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Lubin-Tate representations over nontrivial finite Galois extensions of $\mathbb{Q}_p$ are not Aut-intrinsically Hodge-Tate

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LUBIN-TATE REPRESENTATIONS OVER NONTRIVIAL FINITE GALOIS EXTENSIONS OF $\mathbb{Q}_p$ ARE NOT AUT-INTRINSICALLY HODGE-TATE

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Abstract. In the present paper, we show that, for an odd prime number p and a nontrivial finite Galois extension k of $\mathbb{Q}_p$, the p-adic representation of the absolute Galois group of k determined by a Lubin-Tate formal group over the ring of integers of k is not Aut-intrinsically Hodge-Tate [in the sense of Hoshi]. This settles the odd-degree cases left open in the previous works of Hoshi and the author and, together with the known even-degree case, completes the picture for finite Galois extensions of $\mathbb{Q}_p$ in the case where p is odd. This exhibits a sharp contrast, from the viewpoint of anabelian geometry, between the p-adic cyclotomic character and other p-adic Lubin-Tate characters.

Introduction

Let p be a prime number, k a finite extension of $\mathbb{Q}_p$, and $\overline{k}$ an algebraic closure of k. We write $G_k \overset{\text{def}}{=} \text{Gal}(\overline{k}/k)$ for the absolute Galois group of k determined by the algebraic closure $\overline{k}$. In anabelian geometry, it is natural to discuss conditions for a continuous automorphism of G_k to be induced by a field automorphism of k [cf., e.g., [2], [7]]. The following theorem studies such conditions from the perspective of p-adic Hodge theory:

Theorem A ([2], Corollary 3.4)**.** *Let $\alpha\colon G_k \xrightarrow{\sim} G_k$ be a continuous automorphism of G_k. Then the following conditions are equivalent:*

1. *The automorphism α is induced by a field automorphism of k.*
2. *For every finite-dimensional continuous representation $\rho\colon G_k \to \text{GL}_n(\mathbb{Q}_p)$ of G_k that is Hodge-Tate, the composite $G_k \overset{\alpha}{\to} G_k \overset{\rho}{\to} \text{GL}_n(\mathbb{Q}_p)$ is Hodge-Tate.*
3. *The automorphism α is HT-qLT-type [cf. [2], Definition 1.3, (ii)].*

It follows from this theorem and [2], Definition 1.3, (ii), that Lubin-Tate characters play an important role in the study of the *geometricity* [i.e., the condition to be induced by a field automorphism] of continuous automorphisms of G_k.

Moreover, motivated by this theorem, Hoshi defined the following notion for continuous p-adic representations.

Definition ([4], Definition 1.3)**.** Let V be a $\mathbb{Q}_p$-vector space of finite dimension and $\rho\colon G_k \to \text{Aut}_{\mathbb{Q}_p}(V)$ a continuous representation. Then we shall say that ρ is *Aut-intrinsically Hodge-Tate* if, for an arbitrary continuous automorphism α of G_k, the composite $\rho \circ \alpha\colon G_k \to \text{Aut}_{\mathbb{Q}_p}(V)$ is Hodge-Tate.

Let $\pi \in \mathcal{O}_k$ be a uniformizer of the ring of integers $\mathcal{O}_k$ of k. In the remainder of the present introduction, we write $\rho_{k,\pi}\colon G_k \to \text{Aut}_{\mathbb{Q}_p}(k_+)$ for the continuous p-adic representation obtained by forming the composite

$$G_k \overset{\chi_{k,\pi}}{\to} \mathcal{O}_k^\times \hookrightarrow \text{Aut}_{\mathbb{Q}_p}(k_+),$$

where the first arrow is the Lubin-Tate character $\chi_{k,\pi}\colon G_k \to \mathcal{O}_k^\times$ [i.e., the continuous character determined by a Lubin-Tate formal group law over $\mathcal{O}_k$ associated to π], and the second arrow is the natural inclusion.

It is natural to study which representations are Aut-intrinsically Hodge-Tate. The following theorem is a typical example motivated by this question:

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Theorem B ([4], Theorem 3.3; [6], Theorem 1.9; [9], Theorem 4.4)**.** *Let $\rho\colon G_k \to \mathrm{Aut}_{\mathbb{Q}_p}(V)$ be a continuous p-adic representation. Then the following assertions hold:*

(1) *Suppose either that ρ is one-dimensional or that ρ is two-dimensional and reducible. Then ρ is Hodge-Tate if and only if ρ is Aut-intrinsically Hodge-Tate.*
(2) *Suppose that p is an odd prime number and that $k/\mathbb{Q}_p$ is a finite Galois extension of even degree. Let $\pi \in \mathcal{O}_k$ be a uniformizer of $\mathcal{O}_k$. If ρ is isomorphic to the continuous p-adic representation $\rho_{k,\pi}$ of G_k, then ρ is not Aut-intrinsically Hodge-Tate.*
(3) *Suppose that $p = 2$ and that k contains a primitive 4-th root of unity. Suppose, moreover, that $k/\mathbb{Q}_p$ is an abelian extension. Let $\pi \in \mathcal{O}_k$ be a uniformizer of $\mathcal{O}_k$. If ρ is isomorphic to the continuous p-adic representation $\rho_{k,\pi}$ of G_k, then ρ is not Aut-intrinsically Hodge-Tate.*

