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Hyperbolic Ordinariness of Tetrapods with Ordinary Legendre Parameters

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Abstract. — One of the basic questions in p-adic Teichmüller theory is as follows: Is every hyperbolic curve in odd characteristic hyperbolically ordinary? In the present paper, we prove the hyperbolic ordinariness of tetrapods with ordinary Legendre parameters.

Contents

Introduction

Let p be an odd prime number, k an algebraically closed field of characteristic p, (g, r) a pair of nonnegative integers such that $2g - 2 + r > 0$, and

$$(X, D)$$

a hyperbolic curve of type (g, r) over k — i.e., a pair consisting of a projective smooth curve X of genus g over k and a reduced closed subscheme $D \subseteq X$ of X of degree r. Write

- (X^F, D^F) for the hyperbolic curve over k obtained by base-changing (X, D) via the absolute Frobenius morphism of k,
- $F \colon X \to X^F$ for the relative Frobenius morphism of X/k, and
- $\tau^{\log}$, $(\tau^{\log})^F$ for the logarithmic tangent sheaves of $(X, D)/k$, $(X^F, D^F)/k$, respectively.

First, let us recall the notion of an indigenous bundle and some properties of an indigenous bundle. We shall say that a pair

$$(\pi \colon \mathbf{P} \to X,\ \nabla_{\mathbf{P}})$$

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that consists of a $\mathbf{P}^1$-bundle $\pi\colon \mathbf{P} \to X$ over X and a connection $\nabla_{\mathbf{P}}$ on $\mathbf{P}$ relative to $(X, D)/k$ is an *indigenous bundle* on $(X, D)/k$ [cf. [10, Chapter I, Definition 2.2]] if the following two conditions are satisfied:

- The monodromy operator of $\nabla_{\mathbf{P}}$ at each point of $D \subseteq X$ is nilpotent.
- There exists a [unique — cf. [10, Chapter I, Proposition 2.4]] nonhorizontal section [i.e., the *Hodge section*] $\sigma\colon X \to \mathbf{P}$ of π of canonical height [cf. the discussion preceding [10, Chapter I, Definition 2.2]] $\deg \tau^{\log}/2$.

Let $(\pi\colon \mathbf{P} \to X, \nabla_{\mathbf{P}})$ be an indigenous bundle on $(X, D)/k$. Then the connection $\nabla_{\mathbf{P}}$ on $\mathbf{P}$ determines a horizontal homomorphism [i.e., the *p-curvature homomorphism*]

$$\mathcal{P}\colon F^*(\tau^{\log})^F \longrightarrow \pi_* \tau_{\mathbf{P}/X}$$

— where we write $\tau_{\mathbf{P}/X}$ for the tangent sheaf of $\mathbf{P}/X$. We shall say that the indigenous bundle $(\pi\colon \mathbf{P} \to X, \nabla_{\mathbf{P}})$ is

- *nilpotent* if the square of $\mathcal{P}$ is zero [cf. [10, Chapter II, Definition 2.4]] and
- *admissible* if the dual of $\mathcal{P}$ is surjective [cf. [10, Chapter II, Definition 2.4]].

Moreover, we shall refer to the composite

$$F^*(\tau^{\log})^F \xrightarrow{\mathcal{P}} \pi_* \tau_{\mathbf{P}/X} \twoheadrightarrow \tau^{\log}$$

of the p-curvature homomorphism $\mathcal{P}$ and the surjective homomorphism $\pi_* \tau_{\mathbf{P}/X} \twoheadrightarrow \tau^{\log}$ determined by the Hodge section of $(\pi\colon \mathbf{P} \to X, \nabla_{\mathbf{P}})$ as the *square Hasse invariant* of $(\pi\colon \mathbf{P} \to X, \nabla_{\mathbf{P}})$ [cf. [10, Chapter II, Proposition 2.6, (1)]]. Then, by applying "H^1" to the square Hasse invariant, one may obtain a k-linear homomorphism $H^1(X^F, (\tau^{\log})^F) \to H^1(X, \tau^{\log})$, i.e., the *Frobenius on* $H^1(X, \tau^{\log})$ *induced by* $(\pi\colon \mathbf{P} \to X, \nabla_{\mathbf{P}})$ [cf. the discussion following [10, Chapter II, Lemma 2.11]]. We shall say that the indigenous bundle $(\pi\colon \mathbf{P} \to X, \nabla_{\mathbf{P}})$ is *ordinary* if the Frobenius on $H^1(X, \tau^{\log})$ induced by $(\pi\colon \mathbf{P} \to X, \nabla_{\mathbf{P}})$ is an isomorphism [cf. [10, Chapter II, Definition 3.1]]. Moreover, we shall say that the hyperbolic curve (X, D) over k is *hyperbolically ordinary* if there exists a nilpotent ordinary indigenous bundle on $(X, D)/k$ [cf. [10, Chapter II, Definition 3.3]]. Nilpotent admissible/ordinary indigenous bundles play important roles in the theory of hyperbolically ordinary curves established by S. Mochizuki [cf. [10], [11]]. Now let us recall that, in [4], L. R. A. Finotti studied nilpotent admissible indigenous bundles on hyperbolic curves of type $(2, 0)$ [cf. also [5, Remark 6.1.2]]. Moreover, in [2], I. I. Bouw and S. Wewers studied nilpotent ordinary indigenous bundles on hyperbolic curves of type $(0, 4)$.

Here, let us recall the following basic question in p-adic Teichmüller theory discussed in [11, Introduction, §2.1] [cf. [11, Introduction, §2.1, (1)]]:

Is every hyperbolic curve hyperbolically ordinary?

Some of the previous results concerning this problem are as follows:

- If

$$(g, r, p) \in \{(0, 3, p), (0, 4, 3), (1, 1, 3), (1, 1, 5), (1, 1, 7), (2, 0, 3)\},$$

then the hyperbolic curve (X, D) over k is hyperbolically ordinary [cf. [10, Chapter II, Theorem 2.3] and [6, Proposition 4.2] for $(0, 3, p)$; [6, Theorem C] for $(0, 4, 3)$, $(1, 1, 3)$, $(1, 1, 7)$; [11, Chapter IV, §1.3] and [6, Corollary 5.6] for $(1, 1, 5)$; [5, Theorem D] for $(2, 0, 3)$]. Note that, in the case where $(g, r, p) = (0, 4, 3)$, concrete descriptions of the canonical liftings of level two of hyperbolic curves, as well as of the canonical Frobenius lifting of level two, were given in [8, Theorem A].

- If $p = 3$, $g \leq 5$, $D = \emptyset$, and the projective curve X is hyperelliptic, then the hyperbolic curve (X, D) over k is hyperbolically ordinary [cf. [7, Theorem A]].

- If $p = 3$, $g = 0$, and there exists a nilpotent admissible indigenous bundle on $(X, D)/k$, then the hyperbolic curve (X, D) over k is hyperbolically ordinary [cf. [9, Corollary 2.5, (1)]].

