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The degree 2 part of the LMO invariant of cyclic branched covers of some genus 1 knots with trivial Alexander polynomial

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Abstract

We give some formulas for the degree 2 part of the LMO invariant of cyclic branched covers of some genus 1 knots with trivial Alexander polynomial. More concretely, we present them by using several Vassiliev invariants, where we use the 3-loop polynomial for their proofs. Furthermore, we show the existence of knots whose values of 3-loop invariant cannot be reduced. This paper is a sequel to the author's previous paper [20].

1 Introduction

The LMO invariant is an invariant of closed oriented 3-manifolds which is universal among all perturbative invariants and all finite type invariants. The value of the LMO invariant is presented by an infinite sum of Jacobi diagrams, and it is well-known that that degree 1 part is equivalent to the Casson-Walker-Lescop invariant.

There are some example of calculation of the LMO invariant or low degree parts of it. For example, the LMO invariants of Lens spaces are calculated in [4], the low degree parts of the LMO invariants of Seifert fibered rational homology spheres are calculated in [16], and some surgery formulas for low degree parts of the LMO invariant are given in [9]. Moreover, the author computed a formula for the degree 2 part of the LMO invariant of cyclic branched covers of knots and calculated an example in [19], which is based on the work of Garoufalidis and Kricker [7] and the author's work [18].

In this paper, we calculate new examples of the degree 2 part of the LMO invariant of cyclic branched covers of knots, by using the methods in [7] and [19]. For its calculation, we use the 3-loop polynomial, which is the polynomial presenting the 3-loop part of the Kontsevich invariant of knots. The 3-loop polynomial of some genus 1 knots with trivial Alexander polynomials is calculated in [20]. Based on this result, we calculate the degree 2 part of the LMO invariant of cyclic branched covers of these knots by using several Vassiliev invariants. Furthermore, in Appendix, we show that there exist infinitely many knots whose values of 3-loop invariant cannot be reduced.

This paper is organized as follows. In Section 2, we review the fundamental notions. In Section 3, we state the calculations about the degree 2 part. In Section 4, we prove the theorems. In Appendix, we show the existence of knots whose values of 3-loop invariant cannot be reduced.

This paper is a sequel to the author's previous paper [20], so we often refer to it.

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2 Cyclic branched covers of knots and its LMO invariant

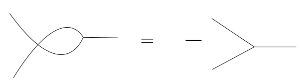
In this section, we review the cyclic branched covers of knots, the LMO invariant of oriented closed 3-manifolds, and the Kontsevich invariant of knots.

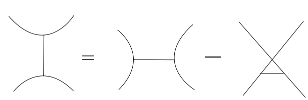
2.1 Cyclic branched covers of knots

Let K be a knot in S^3 and Σ_K^p the p-fold cyclic cover of K. If Σ_K^p is a rational homology sphere, we call the knot K *p-regular*. Also, if K is p-regular for all p, we call the knot K *regular*. Note that a knot is p-regular if and only if its Alexander polynomial has no complex pth root of unity.

2.2 The LMO invariant of closed oriented 3-manifold

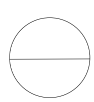
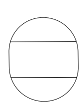
A *Jacobi diagram* on $\emptyset$ is a trivalent graph such that a cyclic order of the three edges around each trivalent vertex is fixed. When drawing a Jacobi diagram on $\emptyset$, we assume that the order of each trivalent vertex of it is given by the counterclockwise order. For a Jacobi diagram, we define the *degree* to be half the number of all vertex of it. Further, we define $\mathcal{A}(\emptyset)$ to be the quotient vector space spanned by Jacobi diagrams on $\emptyset$ subject to the AS and IHX relations.

the AS relation : 

the IHX relation : 

We define the product in $\mathcal{A}(\emptyset)$ by the disjoint union of Jacobi diagrams.

For a closed oriented 3-manifold M, the *LMO invariant* $Z^{LMO}(M)$ is defined to be in the completion of $\mathcal{A}(\emptyset)$ with respect to the degree, and is presented by

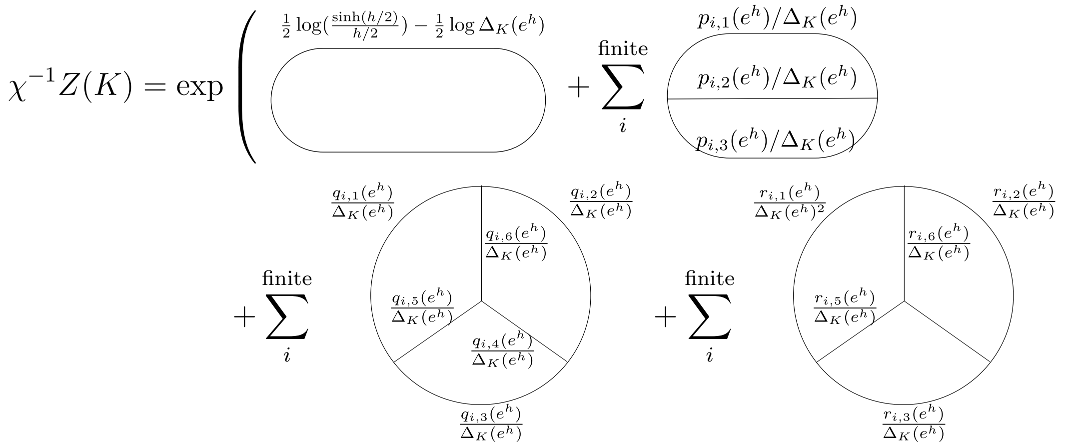
$Z^{LMO}(M) = \exp\Big(c_1(M)$  $+ c_2(M)$  $+ (\text{connected diagrams of degree} > 2)\Big),$

where $c_i(M)$ is a scalar invariant of M. It is known that we have $c_1(M) = (-1)^{b_1(M)}\lambda(M)/2$, where $\lambda(M)$ is the Casson-Walker-Lescop invariant and $b_1(M)$ is the first Betti number of M. For more details about the LMO invariant, see for example [14, 15].