Remark. *We give some remarks on Theorem B.*

(1) *In the proof of Theorem B, (1), it is essential that the p-adic cyclotomic character can be reconstructed from G_k in a group-theoretic manner. This is a significant difference between the p-adic cyclotomic character and other Lubin-Tate characters.*
(2) *In* [4], [6], *and* [9], *Theorem B, (2) is not stated explicitly. However, each of the authors essentially established Theorem B, (2), in* [4], [6], *and* [9]. *Theorem B, (2) was first established in* [4] *for the case where $k/\mathbb{Q}_p$ is an abelian extension of even degree. In* [4], *by making use of the assumption that the extension is of even degree and abelian, the argument was reduced to the case where $k/\mathbb{Q}_p$ is a quadratic extension.*
(3) *In* [4], [6], *and* [9], *each of the authors constructed an explicit continuous automorphism φ of G_k such that $\rho_{k,\pi} \circ \varphi$ is not Hodge-Tate in terms of generators and relations established by Jannsen-Wingberg [cf.* [8], *Theorem 7.5.14].*

In the present paper, we show the following theorem, which is a generalization of Theorem B, (2), above:

Theorem C. *Suppose that p is an odd prime number and that $k/\mathbb{Q}_p$ is a nontrivial finite Galois extension. Let $\pi \in \mathcal{O}_k$ be a uniformizer of $\mathcal{O}_k$. Then the continuous p-adic representation $\rho_{k,\pi}\colon G_k \to \mathrm{Aut}_{\mathbb{Q}_p}(k_+)$ is not Aut-intrinsically Hodge-Tate.*

Remark. *We give some remarks on Theorem C.*

(1) *Here, we note that the novelty of Theorem C lies in the odd-degree cases. Thus, in the proof of Theorem C, we assume that $[k\colon \mathbb{Q}_p]$ is odd. However, the proof of Theorem C in the present paper can also be applied to the case in which $k/\mathbb{Q}_p$ is a finite Galois extension of even degree. From that point of view, the proof of Theorem C in the present paper is more uniform than the proofs of Theorem B, (2), in* [4] *and* [6].
(2) *In the proof of Theorem C, we do not give an explicit continuous automorphism φ of G_k such that $\rho_{k,\pi} \circ \varphi$ is not Hodge-Tate. Thus, in the case where $k/\mathbb{Q}_p$ is of even degree, these two methods have both advantages and disadvantages.*

At the end of Introduction, we describe the outline of the proof of Theorem C. Let α be a continuous automorphism of G_k. We write α_+ for the automorphism of the $\mathbb{Q}_p$-vector space k_+ [obtained by taking the underlying $\mathbb{Q}_p$-vector space of k] induced by mono-anabelian reconstruction algorithms [cf. [3], Proposition 3.10, (vi); [3], Proposition 3.11, (iv); [5], Lemma 1.2]. Under the above notation and definition, we show that if $\rho_{k,\pi} \circ \alpha$ is Hodge-Tate, then $\alpha_+ \in \mathbb{Q}_p[\mathrm{Gal}(k/\mathbb{Q}_p)]$. Then we obtain Theorem C by combining this observation with the theory of mapping class groups and the theory of p-adic Lie groups.

Notational conventions

Topological groups. Let G be a topological group and α a continuous automorphism of the topological group G. Then we shall write G^{ab} for the *abelianization* of G [i.e., the quotient of G by the closure of the commutator subgroup of G] and α^{ab} for the continuous automorphism of the topological group G^{ab} induced by α via the functoriality of abelianization.

Rings. In the present paper, every "ring" is assumed to be unital, associative, and commutative. If R is a ring, then we shall write R_+ for the underlying additive group of R and $R^\times \subset R$ for the multiplicative group of units of R.