Moreover, in [2], Bouw and Wewers "proved" that if $(g, r) = (0, 4)$, then the hyperbolic curve (X, D) over k is hyperbolically ordinary [cf. [2, Proposition 6.4]]. However, unfortunately, the "proof" of [2, Lemma 6.3] — which implies [2, Proposition 6.4] — contains an error [cf. [6, Remark 4.7.1]]. In the present paper, we study the hyperbolic ordinariness of *tetrapods*, i.e., hyperbolic curves of type $(0, 4)$ [cf. Definition 3.1, (i)].

Suppose that $(g, r) = (0, 4)$. Write $p^* \stackrel{\text{def}}{=} (p-1)/2$ and $\chi_{\text{Hss}}(z) \in k[z]$ — where z is an indeterminate — for the *Hasse polynomial* in characteristic p, i.e.,

$$\chi_{\text{Hss}}(z) \stackrel{\text{def}}{=} \sum_{i=0}^{p^*} \binom{p^*}{i}^2 z^i$$

[cf. Definition 1.10]. Then we shall say that the tetrapod (X, D) over k has an *ordinary Legendre parameter* if there exist an element λ of $k \setminus \{0, 1\}$ and an isomorphism over k

$$\operatorname{Spec}\left(k\left[t, \frac{1}{t(t-1)(t-\lambda)}\right]\right) \xrightarrow{\sim} X \setminus D$$

— where t is an indeterminate — such that $\chi_{\text{Hss}}(\lambda)$ is nonzero [cf. Definition 3.1, (ii)]. The main theorem of the present paper is then as follows [cf. Theorem 3.3]:

Theorem A. — *If the equality $(g, r) = (0, 4)$ holds, and the tetrapod (X, D) over k has an ordinary Legendre parameter, then the tetrapod (X, D) over k is hyperbolically ordinary.*

The present paper is organized as follows: In §1, we prove a "coefficient gap lemma" in positive characteristic [cf. Theorem 1.8]. Note that a consequence of this lemma [cf. Corollary 1.9] may be regarded as a positive characteristic analogue of the classical phenomenon concerning sequences of polynomials orthogonal with respect to positive Borel measures on the field of real numbers [cf. Remark 1.9.1]. In §2, we study some properties of solutions of Heun's differential equations in positive characteristic. In §3, we prove the hyperbolic ordinariness of tetrapods with ordinary Legendre parameters.

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1. A Coefficient Gap Lemma in Positive Characteristic

In the present §1, we prove a "coefficient gap lemma" in positive characteristic [cf. Theorem 1.8 below]. Note that a consequence of this lemma [cf. Corollary 1.9 below] may be regarded as a positive characteristic analogue of the classical phenomenon concerning sequences of polynomials orthogonal with respect to positive Borel measures on the field of real numbers [cf. Remark 1.9.1 below]. In the present §1, let

- p be an odd prime number,
- R an algebra over $\mathbb{F}_p$,
- g, d nonnegative integers, and
- $\Gamma(s)$, $\Delta(s) \in R[s]$ — where s is an indeterminate — polynomials of degree $\leq g$, $\leq d$, respectively.

Definition 1.1.

(i) We define the polynomials $P_{-1}(x), P_0(x), \ldots, P_p(x) \in R[x]$ — where x is an indeterminate — as follows:

$$P_{-1}(x) \stackrel{\mathrm{def}}{=} 0, \quad P_0(x) \stackrel{\mathrm{def}}{=} 1,$$

$$P_{i+1}(x) \stackrel{\mathrm{def}}{=} \bigl(x + \Gamma(i)\bigr)P_i(x) - \Delta(i)P_{i-1}(x) \quad (i \in \{0, \ldots, p-1\}).$$

Thus, one verifies easily that, for each $i \in \{1, \ldots, p\}$, the polynomial $P_i(x) \in R[x]$ is monic and of degree i.

(ii) We shall write

$$\Phi(x) \stackrel{\mathrm{def}}{=} P_p(x)$$

and $c_0, \ldots, c_p \in R$ for the elements of R uniquely determined by the equality

$$\Phi(x) = \sum_{i=0}^{p} c_i x^{p-i} = c_p + c_{p-1}x + \cdots + c_0 x^p.$$

Definition 1.2.

(i) Let i, j be integers. Then we shall write

$$J_{i,j} \stackrel{\mathrm{def}}{=} \begin{cases} -\Gamma(i-1) & \text{if } \{i,j\} \subseteq \{1, \ldots, p\} \text{ and } i = j, \\ \Delta(i-1) & \text{if } \{i,j\} \subseteq \{1, \ldots, p\} \text{ and } j = i-1, \\ 1 & \text{if } \{i,j\} \subseteq \{1, \ldots, p\} \text{ and } j = i+1, \\ 0 & \text{if } \{i,j\} \not\subseteq \{1, \ldots, p\} \text{ or } |i-j| > 1. \end{cases}$$

(ii) We define the $p \times p$ matrix J with entries in R to be $(J_{i,j})_{1 \le i,j \le p}$, i.e.,

$$J \stackrel{\text{def}}{=} \begin{pmatrix} -\Gamma(0) & 1 & & & \\ \Delta(1) & -\Gamma(1) & 1 & & \\ & \Delta(2) & -\Gamma(2) & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & \Delta(p-1) & -\Gamma(p-1) \end{pmatrix}.$$

Lemma 1.3. — *The following assertions hold:*

(i) *The equality* $\det(x - J) = \Phi(x)$ *in* $R[x]$ *holds.*

(ii) *For each positive integer* $n \le p$*, the equality* $-nc_n = \sum_{i=1}^{n} c_{n-i}\mathrm{tr}(J^i)$ *holds.*

Proof. — First, we verify assertion (i). For each $n \in \{1, \dots, p\}$, write $\delta_n \stackrel{\text{def}}{=} \det(x - (J_{i,j})_{1 \le i,j \le n})$. Write, moreover, $\delta_0 \stackrel{\text{def}}{=} 1$ and $\delta_{-1} \stackrel{\text{def}}{=} 0$. Then it follows immediately from the definition of the $J_{i,j}$'s that, for each $n \in \{0, \dots, p-1\}$, the equality

$$\delta_{n+1} = \big(x + \Gamma(n)\big)\delta_n - \Delta(n)\delta_{n-1}$$

holds. Thus, assertion (i) follows from the definition of the $P_i(x)$'s. This completes the proof of assertion (i). Assertion (ii) follows from assertion (i) and the well-known Newton identities. This completes the proof of Lemma 1.3. □

Definition 1.4. — Let n be a positive integer.

(i) Let i be an element of $\{1, \dots, p\}$. Then we shall write

$$V_{n,i} \subseteq \mathbb{Z}^{\times(n+1)}$$

for the subset of the direct product of $n+1$ copies of $\mathbb{Z}$ that consists of $(v_0, \dots, v_n)$ such that $|v_j - v_{j-1}| \le 1$ for each $j \in \{1, \dots, n\}$ and $v_0 = v_n = i$.