2.3 The Kontsevich invariant of knots

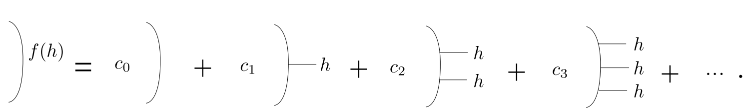
An h-marked open Jacobi diagrams is a uni-trivalent graph such that each univalent vertex is labeled by h and a cyclic order of the three edges around each trivalent vertex is fixed. (We may sometimes omit "h" in figures.) We define $\mathcal{A}(*_h)$ to be the quotient vector space spanned by h-marked open Jacobi diagrams subject to the AS and IHX relations. We also define the product in $\mathcal{A}(*_h)$ by the disjoint union of Jacobi diagrams. We denote by $\mathcal{A}(*_h)^{(n\text{-loop})}_{\text{conn}}$ the subspace of $\mathcal{A}(*_h)$ spanned by connected n-loop open Jacobi diagrams, where for a Jacobi diagram, "n-loop" means that the first Betti number of the diagram is n.

For an oriented framed knot K, the *Kontsevich invariant* $\chi^{-1}Z(K)$ is defined to be in the completion of $\mathcal{A}(*_h)$ with respect to the degree. It is shown [6, 13] that for a 0-framed oriented knot K the value $\chi^{-1}Z(K)$ can be presented as follows,



$$+(\text{terms of } (>3)\text{-loop parts})\Big), \qquad (1)$$

where $\Delta_K(t)$ denotes the Alexander polynomial of K (which satisfies that $\Delta_K(t) = \Delta_K(t^{-1}), \Delta_K(1) = 1$), and $p_{i,j}(e^h)$ and $q_{i,j}(e^h)$ are polynomials in $e^{\pm h}$. Further, a labeling on one side of an edge of a Jacobi diagram by a power series $f(h) = c_0 + c_1 h + c_2 h^2 + c_3 h^3 + \cdots$ means a formal power series of h-marked open Jacobi diagrams given as follows,



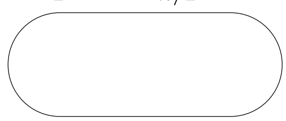
3 The degree 2 part of the LMO invariant of the cyclic branched covers of some genus 1 knots

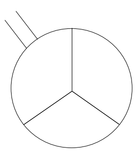
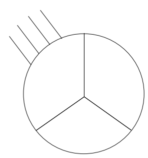
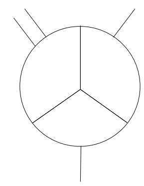
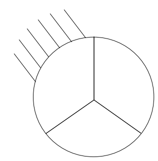
In this section, we calculate new examples of the degree 2 part of the LMO invariant of cyclic branched covers of two types of genus 1 knots with trivial Alexander polynomial.

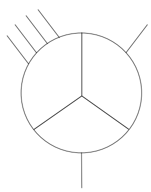
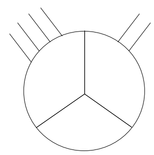
3.1 For genus 1 knots with trivial (≤ 2)-loop parts

Let K be a genus 1 knot with trivial (≤ 2)-loop parts, namely, with trivial Alexander polynomial and trivial 2-loop part. Note that K is regular, and the Casson-Walker invariant of Σ_K^p is trivial. We denote

$$\chi^{-1}Z(K) = \exp\left(\quad \frac{1}{2}\log\left(\frac{\sinh(h/2)}{h/2}\right) \quad + \sum_{i\geq 1} a_i\lambda_i + (\text{terms of } (>3)\text{-loop parts}) \right),$$



where $\lambda_1 =$ , $\lambda_2 =$ , $\lambda_3 =$ , $\lambda_4 =$ ,

$\lambda_5 =$ , $\lambda_6 =$ , and λ_i's $(i > 6)$ are basis vectors of $\mathcal{A}(*_h)^{(\text{3-loop})}_{\text{conn}}$ of degree>8. It is known [5] that λ_i's $(1 \leq i \leq 6)$ are basis vectors of $\mathcal{A}(*_h)^{(\text{3-loop})}_{\text{conn}}$ of degree$\leq$8.

Then, we calculate the degree 2 part of the LMO invariant of the p-fold cyclic branched cover of K.

Theorem 3.1. *For all $p > 1$, we have*

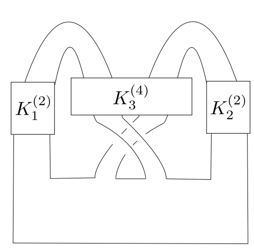
$$c_2(\Sigma_K^p) = \Big(-\frac{22}{15}a_1 + 4a_2 - a_3 + 48a_4 + 12a_5\Big)p.$$

Remark 3.2. Since K is genus 1 knot with trivial (≤ 2)-loop parts, it has a Seifert surface of genus 1 which satisfies the condition in the proof of Lemma 4.1 below. By using the result in [20], we can present $c_2(\Sigma_K^p)$ by finite type invariants a tangle T in the proof of Lemma 4.1 concretely, but it would be a little complicated, so we do not give the formula.

3.2 For $D(K_1, K_2, K_3)$

Let K_1, K_2, and K_3 be 0-framed long knots (1-tangles). We define $D(K_1, K_2, K_3)$ to be the following knot,

$$D(K_1, K_2, K_3) = $$



,

where, $K_1^{(2)}$, $K_2^{(2)}$ are the doubles of K_1, K_2, respectively and $K_3^{(4)}$ is the double of double of K_3. Note that $D(K_1, K_2, K_3)$ is a genus 1 knot with trivial Alexander polynomial, and hence $D(K_1, K_2, K_3)$ is regular. Also, it can be shown that the Casson invariant of $\Sigma^p_{D(K_1,K_2,K_3)}$ depends only on K_3. Then, we denote

$$a_i = -\frac{1}{2}d_2^i, \quad b_i = -\frac{1}{24}j_3^i, \quad c_i = \frac{1}{24}(-12d_4^i + 6(d_2^i)^2 - d_2^i).$$

Here d_n^i are coefficients of the Conway polynomial $\nabla_{\hat{K}_i}(z) = \sum d_n^i z^n$ $\big($defined by $\nabla_{\hat{K}_i}(t^{\frac{1}{2}} - t^{-\frac{1}{2}}) = \Delta_{\hat{K}_i}(t)\big)$ and j_n^i are coefficients of the Jones polynomial $J_{\hat{K}_i}(e^t) = \sum j_n^i t^n$, where $\hat{K}_i$ is the closure of K_i. Note that $a_i, b_i,$ and c_i are Vassiliev invariants of knots which can be appeared in coefficients of the Kontsevich invariant of knots, see [17], or [18].