Mixed-characteristic local fields. We shall refer to a field isomorphic to a finite extension of $\mathbb{Q}_p$, for some prime number p, as an *MLF*. Here, "MLF" is to be understood as an abbreviation for "mixed-characteristic local field". Let k be an MLF and $\overline{k}$ an algebraic closure of k. Then we shall write

- $\mathcal{O}_k$ for the ring of integers of k,
- $\mathfrak{m}_k \subset \mathcal{O}_k$ for the maximal ideal of $\mathcal{O}_k$,
- $\underline{k} \stackrel{\mathrm{def}}{=} \mathcal{O}_k/\mathfrak{m}_k$ for the residue field of $\mathcal{O}_k$,
- $k^{(d=1)} \subset k$ for the [uniquely determined] minimal MLF contained in k,
- p_k for the residue characteristic of k,
- $d_k \stackrel{\mathrm{def}}{=} [k\colon k^{(d=1)}]$ for the degree of the finite extension $k/k^{(d=1)}$,
- $f_k \stackrel{\mathrm{def}}{=} [\underline{k}\colon \underline{k}^{(d=1)}]$ for the degree of the finite extension $\underline{k}/\underline{k}^{(d=1)}$ [where we write $\underline{k}^{(d=1)}$ for the residue field of the ring of integers of the MLF $k^{(d=1)}$],
- $G_k \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k)$ for the absolute Galois group of k determined by the algebraic closure $\overline{k}$,
- $I_k \subset G_k$ for the inertia subgroup of G_k,
- $P_k \subset I_k$ for the wild inertia subgroup of G_k,
- $\log_k\colon \mathcal{O}_k^\times \to k_+$ for the p_k-adic logarithm,
- $\widehat{k^\times}$ for the profinite completion of the multiplicative group $k^\times$ of k, and
- $\mathrm{rec}_k\colon \widehat{k^\times} \xrightarrow{\sim} G_k^{\mathrm{ab}}$ for the isomorphism induced by the reciprocity homomorphism $k^\times \hookrightarrow G_k^{\mathrm{ab}}$ in local class field theory.

Groups of MLF-type. We shall refer to a topological group isomorphic to the absolute Galois group of an MLF as a *group of MLF-type*.

Let us recall [cf. [3], Definition 3.5; [3], Proposition 3.6; [3], Definition 3.10; [3], Proposition 3.11] that there exist functorial group-theoretic algorithms for constructing, from a group of MLF-type G,

- a prime number $p(G)$,
- positive integers $d(G)$, $f(G)$,
- subgroups $P(G) \subset I(G) \subset G$ of G,
- subgroups $\mathcal{O}^\times(G) \subset k^\times(G) \subset G^{\mathrm{ab}}$, whose final inclusion $k^\times(G) \subset G^{\mathrm{ab}}$ we denote by rec_G, and
- a topological group $k_+(G)$

which "correspond" to

- the prime number p_k,
- the positive integers d_k, f_k,
- the subgroups $P_k \subset I_k \subset G_k$ of G_k,
- the subgroups $\mathcal{O}_k^\times \subset k^\times \stackrel{\mathrm{rec}_k}{\hookrightarrow} G_k^{\mathrm{ab}}$, and
- the topological group k_+,

respectively.

Moreover, it follows from [3], Proposition 3.11, (i), (iv), that we have natural homomorphisms

$$\mathrm{Aut}(G) \to \mathrm{Aut}(k^\times(G)), \quad \mathrm{Aut}(G) \to \mathrm{Aut}(k_+(G)).$$

Since the automorphism of G^{ab} induced by an inner automorphism of G is trivial, it follows from the constructions of $k^\times(G)$ and $k_+(G)$ that the automorphisms of $k^\times(G)$ and $k_+(G)$ induced by an inner automorphism of G are trivial. Thus, the above two homomorphisms determine group homomorphisms

$$\mathrm{Out}(G) \to \mathrm{Aut}(k^\times(G)), \quad \mathrm{Out}(G) \to \mathrm{Aut}(k_+(G)).$$

Let α be an element of $\mathrm{Out}(G)$. We write $\alpha^\times$ [respectively, α_+] for the image of α by this homomorphism $\mathrm{Out}(G) \to \mathrm{Aut}(k^\times(G))$ [respectively, $\mathrm{Out}(G) \to \mathrm{Aut}(k_+(G))$]. In the present paper, we call $\alpha^\times$ and α_+ *the automorphisms induced from* α *by the mono-anabelian reconstruction algorithms.*

Let k be an MLF, $\overline{k}$ an algebraic closure of k, and α an element of $\mathrm{Out}(G_k)$. By abuse of notation, we shall denote by $\alpha_+ \colon k_+ \xrightarrow{\sim} k_+$, $\alpha^\times \colon k^\times \xrightarrow{\sim} k^\times$ the respective images of $\alpha_+ \colon k_+(G_k) \xrightarrow{\sim} k_+(G_k)$, $\alpha^\times \colon k^\times(G_k) \xrightarrow{\sim} k^\times(G_k)$ by the isomorphisms $\mathrm{Aut}(k_+(G_k)) \xrightarrow{\sim} \mathrm{Aut}(k_+)$, $\mathrm{Aut}(k^\times(G_k)) \xrightarrow{\sim} \mathrm{Aut}(k^\times)$ induced by the isomorphisms $k_+ \xrightarrow{\sim} k_+(G_k)$, $k^\times \xrightarrow{\sim} k^\times(G_k)$ of [3], Proposition 3.11, (i), (iv).