(ii) We shall write

$$S_n \subseteq \{-1, 0, 1\}^{\times n}$$

for the subset of the direct product of n copies of $\{-1, 0, 1\}$ that consists of $(\sigma_1, \dots, \sigma_n)$ such that $\sum_{j=1}^{n} \sigma_j = 0$. Thus, for each $i \in \{1, \dots, p\}$, we have a bijective map

$$v_{n,i} \colon S_n \xrightarrow{\sim} V_{n,i}$$

$$(\sigma_1, \dots, \sigma_n) \longmapsto (i, i + \sigma_1, \dots, i + \sigma_1 + \cdots + \sigma_n).$$

(iii) For each element σ of S_n, we shall define the polynomial

$$w_\sigma(s) \in R[s]$$

to be

$$\prod_{i \in \{1,\dots,n\}, \sigma_i = 0} \big(-\Gamma(s + \sigma_1 + \cdots + \sigma_{i-1} - 1)\big) \cdot \prod_{i \in \{1,\dots,n\}, \sigma_i = -1} \Delta(s + \sigma_1 + \cdots + \sigma_{i-1} - 1).$$

Moreover, we shall write

$$W_n(s) \stackrel{\text{def}}{=} \sum_{\sigma \in S_n} w_\sigma(s) \in R[s].$$

(iv) For each $i \in \{1, \ldots, p-1\}$, we shall write

$$d_i \stackrel{\text{def}}{=} gi + \max\{d - 2g, 0\} \cdot \lfloor i/2 \rfloor \in \mathbb{Z}.$$

Moreover, we shall write

$$M_{g,d} \stackrel{\text{def}}{=} \sup\{\, i \in \{1, \ldots, p-1\} \mid d_i \leq p-2 \,\}$$

— where we write $\sup \emptyset \stackrel{\text{def}}{=} 0$.

LEMMA 1.5. — *Let $n \leq p-1$ be a positive integer. Then the following assertions hold:*

(i) *Let σ be an element of S_n. Then the polynomial $w_\sigma(s) \in R[s]$ is of degree $\leq d_n$.*

(ii) *The polynomial $W_n(s) \in R[s]$ is of degree $\leq d_n$.*

PROOF. — First, we verify assertion (i). For each $\Box \in \{-1, 0, 1\}$, write $\sigma^{\#}_{\Box} \stackrel{\text{def}}{=} \#\{\, i \in \{1, \ldots, n\} \mid \sigma_i = \Box \,\}$. Then it follows from the definition of S_n that the equalities

$$\sigma^{\#}_1 = \sigma^{\#}_{-1}, \quad n = 2\sigma^{\#}_1 + \sigma^{\#}_0$$

hold. Thus, since $\Gamma(s)$, $\Delta(s)$ are of degree $\leq g$, $\leq d$, respectively, assertion (i) follows from the definition of the polynomial $w_\sigma(s)$. This completes the proof of assertion (i). Assertion (ii) is a formal consequence of assertion (i). This completes the proof of Lemma 1.5. $\Box$

LEMMA 1.6. — *Let $n \leq p-1$ be a positive integer, and let i be an element of $\{1, \ldots, p\}$. Suppose that the equality $\Delta(0) = 0$ holds. Then the following assertions hold:*

(i) *Let $\sigma = (\sigma_1, \ldots, \sigma_n)$ be an element of S_n. Write $(v_0, \ldots, v_n) \stackrel{\text{def}}{=} v_{n,i}(\sigma) \in V_{n,i}$ and*

$$J_\sigma \stackrel{\text{def}}{=} \prod_{j=0}^{n-1} J_{v_j, v_{j+1}}.$$

Then the equality $J_\sigma = w_\sigma(i)$ holds.

(ii) *The (i,i)-component of the matrix J^n coincides with $W_n(i)$.*

PROOF. — First, we verify assertion (i). If the inclusion $v_j \in \{1, \ldots, p\}$ holds for every $j \in \{0, \ldots, n\}$, the equality of assertion (i) follows immediately from the various definitions involved. Suppose that $v_j \notin \{1, \ldots, p\}$ for some $j \in \{0, \ldots, n\}$. Then one verifies easily that

(a) there exists $j_0 \in \{1, \ldots, n\}$ such that $\sigma_{j_0} = -1$ and $v_{j_0-1} = 1$ or

(b) there exists $j_0 \in \{1, \ldots, n\}$ such that $\sigma_{j_0} = -1$ and $v_{j_0-1} = p+1$.

If (a) is satisfied, then since the equalities $J_{v_{j_0-1}, v_{j_0}} = J_{1,0} = 0$ and $\Delta(0) = 0$ hold, one concludes that $J_\sigma = 0 = w_\sigma(i)$. If (b) is satisfied, then since the equalities $J_{v_{j_0-1}, v_{j_0}} = J_{p+1,p} = 0$ and $\Delta(p) = 0$ [note that R is an algebra over $\mathbb{F}_p$] hold, one concludes that $J_\sigma = 0 = w_\sigma(i)$. This completes the proof of assertion (i).

Next, we verify assertion (ii). First, observe that it follows immediately from the definition of the $J_{i,j}$'s that the (i,i)-component of the matrix J^n is given by

$$\sum_{(v_0,\ldots,v_n)\in V_{n,i}} \prod_{j=0}^{n-1} J_{v_j,v_{j+1}} = \sum_{\sigma\in S_n} J_\sigma$$

[cf. the statement of assertion (i)]. Thus, assertion (ii) follows from assertion (i). This completes the proof of assertion (ii), hence also of Lemma 1.6. $\square$

Lemma 1.7. — *Let n be a positive integer. Suppose that the equality $\Delta(0) = 0$ holds, and that the inequality $n \leq M_{g,d}$ holds. Then the equality $\mathrm{tr}(J^n) = 0$ holds.*

Proof. — Observe that since [we have assumed that] the inequality $n \leq M_{g,d}$ holds, it follows from the definition of $M_{g,d}$ that the inequality $d_n \leq p-2$ holds, which implies [cf. Lemma 1.5, (ii)] that the polynomial $W_n(s) \in R[s]$ is of degree $\leq p-2$. Let $a_0, \ldots, a_{p-2} \in R$ be such that $W_n(s) = \sum_{j=0}^{p-2} a_j s^j$. Then it follows from Lemma 1.6, (ii), that

$$\mathrm{tr}(J^n) = \sum_{i=1}^{p} W_n(i) = \sum_{i=1}^{p}\sum_{j=0}^{p-2} a_j i^j = \sum_{j=0}^{p-2} a_j \sum_{i=1}^{p} i^j.$$

On the other hand, it is well-known that, for each $j \in \{0, \ldots, p-2\}$, the equality $\sum_{i\in\mathbb{F}_p} i^j = 0$ holds. Thus, Lemma 1.7 holds. This completes the proof of Lemma 1.7. $\square$

Theorem 1.8. — *Suppose that the equality $\Delta(0) = 0$ holds. Then, for each $n \in \{1, \ldots, M_{g,d}\}$, the equality $c_n = 0$ holds.*

Proof. — Since [it is immediate that] the inequalities $n \leq M_{g,d} \leq p-1$ hold, n is invertible in $\mathbb{F}_p$. Thus, Theorem 1.8 follows from Lemma 1.3, (ii), and Lemma 1.7. $\square$

Corollary 1.9. — *Suppose that the following four conditions are satisfied:*

(1) *The algebra R over $\mathbb{F}_p$ is an algebraically closed field.*

(2) *The derivative $d\Phi(x)/dx$ is nonzero.*

(3) *The equality $\Delta(0) = 0$ holds.*

(4) *The inequality $2M_{g,d} \geq p-3$ holds.*

Then the equation $\Phi(x) = 0$ has a simple root.

