. the assertion (2)), there exists no edge which abuts both to $\tilde{c}$ and to $\tilde{d}$.*

Proof. The assertion (1) follows immediately from [NodNon], Lemma 1.5. The assertion (2) follows immediately from [NodNon], Lemma 1.7 (cf. also [CbGC], Remark 1.1.3, for the "$\neq$" of "$\subsetneq$"). The assertion (3) follows, in light of the assertion (1), immediately from [NodNon], Lemma 1.8 and Lemma 1.9. This completes the proof of Lemma 1.2. □

Remark 1.2.1. It is also well-known that, for every $\tilde{e} \in \mathrm{Node}(\widetilde{\mathcal{G}})$, there exist precisely two vertices of $\widetilde{\mathcal{G}}$ to which $\tilde{e}$ abuts, i.e., the two branches of $\tilde{e}$ abut to two distinct vertices (cf. [NodNon], Remark 1.2.1, (iii), or [HmCbGCII], Remark 4.5.1). Combining this fact with Lemma 1.2, (3), we deduce that a nodal subgroup of $\Pi_{\mathcal{G}}$ is uniquely presented as the intersection of two verticial subgroups of $\Pi_{\mathcal{G}}$.

Lemma 1.3. *Recall that we have a natural right action $\widetilde{\mathcal{G}} \curvearrowleft \Pi_{\mathcal{G}}$ which, in particular, determines a natural action $\mathrm{VCN}(\widetilde{\mathcal{G}}) \curvearrowleft \Pi_{\mathcal{G}}$. For a connected finite étale Galois subcovering $\mathcal{G}' \longrightarrow \mathcal{G}$ of $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$, define a right action* $\big($*the set of VCN-subgroups of* $\Pi_{\mathcal{G}'}\big) \curvearrowleft \Pi_{\mathcal{G}}$ *by conjugation, i.e., by the formula* $(\Pi_{\tilde{c}} \cap \Pi_{\mathcal{G}'}) \cdot \gamma \stackrel{\mathrm{def}}{=} \gamma^{-1}\Pi_{\tilde{c}}\gamma \cap \Pi_{\mathcal{G}'}$ *(cf. Remark 1.5.2). Relative to these actions, the following assertions hold.*

(1) For any $\tilde{c} \in \mathrm{VCN}(\widetilde{\mathcal{G}})$, the VCN-subgroup $\Pi_{\tilde{c}} \subset \Pi_{\mathcal{G}}$ coincides with the stabilizer of $\tilde{c} \in \mathrm{VCN}(\widetilde{\mathcal{G}})$.

(2) For any connected finite étale Galois subcovering $\mathcal{G}' \longrightarrow \mathcal{G}$ of $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$, we have a $\Pi_{\mathcal{G}}$-equivariant bijection

$$\mathrm{VCN}(\widetilde{\mathcal{G}}) \stackrel{\cong}{\longrightarrow} \Big(\textit{the set of VCN-subgroups of } \Pi_{\mathcal{G}'}\Big);\ \tilde{c} \longmapsto \Pi_{\tilde{c}} \cap \Pi_{\mathcal{G}'}.$$

Proof. The assertion (1) follows immediately from the definitions (cf. [SemiAn], Remark 2.2.1).

Let us verify the assertion (2). It follows immediately from the definitions that the map under consideration is a $\Pi_{\mathcal{G}}$-equivariant surjection. Moreover, in light of Remark 1.5.2, it follows from Lemma 1.2, applied to "$\mathcal{G}$" $= \mathcal{G}'$, that the map under consideration is injective. This completes the proof of the assertion (2), hence also the proof of Lemma 1.3. □

Remark 1.3.1. It follows immediately from Lemma 1.3, (1), that, for every $\tilde{c} \in \mathrm{VCN}(\widetilde{\mathcal{G}})$ and $\gamma \in \Pi_{\mathcal{G}}$, it holds that $\Pi_{\tilde{c}^{\gamma}} = \gamma^{-1}\Pi_{\tilde{c}}\gamma$.

Remark 1.3.2. It also follows from Lemma 1.3, (1), that, for $c \in \mathrm{VCN}(\mathcal{G})$ and $\tilde{c} \in \mathrm{VCN}(\widetilde{\mathcal{G}})$ such that $\tilde{c}(\mathcal{G}) = c$, the $\Pi_{\mathcal{G}}$-conjugacy class of $\Pi_{\tilde{c}}$ is completely determined by c. We shall write Π_c for (some representative of) this $\Pi_{\mathcal{G}}$-conjugacy class. Note that (every representative of) Π_c is precisely the image of (some representative of) the natural outer homomorphism $\pi_1(\mathcal{G}_c) \longrightarrow \Pi_{\mathcal{G}}$, where we write $\mathcal{G}_c$ for the consituent anabelioid of $\mathcal{G}$ at c.

Remark 1.3.3. Unfortunately, the equality $\Pi_{\tilde{c}^{\gamma}} = \gamma^{-1}\Pi_{\tilde{c}}\gamma$ pointed out in Remark 1.3.1 is inconsistent with the equality $\Pi_{\tilde{v}^{\gamma}} = \gamma\Pi_{\tilde{v}}\gamma^{-1}$ (where $\tilde{v} \in \mathrm{Vert}(\widetilde{\mathcal{G}})$) implicitly given in the proof of [NodNon], Lemma 3.6. This inconsistency is not because of the logical flaw of either *loc. cit.* or the present paper, but just because of the difference of the definitions. The author decided to adopt a different notation from that of *loc. cit.* in order to avoid the confusing equality "$\tilde{c}^{ab} = (\tilde{c}^{b})^{a}$" caused by the notation of *loc. cit.*

Lemma 1.4. *Let $\tilde{c} \in \mathrm{VCN}(\widetilde{\mathcal{G}})$ and H a closed subgroup of $\Pi_{\mathcal{G}}$. Then the following assertions hold.*

(1) If $1 \neq H \subset \Pi_{\tilde{c}}$, then $C_{\Pi_{\mathcal{G}}}(H) \subset \Pi_{\tilde{c}}$ (cf. the discussion entitled "Topological Groups" in §0).

(2) If U is an open subgroup of H and $U \subset \Pi_{\tilde{c}}$, then $H \subset \Pi_{\tilde{c}}$.

Proof. First we verify the assertion (1). Note that any open subgroup of H is non-trivial, since H is non-trivial, and $\Pi_{\mathcal{G}}$ is torsion-free (cf. [CbGC], Remark 1.1.3).

Let $\gamma \in C_{\Pi_{\mathcal{G}}}(H)$. Then we have

$$1 \neq H \cap \gamma^{-1} H \gamma \subset \Pi_{\tilde{c}} \cap \gamma^{-1} \Pi_{\tilde{c}} \gamma,$$

where the "$\neq$" follows from the fact that $H \cap \gamma^{-1} H \gamma \subset H$ is open. In particular, the two VCN-subgroups $\Pi_{\tilde{c}}$, $\gamma^{-1}\Pi_{\tilde{c}}\gamma = \Pi_{\tilde{c}^{\gamma}}$ intersect non-trivially. In light of Lemma 1.2, this non-triviality, in fact, implies either the equality $\Pi_{\tilde{c}} = \gamma^{-1}\Pi_{\tilde{c}}\gamma$ or the equality $\Pi_{\tilde{c}} \cap \gamma^{-1}\Pi_{\tilde{c}}\gamma = \Pi_{\tilde{e}}$, where in the latter case $\tilde{c}$ is necessarily a vertex and $\tilde{e} \in \mathrm{Node}(\widetilde{\mathcal{G}})$ is the unique node which abuts both to $\tilde{c}$ and to $\tilde{c}^{\gamma}$.

Suppose that we are in the former case, i.e., that $\Pi_{\tilde{c}} = \gamma^{-1}\Pi_{\tilde{c}}\gamma$. (For instance, this will be the case if $\tilde{c} \in \mathrm{Edge}(\widetilde{\mathcal{G}})$.) Then, by the commensurable terminality of $\Pi_{\tilde{c}}$ in $\Pi_{\mathcal{G}}$ (cf. [CbGC], Proposition 1.2, (ii)), we obtain the desired relation $\gamma \in C_{\Pi_{\mathcal{G}}}(\Pi_{\tilde{c}}) = \Pi_{\tilde{c}}$.

Next, suppose that we are in the latter case, i.e., that $\Pi_{\tilde{c}} \cap \gamma^{-1}\Pi_{\tilde{c}}\gamma = \Pi_{\tilde{e}}$. Then we have the relations $1 \neq H \cap \gamma^{-1} H \gamma \subset \Pi_{\tilde{c}} \cap \gamma^{-1}\Pi_{\tilde{c}}\gamma = \Pi_{\tilde{e}}$. By applying the already verified former case to "$(\tilde{c}, H)$" $= (\tilde{e}, H \cap \gamma^{-1} H \gamma)$, it follows that $C_{\Pi_{\mathcal{G}}}(H \cap \gamma^{-1} H \gamma) \subset \Pi_{\tilde{e}}$. Since $\Pi_{\tilde{e}} \subset \Pi_{\tilde{c}}$ and $C_{\Pi_{\mathcal{G}}}(H) = C_{\Pi_{\mathcal{G}}}(H \cap \gamma^{-1} H \gamma)$ (where we note that $H \cap \gamma^{-1} H \gamma$ is open in H), we obtain the desired inclusion $C_{\Pi_{\mathcal{G}}}(H) \subset \Pi_{\tilde{c}}$. This completes the proof of the assertion (1).

Next, we verify the assertion (2). If H is trivial, obviously we are done. Thus we may assume that H is non-trivial. In this case U is also non-trivial, since $\Pi_{\mathcal{G}}$ is torsion-free (cf. [CbGC], Remark 1.1.3). Thus the assertion (2) follows immediately from the assertion (1) applied to "H" $= U$, in light of the fact that $H \subset C_{\Pi_{\mathcal{G}}}(H) = C_{\Pi_{\mathcal{G}}}(U)$. This completes the proof of the assertion (2), hence also the proof of Lemma 1.4. □

Now we proceed to give some definitions which play the most central roles in the present series of papers.

Definition 1.5. Let $S \subset \mathrm{VCN}(\widetilde{\mathcal{G}})$ and $\mathcal{G}' \longrightarrow \mathcal{G}$ a connected finite étale subcovering of $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$.

(1) We shall write $S(\mathcal{G}')$ for the image of $S \subset \mathrm{VCN}(\widetilde{\mathcal{G}})$ via the natural surjection $\mathrm{VCN}(\widetilde{\mathcal{G}}) \twoheadrightarrow \mathrm{VCN}(\mathcal{G}')$.

(2) An element of $\mathrm{VCN}(\mathcal{G}')$ (resp. $\mathrm{VCN}(\widetilde{\mathcal{G}})$) is said to be *$S$-like* if it belongs to $S(\mathcal{G}')$ (resp. S).

(3) A subgroup of $\Pi_{\mathcal{G}'}$ is said to be *S-like* if it arises as the VCN-subgroup associated to an S-like component of $\widetilde{\mathcal{G}}$.

(4) We shall write $M_{\mathcal{G}'}$ for the abelianization $\Pi_{\mathcal{G}'}^{\mathrm{ab}}$ of $\Pi_{\mathcal{G}'}$. Moreover, we shall write $M_{\mathcal{G}'}^{S}$ for the closed submodule of $M_{\mathcal{G}'}$ generated by the images of the S-like subgroups of $\Pi_{\mathcal{G}'}$ via the natural surjection $\Pi_{\mathcal{G}'} \twoheadrightarrow M_{\mathcal{G}'}$.

(5) We shall write $M_{\mathcal{G}'}^{\mathrm{vert}} \stackrel{\mathrm{def}}{=} M_{\mathcal{G}'}^{\mathrm{Vert}(\widetilde{\mathcal{G}})}$ and $M_{\mathcal{G}'}^{\mathrm{edge}} \stackrel{\mathrm{def}}{=} M_{\mathcal{G}'}^{\mathrm{Edge}(\widetilde{\mathcal{G}})}$.

Remark 1.5.1. The definitions of the notations "$M_{\mathcal{G}'}^{\mathrm{vert}}$" and "$M_{\mathcal{G}'}^{\mathrm{edge}}$" coincide with those of [CbGC], Definition 1.1, (ii).

Remark 1.5.2. Suppose that we are in the situation of Definition 1.5. Then it follows immediately from Lemma 1.3, (1), that the "$\Pi_{\tilde{c}}$" of $\Pi_{\mathcal{G}'}$, i.e., the VCN-subgroup of $\Pi_{\mathcal{G}'}$ corresponding to $\tilde{c} \in \mathrm{VCN}(\widetilde{\mathcal{G}}) = \mathrm{VCN}(\widetilde{\mathcal{G}}')$, is equal to $\Pi_{\tilde{c}} \cap \Pi_{\mathcal{G}'}$. In particular, it holds that a subgroup of $\Pi_{\mathcal{G}'}$ is S-like if and only if it can be written as the intersection of an S-like subgroup of $\Pi_{\mathcal{G}}$ with $\Pi_{\mathcal{G}'}$.

Remark 1.5.3. Suppose that we are in the situation of Definition 1.5. Then it holds, as is sketched below, that $M_{\mathcal{G}}/M_{\mathcal{G}}^{S}$ is free over $\hat{\mathbb{Z}}^{\Sigma_{\mathcal{G}}}$, hence torsion-free. In particular, $M_{\mathcal{G}}^{S} \subset M_{\mathcal{G}}$ coincides with the "saturation" of (i.e., the inverse image via the natural surjection of the torsion part of the quotient module of $M_{\mathcal{G}}$ by) the image of $M_{\mathcal{G}'}^{S}$ via the natural homomorphism $M_{\mathcal{G}'} \longrightarrow M_{\mathcal{G}}$ induced by the inclusion homomorphism $\Pi_{\mathcal{G}'} \longrightarrow \Pi_{\mathcal{G}}$.

One way to verify the freeness of $M_{\mathcal{G}}/M_{\mathcal{G}}^{S}$ over $\hat{\mathbb{Z}}^{\Sigma_{\mathcal{G}}}$ is as follows. Write X for "the compact Riemann surface minus some points corresponding to $\mathcal{G}$". Then it follows from basic algebraic topology, together with the link between topological and algebraic fundamental groups, that $M_{\mathcal{G}}$ is naturally isomorphic to $H_1(X, \emptyset) \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^{\Sigma_{\mathcal{G}}}$, where "$H_1$" denotes the first relative singular homology group. Then one constructs an appropriate subset $Y \subset X$ such that $M_{\mathcal{G}}/M_{\mathcal{G}}^{S}$ injects (over $\hat{\mathbb{Z}}^{\Sigma_{\mathcal{G}}}$) to $H_1(X, Y) \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^{\Sigma_{\mathcal{G}}}$. (For example, if $S \subset$

$\mathrm{Vert}(\mathcal{G})$, then one may take such a "Y" to be the disjoint union inside X of the various "sub Riemann surfaces" of X determined by various $c \in S(\mathcal{G})$.) Finally, one verifies (by excision isomorphism, for example) that $H_1(X, Y)$ is free over $\mathbb{Z}$.