Proof of the main theorem

Let k be an MLF, $\overline{k}$ an algebraic closure of k, and G a group of MLF-type. We begin by giving an overview of the remainder of the present paper. First, we recall various notions introduced in [6] for the convenience of the reader. Next, we review the classification of abelian Hodge-Tate representations and prove a key lemma. Finally, we prove the main theorem of the present paper by combining this lemma with certain "mapping class group and p-adic Lie group techniques" developed in [6].

Theorem 1. *Suppose that* $p(G)$ *is an odd prime number. Then there exist* σ, τ, $x_0, \dots, x_{d(G)} \in G$, *positive integers* s, t, *an element* $x_0' \in \overline{\langle \tau, x_0 \rangle}$, *and an element* $x_1' \in \overline{\langle \sigma, \tau, x_1 \rangle}$, *where "*$\overline{\langle S \rangle}$*" denotes the closed subgroup of* G *topologically generated by "*S*", such that the following conditions hold:*

(1) *The profinite group* G *is presented as the profinite group topologically generated by* σ, τ, $x_0, \dots, x_{d(G)} \in G$ *and subject to the relations described in the conditions (2), (3), and (4) below.*
(2) *The closed normal subgroup* $P(G)$ *of* G *is pro-*$p(G)$ *and topologically normally generated by* $x_0, \dots, x_{d(G)}$.
(3) *The elements* σ, τ *satisfy the relation* $\sigma\tau\sigma^{-1} = \tau^{p(G)^{f(G)}}$.
(4) *In addition, the generators satisfy one further relation:*
   (a) *for even* $d(G)$,
$$\sigma x_0 \sigma^{-1} = (x_0')^t x_1^{p(G)^s} [x_1, x_2][x_3, x_4] \cdots [x_{d(G)-1}, x_{d(G)}];$$
   (b) *for odd* $d(G)$,
$$\sigma x_0 \sigma^{-1} = (x_0')^t x_1^{p(G)^s} [x_1, x_1'][x_2, x_3] \cdots [x_{d(G)-1}, x_{d(G)}].$$

Proof. This assertion follows from [8], Theorem 7.5.14, together with [3], Proposition 3.6. ☐

In the remainder of the present paper, we apply the notational conventions introduced in the statement of Theorem 1 in each of the situations in which $p(G)$ is assumed to be odd. Moreover, if $p(G)$ is odd, then, for each $i = 1, 2, \dots, d(G)$, write $y_i \in k_+(G)$ for the image of x_i in $k_+(G)$ by the composite $P(G) \hookrightarrow I(G) \to \mathcal{O}^\times(G) \to k_+(G)$ [cf. conditions (1), (2) of Theorem 1; [3], Definition 3.10, (i), (ii), (v)].

We recall that the topological group $k_+(G)$ has a natural structure of a $\mathbb{Q}_{p(G)}$-vector space of dimension $d(G)$ [cf. [5], Lemma 1.2] and that, for any continuous automorphism α of G, the induced automorphism α_+ of $k_+(G)$ is an automorphism of $\mathbb{Q}_{p(G)}$-vector spaces.

Lemma 2. *Suppose that* $d(G) > 1$ *and* $p(G)$ *is odd. Then the* $d(G)$ *elements* $y_1, \dots, y_{d(G)}$ *defined in the discussion following Theorem 1 form a basis of the* $\mathbb{Q}_{p(G)}$*-vector space* $k_+(G)$.

Proof. This assertion is none other than [5], Lemma 1.3. ☐

One verifies easily that the isomorphism of topological groups $k_+(G_k) \xrightarrow{\sim} k_+$ of [3], Proposition 3.11, (iv), is also an isomorphism of $\mathbb{Q}_{p_k}$-vector spaces [here, we have $p_k = p(G_k)$ — cf. [3], Proposition 3.6]. By abuse of notation, if $d_k > 1$ and p_k is odd, then, for each integer i satisfying $1 \leq i \leq d_k$, we write $y_i \in k_+$ for the image of $y_i \in k_+(G_k)$ by the isomorphism $k_+(G_k) \xrightarrow{\sim} k_+$ of [3], Proposition 3.11, (iv). In the remainder of the present paper, if $d(G)$ [resp. d_k] is greater than one, and $p(G)$ [resp. p_k] is odd, then we equip $k_+(G)$ [resp. k_+] with this basis, which allows us to identify $\mathrm{Aut}_{\mathbb{Q}_{p(G)}}(k_+(G))$ [resp. $\mathrm{Aut}_{\mathbb{Q}_{p_k}}(k_+)$] with $\mathrm{GL}_{d(G)}(\mathbb{Q}_{p(G)})$ [resp. $\mathrm{GL}_{d_k}(\mathbb{Q}_{p_k})$].