Theorem 3.3. *For all $p > 1$, we have*

$$c_2(\Sigma^p_{D(K_1,K_2,K_3)}) = \Big(\frac{a_3}{4} + 4a_1a_2 + 4a_1a_3 + 4a_2a_3 + 6a_3^2 + 2b_3 + 9c_3\Big)p.$$

4 Proof of Theorem 3.1 and 3.3

In this section, we prove Theorem 3.1 and 3.3.

4.1 The 3-loop invariant of knots

Recall that the Kontsevich invariant of 0-framed oriented knot K is presented by (1). Then, we can define the *3-loop invariant* as follows,

$$\begin{aligned}
&\Lambda_K(t_1, t_2, t_3, t_4) \\
&= \sum_{\substack{i \\ \tau\in\mathfrak{S}_4}} \frac{q_{i,1}(t_{\tau(1)}^{\mathrm{sgn}\tau}t_{\tau(4)}^{-\mathrm{sgn}\tau})q_{i,2}(t_{\tau(2)}^{\mathrm{sgn}\tau}t_{\tau(4)}^{-\mathrm{sgn}\tau})q_{i,3}(t_{\tau(3)}^{\mathrm{sgn}\tau}t_{\tau(4)}^{-\mathrm{sgn}\tau})q_{i,4}(t_{\tau(2)}^{\mathrm{sgn}\tau}t_{\tau(3)}^{-\mathrm{sgn}\tau})q_{i,5}(t_{\tau(3)}^{\mathrm{sgn}\tau}t_{\tau(1)}^{-\mathrm{sgn}\tau})q_{i,6}(t_{\tau(1)}^{\mathrm{sgn}\tau}t_{\tau(2)}^{-\mathrm{sgn}\tau})}{\Delta_K(t_1t_4^{-1})\Delta_K(t_2t_4^{-1})\Delta_K(t_3t_4^{-1})\Delta_K(t_2t_3^{-1})\Delta_K(t_3t_1^{-1})\Delta_K(t_1t_2^{-1})} \\
&+ \sum_{\substack{i \\ \tau\in\mathfrak{S}_4}} \frac{r_{i,1}(t_{\tau(1)}^{\mathrm{sgn}\tau}t_{\tau(4)}^{-\mathrm{sgn}\tau})r_{i,2}(t_{\tau(2)}^{\mathrm{sgn}\tau}t_{\tau(4)}^{-\mathrm{sgn}\tau})r_{i,3}(t_{\tau(3)}^{\mathrm{sgn}\tau}t_{\tau(4)}^{-\mathrm{sgn}\tau})r_{i,5}(t_{\tau(3)}^{\mathrm{sgn}\tau}t_{\tau(1)}^{-\mathrm{sgn}\tau})r_{i,6}(t_{\tau(1)}^{\mathrm{sgn}\tau}t_{\tau(2)}^{-\mathrm{sgn}\tau})}{\Delta_K(t_{\tau(1)}t_{\tau(4)}^{-1})^2\Delta_K(t_{\tau(2)}t_{\tau(4)}^{-1})\Delta_K(t_{\tau(3)}t_{\tau(4)}^{-1})\Delta_K(t_{\tau(3)}t_{\tau(1)}^{-1})\Delta_K(t_{\tau(1)}t_{\tau(2)}^{-1})} \\
&\quad \in \frac{1}{\hat{\Delta}^2}\cdot\mathbb{Q}[t_1^{\pm1}, t_2^{\pm1}, t_3^{\pm1}, t_4^{\pm1}]/(\mathfrak{S}_4, t_1t_2t_3t_4 = 1),
\end{aligned}$$

where we put

$$\hat{\Delta} = \Delta_K(t_1t_4^{-1})\Delta_K(t_2t_4^{-1})\Delta_K(t_3t_4^{-1})\Delta_K(t_2t_3^{-1})\Delta_K(t_3t_1^{-1})\Delta_K(t_1t_2^{-1}).$$

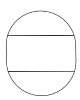
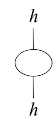
In particular, if the Alexander polynomial of K is trivial, the 3-loop invariant is a polynomial, so in this case, we call it the *3-loop polynomial*. For more details about the 3-loop invariant, see [18].

4.2 Formulas for the degree 2 part of the LMO invariant of cyclic branched covers of knots

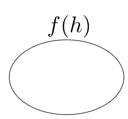
In [7], Garoufalidis and Kricker found a formula for the LMO invariant of cyclic branched covers of knots. Namely, it is shown that for all p and p-regular knot K, the LMO invariant of the p-fold cyclic branched cover Σ_K^p can be presented by the Kontsevich invariant of K and the total p-signature of K. In particular, the degree 2 part of the LMO invariant can be presented as follows [19],

$$c_2(\Sigma_K^p) = \frac{1}{48p^2} \sum_{\omega_1^p=\omega_2^p=\omega_3^p=1} \Lambda_K(\omega_1^{\frac{3}{4}}\omega_2^{-\frac{1}{4}}\omega_3^{-\frac{1}{4}}, \omega_1^{-\frac{1}{4}}\omega_2^{\frac{3}{4}}\omega_3^{-\frac{1}{4}}, \omega_1^{-\frac{1}{4}}\omega_2^{-\frac{1}{4}}\omega_3^{\frac{3}{4}}, \omega_1^{-\frac{1}{4}}\omega_2^{-\frac{1}{4}}\omega_3^{-\frac{1}{4}}) + l_p^K, \qquad (2)$$

where l_p^K is a scalar invariant of K defined by

$$l_p^K$$  $$= \mathrm{Lift}_p\left(\left\langle \exp\left(-\frac{1}{2}\mathrm{tr}^{\circlearrowleft}\log(EL(te^h)EL(t)^{-1})\right), -\frac{p-1}{48p}\right.\right.$$  $$\left.\left.\right\rangle\right). \qquad (3)$$

Here, $EL(t)$ is an equivariant linking matrix of a surgery link in $S^3\backslash K$, and $\mathrm{tr}^{\circlearrowleft}$ is the wheel-valued trace, which is defined by,

$$\mathrm{tr}^{\circlearrowleft}\big(F(h)\big) =$$  $$\big(\text{where } f(h) = \mathrm{tr}\big(F(h)\big)\big),$$