Proof. — Observe that it follows from condition (1) that there exist a positive integer n, pairwise distinct elements $b_1, \ldots, b_n \in R$, and positive integers $e_1, \ldots, e_n$ such that $\Phi(x) = \prod_{i=1}^{n}(x-b_i)^{e_i}$, which implies that $\sum_{i=1}^{n} e_i = \deg\Phi(x) = p$. Assume that $e_i > 1$ for every $i \in \{1, \ldots, n\}$, which implies that

(a) the inequality $2n \leq \sum_{i=1}^{n} e_i = p$ holds.

Then one verifies easily from condition (2) that

(b) the derivative $d\Phi(x)/dx$ is of degree $\geq \sum_{i=1}^{n}(e_i - 1) = p - n$.

On the other hand, it follows from Theorem 1.8, together with conditions (3), (4), that

(c) the derivative $d\Phi(x)/dx$ is of degree $\leq p/2$.

Thus, one concludes from (b), (c) that the inequality $p-n \leq p/2$, hence also the inequality $p \leq 2n$, holds. In particular, since [we have assumed that] p is odd, it follows from (a) that we have a contradiction, as desired. This completes the proof of Corollary 1.9. $\square$

Remark 1.9.1. — Suppose that we are given real numbers a_i, $b_i \in \mathbb{R}$ — where i ranges over the nonnegative integers — such that $b_i > 0$ for each positive integer i and a three-term recurrence relation of polynomials with real coefficients

$$p_{-1}(x) \stackrel{\text{def}}{=} 0, \quad p_0(x) \stackrel{\text{def}}{=} 1, \quad p_{i+1}(x) \stackrel{\text{def}}{=} \big(x + a_i\big)p_i(x) - b_i p_{i-1}(x) \in \mathbb{R}[x]$$

— where x is an indeterminate. Then the well-known theorem of Favard [cf., e.g., [3, Chapter I, Theorem 4.4], [3, Chapter II, Theorem 3.1]] asserts that the polynomials $p_i(x)$ form an orthogonal sequence with respect to some positive Borel measure on $\mathbb{R}$. In this situation, it is also well-known [cf. [3, Chapter I, Theorem 5.2]] that, for each positive integer n, the equation $p_n(x) = 0$ has n pairwise distinct real roots. In particular, one concludes that the equation $p_n(x) = 0$ has a simple root. Corollary 1.9 may be regarded as a positive characteristic analogue of this classical phenomenon over the field of real numbers.

Definition 1.10. — We shall write

$$p^{\divideontimes} \stackrel{\text{def}}{=} (p-1)/2$$

and $\chi_{\mathrm{Hss}}(z) \in R[z]$ — where z is an indeterminate — for the *Hasse polynomial* in characteristic p, i.e.,

$$\chi_{\mathrm{Hss}}(z) \stackrel{\text{def}}{=} \sum_{i=0}^{p^{\divideontimes}} \binom{p^{\divideontimes}}{i}^2 z^i.$$

Lemma 1.11. — *Suppose that R is an algebraically closed field. Let λ be an element of R such that $\lambda \notin \{0, 1\}$. Suppose, moreover, that the equalities*

$$(g, d) = (2, 4), \quad \Gamma(s) = (1 + \lambda)s(s + 1), \quad \Delta(s) = \lambda s^4$$

hold. In particular, one obtains that

$$d_i = 2i, \quad M_{g,d} = p^{\divideontimes} - 1$$

— *where i is an element of $\{1, \ldots, p-1\}$. Then the following assertions hold:*

(i) *Suppose that the derivative $d\Phi(x)/dx$ is nonzero. Then the equation $\Phi(x) = 0$ has a simple root.*

(ii) *The coefficient of $x^{p^{\divideontimes}+1}$ in the polynomial $\Phi(x) \in R[x]$ coincides with*

$$(-1)^{p^{\divideontimes}+1} 2\chi_{\mathrm{Hss}}(\lambda).$$

(iii) *Suppose that $\chi_{\mathrm{Hss}}(\lambda)$ is nonzero. Then the equation $\Phi(x) = 0$ has a simple root.*

PROOF. — In the present proof of Lemma 1.11, for each integer m and each Laurent polynomial $f(z) \in R[z, z^{-1}]$ in the variable z, write $C^{[z]}_m(f(z)) \in R$ for the coefficient of z^m in $f(z)$.

Assertion (i) follows from Corollary 1.9. Next, we verify assertion (ii). First, observe that it follows from Theorem 1.8 that

(a) the equality $C^{[x]}_i(\Phi(x)) = 0$ holds for each $i \in \{p^* + 2, \ldots, p-1\}$.

Next, observe that it follows from Lemma 1.3, (ii), together with (a), that

(b) the equalities

$$-p^* C^{[x]}_{p^*+1}\big(\Phi(x)\big) = \sum_{i=1}^{p^*} C^{[x]}_{p^*+1+i}\big(\Phi(x)\big)\mathrm{tr}(J^i) = \mathrm{tr}(J^{p^*})$$

hold.

Next, observe that it follows from Lemma 1.6, (ii), that

(c) the equality

$$\mathrm{tr}(J^{p^*}) = \sum_{i=1}^{p} W_{p^*}(i)$$

holds.

On the other hand, it follows from Lemma 1.5, (ii), that the polynomial $W_{p^*}(s) \in R[s]$ is of degree $\leq d_{p^*} = p-1$. Thus, since [it is well-known that] the equality $\sum_{i\in\mathbb{F}_p} i^j = 0$ holds for each $j \in \{0, \ldots, p-2\}$, one concludes that

(d) the equality

$$\sum_{i\in\mathbb{F}_p} W_{p^*}(i) = -C^{[s]}_{p-1}\big(W_{p^*}(s)\big)$$

holds.