We equip $\mathrm{GL}_{d(G)}(\mathbb{Q}_{p(G)})$ [resp. $\mathrm{GL}_{d_k}(\mathbb{Q}_{p_k})$] with the natural $p(G)$-adic [resp. p_k-adic] Lie group structure.

Next, we review the profinite group structure of $\mathrm{Out}(G)$. It follows from [8], Theorem 7.4.1, and [10], Proposition 4.4.3, that $\mathrm{Out}(G)$ has a natural profinite group structure. In the remainder of the present paper, we endow $\mathrm{Out}(G)$ with this profinite group structure.

Lemma 3. *The action* $\mathrm{Out}(G) \curvearrowright k_+(G)$ *which is defined via the mono-anabelian reconstruction algorithm is continuous. In particular, the induced map*

$$\Phi\colon \mathrm{Out}(G) \to \mathrm{Aut}_{\mathbb{Q}_{p(G)}}(k_+(G)) \xrightarrow{\sim} \mathrm{GL}_{d(G)}(\mathbb{Q}_{p(G)})$$

is continuous and closed.

Proof. This assertion is none other than [6], Lemma 2.13. □

Lemma 4. *The image of the homomorphism* Φ *of Lemma 3 has a natural* $p(G)$*-adic Lie group structure.*

Proof. This assertion follows immediately from Lemma 3, together with [1], Theorem 9.6. □

In the remainder of the present paper, we assume that p_k [resp. $p(G)$] is an odd prime number and that d_k [resp. $d(G)$] is an odd integer greater than one.

Next, we review the subgroup of $\mathrm{Out}(G)$ that corresponds to a "mapping class group" introduced in the discussion following [6], Lemma 2.15.

Let $g \overset{\text{def}}{=} \frac{d(G)-1}{2}$, S a closed orientable surface of genus g (≥ 1), and P a point on S. We write $\mathrm{Mod}(S \setminus \{P\})$ for the mapping class group of $S \setminus \{P\}$.

In the remainder of the present paper, we regard $\mathrm{Sp}_{2g}(\mathbb{Z}_{p(G)})$ as a subgroup of $\mathrm{GL}_{d(G)}(\mathbb{Q}_{p(G)})$ via the injective group homomorphism that is defined by

$$A \mapsto \begin{bmatrix} 1 & 0_{1\times 2g} \\ 0_{2g\times 1} & A \end{bmatrix},$$

where $0_{1\times 2g}$ [respectively, $0_{2g\times 1}$] denotes the $1\times 2g$ matrix [respectively, the $2g\times 1$ matrix] whose entries are all 0. Then there exists [cf. the discussion following [6], Lemma 2.15] a map

$$\rho\colon \mathrm{Mod}(S \setminus \{P\}) \to \mathrm{Out}(G),$$

which is not necessarily a homomorphism of groups, such that the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Out}(G) & \xrightarrow{\Phi} & \mathrm{GL}_{d(G)}(\mathbb{Q}_{p(G)}) \\ {\scriptstyle\rho}\uparrow & & \uparrow{\scriptstyle\subset} \\ \mathrm{Mod}(S \setminus \{P\}) & \longrightarrow & \mathrm{Sp}_{2g}(\mathbb{Z}). \end{array}$$

Here, the lower horizontal arrow is the surjective homomorphism discussed in [6], Theorem 2.17. Write $D \subset \mathrm{Out}(G)$ for the closed subgroup of $\mathrm{Out}(G)$ that is topologically generated by the image of ρ.

In what follows, for a $p(G)$-adic [resp. p_k-adic] Lie group X, we write $\dim(X)$ for the dimension of X as such a Lie group.

Lemma 5. *The image of the homomorphism* $\Phi\colon \mathrm{Out}(G) \to \mathrm{GL}_{d(G)}(\mathbb{Q}_{p(G)})$ *of Lemma 3 contains* $\mathrm{Sp}_{2g}(\mathbb{Z}_{p(G)})$. *In particular, we have* $\dim(\mathrm{Im}(\Phi)) \geq 2g^2 + g$.

Proof. It follows immediately from Lemma 3, the commutative diagram above, and the fact that the topological closure of $\mathrm{Sp}_{2g}(\mathbb{Z})$ in $\mathrm{GL}_{d(G)}(\mathbb{Q}_{p(G)})$ is $\mathrm{Sp}_{2g}(\mathbb{Z}_{p(G)}) \subset \mathrm{GL}_{d(G)}(\mathbb{Q}_{p(G)})$ that $\Phi(D) \supset \mathrm{Sp}_{2g}(\mathbb{Z}_{p(G)})$. This completes the proof of the first assertion. The second assertion follows immediately from the first assertion, together with the well-known equality $\dim(\mathrm{Sp}_{2g}(\mathbb{Z}_{p(G)})) = 2g^2 + g$. □

Next, we review a classification theorem of abelian Hodge-Tate representations.