Remark 1.5.4. In the situation of Definition 1.5, we shall write $\overline{S}$ for the projective limit of the projective system $\big(S(\mathcal{G}')\big)_{\mathcal{G}' \longrightarrow \mathcal{G}}$, where $\mathcal{G}' \longrightarrow \mathcal{G}$ runs through the connected finite étale subcoverings of $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$. Then it is easily verified that $\overline{S}$ is naturally regarded as a subset of $\mathrm{VCN}(\widetilde{\mathcal{G}})$ and in fact coincides with the closure of S in $\mathrm{VCN}(\widetilde{\mathcal{G}})$ with respect to the profinite topology of $\mathrm{VCN}(\widetilde{\mathcal{G}})$.

Remark 1.5.5. Let $S \subset \mathrm{VCN}(\widetilde{\mathcal{G}})$ and $\mathcal{G}' \longrightarrow \mathcal{G}$ a connected finite étale subcovering of $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$. Then another immediate consequence of the definition of $\overline{S}$ (cf. Remark 1.5.4) is that $S(\mathcal{G}') = \overline{S}(\mathcal{G}')$. Put another way, roughly speaking, the difference between S and $\overline{S}$ is "invisible" in any finite approximation level. It also follows formally from this observation that $M^S_{\mathcal{G}'} = M^{\overline{S}}_{\mathcal{G}'}$.

Remark 1.5.6. Let A be a subset of $\Pi_{\mathcal{G}}$ and $S \subset \mathrm{VCN}(\widetilde{\mathcal{G}})$. Then it follows from Lemma 1.3, (1), and Remark 1.5.5, together with the well-known fact that the projective limit of a projective system consisting of non-empty finite sets is non-empty, that A is included in an $\overline{S}$-like subgroup of $\Pi_{\mathcal{G}}$ if and only if there exists a cofinal subsystem $(\mathcal{G}_\lambda \longrightarrow \mathcal{G})_{\lambda \in \Lambda}$ of (the projective system which gives rises to) the pro-$\Sigma_{\mathcal{G}}$ universal covering $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ constituted by finite étale Galois subcoverings of $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ such that, for every $\lambda \in \Lambda$, there exists $c_\lambda \in S(\mathcal{G}_\lambda)$ stabilized by every element of A with respect to the natural action $\mathrm{VCN}(\mathcal{G}_\lambda) \curvearrowleft \Pi_{\mathcal{G}} \supset A$.

Definition 1.6. Let $S \subset \mathrm{VCN}(\widetilde{\mathcal{G}})$.

(1) We shall write

$$E_S \stackrel{\mathrm{def}}{=} S \cup \{\tilde{e} \in \mathrm{Edge}(\widetilde{\mathcal{G}}) \mid \tilde{e} \text{ abuts to a vertex in } S.\}.$$

(2) We shall say that S is *edge-complete* if it holds that $S = E_S$.

Remark 1.6.1. Let $S \subset \mathrm{VCN}(\widetilde{\mathcal{G}})$ and A a subset of $\Pi_{\mathcal{G}}$. Then it follows immediately from Lemma 1.2, (2), that A is included in an S-like subgroup of $\Pi_{\mathcal{G}}$ if and only if it is included in an E_S-like subgroup of $\Pi_{\mathcal{G}}$.

Remark 1.6.2. It follows immediately from the definitions (cf. Remark 1.5.4) that, for every $S \subset \mathrm{VCN}(\widetilde{\mathcal{G}})$, $E_{\overline{S}} = \overline{E_S}$.

Definition 1.7. Let $\alpha\colon \Pi_{\mathcal{G}} \longrightarrow \Pi_{\mathcal{H}}$ be a continuous homomorphism, $S \subset \mathrm{VCN}(\widetilde{\mathcal{G}})$, and $T \subset \mathrm{VCN}(\widetilde{\mathcal{H}})$.

(1) We shall say that α is (S,T)-*compatible* if, for every $\tilde{c} \in S$, $\alpha(\Pi_{\tilde{c}}) = 1$ or there exists $\tilde{d} \in T$ such that $\alpha(\Pi_{\tilde{c}}) \subset \Pi_{\tilde{d}}$.

(2) We shall say that α is (S,T)-*filtration-preserving* if, for every connected finite étale subcovering $\mathcal{H}' \longrightarrow \mathcal{H}$ of $\widetilde{\mathcal{H}} \longrightarrow \mathcal{H}$, it holds that $\alpha^{\mathrm{ab}}(M^S_{\mathcal{G}'}) \subset M^T_{\mathcal{H}'}$, where we write $\mathcal{G}' \longrightarrow \mathcal{G}$ for the connected finite étale subcovering of $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ corresponding to the open subgroup $\alpha^{-1}(\Pi_{\mathcal{H}'})$ of $\Pi_{\mathcal{G}}$ (i.e., $\Pi_{\mathcal{G}'} = \alpha^{-1}(\Pi_{\mathcal{H}'})$) and α^{ab} for the homomorphism $M_{\mathcal{G}'} \longrightarrow M_{\mathcal{H}'}$ naturally induced by α.

Remark 1.7.1. Suppose that we are in the situation of Definition 1.7. Then the part "$\alpha(\Pi_{\tilde{c}}) = 1$ or" in Definition 1.7, (1), is relevant only when $T = \emptyset$, and we often omit the mention of this case. The reason we allowed such α that $\alpha(\Pi_{\tilde{c}}) = 1$ for every $\tilde{c} \in S$ to be $(S, \emptyset)$-compatible (even when $S \neq \emptyset$) is to simplify things. See Remark 3.2.2 for an example.

Example 1.7.2. Suppose that $\mathcal{G}$ is sturdy (cf. [CbGC], Definition 1.1, (ii), and [CbGC], Remark 1.1.5). Write $\mathcal{G}^{\mathrm{cpt}}$ for the compactification of $\mathcal{G}$ (cf. [CbGC], Remark 1.1.6), which is again a semi-graph of anabelioids of pro-$\Sigma_{\mathcal{G}}$ PSC-type. Then it is immediate that the natural surjection $\alpha\colon \Pi_{\mathcal{G}} \twoheadrightarrow \Pi_{\mathcal{G}^{\mathrm{cpt}}}$ is $(\mathrm{Vert}(\widetilde{\mathcal{G}}), \mathrm{Vert}(\widetilde{\mathcal{G}^{\mathrm{cpt}}}))$-compatible, $(\mathrm{Node}(\widetilde{\mathcal{G}}), \mathrm{Node}(\widetilde{\mathcal{G}^{\mathrm{cpt}}}))$-compatible, and $(\mathrm{Cusp}(\widetilde{\mathcal{G}}), \mathrm{Cusp}(\widetilde{\mathcal{G}^{\mathrm{cpt}}}))$-compatible. Note, however, that if assuming the part "$\alpha(\Pi_{\tilde{c}}) = 1$ or" in Definition 1.7, (1), is omitted, then $(\mathrm{Cusp}(\widetilde{\mathcal{G}}), \mathrm{Cusp}(\widetilde{\mathcal{G}^{\mathrm{cpt}}}))$-compatibility of α is not satisfied unless $\mathrm{Cusp}(\mathcal{G}) = \emptyset$.

Remark 1.7.3. Suppose that we are in the situation of Definition 1.7. Then it follows immediately from Remark 1.5.3 that, in order to verify that α is (S,T)-filtration-preserving, it suffices to verify the following (apparently weaker) condition: there exists a cofinal subsystem $(\mathcal{H}_\lambda \longrightarrow \mathcal{H})_{\lambda \in \Lambda}$ of (the projective system which gives rise to) the pro-$\Sigma_{\mathcal{H}}$ universal covering $\widetilde{\mathcal{H}} \longrightarrow \mathcal{H}$ constituted by connected finite étale subcoverings of $\widetilde{\mathcal{H}} \longrightarrow \mathcal{H}$ such that $\alpha^{\mathrm{ab}}(M^S_{\mathcal{G}_\lambda}) \subset M^T_{\mathcal{H}_\lambda}$ for every $\lambda \in \Lambda$, where we write $\mathcal{G}_\lambda \longrightarrow \mathcal{G}$ for the connected finite étale subcovering of $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ corresponding to the open subgroup $\alpha^{-1}(\Pi_{\mathcal{H}_\lambda})$ of $\Pi_{\mathcal{G}}$ (i.e., $\Pi_{\mathcal{G}_\lambda} = \alpha^{-1}(\Pi_{\mathcal{H}_\lambda})$).

Remark 1.7.4. As is noted in Remark 1.7.5 below, the property of being (S,T)-compatible, defined for arbitrary continuous homomorphisms $\alpha\colon \Pi_{\mathcal{G}} \longrightarrow \Pi_{\mathcal{H}}$, can be considered as the unified generalized Hom-version of the properties of being "group-theoretically verticial", "group-theoretically edge-like", "group-theoretically nodal", and "group-theoretically cuspidal", defined only for isomorphisms $\alpha\colon \Pi_{\mathcal{G}} \xrightarrow{\cong} \Pi_{\mathcal{H}}$ (cf. [CbGC], Definition 1.4, (iv), and [NodNon], Definition 1.12). In light of this observation, we shall collectively refer to the properties of being (S,T)-compatible for some S, T as the *group-theoretic compatibility properties*.

Similarly, we shall refer to the properties of being (S,T)-filtration-preserving for some S, T as the *filtration-preservation properties*.

Remark 1.7.5. Let $\alpha\colon \Pi_{\mathcal{G}} \xrightarrow{\cong} \Pi_{\mathcal{H}}$ be an isomorphism of profinite groups. Then it holds that, for α to be group-theoretically verticial in the sense of [CbGC], Definition 1.4, (iv), it is necessary and sufficient that α is $\big(\mathrm{Vert}(\widetilde{\mathcal{G}}), \mathrm{Vert}(\widetilde{\mathcal{H}})\big)$-compatible and α^{-1} is $\big(\mathrm{Vert}(\widetilde{\mathcal{H}}), \mathrm{Vert}(\widetilde{\mathcal{G}})\big)$-compatible. Similar remarks also hold for the terms "group-theoretically edge-like", "group-theoretically nodal", "group-theoretically cuspidal", "verticially filtration-preserving", and "edgewise filtration-preserving" (cf. [CbGC], Definition 1.4, (iii), (iv), and [NodNon], Definition 1.12). Indeed, the equivalences on the filtration-preservation properties follow formally from the definitions. The equivalences on the group-theoretic compatibility properties are also easy, though we include the proof in Lemma 1.9 below.

Remark 1.7.6. Let $\alpha\colon \Pi_{\mathcal{G}} \longrightarrow \Pi_{\mathcal{H}}$ be a continuous homomorphism. Then it follows from [NodNon], Proposition 1.13, that, if α is an isomorphism, then the group-theoretic verticiality (cf. [CbGC], Definition 1.4, (iv)) of α implies the group-theoretic nodality (cf. [NodNon], Definition 1.12) of α. However, one may verify that the $\big(\mathrm{Vert}(\widetilde{\mathcal{G}}), \mathrm{Vert}(\widetilde{\mathcal{H}})\big)$-compatibility of α does not imply the $\big(\mathrm{Node}(\widetilde{\mathcal{G}}), \mathrm{Node}(\widetilde{\mathcal{H}})\big)$-compatibility of α in general. A counter-example is given in Example 1.7.7 below.

Here, we explain what a problem occurs when one tries to apply to our situation a similar argument to the proof of [NodNon], Proposition 1.13. Suppose that α is $\big(\mathrm{Vert}(\widetilde{\mathcal{G}}), \mathrm{Vert}(\widetilde{\mathcal{H}})\big)$-compatible. Let $\tilde{e} \in \mathrm{Node}(\widetilde{\mathcal{G}})$ and $\tilde{v}_1$, $\tilde{v}_2$ the two distinct vertices to which $\tilde{e}$ abuts (cf. Remark 1.2.1). By assumption, for each $i \in \{1,2\}$, there exists $\tilde{w}_i \in \mathrm{Vert}(\widetilde{\mathcal{H}})$ such that $\alpha(\Pi_{\tilde{v}_i}) \subset \Pi_{\tilde{w}_i}$. By Lemma 1.2, (3), it holds that $\Pi_{\tilde{e}} = \Pi_{\tilde{v}_1} \cap \Pi_{\tilde{v}_2}$, hence $\alpha(\Pi_{\tilde{e}}) \subset \Pi_{\tilde{w}_1} \cap \Pi_{\tilde{w}_2}$. However, it is still possible that $\tilde{w}_1 = \tilde{w}_2$, which implies that $\Pi_{\tilde{w}_1} \cap \Pi_{\tilde{w}_2}$ is not a nodal subgroup (but a verticial subgroup) of $\Pi_{\mathcal{H}}$. In particular, one cannot conclude that $\alpha(\Pi_{\tilde{e}})$ is included in a nodal subgroup.

Example 1.7.7. Let $\mathcal{G}$ be a semi-graph of anabelioids of PSC-type such that $\mathrm{Node}(\mathcal{G}) \neq \emptyset$, $\mathcal{H}$ its generization associated to the full subset $\mathrm{Node}(\mathcal{G}) \subset \mathrm{Node}(\mathcal{G})$ (cf. [CbTpI], Definition 2.8), and $\alpha\colon \Pi_{\mathcal{G}} \xrightarrow{\cong} \Pi_{\mathcal{H}}$ the inverse of (some representative of) the specialization outer isomorphism $\Pi_{\mathcal{H}} \xrightarrow{\cong} \Pi_{\mathcal{G}}$ (cf. [CbTpI], Definition 2.10). Then it is clear that α is $\big(\mathrm{Vert}(\widetilde{\mathcal{G}}), \mathrm{Vert}(\widetilde{\mathcal{H}})\big)$-compatible while it is not $\big(\mathrm{Node}(\widetilde{\mathcal{G}}), \mathrm{Node}(\widetilde{\mathcal{H}})\big)$-compatible.