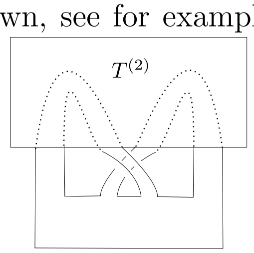
see for example [12, Definition 3], and, for the definition of the map Lift_p, see [7]. Further, the bracket $\langle \quad , \quad \rangle$ is defined by

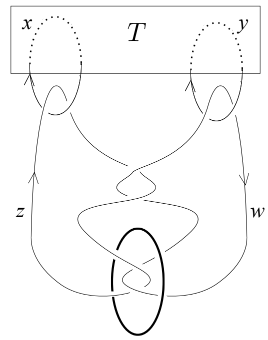
$$\langle C_1, C_2\rangle = \left(\begin{array}{l}\text{sum of all ways gluing the } h\text{-marked univalent vertices of } C_1 \\ \text{to the } h\text{-marked univalent vertices of } C_2\end{array}\right).$$

It can be shown that l_p^K does not change by Kirby moves in $S^3\backslash K$, hence it is a scalar invariant K. For details, see [7].

We have the following lemma for l_p^K.

Lemma 4.1. *Let K be a genus 1 knot with trivial Alexander polynomial. Then, we have $l_p^K = 0$.*

Proof. It is known, see for example [8], that there exists a 2-component tangle T of K such that $K =$  , and each strand in T has 0-framing, and the linking number of the two strands in T is equal to 0, where $T^{(2)}$ is the doubles of T. Thus, we obtain the following surgery presentation of K,

$K_0 \cup L =$  ,

where K_0 is depicted by a thick line and L is depicted by thin lines, and "surgery presentation" means that K is obtained from $K_0 \cup L$ by surgery along L. Hence, an equivariant linking matrix $EL(t)$ of $L \subset S^3 \backslash K_0$ is given by

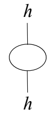
$$EL(t) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & t-1 \\ 0 & 1 & t^{-1}-1 & 0 \end{pmatrix},$$

and we get

$$EL(t)^{-1} = \begin{pmatrix} 0 & -t+1 & 1 & 0 \\ -t^{-1}+1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Thus, we have

$$\begin{aligned} \log(EL(te^h)EL(t)^{-1}) &= \log \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & t(e^h-1) & 1 & 0 \\ t^{-1}(e^{-h}-1) & 0 & 0 & 1 \end{pmatrix} \\ &= \log \Bigg(I_4 + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & th & 0 & 0 \\ -t^{-1}h & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}th^2 & 0 & 0 \\ \frac{1}{2}t^{-1}h^2 & 0 & 0 & 0 \end{pmatrix} \\ &\qquad + \big(\text{linear sum of matrices with (degree of } h) > 2\big)\Bigg) \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & th & 0 & 0 \\ -t^{-1}h & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{2}th^2 & 0 & 0 \\ \frac{1}{2}t^{-1}h^2 & 0 & 0 & 0 \end{pmatrix} \\ &\qquad + \big(\text{linear sum of matrices with (degree of } h) > 2\big). \end{aligned}$$

Hence, the coefficients of h and h^2 of $\operatorname{tr}\log(EL(te^h)EL(t)^{-1})$ are equal to 0, which implies that the coefficient of 

in $\exp\left(-\frac{1}{2}\operatorname{tr}^{\circlearrowleft}\log(EL(te^h)EL(t)^{-1})\right)$ is equal to 0. Therefore, by (3), we obtain $l_p^K = 0$. □

4.3 Proof of Theorem 3.1

Let K be a genus 1 knot with trivial (≤ 2)-loop parts.

Lemma 4.2. *[20] We have*

$$\begin{aligned}
&\Lambda_K(t_1,t_2,t_3,t_4)\\
&= 4a_1(u_{12}+u_{13}+u_{14}+u_{23}+u_{24}+u_{34})\\
&\quad + \left(-\frac{a_1}{3}+4a_2+2a_3\right)(u_{14}u_{24}+u_{14}u_{34}+u_{24}u_{34}+u_{13}u_{23}+u_{13}u_{43}+u_{23}u_{43}\\
&\qquad + u_{12}u_{32}+u_{12}u_{42}+u_{32}u_{42}+u_{21}u_{31}+u_{21}u_{41}+u_{31}u_{41})\\
&\quad - 4a_3(u_{12}u_{34}+u_{13}u_{24}+u_{14}u_{23})\\
&\quad + \left(\frac{a_1}{10}+a_3-36a_4-6a_5\right)(u_{14}u_{24}u_{34}+u_{13}u_{23}u_{43}+u_{12}u_{32}u_{42}+u_{21}u_{31}u_{41})\\
&\quad + \left(\frac{a_1}{30}+\frac{2}{3}a_3-12a_4-4a_5\right)(u_{13}u_{43}u_{42}+u_{23}u_{43}u_{41}+u_{12}u_{42}u_{43}+u_{32}u_{42}u_{41}\\
&\qquad + u_{21}u_{41}u_{43}+u_{31}u_{41}u_{42}+u_{12}u_{32}u_{34}+u_{42}u_{32}u_{31}+u_{21}u_{31}u_{34}\\
&\qquad + u_{41}u_{31}u_{32}+u_{31}u_{21}u_{24}+u_{41}u_{21}u_{23}),
\end{aligned}$$

where we denote $u_{mn} = t_m t_n^{-1} + t_m^{-1} t_n - 2$, and a_i's are invariants of K defined is Section 3.1.