Next, for each $\square \in \{-1, 0, 1\}$ and each $\sigma \in S_{p^*}$, write $\sigma^{\#}_{\square} \stackrel{\mathrm{def}}{=} \#\{\, i \in \{1, \ldots, p^*\} \mid \sigma_i = \square \,\}$. Then it follows from the definition of S_{p^*} that the equalities

$$\sigma^{\#}_1 = \sigma^{\#}_{-1}, \quad p^* = 2\sigma^{\#}_1 + \sigma^{\#}_0$$

hold. Thus, since the coefficient of s^2 in $-(1+\lambda)s(s+1)$ is $-(1+\lambda)$, and the coefficient of s^4 in λs^4 is λ, one verifies immediately that, for each $\sigma \in S_{p^*}$, the polynomial $w_\sigma(s) \in R[s]$ is of degree $\leq p-1$, and, moreover, the equality

$$C^{[s]}_{p-1}\big(w_\sigma(s)\big) = \big(-(1+\lambda)\big)^{\sigma^{\#}_0} \lambda^{\sigma^{\#}_{-1}}$$

holds. Thus, one concludes immediately from the various definitions involved that

(e) the equalities

$$C^{[s]}_{p-1}\big(W_{p^*}(s)\big) = \sum_{\sigma\in S_{p^*}} C^{[s]}_{p-1}\big(w_\sigma(s)\big)$$

$$= \sum_{i=0}^{\lfloor p^*/2 \rfloor} \frac{p^*!}{i!i!(p^*-2i)!}\big(-(1+\lambda)\big)^{p^*-2i}\lambda^i = C^{[s]}_0\left(\left(s-(1+\lambda)+\frac{\lambda}{s}\right)^{p^*}\right)$$

$$= C_0^{[s]}\left(\left(\frac{(s-1)(s-\lambda)}{s}\right)^{p^*}\right) = C_{p^*}^{[s]}\left(\left((s-1)(s-\lambda)\right)^{p^*}\right)$$

$$= (-1)^{p^*}\sum_{i=0}^{p^*}\binom{p^*}{i}^2\lambda^i = (-1)^{p^*}\chi_{\mathrm{Hss}}(\lambda)$$

hold.

Then assertion (ii) follows from (b), (c), (d), (e). This completes the proof of assertion (ii). Assertion (iii) follows from assertions (i), (ii). This completes the proof of Lemma 1.11.

□

2. Solutions of Heun's Differential Equations in Positive Characteristic

In the present §2, we study some properties of solutions of Heun's differential equations in positive characteristic. In the present §2, let

- p be an odd prime number,
- B an integral domain over $\mathbb{F}_p$, and
- λ, β elements of B such that both λ and $1-\lambda$ are invertible in B.

Definition 2.1.

(i) We shall write

$$f_0(t) \stackrel{\text{def}}{=} t(t-1)(t-\lambda) \in B[t]$$

— where t is an indeterminate.

(ii) We shall write

$$L_{\lambda,\beta}\colon B[t] \longrightarrow B[t]$$

for the B-linear endomorphism of $B[t]$ given by the differential operator

$$f_0(t)\frac{d^2}{dt^2} + \frac{df_0(t)}{dt}\frac{d}{dt} + (t-\beta)$$

$$= f_0(t)\left(\frac{d^2}{dt^2} + \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{t-\lambda}\right)\frac{d}{dt} + \frac{t-\beta}{t(t-1)(t-\lambda)}\right).$$

Lemma 2.2. — *Let $u(t) = \sum_{i=0}^{d} u_i t^i$ be an element of $B[t]$. Then the following assertions hold:*

(i) *Let j be a nonnegative integer. Then the coefficient of t^j in $L_{\lambda,\beta}(u(t)) \in B[t]$ coincides with*

$$\lambda(j+1)^2u_{j+1} - \big((1+\lambda)j(j+1)+\beta\big)u_j + j^2u_{j-1} \in B$$

— where we write $u_i \stackrel{\text{def}}{=} 0$ for each integer i such that $i \notin \{0,\ldots,d\}$.

(ii) *The equality*

$$L_{\lambda,\beta}\big(u(t)\big) = \frac{d}{dt}\left(f_0(t)\frac{du(t)}{dt}\right) + (t-\beta)u(t)$$

holds.

PROOF. — These assertions may be verified by straightforward computation. □

LEMMA 2.3. — *Let $u_0, \ldots, u_{p-1}$ be elements of B. Write $u(t) \stackrel{\text{def}}{=} \sum_{i=0}^{p-1} u_i t^i \in B[t]$. Then the following two conditions are equivalent:*

(1) *The equality $L_{\lambda,\beta}(u(t)) = 0$ holds.*

(2) *For each $i \in \{0, \ldots, p-1\}$, the equality*

$$\lambda(i+1)^2 u_{i+1} = \big((1+\lambda)i(i+1) + \beta\big)u_i - i^2 u_{i-1}$$

— where we write $u_{-1} \stackrel{\text{def}}{=} 0$ and $u_p \stackrel{\text{def}}{=} 0$ — holds.

PROOF. — This assertion is a formal consequence of Lemma 2.2, (i). □

DEFINITION 2.4. — Let $u_0, \ldots, u_{p-1}$ be elements of B. Then we shall say that $(u_0, \ldots, u_{p-1})$ is *of Heun type* if at least one of $u_0, \ldots, u_{p-1}$ is nonzero, and, moreover, the equality $L_{\lambda,\beta}(\sum_{i=0}^{p-1} u_i t^i) = 0$ holds [cf. condition (1) of Lemma 2.3].

DEFINITION 2.5.

(i) We define the polynomials $Q_{-1}(x), Q_0(x), \ldots, Q_p(x) \in B[x]$ — where x is an indeterminate — as follows:

$$Q_{-1}(x) \stackrel{\text{def}}{=} 0, \quad Q_0(x) \stackrel{\text{def}}{=} 1,$$

$$Q_{i+1}(x) \stackrel{\text{def}}{=} \big(x + (1+\lambda)i(i+1)\big)Q_i(x) - \lambda i^4 Q_{i-1}(x) \quad (i \in \{0, \ldots, p-1\}).$$

Thus, one verifies easily that, for each $i \in \{1, \ldots, p\}$, the polynomial $Q_i(x) \in B[x]$ is monic and of degree i. Moreover, we shall write

$$\Phi_\lambda(x) \stackrel{\text{def}}{=} Q_p(x).$$

(ii) We shall say that the element $\beta \in B$ is λ*-null* (respectively, *simply* λ*-null*) if $\beta \in B$ is a root (respectively, simple root) of the equation $\Phi_\lambda(x) = 0$.

REMARK 2.5.1. — Note that it is immediate that the polynomial $\Phi_\lambda(x) \in B[x]$ coincides with the polynomial "$\Phi(x)$" of Lemma 1.11, i.e., in the case where the "(R, x, λ)" of Lemma 1.11 is taken to be (B, x, λ).