Definition 6. We shall say that the MLF k is an *absolutely Galois* MLF if the extension $k/k^{(d=1)}$ is a Galois extension.

Definition 7. Suppose that k is an absolutely Galois MLF. Let $\pi \in \mathcal{O}_k$ be a uniformizer of $\mathcal{O}_k$ and σ an element of $\mathrm{Gal}(k/k^{(d=1)})$. Then we shall write

$$\chi_{\pi,\sigma} \colon G_k^{\mathrm{ab}} \overset{\mathrm{rec}_k^{-1}}{\to} \widehat{k^\times} \twoheadrightarrow \mathcal{O}_k^\times \overset{\sigma}{\to} \mathcal{O}_k^\times,$$

where the second arrow is the projection determined by π.

Theorem 8. *Suppose that k is an absolutely Galois MLF. Let $\pi \in \mathcal{O}_k$ be a uniformizer of $\mathcal{O}_k$ and $\phi \colon G_k^{\mathrm{ab}} \to \mathcal{O}_k^\times$ a continuous homomorphism. Then the following two conditions are equivalent:*

(1) *The continuous representation obtained by forming the composite*

$$G_k \twoheadrightarrow G_k^{\mathrm{ab}} \overset{\phi}{\to} \mathcal{O}_k^\times \hookrightarrow \mathrm{Aut}_{\mathbb{Q}_{p_k}}(k_+)$$

— where the first arrow is the natural surjective continuous homomorphism, and the third arrow is the natural inclusion — is Hodge-Tate.

(2) *There exist an integer i_σ for each $\sigma \in \mathrm{Gal}(k/k^{(d=1)})$ and an open subgroup $J \subset I_k$ such that*
- *the restriction to J of the composite of the natural surjective continuous homomorphism $G_k \twoheadrightarrow G_k^{\mathrm{ab}}$ and the given homomorphism $\phi \colon G_k^{\mathrm{ab}} \to \mathcal{O}_k^\times$*

coincides with
- *the restriction to J of the composite of the natural surjective continuous homomorphism $G_k \twoheadrightarrow G_k^{\mathrm{ab}}$ and the homomorphism*

$$\prod_{\sigma \in \mathrm{Gal}(k/k^{(d=1)})} \chi_{\pi,\sigma}^{i_\sigma} \colon G_k^{\mathrm{ab}} \to \mathcal{O}_k^\times.$$

Proof. This assertion is none other than [4], Lemma 1.8. □

Definition 9. Let $\pi \in \mathcal{O}_k$ be a uniformizer of $\mathcal{O}_k$. Then we write $\rho_{k,\pi} \colon G_k \to \mathrm{Aut}_{\mathbb{Q}_{p_k}}(k_+)$ for the continuous p_k-adic representation of G_k obtained by forming the composite

$$G_k \twoheadrightarrow G_k^{\mathrm{ab}} \overset{\chi_{\pi,\mathrm{id}_k}}{\to} \mathcal{O}_k^\times \hookrightarrow \mathrm{Aut}_{\mathbb{Q}_{p_k}}(k_+),$$

where the first arrow is the natural surjective continuous homomorphism, and the third arrow is the natural inclusion.

Remark 10. It follows from [11], Ⅲ, §A.4, Proposition 4, that $\rho_{k,\pi}$ is isomorphic to the continuous p_k-adic representation determined by a Lubin-Tate character [i.e., a continuous character determined by a Lubin-Tate formal group over $\mathcal{O}_k$].

Remark 11. It follows from [11], Ⅲ, §A.1, Corollary 2, that the Hodge-Tate-ness of continuous p_k-adic representations of G_k is independent of the choice of representatives of an inertial equivalence class [cf., e.g., [2], Definition 1.2, (i)]. Moreover, one verifies easily that the inertial equivalence class of χ_{π,id_k} is independent of the choice of a uniformizer of $\mathcal{O}_k$. Thus, the choice of a uniformizer of $\mathcal{O}_k$ is inessential for the discussion that follows.

The following proposition is one of the key ingredients of the present paper:

Proposition 12. *Suppose that k is an absolutely Galois MLF. Let $\pi \in \mathcal{O}_k$ be a uniformizer of $\mathcal{O}_k$ and α a continuous automorphism of G_k. If $\rho_{k,\pi} \circ \alpha$ is Hodge-Tate, then there exists an integer i_σ for each $\sigma \in \mathrm{Gal}(k/k^{(d=1)})$ such that*

$$\alpha_+ = \sum_{\sigma \in \mathrm{Gal}(k/k^{(d=1)})} i_\sigma \cdot \sigma.$$

In particular, if $\rho_{k,\pi} \circ \alpha$ is Hodge-Tate, then $\alpha_+ \in \mathbb{Q}_{p_k}[\mathrm{Gal}(k/k^{(d=1)})] \cap \mathrm{Aut}_{\mathbb{Q}_{p_k}}(k_+) \subset \mathrm{End}_{\mathbb{Q}_{p_k}}(k_+)$.