Proof of Theorem 3.1. By (2), we have

$$\begin{aligned}
&c_2(\Sigma_K^p)\\
&= \frac{1}{48p^2}\sum_{\omega_1^p=\omega_2^p=\omega_3^p=1}\Lambda_K(\omega_1^{\frac34}\omega_2^{-\frac14}\omega_3^{-\frac14},\omega_1^{-\frac14}\omega_2^{\frac34}\omega_3^{-\frac14},\omega_1^{-\frac14}\omega_2^{-\frac14}\omega_3^{\frac34},\omega_1^{-\frac14}\omega_2^{-\frac14}\omega_3^{-\frac14}) + l_p^K.
\end{aligned}$$

However, by Lemma 4.1, we have $l_p^K = 0$. Moreover, we note that

$$e^{\frac{2\pi i l}{p}} + e^{-\frac{2\pi i l}{p}} - 2 = 2\cos\frac{2l\pi}{p} - 2 =: r_l \quad (l = 0,1,\cdots,p-1)$$

Thus, by Lemma 4.2, we get

$$\begin{aligned}
&c_2(\Sigma_K^p)\\
&= \frac{1}{48p^2}\sum_{\substack{0\leq l<p\\ 0\leq m<p\\ 0\leq n<p}}\Big(4a_1(r_l+r_m+r_n+r_{l-m}+r_{m-n}+r_{n-l})
\end{aligned}$$

$$
\begin{aligned}
&+\Big(-\frac{a_1}{3}+4a_2+2a_3\Big)(r_lr_m+r_mr_n+r_nr_l+r_lr_{l-m}+r_lr_{n-l}+r_mr_{m-n}+r_mr_{l-m}\\
&\qquad +r_nr_{n-l}+r_nr_{m-n}+r_{l-m}r_{m-n}+r_{m-n}r_{n-l}+r_{n-l}r_{l-m})\\
&-4a_3(r_lr_{m-n}+r_mr_{n-l}+r_nr_{l-m})\\
&+\Big(\frac{a_1}{10}+a_3-36a_4-6a_5\Big)(r_lr_mr_n+r_lr_{n-l}r_{l-m}+r_mr_{l-m}r_{m-n}+r_nr_{m-n}r_{n-l})\\
&+\Big(\frac{a_1}{30}+\frac{2}{3}a_3-12a_4-4a_5\Big)(r_lr_mr_{m-n}+r_mr_nr_{n-l}+r_nr_lr_{l-m}+r_lr_mr_{n-l}\\
&\qquad +r_mr_nr_{l-m}+r_nr_lr_{m-n}+r_lr_{l-m}r_{m-n}+r_mr_{m-n}r_{n-l}+r_nr_{n-l}r_{l-m}+r_lr_{m-n}r_{n-l}\\
&\qquad +r_mr_{n-l}r_{l-m}+r_nr_{l-m}r_{m-n})\Big).
\end{aligned}
$$

Since the sum of p-th root of unity is equal to zero, we have

$$
\begin{aligned}
&\sum_{\substack{0\le l<p\\0\le m<p\\0\le n<p}}(r_l+r_m+r_n+r_{l-m}+r_{m-n}+r_{n-l})=-12p^3,\\
&\sum_{\substack{0\le l<p\\0\le m<p\\0\le n<p}}(r_lr_m+r_mr_n+r_nr_l+r_lr_{l-m}+r_lr_{n-l}+r_mr_{m-n}+r_mr_{l-m}+r_nr_{n-l}+r_nr_{m-n}\\
&\qquad +r_{l-m}r_{m-n}+r_{m-n}r_{n-l}+r_{n-l}r_{l-m})=48p^3,\\
&\sum_{\substack{0\le l<p\\0\le m<p\\0\le n<p}}(r_lr_{m-n}+r_mr_{n-l}+r_nr_{l-m})=12p^3,\\
&\sum_{\substack{0\le l<p\\0\le m<p\\0\le n<p}}(r_lr_mr_n+r_lr_{n-l}r_{l-m}+r_mr_{l-m}r_{m-n}+r_nr_{m-n}r_{n-l})=-32p^3,\\
&\sum_{\substack{0\le l<p\\0\le m<p\\0\le n<p}}(r_lr_mr_{m-n}+r_mr_nr_{n-l}+r_nr_lr_{l-m}+r_lr_mr_{n-l}+r_mr_nr_{l-m}+r_nr_lr_{m-n}+r_lr_{l-m}r_{m-n}\\
&\qquad +r_mr_{m-n}r_{n-l}+r_nr_{n-l}r_{l-m}+r_lr_{m-n}r_{n-l}+r_mr_{n-l}r_{l-m}+r_nr_{l-m}r_{m-n})=-96p^3.
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
&c_2(\Sigma_K^p)\\
&=\frac{1}{48p^2}\Big(4a_1\cdot(-12p^3)+\Big(-\frac{a_1}{3}+4a_2+2a_3\Big)\cdot(48p^3)-4a_3\cdot(12p^3)\\
&\qquad +\Big(\frac{a_1}{10}+a_3-36a_4-6a_5\Big)\cdot(-32p^3)+\Big(\frac{a_1}{30}+\frac{2}{3}a_3-12a_4-4a_5\Big)\cdot(-96p^3)\Big)\\
&=\Big(-\frac{22}{15}a_1+4a_2-a_3+48a_4+12a_5\Big)p.
\end{aligned}
$$

□

4.4 Proof of Theorem 3.3

Let $D(K_1, K_2, K_3)$ a knot in Section 3.2.

Lemma 4.3. *[20]*

$$\begin{aligned}
&\Lambda_{D(K_1,K_2,K_3)}(t_1, t_2, t_3, t_4) \\
&= (-a_3 - 16a_1a_2 - 16a_1a_3 - 16a_2a_3 - 8b_3 - 12c_3)\,(u_{12} + u_{13} + u_{14} + u_{23} + u_{24} + u_{34}) \\
&\quad + 24c_3(u_{12}u_{34} + u_{13}u_{24} + u_{14}u_{23}) \\
&\quad + 8a_3^2(u_{12}^2 + u_{13}^2 + u_{14}^2 + u_{2,3}^2 + u_{24}^2 + u_{34}^2).
\end{aligned}$$

where we denote $u_{mn} = t_m t_n^{-1} + t_m^{-1} t_n - 2$, and a_i, b_i, c_i are invariants of K_i defined in Section 3.2.