Corollary 1.8. *Let $\alpha\colon \Pi_{\mathcal{G}} \longrightarrow \Pi_{\mathcal{H}}$ be a continuous homomorphism, $\mathcal{G}' \longrightarrow \mathcal{G}$ (resp. $\mathcal{H}' \longrightarrow \mathcal{H}$) a connected finite étale subcovering of $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ (resp. $\widetilde{\mathcal{H}} \longrightarrow \mathcal{H}$), $S \subset \mathrm{VCN}(\widetilde{\mathcal{G}})$, and $T \subset \mathrm{VCN}(\widetilde{\mathcal{H}})$. Suppose that $\alpha(\Pi_{\mathcal{G}'}) \subset \Pi_{\mathcal{H}'}$. Write $\alpha'\colon \Pi_{\mathcal{G}'} \longrightarrow \Pi_{\mathcal{H}'}$ for the restriction of α. Then α is (S,T)-compatible if and only if α' is (S,T)-compatible.*

Proof. This follows easily from Lemma 1.4, (2), and Remark 1.5.2. □

Lemma 1.9. *Let $\alpha\colon \Pi_{\mathcal{G}} \xrightarrow{\cong} \Pi_{\mathcal{H}}$ be an isomorphism of profinite groups, $\square \in \{\mathrm{Vert}, \mathrm{Edge}\}$, $S \subset \square(\widetilde{\mathcal{G}})$, and $T \subset \square(\widetilde{\mathcal{H}})$. Then the following two conditions are equivalent.*

- *(i) α maps each S-like subgroup of $\Pi_{\mathcal{G}}$ isomorphically onto a T-like subgroup of $\Pi_{\mathcal{H}}$, and, moreover, every T-like subgroup of $\Pi_{\mathcal{H}}$ arises in this fashion.*
- *(ii) α is (S,T)-compatible, and α^{-1} is (T,S)-compatible.*

Proof. The implication (i) $\Longrightarrow$ (ii) is clear. To show the converse, suppose that α is (S,T)-compatible and that α^{-1} is (T,S)-compatible. Let $\tilde{c} \in S$. The assumptions on α imply that there exists $\tilde{d} \in T$ such that $\alpha(\Pi_{\tilde{c}}) \subset \Pi_{\tilde{d}}$. In a similar vein, we have a component $\tilde{c}' \in S$ satisfying the relation $\alpha^{-1}(\Pi_{\tilde{d}}) \subset \Pi_{\tilde{c}'}$, hence the relation $\Pi_{\tilde{c}} \subset \alpha^{-1}(\Pi_{\tilde{d}}) \subset \Pi_{\tilde{c}'}$. Then, if $\square = \mathrm{Edge}$ (resp. $\square = \mathrm{Vert}$), then it follows from Lemma 1.2, (1) (resp. Lemma 1.2, (2), (3)), that $\Pi_{\tilde{c}} = \alpha^{-1}(\Pi_{\tilde{d}}) = \Pi_{\tilde{c}'}$. Thus $\alpha(\Pi_{\tilde{c}})$ is equal to $\Pi_{\tilde{d}}$, hence a T-like subgroup of $\Pi_{\mathcal{H}}$. By switching the roles played by α and α^{-1} in the above argument, it also follows that every T-like subgroup of $\Pi_{\mathcal{H}}$ arises in this fashion. This completes the proof of the implication (ii) $\Longrightarrow$ (i), hence also the proof of Lemma 1.9. □

Lemma 1.10. *Let* $\alpha\colon \Pi_{\mathcal{G}} \longrightarrow \Pi_{\mathcal{H}}$ *be a continuous homomorphism,* $S \subset \mathrm{VCN}(\widetilde{\mathcal{G}})$*, and* $T \subset \mathrm{VCN}(\widetilde{\mathcal{H}})$*. We shall write* $S^{\Pi_{\mathcal{G}}} \stackrel{\mathrm{def}}{=} \{\tilde{c}^{\gamma} \in \mathrm{VCN}(\widetilde{\mathcal{G}}) \mid \tilde{c} \in S,\ \gamma \in \Pi_{\mathcal{G}}\}$ *and we define* $T^{\Pi_{\mathcal{H}}} \subset \mathrm{VCN}(\widetilde{\mathcal{H}})$ *in a similar vein. Then, if* α *is* (S,T)*-compatible, then* α *is* $(S^{\Pi_{\mathcal{G}}}, T^{\Pi_{\mathcal{H}}})$*-compatible.*

Proof. By Remark 1.3.1, the $S^{\Pi_{\mathcal{G}}}$-like subgroups of $\Pi_{\mathcal{G}}$ are precisely the $\Pi_{\mathcal{G}}$-conjugates of the S-like subgroups of $\Pi_{\mathcal{G}}$; a similar assertion holds also for $T^{\Pi_{\mathcal{H}}}$-subgroups of $\Pi_{\mathcal{H}}$. Lemma 1.10 follows immediately from this observation. □

2 Reduction to the Case of a Smaller Set of Prime Numbers

In this section, we give a detailed exposition on the argument of "reduction to the case of a smaller set of prime numbers", which is well-known to experts. As a consequence, we prove that, in order to show a certain group-theoretic compatibility property of a homomorphism between PSC-fundamental groups, it suffices to verify certain group-theoretic compatibility properties of homomorphisms between the maximal pro-Σ quotients of various open subgroups, where Σ is a certain "smaller" set of prime numbers (cf. Lemma 2.5 below). Even though only the particular case where the "smaller set of prime numbers" is given by "$\{l\}$", a singleton, is applied in the present series of papers, the author decided to include the general case because the proof is entirely the same.

The notational and terminological conventions established in the discussion preceding Definition 1.1 remains valid; in particular, the letters "$\mathcal{G}$" and "$\mathcal{H}$" always denote semi-graphs of anabelioids of PSC-type; we do not assume that $\Sigma_{\mathcal{G}} = \Sigma_{\mathcal{H}}$.

First, let us make it explicit how to consider the maximal pro-Σ quotient $\Pi_{\mathcal{G}}^{\Sigma}$ (where $\emptyset \neq \Sigma \subset \Sigma_{\mathcal{G}}$) itself as a PSC-fundamental group.

Lemma 2.1. *Let* Σ *be a non-empty subset of* $\Sigma_{\mathcal{G}}$ *and* $\mathcal{G}' \longrightarrow \mathcal{G}$ *a connected finite étale subcovering of* $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$*. Write* $(\mathcal{G}')^{\Sigma}$ *for the pro-*Σ *completion of* $\mathcal{G}'$*, i.e., the semi-graph of anabelioids obtained by replacing the constituent anabelioids of* $\mathcal{G}'$ *by their pro-*Σ *completions (cf. [SemiAn], Definition 2.9, (ii)).*

(1) The semi-graph of anabelioids $(\mathcal{G}')^\Sigma$ is of pro-Σ PSC-type. Moreover, one can construct the pro-Σ universal covering $\widetilde{(\mathcal{G}')^\Sigma} \longrightarrow (\mathcal{G}')^\Sigma$ from the pro-$\Sigma_\mathcal{G}$ universal covering $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ in a natural way.

(2) The fundamental group $\Pi_{(\mathcal{G}')^\Sigma}$ of $(\mathcal{G}')^\Sigma$ is naturally isomorphic to the maximal pro-Σ quotient $\Pi_{\mathcal{G}'}^\Sigma$ of $\Pi_{\mathcal{G}'}$. Here, we choose the "$\widetilde{(\mathcal{G}')^\Sigma} \longrightarrow (\mathcal{G}')^\Sigma$" constructed naturally from the pro-$\Sigma_\mathcal{G}$ universal covering $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ (cf. the assertion (1)) as the pro-Σ universal covering of $(\mathcal{G}')^\Sigma$, and we think $\Pi_{(\mathcal{G}')^\Sigma}$ as associated to that pro-Σ universal covering.

(3) Suppose that we are in the situation of the assertion (2). Write X for the set of VCN-subgroups of $\Pi_{\mathcal{G}'}$; write Y for the set of VCN-subgroups of $\Pi_{(\mathcal{G}')^\Sigma}$. Then we have a natural commutative diagram

$$\begin{array}{ccc} \mathrm{VCN}(\widetilde{\mathcal{G}}) & \xrightarrow{\ \cong\ } & X \\ \big\downarrow & & \big\downarrow \\ \mathrm{VCN}\big(\widetilde{(\mathcal{G}')^\Sigma}\big) & \xrightarrow{\ \cong\ } & Y, \end{array}$$

where the right-hand vertical arrow $X \twoheadrightarrow Y$ maps a VCN-subgroup of $\Pi_{\mathcal{G}'}$ to its image via the natural surjection $\Pi_{\mathcal{G}'} \twoheadrightarrow \Pi_{(\mathcal{G}')^\Sigma}$ (cf. the assertion (2)).

(4) Suppose that we are in the situation of the assertion (3) and moreover that the connected finite étale covering $\mathcal{G}' \longrightarrow \mathcal{G}$ is Galois. Recall that we have a natural right actions of $\Pi_\mathcal{G}$ on $\mathrm{VCN}(\widetilde{\mathcal{G}})$ and on X (cf. Lemma 1.3). Then we have natural right actions of $\Pi_\mathcal{G}$ on $\mathrm{VCN}\big(\widetilde{(\mathcal{G}')^\Sigma}\big)$ and on Y with respect to which the four arrows in the diagram of the assertion (3) are all $\Pi_\mathcal{G}$-equivariant.

(5) By forming the quotients of (each vertex of) the diagram in the assertion (3) by the actions of $\Pi_{\mathcal{G}'} \subset \Pi_\mathcal{G}$ (cf. the assertion (4)), one obtains a commutative diagram of sets with right $\Pi_\mathcal{G}$ actions and $\Pi_\mathcal{G}$-equivariant maps

$$\begin{array}{ccc} \mathrm{VCN}(\mathcal{G}') & \xrightarrow{\ \cong\ } & X/\Pi_{\mathcal{G}'} \\ \cong\big\downarrow & & \big\downarrow\cong \\ \mathrm{VCN}((\mathcal{G}')^\Sigma) & \xrightarrow{\ \cong\ } & Y/\Pi_{\mathcal{G}'}. \end{array}$$

Moreover, the left-hand vertical arrow of this diagram coincides with the "identity" map $\mathrm{VCN}(\mathcal{G}') \xrightarrow{\cong} \mathrm{VCN}((\mathcal{G}')^{\Sigma})$; $c \longmapsto c$.

(6) *Suppose that we are in the situation of the assertion (2); in particular, by conjugation, we have a natural right action of* $\Pi_{\mathcal{G}}$ *on* $\Pi_{\mathcal{G}'}$, *which thus induces a right action of* $\Pi_{\mathcal{G}}$ *on* $\Pi_{\mathcal{G}'}^{\Sigma} \cong \Pi_{(\mathcal{G}')^{\Sigma}}$. *Then, for every* $\gamma \in \Pi_{\mathcal{G}}$, *the isomorphism* $\Pi_{(\mathcal{G}')^{\Sigma}} \xrightarrow{\cong} \Pi_{(\mathcal{G}')^{\Sigma}}$ *induced by* γ *is graphic in the sense of [CbGC], Definition 1.4, (i). Moreover, the action* $Y \curvearrowleft \Pi_{\mathcal{G}}$ *in the assertion (4) coincides with the action induced by the action* $\Pi_{(\mathcal{G}')^{\Sigma}} \curvearrowleft \Pi_{\mathcal{G}}$.

Proof. First, we verify the assertion (1). The assertion that the semi-graph of anabelioids $(\mathcal{G}')^{\Sigma}$ is of pro-Σ PSC-type follows immediately from the definitions. Thus we have only to construct the pro-Σ universal covering $\widetilde{(\mathcal{G}')^{\Sigma}} \longrightarrow (\mathcal{G}')^{\Sigma}$ naturally from the pro-$\Sigma_{\mathcal{G}}$ universal covering $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$. Since the implicit structure morphism $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}'$ of the subcovering $\mathcal{G}' \longrightarrow \mathcal{G}$ of $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ gives the pro-$\Sigma_{\mathcal{G}'}$ universal covering of $\mathcal{G}'$, we may assume that $\mathcal{G}' = \mathcal{G}$. However, for the sake of the later use (cf. the proof of the assertion (4)), we continue to use the notation "$\mathcal{G}'$".

Suppose that the pro-$\Sigma_{\mathcal{G}'}$ universal covering $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}'$ arises from a pro-object $\widetilde{P} = (P_{\lambda})_{\lambda \in \Lambda}$ of the Galois category $\mathcal{B}(\mathcal{G}')$. Suppose moreover, for simplicity, that P_{λ} is a connected object of $\mathcal{B}(\mathcal{G}')$ for every $\lambda \in \Lambda$ and that every connected object of $\mathcal{B}(\mathcal{G}')$ is isomorphic to P_{λ} for some $\lambda \in \Lambda$. Then we define $M \stackrel{\mathrm{def}}{=} \{\lambda \in \Lambda \mid P_{\lambda}$ is Galois of degree $\in \mathbb{N}_{>0}(\Sigma)$ in $\mathcal{B}(\mathcal{G}').\}$, where we write $\mathbb{N}_{>0}(\Sigma)$ for the (multiplicative) submonoid of $\mathbb{N}_{>0}$ (freely) generated by the elements of Σ. In light of Claim 2.1.A below, it follows immediately that P_{μ} belongs to the full subcategory $\mathcal{B}((\mathcal{G}')^{\Sigma})$ of $\mathcal{B}(\mathcal{G}')$ for every $\mu \in M$ and that the pro-object $(P_{\mu})_{\mu \in M}$ of $\mathcal{B}((\mathcal{G}')^{\Sigma})$ defines a pro-Σ universal covering of $(\mathcal{G}')^{\Sigma}$. Thus it suffices to verify Claim 2.1.A below:

> Claim 2.1.A: Let A be an object of $\mathcal{B}(\mathcal{G}')$. Then A is a connected Galois object of $\mathcal{B}(\mathcal{G}')$ of degree in $\mathbb{N}_{>0}(\Sigma)$ if and only if A belongs to the full subcategory $\mathcal{B}((\mathcal{G}')^{\Sigma})$ and is a connected Galois object of $\mathcal{B}((\mathcal{G}')^{\Sigma})$ of degree in $\mathbb{N}_{>0}(\Sigma)$.

Recall that the finite direct sums in the Galois category $\mathcal{B}(\mathcal{G}')$ (resp. $\mathcal{B}((\mathcal{G}')^{\Sigma})$) are constructed by taking the finite direct sums in (the underlying Galois category of) each constituent anabelioid on each vertex and then gluing them together in an appropriate way; similar constructions are valid also for the quotients by finite group actions. It follows immediately from this observation that the inclusion functor $\mathcal{B}((\mathcal{G}')^{\Sigma}) \longrightarrow \mathcal{B}(\mathcal{G}')$ preserves finite direct sums and quotients by finite group actions. Thus we have only to show that, if A is a connected Galois object of $\mathcal{B}(\mathcal{G}')$ of degree in $\mathbb{N}_{>0}(\Sigma)$, then A belongs to $\mathcal{B}((\mathcal{G}')^{\Sigma})$. Let $v \in \mathrm{Vert}(\mathcal{G}')$. Write $\mathcal{G}'_v$ for (the underlying Galois category of) the constituent anabelioid of $\mathcal{G}'$ at v; write ϕ_v for the natural exact functor $\mathcal{B}(\mathcal{G}') \longrightarrow \mathcal{G}'_v$. Then, in light of the construction of the quotients by a finite group actions in $\mathcal{B}(\mathcal{G}')$, together with the assumption that A is Galois in $\mathcal{B}(\mathcal{G}')$, it follows that $\phi_v(A)$ is a direct sum of some mutually isomorphic connected Galois objects of $\mathcal{G}'_v$. If $\phi_v(A)$ is a direct sum of m pieces of mutually isomorphic connected Galois objects of $\mathcal{G}'_v$ of degree n (where $m, n \in \mathbb{N}_{>0}$), then mn is in $\mathbb{N}_{>0}(\Sigma)$ by assumption, hence n is also in $\mathbb{N}_{>0}(\Sigma)$. This completes the proof of Claim 2.1.A, hence also the proof of the assertion (1).