LEMMA 2.6. — *The following two conditions are equivalent:*

(1) *There exist elements $u_0, \ldots, u_{p-1}$ of B such that $(u_0, \ldots, u_{p-1})$ is of Heun type.*

(2) *The element $\beta \in B$ is λ-null.*

PROOF. — First, we verify the implication (1) $\Rightarrow$ (2). Suppose that condition (1) is satisfied. Then it follows from condition (1), together with the implication (1) $\Rightarrow$ (2) of Lemma 2.3, that there exist elements $u_0, \ldots, u_{p-1}$ of B such that at least one of $u_0, \ldots, u_{p-1}$ is nonzero, and, moreover, condition (2) of Lemma 2.3 is satisfied. Then one verifies immediately from condition (2) of Lemma 2.3 and the definition of the $Q_i(x)$'s that u_i coincides with $u_0\lambda^{-i}(i!)^{-2}Q_i(\beta)$ [which implies that $u_0 \neq 0$ — cf. our assumption that at least one of $u_0, \ldots, u_{p-1}$ is nonzero]. Thus, since the equality $0 = \beta u_{p-1} - u_{p-2}$ holds [cf. condition (2) of Lemma 2.3], one concludes that the equality $\Phi_\lambda(\beta) = 0$ holds. This completes the proof of the implication (1) $\Rightarrow$ (2).

Next, we verify the implication (2) $\Rightarrow$ (1). Suppose that condition (2) is satisfied. For each $i \in \{0, \ldots, p-1\}$, write $u_i \stackrel{\text{def}}{=} \lambda^{-i}(i!)^{-2}Q_i(\beta)$. Then one verifies immediately from the definition of the $Q_i(x)$'s, together with condition (2), that condition (2) of Lemma 2.3 is satisfied. Thus, since $u_0 = Q_0(\beta) = 1$, it follows from the implication (2) $\Rightarrow$ (1) of Lemma 2.3 that $(u_0, \ldots, u_{p-1})$ is of Heun type. This completes the proof of the implication (2) $\Rightarrow$ (1), hence also of Lemma 2.6. $\square$

LEMMA 2.7. — *Let $u_0, \ldots, u_{p-1} \in B$ be such that $(u_0, \ldots, u_{p-1})$ is of Heun type. Write $u(t) \stackrel{\text{def}}{=} \sum_{i=0}^{p-1} u_i t^i \in B[t]$. Then the following assertions hold:*

(i) *Each of $u(0)$, $u(1)$, and $u(\lambda)$ is nonzero. In particular, the element $u_0 \in B$ is nonzero.*

(ii) *The element $u_{p-1} \in B$ is nonzero. In particular, the polynomial $u(t) \in B[t]$ is of degree $p-1$.*

(iii) *Suppose that B is an algebraically closed field. Then the polynomial $u(t)$ is separable.*

(iv) *Let $w(t)$ be an element of $B[t]$. Then the equality*

$$\frac{d}{dt}\left(f_0(t)\left(u(t)\frac{dw(t)}{dt} - \frac{du(t)}{dt}w(t)\right)\right) = u(t)L_{\lambda,\beta}\big(w(t)\big)$$

holds.

(v) *The $p-1$ elements*

$$L_{\lambda,\beta}(1), \quad L_{\lambda,\beta}(t), \quad \ldots, \quad L_{\lambda,\beta}(t^{p-2}) \in B[t]$$

are linearly independent over B.

PROOF. — First, we verify assertion (i). Assume that $u(0) = 0$. Then it follows from the implication (1) $\Rightarrow$ (2) of Lemma 2.3 that $u(t) = 0$, in contradiction to our assumption that at least one of $u_0, \ldots, u_{p-1}$ is nonzero. This completes the proof of the assertion that $u(0)$ is nonzero. Moreover, the assertion that each of $u(1)$ and $u(\lambda)$ is nonzero may be derived from a similar argument to the argument applied in the proof of the assertion that $u(0)$ is nonzero, i.e., may be verified by considering a similar three-term recurrence relation to the three-term recurrence relation of condition (2) of Lemma 2.3. This completes the proof of assertion (i). Moreover, assertion (ii) follows immediately from a similar argument to the argument applied in the proof of the assertion that $u(0)$ is nonzero in assertion (i), i.e., from the implication (1) $\Rightarrow$ (2) of Lemma 2.3.

Next, we verify assertion (iii). Assume that there exist an integer $i \geq 2$ and an element $b \in B$ such that $u(t)$ is divisible by $(t-b)^i$ [which implies that the inequality $i \leq p-1$ holds — cf. assertion (ii)]. Observe that it follows from assertion (i) that b is not contained in $\{0, 1, \lambda\}$. Thus, it follows immediately from the definition of $f_0(t)$ and the definition of $L_{\lambda,\beta}$ that $d^2u(t)/dt^2$ is divisible by $(t-b)^{i-1}$. In particular, since the inequalities $2 \leq i \leq p-1$ hold [which implies that $i(i-1)$ is invertible in $\mathbb{F}_p$], it follows that $u(t)$ is divisible by $(t-b)^{i+1}$. Thus, one may conclude that if $u(t)$ is not separable, then the equality $u(t) = 0$ holds, in contradiction to our assumption that at least one of $u_0, \ldots, u_{p-1}$ is nonzero. This completes the proof of assertion (iii). Since [we have assumed that] $(u_0, \ldots, u_{p-1})$ is of Heun type, assertion (iv) follows immediately from the definition of $L_{\lambda,\beta}$. Assertion (v) is an immediate consequence of Lemma 2.2, (i). This completes the proof of Lemma 2.7. $\square$

Lemma 2.8. — *Suppose that there exists a sub-$\mathbb{F}_p$-algebra $B_0 \subseteq B$ of B such that B is the polynomial ring $B_0[\beta]$ over B_0 in the variable $\beta \in B$, and that $\lambda \in B$ is contained in $B_0 \subseteq B$, i.e., that $\lambda \in B_0$. For each $i \in \{0, \ldots, p-1\}$, write*

$$u_i(\beta) \stackrel{\text{def}}{=} \lambda^{-i}(i!)^{-2}Q_i(\beta) \in B_0[\beta] = B$$

[*cf. Definition* 2.5, (i)]. *Write, moreover,*

$$u(t, \beta) \stackrel{\text{def}}{=} \sum_{i=0}^{p-1} u_i(\beta)t^i \in B_0[\beta, t] = B[t].$$

Then the equality

$$L_{\lambda,\beta}\left(\frac{\partial u(t,\beta)}{\partial \beta}\right) = u(t,\beta) - \lambda^{-(p-1)}\big((p-1)!\big)^{-2}\frac{d\Phi_\lambda(\beta)}{d\beta}t^{p-1}$$

holds.

Proof. — Observe that it follows immediately from Lemma 2.2, (i), and the definition of the polynomial $\Phi_\lambda(x) \in B[x]$ that the equality

$$L_{\lambda,\beta}\big(u(t,\beta)\big) = -\lambda^{-(p-1)}\big((p-1)!\big)^{-2}\Phi_\lambda(\beta)t^{p-1},$$

hence also

(a) the equality

$$\frac{\partial}{\partial \beta}L_{\lambda,\beta}\big(u(t,\beta)\big) = -\lambda^{-(p-1)}\big((p-1)!\big)^{-2}\frac{d\Phi_\lambda(\beta)}{d\beta}t^{p-1},$$

holds.