Proof of Theorem 3.3. We use (2), Lemma 4.1, and Lemma 4.3. In a similar way of the proof of Theorem 3.1, we obtain

$$\begin{aligned}
&c_2(\Sigma^p_{D(K_1,K_2,K_3)}) \\
&= \frac{1}{48p^2} \sum_{\substack{0\le l<p \\ 0\le m<p \\ 0\le n<p}} \Big(\big(- a_3 - 16a_1a_2 - 16a_1a_3 - 16a_2a_3 - 8b_3 - 12c_3\big) \\
&\qquad\qquad\qquad\qquad \times (r_l + r_m + r_n + r_{l-m} + r_{m-n} + r_{n-l}) \\
&\quad + 24c_3(r_l r_{m-n} + r_m r_{n-l} + r_n r_{l-m}) \\
&\quad + 8a_3^2(r_l^2 + r_m^2 + r_n^2 + r_{l-m}^2 + r_{m-n}^2 + r_{n-l}^2)\Big). \\
&= \frac{1}{48p^2}\Big(\big(- a_3 - 16a_1a_2 - 16a_1a_3 - 16a_2a_3 - 8b_3 - 12c_3\big) \cdot (-12p^3) \\
&\quad + 24c_3 \cdot (12p^3) + 8a_3^2 \cdot (36p^3)\Big). \\
&= \Big(\frac{a_3}{4} + 4a_1a_2 + 4a_1a_3 + 4a_2a_3 + 6a_3^2 + 2b_3 + 9c_3\Big)p.
\end{aligned}$$

□

Appendix

A The existence of knots whose values of 3-loop invariant cannot be reduced

In this section, we show that there exist infinitely many knots whose values of 3-loop invariant cannot be reduced.

We recall that for a knot K the 3-loop invariant is given by

$$
\begin{aligned}
&\Lambda_K(t_1,t_2,t_3,t_4)\\
&=\sum_{\substack{i\\ \tau\in\mathfrak{S}_4}} \frac{q_{i,1}(t_{\tau(1)}^{\mathrm{sgn}\tau}t_{\tau(4)}^{-\mathrm{sgn}\tau})q_{i,2}(t_{\tau(2)}^{\mathrm{sgn}\tau}t_{\tau(4)}^{-\mathrm{sgn}\tau})q_{i,3}(t_{\tau(3)}^{\mathrm{sgn}\tau}t_{\tau(4)}^{-\mathrm{sgn}\tau})q_{i,4}(t_{\tau(2)}^{\mathrm{sgn}\tau}t_{\tau(3)}^{-\mathrm{sgn}\tau})q_{i,5}(t_{\tau(3)}^{\mathrm{sgn}\tau}t_{\tau(1)}^{-\mathrm{sgn}\tau})q_{i,6}(t_{\tau(1)}^{\mathrm{sgn}\tau}t_{\tau(2)}^{-\mathrm{sgn}\tau})}{\Delta_K(t_1t_4^{-1})\Delta_K(t_2t_4^{-1})\Delta_K(t_3t_4^{-1})\Delta_K(t_2t_3^{-1})\Delta_K(t_3t_1^{-1})\Delta_K(t_1t_2^{-1})}\\
&+\sum_{\substack{i\\ \tau\in\mathfrak{S}_4}} \frac{r_{i,1}(t_{\tau(1)}^{\mathrm{sgn}\tau}t_{\tau(4)}^{-\mathrm{sgn}\tau})r_{i,2}(t_{\tau(2)}^{\mathrm{sgn}\tau}t_{\tau(4)}^{-\mathrm{sgn}\tau})r_{i,3}(t_{\tau(3)}^{\mathrm{sgn}\tau}t_{\tau(4)}^{-\mathrm{sgn}\tau})r_{i,5}(t_{\tau(3)}^{\mathrm{sgn}\tau}t_{\tau(1)}^{-\mathrm{sgn}\tau})r_{i,6}(t_{\tau(1)}^{\mathrm{sgn}\tau}t_{\tau(2)}^{-\mathrm{sgn}\tau})}{\Delta_K(t_{\tau(1)}t_{\tau(4)}^{-1})^2\Delta_K(t_{\tau(2)}t_{\tau(4)}^{-1})\Delta_K(t_{\tau(3)}t_{\tau(4)}^{-1})\Delta_K(t_{\tau(3)}t_{\tau(1)}^{-1})\Delta_K(t_{\tau(1)}t_{\tau(2)}^{-1})},
\end{aligned}
$$

where $\hat{\Delta}=\Delta_K(t_1t_4^{-1})\Delta_K(t_2t_4^{-1})\Delta_K(t_3t_4^{-1})\Delta_K(t_2t_3^{-1})\Delta_K(t_3t_1^{-1})\Delta_K(t_1t_2^{-1})$. In general the 3-loop invariant belongs to $\frac{1}{\hat{\Delta}^2}\cdot\mathbb{Q}[t_1^{\pm1},t_2^{\pm1},t_3^{\pm1},t_4^{\pm1}]/(\mathfrak{S}_4, t_1t_2t_3t_4=1)$. We say that an element in $\frac{1}{\hat{\Delta}^2}\cdot\mathbb{Q}[t_1^{\pm1},t_2^{\pm1},t_3^{\pm1},t_4^{\pm1}]/(\mathfrak{S}_4, t_1t_2t_3t_4=1)$ *can be reduced* if it belongs to $\frac{1}{\hat{\Delta}}\cdot\mathbb{Q}[t_1^{\pm1},t_2^{\pm1},t_3^{\pm1},t_4^{\pm1}]/(\mathfrak{S}_4, t_1t_2t_3t_4=1)$. There exist infinitely many knots whose values of 3-loop invariant can be reduced, for example, knots with trivial Alexander polynomial. On the other hand, so far we did not know the existence of knots whose values of 3-loop invariant cannot be reduced, namely, $\Lambda_K(t_1,t_2,t_3,t_4)\notin\frac{1}{\hat{\Delta}}\cdot\mathbb{Q}[t_1^{\pm1},t_2^{\pm1},t_3^{\pm1},t_4^{\pm1}]/(\mathfrak{S}_4, t_1t_2t_3t_4=1)$. So we did not know whether the space $\frac{1}{\hat{\Delta}^2}\cdot\mathbb{Q}[t_1^{\pm1},t_2^{\pm1},t_3^{\pm1},t_4^{\pm1}]/(\mathfrak{S}_4, t_1t_2t_3t_4=1)$ is appropriate as a space which the 3-loop invariant of knots belongs to. But now, we found the following proposition.

Proposition A.1. *There exist infinitely many knots whose values of 3-loop invariant cannot be reduced.*

For the proof of Proposition 5.1, we briefly review the rational version of the Aarhus integral [1, 2, 3, 6, 13]. We also refer to Section 4 in [20], but we partially use some notations which are slightly different from those used in Section 4 in [20].