Proof. Suppose that $\rho_{k,\pi} \circ \alpha$ is Hodge-Tate. Then it follows from Theorem 8 that there exist an integer i_σ for each $\sigma \in \mathrm{Gal}(k/k^{(d=1)})$ and an open subgroup $J \subset I_k$ such that
- the restriction to J of the composite of the natural surjective continuous homomorphism $G_k \twoheadrightarrow G_k^{\mathrm{ab}}$ and the homomorphism

$$\chi_{\pi,\mathrm{id}_k} \circ \alpha^{\mathrm{ab}} \colon G_k^{\mathrm{ab}} \to \mathcal{O}_k^\times$$

coincides with

- the restriction to J of the composite of the natural surjective continuous homomorphism $G_k \twoheadrightarrow G_k^{\mathrm{ab}}$ and the homomorphism

$$\prod_{\sigma \in \mathrm{Gal}(k/k^{(d=1)})} \chi_{\pi,\sigma}^{i_\sigma} : G_k^{\mathrm{ab}} \to \mathcal{O}_k^\times.$$

Thus, it follows from the definition of $\alpha^\times$ that there exist an integer i_σ for each $\sigma \in \mathrm{Gal}(k/k^{(d=1)})$ and an open subgroup $U \subset \mathcal{O}_k^\times$ such that

- the restriction to U of the homomorphism

$$\alpha^\times : \mathcal{O}_k^\times \to \mathcal{O}_k^\times$$

coincides with

- the restriction to U of the homomorphism

$$\prod_{\sigma \in \mathrm{Gal}(k/k^{(d=1)})} \sigma_{i_\sigma} : \mathcal{O}_k^\times \to \mathcal{O}_k^\times,$$

where, we write σ_{i_σ} for the endomorphism of $\mathcal{O}_k^\times$ defined by $x \mapsto \sigma(x)^{i_\sigma}$. Let us recall that it follows from the definition of α_+ and [3], Proposition 3.11, (iv), that the following diagram commutes:

$$\begin{array}{ccc} k_+ & \xrightarrow{\alpha_+} & k_+ \\ {\scriptstyle \log_k}\uparrow & & \uparrow{\scriptstyle \log_k} \\ \mathcal{O}_k^\times & \xrightarrow[\alpha^\times]{} & \mathcal{O}_k^\times. \end{array}$$

Thus, it follows from [7], Lemma 4.1, together with the commutativity of the diagram above, that for each $\sigma \in \mathrm{Gal}(k/k^{(d=1)})$ there exists an integer i_σ such that

$$\alpha_+ = \sum_{\sigma \in \mathrm{Gal}(k/k^{(d=1)})} i_\sigma \cdot \sigma.$$

This completes the proof of Proposition 12. □

With the above preparations, we now prove the main theorem of the present paper.

Theorem 13. *Suppose that k is an absolutely Galois MLF, that p_k is odd, and that d_k is odd and greater than one. Then the continuous p_k-adic representation $\rho_{k,\pi}$ is not Aut-intrinsically Hodge-Tate [cf. Definition in Introduction].*

Proof. We shall write Z for the image of $\mathbb{Q}_{p_k}[\mathrm{Gal}(k/k^{(d=1)})]^\times$ in $\mathrm{GL}_{d_k}(\mathbb{Q}_{p_k})$ via the isomorphism $\mathrm{Aut}_{\mathbb{Q}_{p_k}}(k_+) \xrightarrow{\sim} \mathrm{GL}_{d_k}(\mathbb{Q}_{p_k})$ determined by the basis $y_1, \ldots, y_{d_k}$ of k_+.

We prove Theorem 13 by contradiction. Suppose that the continuous p_k-adic representation $\rho_{k,\pi}$ is Aut-intrinsically Hodge-Tate. Then it follows from Proposition 12 that, for any $\alpha \in \mathrm{Out}(G_k)$, it holds that $\alpha_+, \alpha_+^{-1} \in \mathbb{Q}_{p_k}[\mathrm{Gal}(k/k^{(d=1)})]$. In particular, the image of the continuous group homomorphism Φ [cf. Lemma 3] is contained in Z.

We first consider the case where $g \stackrel{\mathrm{def}}{=} \frac{d_k-1}{2} \geq 2$. In light of the fact that $Z \subset \mathrm{GL}_{d_k}(\mathbb{Q}_{p_k})$ is a closed subgroup, we endow Z with the natural p_k-adic Lie group structure [cf. [1], Theorem 9.6]. Then it follows from [6], Lemma 2.14, (2), (3), that $\dim(Z) \leq d_k = 2g+1$. Thus, it follows from the [easily verified] inequality $2g^2+g > 2g+1$ and Lemma 5 that there exists an automorphism α of G_k such that $\Phi(\alpha) \notin Z$. This contradicts the above observation that the image of Φ is contained in Z. This completes the proof of Theorem 13 in the case where $g \geq 2$.