Next, recall from Lemma 2.2, (ii), that the equality

$$L_{\lambda,\beta}\big(u(t,\beta)\big) = \frac{\partial}{\partial t}\left(f_0(t)\frac{\partial u(t,\beta)}{\partial t}\right) + (t-\beta)u(t,\beta),$$

hence also

(b) the equality

$$\frac{\partial}{\partial \beta} L_{\lambda,\beta}\big(u(t,\beta)\big) = \frac{\partial}{\partial t}\left(f_0(t)\frac{\partial u'(t,\beta)}{\partial t}\right) - u(t,\beta) + (t-\beta)u'(t,\beta)$$

— where we write $u'(t,\beta) \stackrel{\text{def}}{=} \partial u(t,\beta)/\partial\beta$ — holds.

Moreover, by applying Lemma 2.2, (ii), to $u'(t,\beta)$, one also obtains

(c) the equality

$$L_{\lambda,\beta}\big(u'(t,\beta)\big) = \frac{\partial}{\partial t}\left(f_0(t)\frac{\partial u'(t,\beta)}{\partial t}\right) + (t-\beta)u'(t,\beta).$$

Thus, Lemma 2.8 follows from (a), (b), (c). This completes the proof of Lemma 2.8. □

Definition 2.9.

(i) For each nonnegative integer d, we shall write

$$B[t]^{\leq d} \stackrel{\text{def}}{=} \{\, g(t) \in B[t] \mid \deg g(t) \leq d \,\}.$$

(ii) We shall write

$$d_0\colon B[t]^{\leq 2p-4} \longrightarrow B[t]^{\leq 2p-2}$$

for the B-linear homomorphism given by

$$g(t) \longmapsto \frac{d}{dt}\big(g(t)\cdot f_0(t)\big).$$

Lemma 2.10. — *Suppose that B is a field. Let $u_0, \ldots, u_{p-1} \in B$ be such that $(u_0, \ldots, u_{p-1})$ is of Heun type. Write $u(t) \stackrel{\text{def}}{=} \sum_{i=0}^{p-1} u_i t^i \in B[t]$. Then the $p-1$ elements*

$$u(t)L_{\lambda,\beta}(1), \quad u(t)L_{\lambda,\beta}(t), \quad \ldots, \quad u(t)L_{\lambda,\beta}(t^{p-2}) \in B[t]$$

form a basis of the subspace

$$\mathrm{Im}(d_0) \cap u(t)\cdot B[t]^{\leq p-1} \subseteq B[t]^{\leq 2p-2}.$$

In particular, the intersection $\mathrm{Im}(d_0) \cap u(t)\cdot B[t]^{\leq p-1}$ is of dimension $p-1$ over B.

Proof. — First, observe that it follows from Lemma 2.7, (iv), that, for each $i \in \{0, \ldots, p-2\}$, the equality

$$u(t)L_{\lambda,\beta}(t^i) = \frac{d}{dt}\left(f_0(t)\left(it^{i-1}u(t) - \frac{du(t)}{dt}t^i\right)\right)$$

holds. Thus, since the polynomial $u(t) \in B[t]$ is of degree $p-1$ [cf. Lemma 2.7, (ii)], the $p-1$ elements under consideration are contained in the intersection $\mathrm{Im}(d_0)\cap u(t)\cdot B[t]^{\leq p-1}$. In particular, it follows from Lemma 2.7, (v), that the intersection $\mathrm{Im}(d_0)\cap u(t)\cdot B[t]^{\leq p-1}$ is of dimension $\geq p-1$ over B. Thus, since [one verifies easily that] $u(t)\cdot B[t]^{\leq p-1}$ is of dimension p, to verify Lemma 2.10, it suffices to verify that $u(t)\cdot B[t]^{\leq p-1}$ is not contained in $\mathrm{Im}(d_0)$. On the other hand, since [one verifies immediately that] the coefficient of t^{p-1}

in every element of $\mathrm{Im}(d_0)$ is zero, and u_0 is nonzero [cf. Lemma 2.7, (i)], the element $u(t) \cdot t^{p-1} \in u(t) \cdot B[t]^{\leq p-1}$ is not contained in $\mathrm{Im}(d_0)$. This completes the proof of Lemma 2.10. $\square$

LEMMA 2.11. — *Suppose that B is a field. Let $u_0, \ldots, u_{p-1} \in B$ be such that $(u_0, \ldots, u_{p-1})$ is of Heun type. Write $u(t) \stackrel{\mathrm{def}}{=} \sum_{i=0}^{p-1} u_i t^i \in B[t]$ and*

$$\kappa_u : B[t]^{\leq p-1} \longrightarrow B$$

for the B-linear homomorphism given by

$$t^i \longmapsto u_0^{-1} \lambda^i u_i$$

[cf. Lemma 2.7, (i)]. Then the following assertions hold:

(i) *The sequence*

$$0 \longrightarrow B[t]^{\leq p-2} \xrightarrow{L_{\lambda,\beta}} B[t]^{\leq p-1} \xrightarrow{\kappa_u} B \longrightarrow 0$$

is exact.

(ii) *The natural inclusion $B \hookrightarrow B[t]^{\leq p-1}$ determines a splitting of the final arrow of the sequence of (i). In particular, we have an isomorphism of B-modules*

$$L_{\lambda,\beta}(B[t]^{\leq p-2}) \oplus B \xrightarrow{\sim} B[t]^{\leq p-1}.$$

(iii) *The element $t^{p-1} \in B[t]^{\leq p-1}$ is not contained in the image $L_{\lambda,\beta}(B[t]^{\leq p-2}) \subseteq B[t]^{\leq p-1}$.*

PROOF. — First, we verify assertions (i), (ii). It follows immediately from Lemma 2.7, (v), that the restriction of $L_{\lambda,\beta}$ to $B[t]^{\leq p-2}$ is injective. Moreover, it follows immediately from Lemma 2.2, (i), that the composite $\kappa_u \circ L_{\lambda,\beta}$ is zero. Thus, since [one verifies easily that] $B[t]^{\leq p-2}$, $B[t]^{\leq p-1}$ are free of rank $p-1$, p, respectively, and [it is immediate that] the natural inclusion $B \hookrightarrow B[t]^{\leq p-1}$ determines a splitting of the homomorphism κ_u, one concludes immediately that assertions (i), (ii) hold. This completes the proofs of assertions (i), (ii). Assertion (iii) follows from assertion (i) and Lemma 2.7, (ii). This completes the proof of Lemma 2.11. $\square$

LEMMA 2.12. — *Suppose that B is a field. Let $u_0, \ldots, u_{p-1} \in B$ be such that $(u_0, \ldots, u_{p-1})$ is of Heun type. Write $u(t) \stackrel{\mathrm{def}}{=} \sum_{i=0}^{p-1} u_i t^i \in B[t]$. Then the following three conditions are equivalent:*