Let K be a 0-framed oriented knot. We can take its surgery presentation $K_0\cup L$ such that K_0 is the unknot with 0 framing and L is a l-components framed link, and the linking number of K_0 and each component of L is equal to 0. The pair (S^3,K) is obtained from the pair (S^3,K_0) by surgery along L. Next, we compute $\chi^{-1}\check{Z}(K_0\cup L)\in\mathcal{A}(*_{\{h\}\cup X})$, where X is a finite set with $|X|=l$. The value $\check{Z}(K_0\cup L)$ is obtained from $Z(K_0\cup L)$ by connected-summing of ν to each component labeled by an element of X, where we denote $\nu=Z(\text{unknot})$, and the mark h corresponds to K_0 and each component in X to each component of L. Then, we denote

$$
\begin{aligned}
&\langle\!\langle\chi^{-1}\check{Z}(K_0\cup L)\rangle\!\rangle^{(3)}\\
&:=\text{the 3-loop part of}\quad \frac{\Big\langle \exp\Big(-\frac{1}{2}\sum_{x_i,x_j\in X}\ \overset{x_j\qquad x_i}{\underset{l^{ij}(t)}{\smile}}\ \Big), P\big(\chi^{-1}\check{Z}(K_0\cup L)\big)\Big\rangle}{N},
\end{aligned}
$$

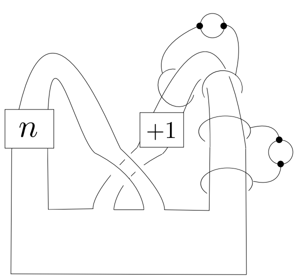
where
- $t = e^h$,
- $\big(l^{ij}(t)\big)$ is the inverse matrix of an equivariant linking matrix $\big(l_{ij}(t)\big)$ of $L \subset S^3 \backslash K_0$ which satisfies that $l_{ji}(t) = l_{ij}(t^{-1})$,
- $P\big(\chi^{-1}\check{Z}(K_0 \cup L)\big)$ is a sum of diagrams which have at least one trivalent vertex on each component,
- N is the normalization term, see [6, 13].

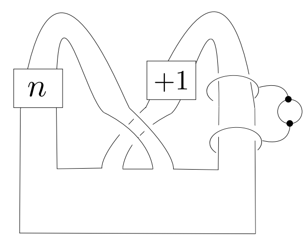
Also, $\langle \quad , \quad \rangle$ is defined by

$$\langle C_1, C_2 \rangle = \left(\begin{array}{l} \text{sum of all ways gluing the } x\text{-marked univalent vertices of } C_1 \\ \text{to the } x\text{-marked univalent vertices of } C_2 \text{ for all } x \in X \end{array} \right).$$

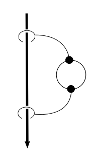
Then, the 3-loop part of $\chi^{-1}Z(K)$, which we denote by $\chi^{-1}Z(K)^{(3)}$, is given by $\langle\!\langle \chi^{-1}\check{Z}(K_0 \cup L) \rangle\!\rangle^{(3)}$.

Proof of Proposition 5.1. Let n be a positive integer and consider the following knot K_n and K'_n,

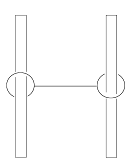
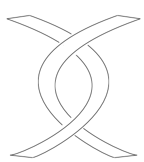
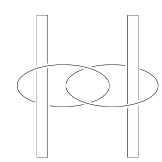
$K_n =$ 

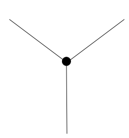
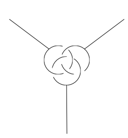
, $K'_n =$ 

,

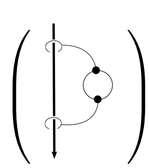
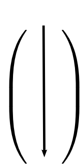
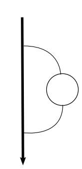
where "n-box" (resp. "+1-box") represents n-full twist (resp. +1-full twist), and 

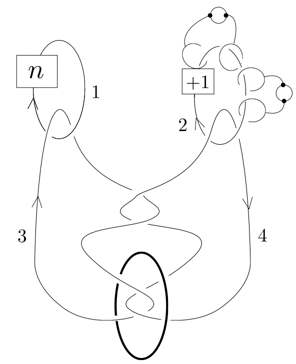
represents a clasper surgery, where we mean

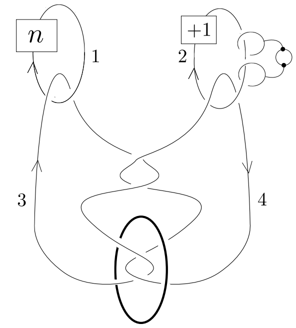
 $=$  $\Big($ = the result obtained by surgery along Hopf link in  $\Big)$,

 $=$  .

It is known that the the clasper surgery formula of the Kontsevich invariant is given as follows,

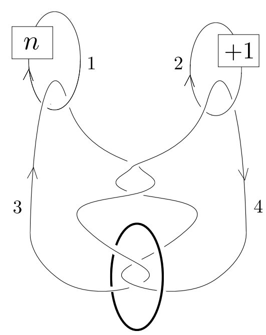
Z  $= Z$  $+$  $+ D$, where D is an infinite sum of Jacobi diagrams with more that 2 trivalent vertices. For more details about claspers, see for example, [10]. It can be shown that K_n and K'_n has the following surgery presentations respectively,

$K_0 \cup L =$ 

, $K_0 \cup L' =$ 

,

where K_0 is depicted by a thick line and L is depicted by thin lines. Namely, K_n can be obtained from $K_0 \cup L$ by surgery along L. Also, we denote

$K_0 \cup L_0 =$ 

,

noting that this is a surgery presentation of the twist knot, which we denote by T_n. We also note that an equivariant linking matrix of $L, L', L_0 \subset S^3 \backslash K_0$ is given by

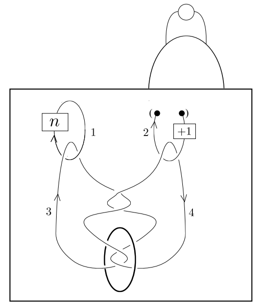
$$\big(l_{ij}(t)\big) = \begin{pmatrix} n & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & t-1 \\ 0 & 1 & t^{-1}-1 & 0 \end{pmatrix}.$$