Next, let us verify the assertion (2). We continue to use the notations applied in the proof of the assertion (1). By definition, $\Pi_{\mathcal{G}'}$ is the opposite group of the (profinite) automorphism group of the pro-object $\widetilde{P} = (P_\lambda)_{\lambda \in \Lambda}$ in $\mathcal{B}(\mathcal{G}')$; hence, by definition of M, $\Pi^{\Sigma}_{\mathcal{G}'}$ is the opposite group of the (profinite) automorphism group of the pro-object $(P_\mu)_{\mu \in M}$ in $\mathcal{B}(\mathcal{G}')$. On the other hand, as is stated essentially in the assertion (2), we think $\Pi_{(\mathcal{G}')^{\Sigma}}$ as the opposite group of the (profinite) automorphism group of the pro-object $\widetilde{P} = (P_\lambda)_{\lambda \in M}$ in $\mathcal{B}((\mathcal{G}')^{\Sigma})$ (cf. the proof of the assertion (1)). Since the inclusion functor $\mathcal{B}((\mathcal{G}')^{\Sigma}) \longrightarrow \mathcal{B}(\mathcal{G}')$ is fully faithful, we obtain a canonical isomorphism $\Pi_{(\mathcal{G}')^{\Sigma}} \xrightarrow{\cong} \Pi^{\Sigma}_{\mathcal{G}'}$. This completes the proof of the assertion (2).

Let us explain the commutative diagram in the assertion (3). The horizontal arrows are given in Lemma 1.3, (2). The left-hand vertical arrow is just the natural projection (cf. the proof of the assertion (4) for more detail). The right-hand vertical arrow is as in the statement. The only nontrivial point is the well-definedness of the right-hand vertical arrow; this is easily verified and left to the reader. Then the commutativity of the diagram follows immediately.

Next, let us verify the assertion (4). It suffices to give a natural right action of $\Pi_{\mathcal{G}}$ on $\mathrm{VCN}\big(\widetilde{(\mathcal{G}')^{\Sigma}}\big)$ which is compatible with the natural projection $\mathrm{VCN}(\widetilde{\mathcal{G}}) \twoheadrightarrow \mathrm{VCN}\big(\widetilde{(\mathcal{G}')^{\Sigma}}\big)$. Suppose that the pro-$\Sigma_{\mathcal{G}}$ universal covering $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ arises from a pro-object $\widetilde{Q} = (Q_\nu)_{\nu \in N}$ of $\mathcal{B}(\mathcal{G})$ and that the factorization

$\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}' \longrightarrow \mathcal{G}$ under consideration corresponds to a factorization $\widetilde{Q} \longrightarrow Q' \longrightarrow$ (terminal object). Suppose moreover that Q_ν is a connected object of $\mathcal{B}(\mathcal{G})$ for every $\nu \in N$ and that every connected object of $\mathcal{B}(\mathcal{G})$ is isomorphic to Q_ν for some $\nu \in N$. Recall that the Galois category $\mathcal{B}(\mathcal{G}')$ is naturally isomorphic to the slice category of $\mathcal{B}(\mathcal{G})$ over Q'. Then $\widetilde{Q} \longrightarrow Q'$ determines a pro-object of $\mathcal{B}(\mathcal{G}')$ (which corresponds to the pro-$\Sigma_{\mathcal{G}'}$ universal covering $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}'$); this is precisely the "$\widetilde{P}$" in the proof of the assertion (1). Write $\widetilde{(Q')^{\Sigma}}$ for the pro-object of $\mathcal{B}(\mathcal{G})$ corresponding to "$\widetilde{(\mathcal{G}')^{\Sigma}}$", i.e., the pro-object of $\mathcal{B}(\mathcal{G})$ obtained as the image of "$(P_\mu)_{\mu \in M}$" in the proof of the assertion (1) via the forgetful functor $\mathcal{B}(\mathcal{G}') \longrightarrow \mathcal{B}(\mathcal{G})$. Then it follows from the various definitions involved that we have a natural factorization $\widetilde{Q} \longrightarrow \widetilde{(Q')^{\Sigma}} \longrightarrow Q'$. Moreover, since Q' is a connected Galois object of $\mathcal{B}(\mathcal{G})$, every $\gamma \in \mathrm{Aut}(\widetilde{Q}) = \Pi_{\mathcal{G}}^{\mathrm{op}}$ induces a commutative diagram

$$\begin{CD}
\widetilde{Q} @>{\gamma}>{\cong}> \widetilde{Q} \\
@V{\text{natural}}VV @VV{\text{natural}}V \\
Q' @>{\text{induced by } \gamma}>{\cong}> Q',
\end{CD}$$

hence, by the definition of $\widetilde{(Q')^{\Sigma}}$, a commutative diagram

$$\begin{CD}
\widetilde{Q} @>{\gamma}>{\cong}> \widetilde{Q} \\
@V{\text{natural}}VV @VV{\text{natural}}V \\
\widetilde{(Q')^{\Sigma}} @>{\text{induced by } \gamma}>{\cong}> \widetilde{(Q')^{\Sigma}} \\
@V{\text{natural}}VV @VV{\text{natural}}V \\
Q' @>{\text{induced by } \gamma}>{\cong}> Q'.
\end{CD}$$

Since, for every $\mu \in M$ in the proof of the assertion (1), the three semi-graphs of anabelioids corresponding to P_μ in $\mathcal{B}((\mathcal{G}')^{\Sigma})$, to P_μ in $\mathcal{B}(\mathcal{G}')$, and to the image of P_μ in $\mathcal{B}(\mathcal{G})$, respectively, have mutually canonically isomorphic underlying semi-graphs, we obtain the desired action $\mathrm{VCN}\big(\widetilde{(\mathcal{G}')^{\Sigma}}\big) \curvearrowleft \Pi_{\mathcal{G}}$. By the commutativity of the upper half square of the second diagram above, it also follows that the natural projection $\mathrm{VCN}(\widetilde{\mathcal{G}}) \twoheadrightarrow \mathrm{VCN}\big(\widetilde{(\mathcal{G}')^{\Sigma}}\big)$ is $\Pi_{\mathcal{G}}$-equivariant with respect to this action. This completes the proof of the assertion (4).

The assertion (5) follows immediately from the various definitions involved, in light of the assumption that the covering $\mathcal{G}' \longrightarrow \mathcal{G}$ is Galois.

Finally, let us verify the assertion (6). The graphicity of the isomorphism $\Pi_{(\mathcal{G}')^\Sigma} \xrightarrow{\cong} \Pi_{(\mathcal{G}')^\Sigma}$ induced by $\gamma \in \Pi_{\mathcal{G}}$ follows from the graphicity of the conjugation action of $\Pi_{\mathcal{G}}$ on $\Pi_{\mathcal{G}'}$, together with the well-definedness of the right-hand vertical arrow of the diagram in the assertion (3) (the proof of which we left to the reader). To show that the action $Y \curvearrowleft \Pi_{\mathcal{G}}$ in the assertion (4) coincides with the action induced by the action $\Pi_{(\mathcal{G}')^\Sigma} \curvearrowleft \Pi_{\mathcal{G}}$, it suffices to show that the right-hand vertical arrow $X \twoheadrightarrow Y$ of the diagram in the assertion (3) is $\Pi_{\mathcal{G}}$-equivariant with respect to the natural action of $X \curvearrowleft \Pi_{\mathcal{G}}$ and the action of $Y \curvearrowleft \Pi_{\mathcal{G}}$ induced by the action $\Pi_{(\mathcal{G}')^\Sigma} \curvearrowleft \Pi_{\mathcal{G}}$. On the other hand, this is immediate from the definitions. This completes the proof of Lemma 2.1. $\square$

Lemma 2.2. *Let $H \subset \Pi_{\mathcal{G}}$ be a closed subgroup. Then the following assertions hold.*

(1) Suppose that $H \neq 1$. Then there exist $l \in \Sigma_{\mathcal{G}}$ and a connected finite étale subcovering $\mathcal{G}' \longrightarrow \mathcal{G}$ of $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ such that the image of $H \cap \Pi_{\mathcal{G}'}$ via the natural surjection $\Pi_{\mathcal{G}'} \longrightarrow \Pi_{\mathcal{G}'}^{(l)}$ is non-trivial.

(2) Let Σ be a non-empty subset of $\Sigma_{\mathcal{G}}$ such that the image of H via the natural surjection $\Pi_{\mathcal{G}} \longrightarrow \Pi_{\mathcal{G}}^{\Sigma}$ is non-trivial. Then, for every connected finite étale subcovering $\mathcal{G}' \longrightarrow \mathcal{G}$ of $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$, the image of $H \cap \Pi_{\mathcal{G}'}$ via the natural surjection $\Pi_{\mathcal{G}'} \longrightarrow \Pi_{\mathcal{G}'}^{\Sigma}$ is non-trivial.

Proof. First, let us verify the assertion (1). We may assume that H is pro-cyclic. Write J for the H assumed to be non-trivial and pro-cyclic. Since J is non-trivial, there exists an open normal subgroup N of $\Pi_{\mathcal{G}}$ such that $J \cap N \subsetneq J$. Write $V = JN$; $\mathcal{G}' \longrightarrow \mathcal{G}$ for the connected finite étale subcovering $\mathcal{G}' \longrightarrow \mathcal{G}$ of $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ corresponding to V (i.e., $V = \Pi_{\mathcal{G}'}$). Let l be a prime number which divides the order of $J/(J \cap N) = V/N$ (hence is contained in $\Sigma_{\mathcal{G}}$). Then we obtain a natural commutative diagram

$$\begin{array}{ccccc} J & \hookrightarrow & V & \twoheadrightarrow & V^{(l)} \\ \downarrow & & \downarrow & & \downarrow \\ J/(J \cap N) & = & V/N & \twoheadrightarrow & (V/N)^{(l)} \quad \neq \quad 1, \end{array}$$

where the "$\neq$" follows from the choice of l, together with the cyclicity of $J/J\cap N$. Then the desired non-triviality of $J \longrightarrow V^{(l)}$ follows formally from the easily verified non-triviality of the composition $J \twoheadrightarrow V/N \twoheadrightarrow (V/N)^{(l)}$. This completes the proof of the assertion (1).

The assertion (2) follows immediately from the natural commutative diagram

$$\begin{array}{ccccc} H & \hookrightarrow & \Pi_{\mathcal{G}} & \twoheadrightarrow & \Pi_{\mathcal{G}}^{\Sigma} \\ \uparrow & & \uparrow & & \uparrow \\ H\cap\Pi_{\mathcal{G}'} & \hookrightarrow & \Pi_{\mathcal{G}'} & \twoheadrightarrow & \Pi_{\mathcal{G}'}^{\Sigma}, \end{array}$$

together with the assumption that the image of H in $\Pi_{\mathcal{G}}^{\Sigma}$ is non-trivial and the fact that $\Pi_{\mathcal{G}}^{\Sigma}$ is torsion-free (cf. Lemma 2.1, (2), and [CbGC], Remark 1.1.3). This completes the proof of the assertion (2), hence also the proof of Lemma 2.2. □

Definition 2.3. Let Σ be a non-empty subset of $\Sigma_{\mathcal{G}}$ and $S \subset \mathrm{VCN}(\widetilde{\mathcal{G}})$. Then, for every connected finite étale subcovering $\mathcal{G}' \longrightarrow \mathcal{G}$ of $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$, we shall write $\widetilde{S}(\mathcal{G}',\Sigma) \subset \mathrm{VCN}\big(\widetilde{(\mathcal{G}')^{\Sigma}}\big)$ for the inverse image of $S(\mathcal{G}') \subset \mathrm{VCN}(\mathcal{G}') \cong \mathrm{VCN}\big((\mathcal{G}')^{\Sigma}\big)$ via the natural surjection $\mathrm{VCN}\big(\widetilde{(\mathcal{G}')^{\Sigma}}\big) \twoheadrightarrow \mathrm{VCN}\big((\mathcal{G}')^{\Sigma}\big)$.

Lemma 2.4. *Let Σ be a non-empty subset of $\Sigma_{\mathcal{G}}$, $S \subset \mathrm{VCN}(\widetilde{\mathcal{G}})$, and $H \subset \Pi_{\mathcal{G}}$ a closed subgroup. For every connected finite étale subcovering $\mathcal{G}' \longrightarrow \mathcal{G}$ of $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$, write $\mathrm{Im}(H\cap\Pi_{\mathcal{G}'})$ for the image of $H\cap\Pi_{\mathcal{G}'}$ via the natural surjection $\Pi_{\mathcal{G}'} \twoheadrightarrow \Pi_{\mathcal{G}'}^{\Sigma} \cong \Pi_{(\mathcal{G}')^{\Sigma}}$ (cf. Lemma 2.1, (2)). Suppose that the following condition holds:*

> *there exists a connected finite étale subcovering $\mathcal{G}' \longrightarrow \mathcal{G}$ of $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ such that $\mathrm{Im}(H\cap\Pi_{\mathcal{G}'}) \neq 1$.*

Then the following conditions are equivalent.

(i) H is included in an $\overline{S}$-like subgroup of $\Pi_{\mathcal{G}}$.

(ii) There exists a cofinal subsystem $(\mathcal{G}_\lambda \longrightarrow \mathcal{G})_{\lambda\in\Lambda}$ of (the projective system which gives rise to) the pro-$\Sigma_{\mathcal{G}}$ universal covering $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ constituted by connected finite étale subcoverings of $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ such that, for every $\lambda\in\Lambda$, $\mathrm{Im}(H\cap\Pi_{\mathcal{G}_\lambda})$ is included in an $\widetilde{S}(\mathcal{G}_\lambda,\Sigma)$-like subgroup of $\Pi_{(\mathcal{G}_\lambda)^{\Sigma}}$.

Proof. The implication (i) $\Longrightarrow$ (ii) is easy.

Let us verify the converse. We may assume that S is edge-complete (cf. Remark 1.6.1 and Remark 1.6.2).