Finally, we consider the case where $g = 1$ [i.e., $d_k = 3$]. In this case, since [it is immediate that] the Galois group $\mathrm{Gal}(k/k^{(d=1)})$ is abelian, it follows that the group Z is abelian. On the other hand, one verifies easily that the group $\mathrm{Sp}_2(\mathbb{Z}_{p_k})$ is not abelian. Thus, it follows from Lemma 5 that the $\mathrm{Im}(\Phi)$ is not abelian. In particular, there exists an automorphism α of G_k such that $\Phi(\alpha) \notin Z$. This contradicts the above observation that the image of Φ is contained in Z. This completes the proof of Theorem 13 in the case where $g = 1$, hence also of Theorem 13. □

Remark 14. Let p be a prime number and $\overline{\mathbb{Q}_p}$ an algebraic closure of $\mathbb{Q}_p$. Then it is well-known that the Lubin-Tate character over $\mathbb{Z}_p$ determined by the uniformizer p coincides with the p-adic cyclotomic character. Thus, the continuous p-adic representations $G_{\mathbb{Q}_p} \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \to \mathbb{Q}_p^{\times}$ determined by the Lubin-Tate characters over $\mathbb{Z}_p$ [i.e., with respect to arbitrary choices of uniformizers of $\mathbb{Z}_p$] are Aut-intrinsically Hodge-Tate [cf. Remark 11; [7], Proposition 1.1]. This reflects a fundamental distinction between $\mathbb{Q}_p$ and its nontrivial finite Galois extensions from the point of view of anabelian geometry in the case where p is odd.

Remark 15. It is straightforward to see that a similar proof strategy applied in the proof of Theorem 13 may also be applied in the case where k is an absolutely Galois MLF with even d_k. We leave the routine details to the interested reader.

Remark 16. In [6], the author of the present paper proved the following assertion:

> Suppose that p_k is odd, that d_k is even, and that k is an absolutely Galois MLF. Let φ be the automorphism of G_k defined by the following equalities [cf. Theorem 1]:
>
> $$\varphi(\sigma) = \sigma,\ \varphi(\tau) = \tau,\ \varphi(x_2) = x_2 x_1,\ \varphi(x_i) = x_i\ (i \neq 2).$$
>
> Then the continuous p_k-adic representation $\rho_{k,\pi} \circ \varphi$ is not Hodge-Tate.

On the other hand, we cannot obtain an explicit automorphism of G_k that violates the Aut-intrinsic Hodge-Tate-ness of Lubin-Tate characters in the above proof of Theorem 13.

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References

[1] J. D. Dixon, M. P. F. du Sautoy, A. Mann, and D. Segal, *Analytic Pro-p Groups*, 2nd ed., Cambridge Stud. Adv. Math. **61**, Cambridge University Press, Cambridge, 1999.

[2] Y. Hoshi, A note on the geometricity of open homomorphisms between the absolute Galois groups of p-adic local fields, *Kodai Math. J.* **36** (2013), no. **2**, 284–298.

[3] Y. Hoshi, Introduction to mono-anabelian geometry, *Publ. Math. Besançon Algèbre Théorie Nr., 2021,* 5–44, Presses Univ. Franche-Comté, Besançon, 2022.

[4] Y. Hoshi, On intrinsic Hodge-Tate-ness of Galois representations of dimension two, *Kodai Math. J.* **47** (2024), no. **1**, 99–111.

[5] Y. Hoshi and Y. Nishio, On the outer automorphism groups of the absolute Galois groups of mixed-characteristic local fields, *Res. Number Theory* **8** (2022), no. **3**, Paper No. **56**, 13 pp.

[6] K. Kondo, *Anabelian aspects of the outer automorphism groups of the absolute Galois groups of mixed-characteristic local fields*, RIMS Preprint **1991** (December, 2025).

[7] S. Mochizuki, A version of the Grothendieck conjecture for p-adic local fields, *Internat. J. Math.* **8** (1997), 499–506.

[8] J. Neukirch, A. Schmidt, and K. Wingberg, *Cohomology of number fields*, 2nd ed., Grundlehren der Mathematischen Wissenschaften, vol. **323**, Springer, 2008.

[9] Y. Nishio, *On the Outer Automorphism Groups of the Absolute Galois Groups of 2-adic local Fields*, arXiv preprint, arXiv:2512.05095 (2025).

[10] L. Ribes and P. Zalesskii, *Profinite Groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete **3**, second edition, Springer-Verlag (2010).

[11] J.-P. Serre, *Abelian l-adic representations and elliptic curves*, With the collaboration of Willem Kuyk and John Labute, Revised reprint of the 1968 original, Research Notes in Mathematics, **7**, A K Peters, Ltd., Wellesley, MA, 1998.

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