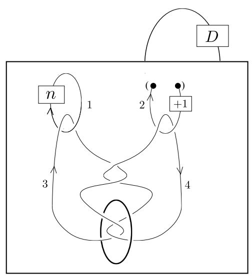
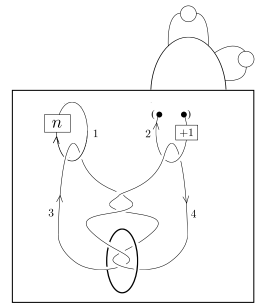
Hence,

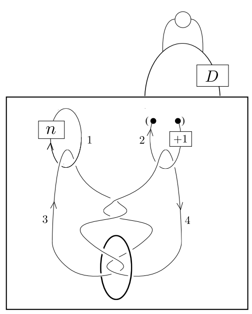
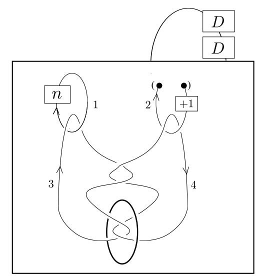
$$-\frac{1}{2}\big(l^{ij}(t)\big) = \frac{1}{2\Delta(t)} \begin{pmatrix} -(t+t^{-1}-2) & t-1 & -1 & -(t-1) \\ t^{-1}-1 & -n(t+t^{-1}-2) & -n(t^{-1}-1) & -1 \\ -1 & -n(t-1) & n & n(t-1) \\ -(t^{-1}-1) & -1 & n(t^{-1}-1) & 1 \end{pmatrix},$$

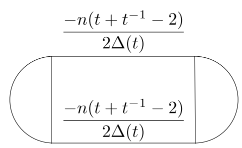
where $\Delta(t) = 1 + n(t + t^{-1} - 2)$. Since we consider the 3-loop part, we can ignore the terms with more that 4 trivalent vertices in the Kontsevich invariant of the surgery presentations. Thus, we have

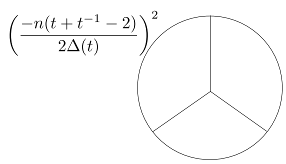
$\chi^{-1} Z(K_n)^{(3)} = \langle\!\langle \chi^{-1} \check{Z}(K_0 \cup L) \rangle\!\rangle^{(3)}$

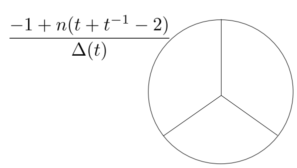
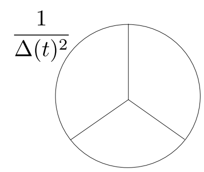
$= \langle\!\langle \chi^{-1}\check{Z}(K_0 \cup L_0)\rangle\!\rangle^{(3)} + 2\Big\langle\!\Big\langle \chi^{-1} \Big(\check{Z}$  $\Big) \Big\rangle\!\Big\rangle^{(3)}$

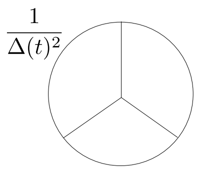
$+ 2\Big\langle\!\Big\langle \chi^{-1} \Big(\check{Z}$  $\Big) \Big\rangle\!\Big\rangle^{(3)} + \Big\langle\!\Big\langle \chi^{-1} \Big(\check{Z}$  $\Big) \Big\rangle\!\Big\rangle^{(3)}$

$+ 2\Big\langle\!\Big\langle \chi^{-1} \Big(\check{Z}$  $\Big) \Big\rangle\!\Big\rangle^{(3)} + \Big\langle\!\Big\langle \chi^{-1} \Big(\check{Z}$  $\Big) \Big\rangle\!\Big\rangle^{(3)}$

$= 2\chi^{-1}Z(K'_n)^{(3)} - \chi^{-1}Z(T_n)^{(3)} + 16$ 

$= 2\chi^{-1}Z(K'_n)^{(3)} - \chi^{-1}Z(T_n)^{(3)} + 32$ 

$= 2\chi^{-1}Z(K'_n)^{(3)} - \chi^{-1}Z(T_n)^{(3)} + 8$  $+ 8$  .

Thus, we can say that the 3-loop polynomial of at least one of K_n, K'_n and T_n cannot be reduced. Indeed, if the values of 3-loop polynomial of all of those three knots can be reduced, 

corresponds to a polynomial which can be reduced, namely it holds that

$$4\left(\frac{1}{\Delta(t_1t_4^{-1})^2}+\frac{1}{\Delta(t_2t_4^{-1})^2}+\frac{1}{\Delta(t_3t_4^{-1})^2}+\frac{1}{\Delta(t_2t_3^{-1})^2}+\frac{1}{\Delta(t_3t_1^{-1})^2}+\frac{1}{\Delta(t_1t_2^{-1})^2}\right)$$
$$=\frac{s(t_1,t_2,t_3,t_4)}{\Delta(t_1t_4^{-1})\Delta(t_2t_4^{-1})\Delta(t_3t_4^{-1})\Delta(t_2t_3^{-1})\Delta(t_3t_1^{-1})\Delta(t_1t_2^{-1})},$$

for some polynomial $s(t_1,t_2,t_3,t_4)$. Hence, we have

$$\begin{aligned}&\Delta(t_2t_4^{-1})^2\Delta(t_3t_4^{-1})^2\cdots\Delta(t_1t_2^{-1})^2+\Delta(t_1t_4^{-1})^2\Delta(t_3t_4^{-1})^2\cdots\Delta(t_1t_2^{-1})^2+\cdots\\&\quad\cdots+\Delta(t_1t_4^{-1})^2\Delta(t_2t_4^{-1})^2\cdots\Delta(t_3t_1^{-1})^2\\&=\Delta(t_1t_4^{-1})\Delta(t_2t_4^{-1})\Delta(t_3t_4^{-1})\Delta(t_2t_3^{-1})\Delta(t_3t_1^{-1})\Delta(t_1t_2^{-1})\cdot\frac{1}{4}s(t_1,t_2,t_3,t_4),\end{aligned}\tag{4}$$

However, when we substitute $t_1=1, t_2=\frac{1}{2n}, t_3=2n-1-\sqrt{-4n+1}, t_4=\frac{2n-1+\sqrt{-4n+1}}{2n}$, the right hand side of (4) is equal to 0, but the left hand side is not. □

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