First, we claim that we may assume that $\mathcal{G}_\lambda \longrightarrow \mathcal{G}$ is Galois for every $\lambda \in \Lambda$. To verify this claim, it suffices to show that, for every connected finite étale Galois subcovering $\mathcal{G}' \longrightarrow \mathcal{G}$ of $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$, $\mathrm{Im}(H \cap \Pi_{\mathcal{G}'}) \subset \Pi^{\Sigma}_{\mathcal{G}'}$ is included in a $\widetilde{S}(\mathcal{G}', \Sigma)$-like subgroup of $\Pi_{(\mathcal{G}')^\Sigma}$. (In fact, as is evident from the following proof, the Galois assumption on $\mathcal{G}' \longrightarrow \mathcal{G}$ is even superfluous.) By the cofinality assumption on $(\mathcal{G}_\lambda \longrightarrow \mathcal{G})_{\lambda \in \Lambda}$, there exists $\lambda \in \Lambda$ such that the morphism $\mathcal{G}_\lambda \longrightarrow \mathcal{G}$ factors through $\mathcal{G}' \longrightarrow \mathcal{G}$. Then we form a natural commutative diagram

$$\begin{array}{ccccc}
H \cap \Pi_{\mathcal{G}'} & \hookrightarrow & \Pi_{\mathcal{G}'} & \twoheadrightarrow & \Pi^{\Sigma}_{\mathcal{G}'} \\
\uparrow & & \uparrow & & \uparrow \\
H \cap \Pi_{\mathcal{G}_\lambda} & \hookrightarrow & \Pi_{\mathcal{G}_\lambda} & \twoheadrightarrow & \Pi^{\Sigma}_{\mathcal{G}_\lambda},
\end{array}$$

where we note that the image of the composite of the upper (resp. lower) horizontal arrows is precisely $\mathrm{Im}(H \cap \Pi_{\mathcal{G}'}) \subset \Pi^{\Sigma}_{\mathcal{G}'}$ (resp. $\mathrm{Im}(H \cap \Pi_{\mathcal{G}_\lambda}) \subset \Pi^{\Sigma}_{\mathcal{G}_\lambda}$). Now it follows from the assumption that $\mathrm{Im}(H \cap \Pi_{\mathcal{G}_\lambda}) \subset \Pi^{\Sigma}_{\mathcal{G}_\lambda}$ is included in a $\widetilde{S}(\mathcal{G}_\lambda, \Sigma)$-like subgroup of $\Pi_{(\mathcal{G}_\lambda)^\Sigma}$, together with the commutativity of the above diagram, that an open subgroup of $\mathrm{Im}(H \cap \Pi_{\mathcal{G}'}) \subset \Pi^{\Sigma}_{\mathcal{G}'}$, hence also $\mathrm{Im}(H \cap \Pi_{\mathcal{G}'})$ itself (cf. Lemma 1.4, (2)), is included in a $\widetilde{S}(\mathcal{G}', \Sigma)$-like subgroup of $\Pi_{(\mathcal{G}')^\Sigma}$. This competes the proof of the reduction to the case where $\mathcal{G}_\lambda \longrightarrow \mathcal{G}$ is Galois for every $\lambda \in \Lambda$.

Also, we observe that it follows from Lemma 2.2, (2), applied to "$(\mathcal{G}, H, \Sigma)$" $=$ $(\mathcal{G}', H \cap \Pi_{\mathcal{G}'}, \Sigma)$, that there exists a cofinal subsystem $(\mathcal{G}_\lambda \longrightarrow \mathcal{G})_{\lambda \in \Lambda'}$ of $(\mathcal{G}_\lambda \longrightarrow \mathcal{G})_{\lambda \in \Lambda}$ (where $\Lambda' \subset \Lambda$) such that, for every $\lambda \in \Lambda'$, $\mathrm{Im}(H \cap \Pi_{\mathcal{G}_\lambda}) \neq 1$. By replacing $(\mathcal{G}_\lambda \longrightarrow \mathcal{G})_{\lambda \in \Lambda}$ by $(\mathcal{G}_\lambda \longrightarrow \mathcal{G})_{\lambda \in \Lambda'}$, we may assume that, for every $\lambda \in \Lambda$, it holds that $\mathrm{Im}(H \cap \Pi_{\mathcal{G}_\lambda}) \neq 1$.

Now we show the condition (i). Here, the readers are recommended to recall the contents of Lemma 2.1, because the group actions, identifications, and notational conventions given in Lemma 2.1 will be applied without further mention below. It follows from Remark 1.5.6 that it suffices to show that there exists an element of $S(\mathcal{G}_\lambda)$ stabilized by the action of $H \subset \Pi_{\mathcal{G}}$. For this, in light of Lemma 2.1, (3), (4), and (5), it suffices to verify the following claim:

Claim 2.4.A: For every $\lambda \in \Lambda$, there exists $\widetilde{c_\lambda} \in \widetilde{S}(\mathcal{G}_\lambda, \Sigma) \subset \mathrm{VCN}\big(\widetilde{(\mathcal{G}_\lambda)^\Sigma}\big)$ such that the subgroup $\Pi_{\widetilde{c_\lambda}} \subset \Pi_{(\mathcal{G}_\lambda)^\Sigma}$ is stabilized by the action of $H \subset \Pi_{\mathcal{G}}$.

To this end, take (by assumption) $\tilde{c} \in \widetilde{S}(\mathcal{G}_\lambda, \Sigma)$ such that $\mathrm{Im}(H \cap \Pi_{\mathcal{G}_\lambda}) \subset \Pi_{(\mathcal{G}_\lambda)^\Sigma}$ is included in $\Pi_{\tilde{c}} \subset \Pi_{(\mathcal{G}_\lambda)^\Sigma}$. Since (it follows immediately from the definition of the action of $\Pi_{\mathcal{G}}$ on $\Pi_{(\mathcal{G}_\lambda)^\Sigma}$ that) the equality $\mathrm{Im}(H \cap \Pi_{\mathcal{G}_\lambda})^\gamma = \mathrm{Im}(H \cap \Pi_{\mathcal{G}_\lambda})$ holds for every $\gamma \in H$, the inclusion $\mathrm{Im}(H \cap \Pi_{\mathcal{G}_\lambda}) \subset \Pi_{\tilde{c}} \cap \Pi_{\tilde{c}}^\gamma$ holds for every $\gamma \in H$. In light of Lemma 1.2, (1) and (3), together with our further assumption that $\mathrm{Im}(H \cap \Pi_{\mathcal{G}_\lambda}) \neq 1$ (cf. the sentence preceding the sentence "Now we show the condition (i)."), the inclusion $\mathrm{Im}(H \cap \Pi_{\mathcal{G}_\lambda}) \subset \Pi_{\tilde{c}} \cap \Pi_{\tilde{c}}^\gamma$ implies the following claim:

Claim 2.4.B: For every $\gamma \in H$, either the equality

$$\Pi_{\tilde{c}} = \Pi_{\tilde{c}}^\gamma$$

or the equality

$$\Pi_{\tilde{c}} \cap \Pi_{\tilde{c}}^\gamma = \Pi_{\tilde{e}}$$

holds, where in the latter case $\tilde{c}$ is necessarily a vertex and $\tilde{e} \in \mathrm{Node}\big(\widetilde{(\mathcal{G}_\lambda)^\Sigma}\big)$ is the unique node which abuts both to $\tilde{c}$ and to $\tilde{c}^\gamma$.

If, for every $\gamma \in H$, the former case of Claim 2.4.B holds, then $\tilde{c} \in \widetilde{S}(\mathcal{G}_\lambda, \Sigma)$ gives the desired "$\widetilde{c_\lambda}$" of Claim 2.4.A. Thus we may assume that there exists $\gamma' \in H$ for which the latter case of Claim 2.4.B holds. Take $\tilde{e} \in \mathrm{Node}\big(\widetilde{(\mathcal{G}_\lambda)^\Sigma}\big)$ such that $\Pi_{\tilde{c}} \cap \Pi_{\tilde{c}}^{\gamma'} = \Pi_{\tilde{e}}$. It holds that $\tilde{e} \in \widetilde{S}(\mathcal{G}_\lambda, \Sigma)$ because of the edge-completeness assumption on S (cf. the second sentence of the second paragraph of this proof). Moreover, we observe, as in the argument preceding Claim 2.4.B, that $1 \neq \mathrm{Im}(H \cap \Pi_{\mathcal{G}_\lambda}) \subset \Pi_{\tilde{e}} \cap \Pi_{\tilde{e}}^\gamma$. In light of Lemma 1.2, (1), it follows that $\Pi_{\tilde{e}} = \Pi_{\tilde{e}}^\gamma$ for every $\gamma \in H$. Thus $\tilde{e} \in \widetilde{S}(\mathcal{G}_\lambda, \Sigma)$ gives the desired "$\widetilde{c_\lambda}$" of Claim 2.4.A. This completes the proof of Claim 2.4.A, hence also the proof of the implication (ii) $\Longrightarrow$ (i). This completes the proof of Lemma 2.4. $\square$

Lemma 2.5. *Let* $\alpha\colon \Pi_{\mathcal{G}} \longrightarrow \Pi_{\mathcal{H}}$ *be a continuous homomorphism,* $S \subset \mathrm{VCN}(\widetilde{\mathcal{G}})$*,* $T \subset \mathrm{VCN}(\widetilde{\mathcal{H}})$*, and* $\emptyset \neq \Sigma' \subset \Sigma_{\mathcal{G}} \cap \Sigma_{\mathcal{H}}$*. Moreover, let* $(\mathcal{H}_\lambda \longrightarrow$

$\mathcal{H})_{\lambda\in\Lambda}$ be a cofinal subsystem of (the projective system which gives rise to) the pro-$\Sigma_{\mathcal{H}}$ universal covering $\widetilde{\mathcal{H}} \longrightarrow \mathcal{H}$ constituted by connected finite étale subcoverings of $\widetilde{\mathcal{H}} \longrightarrow \mathcal{H}$. For every $\lambda \in \Lambda$, write $\mathcal{G}_\lambda \longrightarrow \mathcal{G}$ for the connected finte étale subcovering of $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ corresponding to the open subgroup $\alpha^{-1}(\Pi_{\mathcal{H}_\lambda}) \subset \Pi_{\mathcal{G}}$; write $\alpha_\lambda \colon \Pi_{\mathcal{G}_\lambda} \longrightarrow \Pi_{\mathcal{H}_\lambda}$ for the continuous homomorphism induced by α. Suppose that the following condition holds:

> *for every $\tilde{c} \in S$, there exists a connected finite étale subcovering $\mathcal{H}' \longrightarrow \mathcal{H}$ of $\widetilde{\mathcal{H}} \longrightarrow \mathcal{H}$ such that the image of $\alpha(\Pi_{\tilde{c}}) \cap \Pi_{\mathcal{H}'}$ in $\Pi_{(\mathcal{H}')^\Sigma}$ via the natural surjection $\Pi_{\mathcal{H}'} \twoheadrightarrow \Pi_{\mathcal{H}'}^{\Sigma} \cong \Pi_{(\mathcal{H}')^\Sigma}$ (cf. Lemma 2.1, (2)) is non-trivial.*

Then the following conditions are equivalent.

(i) α is $(\overline{S}, \overline{T})$-compatible.

(ii) For every $\lambda \in \Lambda$, the continuous homomorphism $(\alpha_\lambda)^\Sigma \colon \Pi_{(\mathcal{G}_\lambda)^\Sigma} \longrightarrow \Pi_{(\mathcal{H}_\lambda)^\Sigma}$ is $\big(\widetilde{S}(\mathcal{G}_\lambda, \Sigma), \widetilde{T}(\mathcal{H}_\lambda, \Sigma)\big)$-compatible.

Proof. The implication (i) $\Longrightarrow$ (ii) is immediate. The implication (ii) $\Longrightarrow$ (i) follows immediately from Lemma 2.4, (ii) $\Longrightarrow$ (i). □

Remark 2.5.1. In Lemma 2.5, the condition (i) is equivalent to saying that, for every $\tilde{c} \in \overline{S}$, there exists $\tilde{d} \in \overline{T}$ such that $\alpha(\Pi_{\tilde{c}}) \subset \Pi_{\tilde{d}}$, because of the non-triviality assumption. That is to say, the part "$\alpha(\Pi_{\tilde{c}}) = 1$ or" in Definition 1.7, (1), is irrelevant here.

3 Filtration-preservation and Group-theoretic Compatibility

In this section, we study the relationship between the filtration-preservation properties and the group-theoretic compatibility properties (cf. Theorem 3.2). This result is not only of use in the proof of the main result of the present paper, Theorem 4.4, but also of independent interest. The key step is Proposition 3.1, where, roughly, we extract a characteristic property of the subgroups of S-like subgroups which can be described in a purely "abelianized" way.

The notational and terminological conventions established in the discussion preceding Definition 1.1 remains valid; in particular, the letters "$\mathcal{G}$"

and “$\mathcal{H}$” always denote semi-graphs of anabelioids of PSC-type; we do not assume that $\Sigma_{\mathcal{G}} = \Sigma_{\mathcal{H}}$.

Proposition 3.1. *Let S be a non-empty subset of $\mathrm{VCN}(\widetilde{\mathcal{G}})$ and H a closed subgroup of $\Pi_{\mathcal{G}}$. Recall here that we write $\overline{S}$ for the closure of $S \subset \mathrm{VCN}(\widetilde{\mathcal{G}})$ in $\mathrm{VCN}(\widetilde{\mathcal{G}})$ with respect to the profinite topology (cf. Remark 1.5.4). Let us consider the following seven conditions:*

(i) H is included in an S-like subgroup of $\Pi_{\mathcal{G}}$.

(ii) An open subgroup of H is included in an S-like subgroup of $\Pi_{\mathcal{G}}$.

(iii) Every element of H is contained in an S-like subgroup of $\Pi_{\mathcal{G}}$.

(iv) H is included in an $\overline{S}$-like subgroup of $\Pi_{\mathcal{G}}$.

(v) For any connected finite étale subcovering $\mathcal{G}' \longrightarrow \mathcal{G}$ of $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$, the image of the composite of the natural homomorphisms

$$H \cap \Pi_{\mathcal{G}'} \hookrightarrow \Pi_{\mathcal{G}'} \twoheadrightarrow M_{\mathcal{G}'}/M^S_{\mathcal{G}'}$$

is trivial.

(vi) There exists a connected finite étale subcovering $\mathcal{G}^\dagger \longrightarrow \mathcal{G}$ of $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ such that, for every connected finite étale subcovering $\mathcal{G}' \longrightarrow \mathcal{G}$ of $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ which factors through $\mathcal{G}^\dagger \longrightarrow \mathcal{G}$, the image of the composite of the natural homomorphisms

$$H \cap \Pi_{\mathcal{G}'} \hookrightarrow \Pi_{\mathcal{G}'} \twoheadrightarrow M_{\mathcal{G}'}/M^S_{\mathcal{G}'}$$

is trivial.

(vii) There exists a cofinal subsystem $(\mathcal{G}_\lambda \longrightarrow \mathcal{G})_{\lambda\in\Lambda}$ of (the projective system which gives rise to) the pro-$\Sigma_{\mathcal{G}}$ universal covering $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ constituted by finite étale Galois subcoverings of $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ such that, for every $\lambda \in \Lambda$, the image of the composite of the natural homomorphisms

$$H \cap \Pi_{\mathcal{G}_\lambda} \hookrightarrow \Pi_{\mathcal{G}_\lambda} \twoheadrightarrow M_{\mathcal{G}_\lambda}/M^S_{\mathcal{G}_\lambda}$$

is trivial.

Then the implications

$$\text{(i)} \iff \text{(ii)} \iff \text{(iii)} \implies \text{(iv)} \iff \text{(v)} \iff \text{(vi)} \iff \text{(vii)}$$

hold. In particular, if we suppose further that $S = \overline{S}$, then the conditions above are all equivalent.

Proof. The implication (i) $\implies$ (ii) is clear, while the converse follows immediately from Lemma 1.4, (2). Thus the equivalence (i) $\iff$ (ii) holds.

Next, let us verify the equivalence (i) $\iff$ (iii). The implication (i) $\implies$ (iii) is clear. Conversely, suppose that the condition (iii) holds. We may assume that $H \neq 1$ and S is edge-complete (cf. Remark 1.6.1). If every non-trivial element of H is contained in an edge-like subgroup of $\Pi_{\mathcal{G}}$, it follows from Lemma 1.2, (1) and (2), and the edge-completeness assumption that every non-trivial element of H is contained in an S-like edge-like subgroup of $\Pi_{\mathcal{G}}$. Then it follows from Claim 1.4.A in the proof of [CbTpII], Proposition 1.4 that H is included in an S-like subgroup. Thus we may assume that there exists an element $\gamma \in H$ which is not contained in any edge-like subgroup. In this case, there is a(n) (necessarily unique — cf. Lemma 1.2, (3)) S-like vertex $\tilde{v}$ such that $\gamma \in \Pi_{\tilde{v}}$. Note that we are now in the situation where

- every element of H is contained in a verticial subgroup of $\Pi_{\mathcal{G}}$,
- $\gamma \in H$ is not contained in any edge-like subgroup of $\Pi_{\mathcal{G}}$, and
- $\tilde{v}$ is the unique vertex of $\widetilde{\mathcal{G}}$ such that $\gamma \in \Pi_{\tilde{v}}$,

to which we can apply Claim 1.5.A in the proof of [CbTpII], Proposition 1.5. This shows $H \subset \Pi_{\tilde{v}}$, as desired. This completes the proof of the implication (iii) $\implies$ (i).

The implications (i) $\implies$ (iv) $\implies$ (v) $\implies$ (vi) $\implies$ (vii) are immediate (cf. also Remark 1.5.5), while the implication (vii) $\implies$ (v) follows immediately from Remark 1.5.3. Now we have only to show the implication (v) $\implies$ (iv). For this, we may assume that $S = \overline{S}$ (cf. Remark 1.5.5). Then, in light of the already verified equivalence (i) $\iff$ (iii), we may assume that H is pro-cyclic. Moreover, since (we have assumed that) $S \neq \emptyset$, we may also assume that H is non-trivial. We shall write J for the H assumed to be non-trivial and pro-cyclic.

To show the implication (v) $\implies$ (iv) for $H = J$, first we concentrate on the case where $\Sigma_{\mathcal{G}}$ is equal to $\{l\}$, a singleton. The general case will be treated after that. For U an open subgroup of $\Pi_{\mathcal{G}}$, write $\mathcal{G}_U$ for the connected finite étale subcovering of $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ corresponding to U, i.e., $\Pi_{\mathcal{G}_U} = U \subset \Pi_{\mathcal{G}}$. Then we claim as follows:

> Claim 3.1.A: For any normal open subgroup N of $\Pi_{\mathcal{G}}$, there exists an element of $S(\mathcal{G}_{J\cdot N})$ at which the connected finite étale covering $\mathcal{G}_N \longrightarrow \mathcal{G}_{J\cdot N}$ is totally ramified, i.e., there exists an element $c \in S(\mathcal{G}_{J\cdot N})$ that satisfies the condition that the composite of the natural homomorphisms
>
> $$\Pi_c \hookrightarrow J \cdot N \twoheadrightarrow (J \cdot N)/N$$
>
> is surjective. (Note that, even though the subgroup "Π_c" $\subset J \cdot N = \Pi_{\mathcal{G}_{J\cdot N}}$ is determined by c only up to $J \cdot N$-conjugation, the composite morphism under consideration is completely determined by c since $(J \cdot N)/N$ is abelian.)

We verify Claim 3.1.A as follows. Since J is pro-cyclic, and $(J \cdot N)/N$ is its quotient, $(J \cdot N)/N$ is a finite cyclic (hence abelian) l-group; in particular, we obtain a natural surjection $\Pi^{\mathrm{ab}}_{\mathcal{G}_{J\cdot N}} \twoheadrightarrow (J \cdot N)/N$. Moreover, again since $(J \cdot N)/N$ is a quotient of J, it follows from the condition (v) that the composite of the natural homomorphisms

$$\prod_{c\in S(\mathcal{G}_{J\cdot N})} \Pi_c \longrightarrow \Pi^{\mathrm{ab}}_{\mathcal{G}_{J\cdot N}} \twoheadrightarrow (J \cdot N)/N$$

is surjective. Therefore, it follows from the fact that $(J \cdot N)/N$ is a cyclic l-group and $S(\mathcal{G}_{J\cdot N}) \neq \emptyset$ that there exists $c \in S(\mathcal{G}_{J\cdot N})$ such that the composite of the natural homomorphisms $\Pi_c \hookrightarrow \Pi_{\mathcal{G}_{J\cdot N}} \twoheadrightarrow (J \cdot N)/N$ is surjective, as desired. This completes the proof of Claim 3.1.A.

We apply Claim 3.1.A as follows. In light of Remark 1.5.6, it suffices to give, for every connected finite étale Galois subcovering $\mathcal{G}' \longrightarrow \mathcal{G}$ of $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$, an element $c' \in S(\mathcal{G}')$ stabilized by the action of $J \subset \Pi_{\mathcal{G}}$ on $\mathrm{VCN}(\mathcal{G}')$. On the other hand, if we apply Claim 3.1.A to the open normal subgroup $N = \Pi_{\mathcal{G}'} \subset \Pi_{\mathcal{G}}$, we immediately obtain the desired "c'" as the unique component of $\mathcal{G}' = \mathcal{G}_N$ lying over the "c" in Claim 3.1.A. This completes the proof of the implication (v) $\Longrightarrow$ (iv) for $H = J$ and $\Sigma_{\mathcal{G}} = \{l\}$.

Now we treat the general case of the implication (v) $\Longrightarrow$ (iv) for $H = J$. Here, the readers are recommended to recall the contents of Lemma 2.1, because the group actions, identifications, and notational conventions given in Lemma 2.1 will be applied without further mention below. Suppose that the condition (v) holds for $H = J$. It follows immediately from Lemma 2.2, (1), that there exist $l \in \Sigma_{\mathcal{G}}$ and a connected finite étale subcovering $\mathcal{G}' \longrightarrow \mathcal{G}$ of $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ such that $\mathrm{Im}(J \cap \Pi_{\mathcal{G}'}) \subset \Pi_{(\mathcal{G}')^{(l)}}$ is non-trivial, where we write $(\mathcal{G}')^{(l)}$ for $(\mathcal{G}')^{\{l\}}$ and $\mathrm{Im}(J \cap \Pi_{\mathcal{G}'})$ for the image of $J \cap \Pi_{\mathcal{G}'} \subset \Pi_{\mathcal{G}'}$ via the natural surjection $\Pi_{\mathcal{G}'} \twoheadrightarrow \Pi_{(\mathcal{G}')^{(l)}}$. We fix such $l \in \Sigma_{\mathcal{G}}$. Then, in light of Lemma 2.4, (ii) $\Longrightarrow$ (i), it suffices to show that, for every connected finite étale Galois subcovering $\mathcal{G}' \longrightarrow \mathcal{G}$ of $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$, $\mathrm{Im}(J \cap \Pi_{\mathcal{G}'}) \subset \Pi_{(\mathcal{G}')^{(l)}}$ is included in a $\widetilde{S}(\mathcal{G}', \{l\})$-like subgroup of $\Pi_{(\mathcal{G}')^{(l)}}$ (cf. Definition 2.3). On the other hand, this follows immediately from the already verified "$\Sigma_{\mathcal{G}} = \{l\}$" case of the implication (v) $\Longrightarrow$ (iv) applied to the following Claim 3.1.B:

> Claim 3.1.B: The condition (v) holds for "$(\mathcal{G}, H = J, S)$" $= \big((\mathcal{G}')^{(l)}, \mathrm{Im}(J \cap \Pi_{\mathcal{G}''}), \widetilde{S}(\mathcal{G}', \{l\})\big)$. That is to say, for every connected finite étale subcovering $\mathcal{K} \longrightarrow (\mathcal{G}')^{(l)}$ of $\widetilde{(\mathcal{G}')^{(l)}} \longrightarrow (\mathcal{G}')^{(l)}$, the image of the composite of the natural homomorphisms
>
> $$\mathrm{Im}(J \cap \Pi_{\mathcal{G}'}) \cap \Pi_{\mathcal{K}} \hookrightarrow \Pi_{\mathcal{K}} \twoheadrightarrow M_{\mathcal{K}}/M_{\mathcal{K}}^{\widetilde{S}(\mathcal{G}', \{l\})}$$
>
> is trivial.

Finally, Claim3.1.B, i.e., the triviality of the composite of the lower horizontal arrows of the following natural commutative diagram, follows immediately from the triviality of the composite of the upper horizontal arrows of the following natural commutative diagram, where we write p for the natural surjection $\Pi_{\mathcal{G}'} \twoheadrightarrow \Pi_{\mathcal{G}'}^{(l)} \cong \Pi_{(\mathcal{G}')^{(l)}}$ and $\mathcal{L} \longrightarrow \mathcal{G}'$ for the connected finite étale subcovering of $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}'$ corresponding to the inverse image $p^{-1}(\Pi_{\mathcal{K}}) \subset \Pi_{\mathcal{G}'}$, i.e., $\Pi_{\mathcal{L}} = p^{-1}(\Pi_{\mathcal{K}})$:

$$\begin{array}{ccccc}
J \cap \Pi_{\mathcal{L}} & \hookrightarrow & \Pi_{\mathcal{L}} & \twoheadrightarrow & M_{\mathcal{L}}/M_{\mathcal{L}}^{S} \\
\downarrow {\scriptstyle \text{induced by } p} & & \downarrow {\scriptstyle \text{induced by } p} & & \downarrow {\scriptstyle \text{induced by } p} \\
\mathrm{Im}(J \cap \Pi_{\mathcal{G}'}) \cap \Pi_{\mathcal{K}} & \hookrightarrow & \Pi_{\mathcal{K}} & \twoheadrightarrow & M_{\mathcal{K}}/M_{\mathcal{K}}^{\widetilde{S}(\mathcal{G}', \{l\})}.
\end{array}$$

This completes the proof of the implication (v) $\Longrightarrow$ (iv), hence also the proof of Proposition 3.1. □

Remark 3.1.1. Proposition 3.1, along with its proof, should be thought of as a unified generalization of [CbTpII], Proposition 1.4, [CbTpII], Proposition 1.5, and [GrphPIPSC], Lemma 1.2. Indeed, if one takes S to be $\mathrm{Edge}(\widetilde{\mathcal{G}})$ (resp. $\mathrm{Vert}(\widetilde{\mathcal{G}})$; $\mathrm{Node}(\widetilde{\mathcal{G}})$), which is manifestly closed in $\mathrm{VCN}(\widetilde{\mathcal{G}})$, then the equivalences (i) $\iff$ (ii) $\iff$ (iii) $\iff$ (viii) in Proposition 3.1 are precisely the statement of [CbTpII], Proposition 1.4 (resp. [CbTpII], Proposition 1.5; [GrphPIPSC], Lemma 1.2). Note that the assumption $S \neq \emptyset$ is automatic for $S = \mathrm{Vert}(\widetilde{\mathcal{G}})$, while, in fact, it should have been assumed that $\mathrm{Edge}(\widetilde{\mathcal{G}}) \neq \emptyset$ (resp. $\mathrm{Node}(\widetilde{\mathcal{G}}) \neq \emptyset$) in [CbTpII], Proposition 1.4 (resp. [GrphPIPSC], Lemma 1.2), as is noted in Remark 3.1.2 below.

Remark 3.1.2. We observe that the assumption "$S \neq \emptyset$" in Proposition 3.1 is used only in the verification of the implication (v) $\Longrightarrow$ (iv). If one takes $S \subset \mathrm{VCN}(\widetilde{\mathcal{G}})$ to be empty, each of the conditions (i), (ii), (iii), (iv) of Proposition 3.1 can never be true, while (it is easily verified by an argument entirely similar to the proof of Lemma 2.2, (1) and (2), that) each of the conditions (v), (vi), (vii) is true if and only if $H = 1$.

For essentially the same reason, in [CbTpII], Proposition 1.4 (resp. [GrphPIPSC], Lemma 1.2), it should have been assumed that $\mathrm{Edge}(\widetilde{\mathcal{G}}) \neq \emptyset$ (resp. $\mathrm{Node}(\widetilde{\mathcal{G}}) \neq \emptyset$). The author believes that these slight errors in [CbTpII], Proposition 1.4, and [GrphPIPSC], Lemma 1.2, may not cause any serious problems in their applications obtained so far, since the trivial case $H = 1$ is the only one possible counter-example.

Theorem 3.2. *Let $\alpha\colon \Pi_{\mathcal{G}} \longrightarrow \Pi_{\mathcal{H}}$ be a continuous homomorphism, $S \subset \mathrm{VCN}(\widetilde{\mathcal{G}})$, and $T \subset \mathrm{VCN}(\widetilde{\mathcal{H}})$. Let us consider the following three conditions:*

(i) α is (S, T)-compatible.

(ii) α is (S, T)-filtration-preserving.

(iii) α is $(\overline{S}, \overline{T})$-compatible.

Then the implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) holds. In particular, if we suppose further that $S = \overline{S}$ and $T = \overline{T}$, then the conditions above are all equivalent.

Proof. The implication (i) $\Longrightarrow$ (ii) follows immediately from the definitions. If $T \neq \emptyset$, then the implication (ii) $\Longrightarrow$ (iii) follows immediately from Proposition 3.1, (v) $\Longrightarrow$ (iv), and Remark 1.5.5. If $T = \emptyset$, then the implication (ii) $\Longrightarrow$ (iii) follows immediately from the observation given in Remark 3.1.2, where we recall the part "$\alpha(\Pi_{\tilde{c}})$=1 or" of Definition 1.7, (1). This completes the proof of Theorem 3.2. $\square$

Remark 3.2.1. The last assertion of Theorem 3.2 can be thought of as a unified generalized Hom-version of [CbGC], Theorem 1.6, (ii). Indeed, to deduce [CbGC], Theorem 1.6, (ii), from Theorem 3.2, (4), we have only to apply the equivalences given in Remark 1.7.5 to the isomorphism $\alpha\colon \Pi_{\mathcal{G}} \xrightarrow{\cong} \Pi_{\mathcal{H}}$ under consideration, and set "(S,T)" in Theorem 3.2, (4), to be, for instance, $\big(\mathrm{Vert}(\widetilde{\mathcal{G}}), \mathrm{Vert}(\widetilde{\mathcal{H}})\big)$.

On the other hand, the proof of Theorem 3.2 is substantially different from the original proof of [CbGC], Theorem 1.6, (ii). Indeed, it does not use the technique of reconstruction of various VCN-subgroups involving numerical data concerning ramification, which plays a central role in the original proof of [CbGC], Theorem 1.6, (ii). Rather, Theorem 3.2 is essentially a formal consequence of Proposition 3.1, and the proof of Proposition 3.1 relies deeply on the technique established in the proof of [CbTpII], Proposition 1.4 and [CbTpII], Proposition 1.5 (cf. Remark 3.1.1). It seems to the author that it is difficult to apply the argument involving numerical data to our "Hom-version" situation.

Remark 3.2.2. Suppose that we are in the situation of Theorem 3.2. Then we observe that, if, in Definition 1.7, (1), we had not allowed such α that $\alpha(\Pi_{\tilde{c}}) = 1$ for every $\tilde{c} \in S$ to be $(S, \emptyset)$-compatible when $S \neq \emptyset$ (cf. Remark 1.7.1), then the implication (ii) $\Longrightarrow$ (iii) of Theorem 3.2 would be false. Indeed, under this altered definition, if $S \neq \emptyset$, $T = \emptyset$, and α is the trivial homomorphism, then α satisfies the condition (ii) while α would not satisfy the condition (iii).

4 Hom-version of the Combinatorial Grothendieck Conjecture for Graphically Full Outer Representations

In this section, we prove that a continuous homomorphism between PSC-fundamental groups $\Pi_{\mathcal{G}} \longrightarrow \Pi_{\mathcal{H}}$ satisfying a certain compatibility with "l-graphically full outer representations" is $(\mathrm{Vert}(\widetilde{\mathcal{G}}), \mathrm{Vert}(\widetilde{H}))$- and $(\mathrm{Edge}(\widetilde{\mathcal{G}}), \mathrm{Edge}(\widetilde{\mathcal{H}}))$-compatible. This result may be considered as a Hom-version of the combinatorial Grothendieck conjecture given in [CbGC], Theorem 2.7, (ii). Before that, we also give a short review of l-graphically full outer representations.

The notational and terminological conventions established in the discussion preceding Definition 1.1 remains valid; in particular, the letter "$\mathcal{G}$" always denotes a semi-graph of anabelioids of PSC-type.

Before proceeding, let us recall the issue concerning the profinite topology on $\mathrm{Aut}(\mathcal{G})$. Since the fundamental group $\Pi_{\mathcal{G}}$ of $\mathcal{G}$ is topologically finitely generated (cf. [CbGC], Remark 1.1.3), the profinite topology of $\Pi_{\mathcal{G}}$ induces (profinite) topologies on $\mathrm{Aut}(\Pi_{\mathcal{G}})$ and $\mathrm{Out}(\Pi_{\mathcal{G}})$ (cf. the discussion entitled "Topological Groups" in §0). Moreover, if we write $\mathrm{Aut}(\mathcal{G})$ for the group of automorphisms of $\mathcal{G}$, then by the discussion preceding [CbGC], Lemma 2.1, the natural homomorphism $\mathrm{Aut}(\mathcal{G}) \longrightarrow \mathrm{Out}(\Pi_{\mathcal{G}})$ is an injection with closed image. (Here, we recall that an automorphism of a semi-graph of anabelioids consists of an automorphism of the underlying semi-graph, together with a compatible system of isomorphisms between the various anabelioids at each of the vertices and edges of the underlying semi-graph which are compatible with the various morphisms of anabelioids associated to the branches of the underlying semi-graph — cf. [SemiAn], Definition 2.1.) Thus, by equipping $\mathrm{Aut}(\mathcal{G})$ with the topology induced via this homomorphism by the topology of $\mathrm{Out}(\Pi_{\mathcal{G}})$, we may regard $\mathrm{Aut}(\mathcal{G})$ as a profinite group.

Let I be a profinite group and $\rho\colon I \longrightarrow \mathrm{Aut}(\mathcal{G})$ a continuous homomorphism. Then, as is defined in [NodNon], Definition 2.1, (i), the pair $(\mathcal{G}, \rho\colon I \longrightarrow \mathrm{Aut}(\mathcal{G}))$ is referred to as an *outer representation of PSC-type*. Moreover, for $l \in \Sigma_{\mathcal{G}}$, if I is l-cyclotomically full (resp. l-graphically full) with respect to $\rho\colon I \longrightarrow \mathrm{Aut}(\mathcal{G})$ in the sense of [CbGC], Definition 2.3 (ii) (resp. (iii)), then we shall say that the outer representation of PSC-type $(\mathcal{G}, \rho\colon I \longrightarrow \mathrm{Aut}(\mathcal{G}))$ is *l-cyclotomically full* (resp. *l-graphically full*).

In the remainder of the present paper, we are concerned with the exact sequence of profinite groups and continuous homomorphisms

$$1 \longrightarrow \Pi_{\mathcal{G}} \longrightarrow \Pi_{\rho} \longrightarrow I \longrightarrow 1, \qquad (*)$$

where we write $\Pi_{\rho} \stackrel{\mathrm{def}}{=} \Pi_{\mathcal{G}} \stackrel{\mathrm{out}}{\rtimes} I$, associated to an outer representation of PSC-type $(\mathcal{G}, \rho\colon I \longrightarrow \mathrm{Aut}(\mathcal{G}))$. In other words, we are interested in the exact sequence obtained by pulling back the exact sequence

$$1 \longrightarrow \Pi_{\mathcal{G}} \longrightarrow \mathrm{Aut}(\Pi_{\mathcal{G}}) \longrightarrow \mathrm{Out}(\Pi_{\mathcal{G}}) \longrightarrow 1$$

via the continuous homomorphism $I \xrightarrow{\rho} \mathrm{Aut}(\mathcal{G}) \overset{\text{natural}}{\hookrightarrow} \mathrm{Out}(\Pi_{\mathcal{G}})$ (cf. the discussion entitled "Topological Groups" in §0). Here, we note that $\Pi_{\mathcal{G}}$ is topologically finitely generated and center-free (cf. [CbGC], Remark 1.1.3).

Lemma 4.1. *Let $(\mathcal{G}, \rho\colon I \longrightarrow \mathrm{Aut}(\mathcal{G}))$ be an outer representation of PSC-type. Let us consider the exact sequence $(*)$ associated to $\big(\mathcal{G}, \rho\colon I \longrightarrow \mathrm{Aut}(\mathcal{G})\big)$. Let $\Pi_{\rho'}$ be an open subgroup of Π_ρ. Write I' for the image of Π_ρ via the surjection $\Pi_\rho \twoheadrightarrow I$ and $\mathcal{G}' \longrightarrow \mathcal{G}$ for the connected finite étale subcovering of $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ corresponding to the open subgroup $\Pi_{\rho'} \cap \Pi_{\mathcal{G}}$ of $\Pi_{\mathcal{G}}$, i.e., $\Pi_{\mathcal{G}'} = \Pi_{\rho'} \cap \Pi_{\mathcal{G}}$. Then the following assertions hold.*

(1) The exact sequence $(*)$ naturally induces the following exact sequence:

$$1 \longrightarrow \Pi_{\mathcal{G}'} \longrightarrow \Pi_{\rho'} \longrightarrow I' \longrightarrow 1.$$

Moreover, this exact sequence arises from a unique outer representation of PSC-type $\big(\mathcal{G}', \rho'\colon I' \longrightarrow \mathrm{Aut}(\mathcal{G}')\big)$.

(2) Let $l \in \Sigma_{\mathcal{G}}$. Then, if $\big(\mathcal{G}, \rho\colon I \longrightarrow \mathrm{Aut}(\mathcal{G})\big)$ is l-cyclotomically full (resp. l-graphically full) (cf. [CbGC], Definition 2.3, (ii) (resp.(iii))), then $\big(\mathcal{G}', \rho'\colon I' \longrightarrow \mathrm{Aut}(\mathcal{G}')\big)$ in (1) is also l-cyclotomically full (resp. l-graphically full).

Proof. Let us verify the assertion (1). The well-definedness and the exactness of the sequence are immediate. Thus it suffices to show that the homomorphism $I' \longrightarrow \mathrm{Out}(\Pi_{\mathcal{G}'})$ factors through the natural closed immersion $\mathrm{Aut}(\mathcal{G}') \hookrightarrow \mathrm{Out}(\Pi_{\mathcal{G}'})$, or, equivalently, that the conjugation action of any element of $\Pi_{\rho'}$ on the closed normal subgroup $\Pi_{\mathcal{G}'} \subset \Pi_{\rho'}$ is graphic in the sense of [CbGC], Definition 1.4, (i) (cf. the discussion preceding [CbGC], Lemma 2.1). On the other hand, in light of [CbGC], Proposition 1.5, (ii), this follows immediately from the assumption and Remark 1.5.2. This completes the proof of the assertion (1).

Next, we verify the non-resp'd case of the assertion (2). We apply the notation of [CbGC], Definition 2.3. Moreover, we recall the pro-l cyclotomic character $\mathrm{Aut}(\mathcal{G}) \xrightarrow{\text{cyc.}} \mathbb{Z}_l^\times$ defined in [CbGC], Lemma 2.1. It suffices to show that the restriction of the composite $\Pi_\rho \twoheadrightarrow I \xrightarrow{\rho} \mathrm{Aut}(\mathcal{G}) \xrightarrow{\text{cyc.}} \mathbb{Z}_l^\times$ (whose image is assumed to be open) to the open subgroup $\Pi_{\rho'} \subset \Pi_\rho$ coincides with the composite $\Pi_{\rho'} \twoheadrightarrow I' \xrightarrow{\rho'} \mathrm{Aut}(\mathcal{G}') \xrightarrow{\text{cyc.}} \mathbb{Z}_l^\times$. It follows from the definitions that the composite morphism $\Pi_\rho \longrightarrow \mathbb{Z}_l^\times$ can be computed as follows (and entirely similar statement holds also for the composite morphism $\Pi_{\rho'} \longrightarrow \mathbb{Z}_l^\times$).

(i) Take an arbitrary connected finite étale subcovering $\mathcal{G}'' \longrightarrow \mathcal{G}$ of $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ such that $\mathcal{G}''$ is sturdy (cf. [CbGC], Definition 1.1, (ii), and [CbGC], Remark 1.1.5) and such that the corresponding open subgroup $\Pi_{\mathcal{G}''} \subset \Pi_{\mathcal{G}}$ is stable (as a subgroup) under the conjugation action of Π_ρ on $\Pi_{\mathcal{G}}$. Then Π_ρ acts graphically (cf. [CbGC], Definition 1.4, (i), and [CbGC], Proposition 1.5, (ii), for the terminology "graphic") on $\Pi_{\mathcal{G}''}$, hence also on $\Pi_{(\mathcal{G}'')^{\mathrm{cpt}}}$ (cf. Example 1.7.2 for the notation "$\Pi_{(\mathcal{G}'')^{\mathrm{cpt}}}$").

(ii) The action of Π_ρ on $\Pi_{(\mathcal{G}'')^{\mathrm{cpt}}}$ defined in (i) determines a continuous action of Π_ρ on $H^2(\Pi_{(\mathcal{G}'')^{\mathrm{cpt}}}, \mathbb{Z}_l)$, which is a free $\mathbb{Z}_l$-module of rank 1, hence a continuous homomorphism $\Pi_\rho \longrightarrow \mathbb{Z}_l^\times$. Finally, we take the inverse (i.e., "$(-)^{-1}$") of this homomorphism.

Here, we emphasize that, in (i) above, we may take an open subgroup which is not necessarily characteristic in $\Pi_{\mathcal{G}}$, despite that only the characteristic open subgroups of $\Pi_{\mathcal{G}}$ are considered in (the discussion preceding) [CbGC], Lemma 2.1. The verification of this slight generalization is immediate from an entirely similar argument to the proof of [CbGC], Lemma 2.1. Now the desired coincidence follows immediately from this description of the composite morphisms $\Pi_\rho \longrightarrow \mathbb{Z}_l^\times$ and $\Pi_{\rho'} \longrightarrow \mathbb{Z}_l^\times$.

Finally, we verify the resp'd case of the assertion (2). Since we have already shown the non-resp'd case of the assertion (2), we have only to show that, for every connected finite étale subcovering $\mathcal{G}'' \longrightarrow \mathcal{G}'$ of $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}'$ corresponding to a characteristic open subgroup $\Pi_{\mathcal{G}''} \subset \Pi_{\mathcal{G}'}$, $\underline{w}_l\big((M^{\mathrm{vert}}_{\mathcal{G}''}/M^{\mathrm{edge}}_{\mathcal{G}''}) \otimes_{\hat{\mathbb{Z}}^{\Sigma_{\mathcal{G}}}} \mathbb{Z}_l\big)$ is included in $(0,2)_{\mathbb{Q}} \stackrel{\mathrm{def}}{=} \{q \in \mathbb{Q} \mid 0 < q < 2.\}$, where the "$\underline{w}_l$" is taken with respect to the natural action of $\Pi_{\rho'}$. To this end, take a connected finite étale subcovering $\mathcal{G}''' \longrightarrow \mathcal{G}$ of $\widetilde{\mathcal{G}} \longrightarrow \mathcal{G}$ corresponding to a characteristic open subgroup $\Pi_{\mathcal{G}'''} \subset \Pi_{\mathcal{G}}$ such that it holds that $\Pi_{\mathcal{G}'''} \subset \Pi_{\mathcal{G}''}$. Then it follows from the assumption that it holds that $\underline{w}_l\big((M^{\mathrm{vert}}_{\mathcal{G}'''}/M^{\mathrm{edge}}_{\mathcal{G}'''}) \otimes_{\hat{\mathbb{Z}}^{\Sigma_{\mathcal{G}}}} \mathbb{Z}_l\big) \subset (0,2)_{\mathbb{Q}}$, where the "$\underline{w}_l$" is taken with respect to the natural action of Π_ρ. Moreover, by the "coincidence of the cyclotomic character" demonstrated in the proof of the non-resp'd case, the set $\underline{w}_l\big((M^{\mathrm{vert}}_{\mathcal{G}'''}/M^{\mathrm{edge}}_{\mathcal{G}'''}) \otimes_{\hat{\mathbb{Z}}^{\Sigma_{\mathcal{G}}}} \mathbb{Z}_l\big)$ is invariant if one changes the acting group from Π_ρ to $\Pi_{\rho'}$. Finally, since we have a natural $\Pi_{\rho'}$-equivariant open homomorphism $(M^{\mathrm{vert}}_{\mathcal{G}'''}/M^{\mathrm{edge}}_{\mathcal{G}'''}) \otimes_{\hat{\mathbb{Z}}^{\Sigma_{\mathcal{G}}}} \mathbb{Z}_l \longrightarrow (M^{\mathrm{vert}}_{\mathcal{G}''}/M^{\mathrm{edge}}_{\mathcal{G}''}) \otimes_{\hat{\mathbb{Z}}^{\Sigma_{\mathcal{G}}}} \mathbb{Z}_l$, we have $\underline{w}_l\big((M^{\mathrm{vert}}_{\mathcal{G}''}/M^{\mathrm{edge}}_{\mathcal{G}''}) \otimes_{\hat{\mathbb{Z}}^{\Sigma_{\mathcal{G}}}} \mathbb{Z}_l\big) \subset \underline{w}_l\big((M^{\mathrm{vert}}_{\mathcal{G}'''}/M^{\mathrm{edge}}_{\mathcal{G}'''}) \otimes_{\hat{\mathbb{Z}}^{\Sigma_{\mathcal{G}}}}$

$\mathbb{Z}_l)$. This completes the proof of the resp'd case of the assertion (2), hence also the proof of Lemma 4.1. □

Lemma 4.2. *Let* $l \in \Sigma_{\mathcal{G}}$ *and* $(\mathcal{G}, \rho\colon I \longrightarrow \mathrm{Aut}(\mathcal{G}))$ *an* l*-graphically full outer representation of PSC-type. Let us consider the outer representation of PSC-type* $(\mathcal{G}^{(l)}, \rho^{(l)}\colon I \longrightarrow \mathrm{Aut}(\mathcal{G}^{(l)}))$ *naturally determined by* $(\mathcal{G}, \rho\colon I \longrightarrow \mathrm{Aut}(\mathcal{G}))$. *Then the following assertions hold.*

(1) The pro-l *cyclotomic character* $I \longrightarrow \mathbb{Z}_l^{\times}$ *associated to* $(\mathcal{G}^{(l)}, \rho^{(l)}\colon I \longrightarrow \mathrm{Aut}(\mathcal{G}^{(l)}))$ *coincides with the pro-*l *cyclotomic character* $I \longrightarrow \mathbb{Z}_l^{\times}$ *associated to* $(\mathcal{G}, \rho\colon I \longrightarrow \mathrm{Aut}(\mathcal{G}))$ *(cf. [CbGC], Lemma 2.1 for the definition of pro-*l *cyclotomic character).*

(2) If $(\mathcal{G}, \rho\colon I \longrightarrow \mathrm{Aut}(\mathcal{G}))$ *is* l*-graphically full, then* $(\mathcal{G}^{(l)}, \rho^{(l)}\colon I \longrightarrow \mathrm{Aut}(\mathcal{G}^{(l)}))$ *is also* l*-graphically full.*

(3) The exact sequence $(*)$ *in the case where we take "*$(\mathcal{G}, \rho\colon I \longrightarrow \mathrm{Aut}(\mathcal{G}))$*" to be* $(\mathcal{G}^{(l)}, \rho^{(l)}\colon I \longrightarrow \mathrm{Aut}(\mathcal{G}^{(l)}))$ *coincides with the exact sequence obtained by taking the quotients of* $\Pi_{\mathcal{G}}$ *and* Π_{ρ} *of the exact sequence* $(*)$ *by the kernel of the natural homomorphism* $\Pi_{\mathcal{G}} \twoheadrightarrow \Pi_{\mathcal{G}}^{(l)} \cong \Pi_{\mathcal{G}^{(l)}}$ *(cf. Lemma 2.1, (2)). Here, we note that the kernel of the natural homomorphism* $\Pi_{\mathcal{G}} \twoheadrightarrow \Pi_{\mathcal{G}}^{(l)} \cong \Pi_{\mathcal{G}^{(l)}}$ *is characteristic in* $\Pi_{\mathcal{G}}$ *and thus normal in* Π_{ρ_I}.

Proof. The assertions (1) and (2) follow from a similar argument to the proof of Lemma 4.1, (2). The assertion (3) follows immediately from the definitions. This completes the proof of Lemma 4.2. □

Next, we recall the following result from [CbGC], which plays the most crucial role in the proof of Theorem 4.4.

Lemma 4.3. *Let* $l \in \Sigma_{\mathcal{G}}$ *and* $(\mathcal{G}, \rho\colon I \longrightarrow \mathrm{Aut}(\mathcal{G}))$ *an* l*-graphically full outer representation of PSC-type. Then the following assertions hold.*

(1) The quotient $M_{\mathcal{G}} \otimes_{\hat{\mathbb{Z}}^{\Sigma_{\mathcal{G}}}} \mathbb{Z}_l \twoheadrightarrow (M_{\mathcal{G}}/M_{\mathcal{G}}^{\mathrm{vert}}) \otimes_{\hat{\mathbb{Z}}^{\Sigma_{\mathcal{G}}}} \mathbb{Z}_l$ *is characterized as the maximal torsion-free quasi-trivial* $\mathbb{Z}_l[I]$*-quotient module of* $M_{\mathcal{G}} \otimes_{\hat{\mathbb{Z}}^{\Sigma_{\mathcal{G}}}} \mathbb{Z}_l$ *(cf. [CbGC], Definition 2.3, (i)).*

(2) The submodule $M_{\mathcal{G}}^{\mathrm{edge}} \otimes_{\hat{\mathbb{Z}}^{\Sigma_{\mathcal{G}}}} \mathbb{Z}_l \subset M_{\mathcal{G}} \otimes_{\hat{\mathbb{Z}}^{\Sigma_{\mathcal{G}}}} \mathbb{Z}_l$ *is characterized as the maximal quasi-toral* $\mathbb{Z}_l[I]$*-submodule of* $M_{\mathcal{G}} \otimes_{\hat{\mathbb{Z}}^{\Sigma_{\mathcal{G}}}} \mathbb{Z}_l$ *(cf. [CbGC], Definition 2.3, (i)).*

Proof. This is [CbGC], Proposition 2.4, (viii) and (ix). $\square$

This crucial result, combined with various results obtained in the present paper, naturally leads to the following theorem.

Theorem 4.4. *Let $(\mathcal{G}, \rho_I \colon I \longrightarrow \mathrm{Aut}(\mathcal{G}))$ and $(\mathcal{H}, \rho_J \colon J \longrightarrow \mathrm{Aut}(\mathcal{H}))$ be outer representations of PSC-type. Let us consider the following commutative diagram of profinite groups and continuous homomorphisms:*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Pi_{\mathcal{G}} & \longrightarrow & \Pi_{\rho_I} & \longrightarrow & I & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \tilde{\alpha}} & & \downarrow{\scriptstyle \beta} & & \\ 1 & \longrightarrow & \Pi_{\mathcal{H}} & \longrightarrow & \Pi_{\rho_J} & \longrightarrow & J & \longrightarrow & 1, \end{array}$$

where the two horizontal sequences are the respective exact sequences $(*)$ associated to $(\mathcal{G}, \rho_I \colon I \longrightarrow \mathrm{Aut}(\mathcal{G}))$ and to $(\mathcal{H}, \rho_J \colon J \longrightarrow \mathrm{Aut}(\mathcal{H}))$. Let $\square \in \{\mathrm{Vert}, \mathrm{Edge}\}$ and $S \subset \square(\widetilde{\mathcal{G}})$. Suppose further that there exists $l \in \Sigma_{\mathcal{G}} \cap \Sigma_{\mathcal{H}}$ for which the following four conditions hold.

- *(i) For every $\tilde{c} \in S$, there exists a connected finite étale subcovering $\mathcal{H}' \longrightarrow \mathcal{H}$ of $\widetilde{\mathcal{H}} \longrightarrow \mathcal{H}$ such that the image of $\alpha(\Pi_{\tilde{c}}) \cap \Pi_{\mathcal{H}'}$ via the natural surjection $\Pi_{\mathcal{H}'} \twoheadrightarrow \Pi_{\mathcal{H}'}^{(l)} \cong \Pi_{(\mathcal{H}')^{(l)}}$ is non-trivial.*
- *(ii) The outer representation of PSC-type $(\mathcal{G}, \rho_I \colon I \longrightarrow \mathrm{Aut}(\mathcal{G}))$ is l-graphically full.*
- *(iii) The outer representation of PSC-type $(\mathcal{H}, \rho_J \circ \beta \colon I \longrightarrow \mathrm{Aut}(\mathcal{H}))$ is l-graphically full.*
- *(iv) If, moreover, $\square = \mathrm{Edge}$, then the two cyclotomic characters $I \longrightarrow \mathbb{Z}_l^\times$ associated to $(\mathcal{G}, \rho_I \colon I \longrightarrow \mathrm{Aut}(\mathcal{G}))$ and to $(\mathcal{H}, \rho_J \circ \beta \colon I \longrightarrow \mathrm{Aut}(\mathcal{H}))$ coincide.*

Then α is $\big(\overline{S}, \square(\widetilde{\mathcal{H}})\big)$-compatible.

Proof. We fix $l \in \Sigma_{\mathcal{G}} \cap \Sigma_{\mathcal{H}}$ as in the statement. By pulling back the lower horizontal exact sequence of the commutative diagram under consideration via β, we may assume that β is an isomorphism (or even the identity) of profinite groups.

Let $(\Pi_{\rho_J, \lambda} \subset \Pi_{\rho_J})_{\lambda \in \Lambda}$ be the family of all open subgroups of Π_{ρ_J}. This family determines a family $(\Pi_{\mathcal{H}_\lambda} = \Pi_{\rho_J, \lambda} \cap \Pi_{\mathcal{H}})_{\lambda \in \Lambda}$ of open subgroups of $\Pi_{\mathcal{G}}$

such that $\bigcap_{\lambda\in\Lambda}\Pi_{\mathcal{H}_\lambda}=1$, or, equivalently, a cofinal subsystem $(\mathcal{H}_\lambda\longrightarrow\mathcal{H})_{\lambda\in\Lambda}$ of (the projective system which gives rise to) the pro-$\Sigma_{\mathcal{H}}$ universal covering $\widetilde{\mathcal{H}}\longrightarrow\mathcal{H}$. In light of Lemma 2.5, (ii) $\Longrightarrow$ (i), it suffices to show that the continuous homomorphism $(\alpha_\lambda)^{(l)}\colon\Pi_{(\mathcal{G}_\lambda)^{(l)}}\longrightarrow\Pi_{(\mathcal{H}_\lambda)^{(l)}}$ induced by α, where we write $\mathcal{G}_\lambda\longrightarrow\mathcal{G}$ for the connected finite étale subcovering of $\widetilde{\mathcal{G}}\longrightarrow\mathcal{G}$ corresponding to $\alpha^{-1}(\Pi_{\mathcal{H}_\lambda})\subset\Pi_{\mathcal{G}}$, is $\big(\widetilde{S}(\mathcal{G}_\lambda,\{l\}),\square(\widetilde{(\mathcal{H}_\lambda)^{(l)}})\big)$-compatible. Here, we note that the assumption (i) makes it possible for us to apply Lemma 2.5. By replacing, for every $\lambda\in\Lambda$, each profinite group ("Π_{ρ_J}" at the center of the lower horizontal sequence, for example) in the commutative diagram under consideration by an appropriate open subgroup ($\Pi_{\rho_J,\lambda}\subset\Pi_{\rho_J}$ for example), we are reduced to showing that $\Pi_{\mathcal{G}^{(l)}}\longrightarrow\Pi_{\mathcal{H}^{(l)}}$ is $\big(\widetilde{S}(\mathcal{G},\{l\}),\square(\widetilde{\mathcal{H}^{(l)}})\big)$-compatible. Thus, by Lemma 4.2, we may assume that $\Sigma_{\mathcal{G}}=\Sigma_{\mathcal{H}}=\{l\}$.

To verify the $\big(\overline{S},\square(\widetilde{\mathcal{H}})\big)$-compatibility of α in the case where $\Sigma_{\mathcal{G}}=\Sigma_{\mathcal{H}}=\{l\}$, again, we take the family $(\Pi_{\rho_J,\lambda}\subset\Pi_{\rho_J})_{\lambda\in\Lambda}$ of all open subgroups of Π_{ρ_J} and we put $\Pi_{\mathcal{H}_\lambda}=\Pi_{\rho_J,\lambda}\cap\Pi_{\mathcal{H}}\subset\Pi_{\mathcal{H}}$; $\Pi_{\mathcal{G}_\lambda}=\alpha^{-1}(\Pi_{\mathcal{H}_\lambda})\subset\Pi_{\mathcal{G}}$; $\alpha_\lambda\colon\Pi_{\mathcal{G}_\lambda}\longrightarrow\Pi_{\mathcal{H}_\lambda}$ the restriction of α. Then, in light of Theorem 3.2, (ii) $\Longrightarrow$ (iii), and Remark 1.7.3, it suffices to show that $\alpha_\lambda^{\mathrm{ab}}(M^S_{\mathcal{G}_\lambda})\subset M^\circ_{\mathcal{H}_\lambda}$, where we set $\circ=\mathrm{vert}$ (resp. $\circ=\mathrm{edge}$) if $\square=\mathrm{Vert}$ (resp. $\square=\mathrm{Edge}$). On the other hand, this follows, in light of the assumption that $S\subset\square(\widetilde{\mathcal{G}})$, easily from Lemma 4.3 (resp. Lemma 4.3 and the assumption (iv)), where the "I" in Lemma 4.3 is taken as the image of $\Pi_{\rho_J,\lambda}$ via the natural surjection $\Pi_{\rho_J}\twoheadrightarrow J$. This completes the proof of Theorem 4.4. □

Remark 4.4.1. Theorem 4.4, along with its proof, can be considered as a Hom-version of [CbGC], Corollary 2.7, (ii). Indeed, they share the technical core of the proof, say, the characterization of $M^{\mathrm{vert}}_{\mathcal{G}}$ and $M^{\mathrm{edge}}_{\mathcal{G}}$ (in the case where $\Sigma_{\mathcal{G}}=\{l\}$) in terms of the l-graphically full actions (cf. Lemma 4.3).

Also, we observe that [CbGC], Corollary 2.7, (ii), follows, at least if we (inappropriately) ignore the necessity of the assumption (iv) of Theorem 4.4, formally from Theorem 4.4 and Lemma 1.9. Thus Theorem 4.4 is, in a certain sense, a generalization of [CbGC] Corollary 2.7, (ii), though not in a straightforward manner.

Regarding the assumption (iv) of Theorem 4.4, which we ignored in the above discussion, we note that, in the situation of [CbGC], Corollary 2.7, (ii), one may verify (an "Isom-version" counterpart of) the assumption (iv) of Theorem 4.4 by an argument involving numerical data. In particular, we can indeed prove [CbGC], Corollary 2.7, (ii), using Theorem 4.4 in an essential way.

The author decided to impose the assumption (iv) in Theorem 4.4 precisely due to the difficulty of applying such an argument to our "Hom-version" situation (cf. the proof of [CbGC], Corollary 2.7, (i); also Remark 3.2.1). It is an interesting remaining work to examine the extent to which assumption (iv) can be relaxed.

Remark 4.4.2. It follows immediately from Lemma 1.10 that one can strengthen the conclusion of Theorem 4.4 from α being $(\overline{S}, \square(\widetilde{\mathcal{H}}))$-compatible to α being $(S^{\Pi_{\mathcal{G}}}, \square(\widetilde{\mathcal{H}}))$-compatible (cf. Lemma 1.10 for the notation "$S^{\Pi_{\mathcal{G}}}$").

Remark 4.4.3. In Theorem 4.4, the conclusion "α is $(\overline{S}, \square(\widetilde{\mathcal{H}}))$-compatible" is equivalent to saying that, for every $\tilde{c} \in \overline{S}$, there exists $\tilde{d} \in \square(\widetilde{\mathcal{H}})$ such that $\alpha(\Pi_{\tilde{c}}) \subset \Pi_{\tilde{d}}$, i.e., the part "$\alpha(\Pi_{\tilde{c}}) = 1$ or" in Definition 1.7, (1), is irrelevant, because of the non-triviality assumption (i).

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