(1) *The inclusion $u(t)^2 \in \mathrm{Im}(d_0)$ holds.*

(2) *The inclusion $u(t) \in L_{\lambda,\beta}(B[t]^{\leq p-2})$ holds.*

(3) *The element $\beta \in B$ is not simply λ-null.*

PROOF. — First, we verify the implication (1) $\Rightarrow$ (2). Suppose that condition (1) is satisfied. Then since the polynomial $u(t) \in B[t]$ is of degree $p-1$ [cf. Lemma 2.7, (ii)], one verifies easily from condition (1) that $u(t)^2$ is contained in the intersection $\mathrm{Im}(d_0) \cap u(t) \cdot B[t]^{\leq p-1}$. Thus, it follows from Lemma 2.10 that there exists a polynomial $g(t) \in B[t]$ of degree $\leq p-2$ such that the equality $u(t)^2 = u(t) L_{\lambda,\beta}(g(t))$ holds. Thus, since [we have assumed that] B is a field, one concludes that the equality $u(t) = L_{\lambda,\beta}(g(t))$ holds, which implies condition (2). This completes the proof of the implication (1) $\Rightarrow$ (2).

Next, we verify the implication (2) $\Rightarrow$ (1). Suppose that condition (2) is satisfied, i.e., that there exists an element $g(t) \in B[t]^{\leq p-2}$ such that the equality $u(t) = L_{\lambda,\beta}(g(t))$ holds. In particular, it follows from Lemma 2.7, (iv), that

$$u(t)^2 = u(t)L_{\lambda,\beta}\big(g(t)\big) = \frac{d}{dt}\left(f_0(t)\left(u(t)\frac{dg(t)}{dt} - \frac{du(t)}{dt}g(t)\right)\right) \in \mathrm{Im}(d_0),$$

as desired. This completes the proof of the implication (2) $\Rightarrow$ (1). The equivalence (2) $\Leftrightarrow$ (3) is a formal consequence of Lemma 2.8 and Lemma 2.11, (iii). This completes the proof of Lemma 2.12. $\square$

Lemma 2.13. — *Suppose that B is an algebraically closed field, and that the element $\beta \in B$ is simply λ-null. Then there exists a monic separable polynomial $u(t) \in B[t]$ of degree $p-1$ such that $0 \notin \{u(0), u(1), u(\lambda)\}$, and, moreover, the subspace*

$$u(t) \cdot B[t]^{\leq p-1} \subseteq B[t]^{\leq 2p-2}$$

is contained in the subspace of $B[t]^{\leq 2p-2}$ generated by the two subspaces

$$u(t)^2 \cdot B, \quad \mathrm{Im}(d_0) \subseteq B[t]^{\leq 2p-2}.$$

Proof. — First, observe that since [we have assumed that] the element $\beta \in B$ is simply λ-null, it follows from the implication (2) $\Rightarrow$ (1) of Lemma 2.6 that there exist elements $u_0, \ldots, u_{p-1}$ of B such that $(u_0, \ldots, u_{p-1})$ is of Heun type, which implies [cf. Lemma 2.7, (ii)] that u_{p-1} is nonzero. Write $u(t) \stackrel{\mathrm{def}}{=} u_{p-1}^{-1} \cdot \sum_{i=0}^{p-1} u_i t^i \in B[t]$. Then it follows from Lemma 2.7, (i), (iii), that the monic polynomial $u(t) \in B[t]$ is separable and of degree $p-1$ and satisfies that $0 \notin \{u(0), u(1), u(\lambda)\}$.

Now observe that one verifies easily that $u(t) \cdot B[t]^{\leq p-1}$ is of dimension p over B. Recall that it follows from Lemma 2.10 that the intersection $\mathrm{Im}(d_0) \cap u(t) \cdot B[t]^{\leq p-1}$ is of dimension $\geq p-1$ over B. Next, observe that it follows from the implication (1) $\Rightarrow$ (3) of Lemma 2.12 that $u(t)^2 \notin \mathrm{Im}(d_0)$. Thus, since [it is immediate that] $u(t)^2 \in u(t) \cdot B[t]^{\leq p-1}$, one concludes that Lemma 2.13 holds. This completes the proof of Lemma 2.13. $\square$

3. Hyperbolic Ordinariness of Tetrapods

In the present §3, we prove the hyperbolic ordinariness of tetrapods with ordinary Legendre parameters [cf. Theorem 3.3 below].

Definition 3.1. — Let S be a scheme.

(i) We shall refer to a hyperbolic curve of type $(0, 4)$ over S as a *tetrapod* over S.

(ii) Let p be an odd prime number. Suppose that S is a scheme in characteristic p. Then we shall say that a tetrapod (X, D) over S has an *ordinary Legendre parameter* if the following condition is satisfied: Let k be an algebraically closed field, and let $\overline{s} \stackrel{\mathrm{def}}{=}$

$\mathrm{Spec}(k) \to S$ be a morphism of schemes. Then there exist an element λ of $k \setminus \{0,1\}$ and an isomorphism over k

$$\mathrm{Spec}\left(k\left[t, \frac{1}{t(t-1)(t-\lambda)}\right]\right) \xrightarrow{\sim} (X \setminus D) \times_S \overline{s}$$

— where t is an indeterminate — such that $\chi_{\mathrm{Hss}}(\lambda)$ [cf. Definition 1.10] is nonzero.

Lemma 3.2. — *Let p be an odd prime number, k an algebraically closed field of characteristic p, and λ an element of $k \setminus \{0,1\}$. Write*

$$U \stackrel{\mathrm{def}}{=} \mathrm{Spec}\left(k\left[t, \frac{1}{t(t-1)(t-\lambda)}\right]\right)$$

— where t is an indeterminate — X for the smooth compactification over k of the smooth curve U over k, and $D \subseteq X$ for the reduced closed subscheme of X whose underlying closed subset is given by the complement of U in X. Thus, the pair

$$(X, D)$$

forms a tetrapod over k. Then the following two conditions are equivalent:

(1) *The tetrapod (X,D) over k is hyperbolically ordinary* [*cf.* [10, Chapter II, Definition 3.3]].

(2) *There exists a monic separable polynomial $f_E(t) \in k[t]$ of degree $p-1$ such that $0 \notin \{f_E(0), f_E(1), f_E(\lambda)\}$, and, moreover, the subspace*

$$f_E(t) \cdot k[t]^{\leq p-1} \subseteq k[t]^{\leq 2p-2}$$

[*cf. Definition* 2.9, (i) *— where we take the "(B,λ)" of* §2 *to be* (k,λ)] *is contained in the subspace of $k[t]^{\leq 2p-2}$ generated by the two subspaces*

$$f_E(t)^2 \cdot k, \quad \mathrm{Im}(d_0) \subseteq k[t]^{\leq 2p-2}$$

[*cf. Definition* 2.9, (ii) *— where we take the "(B,λ)" of* §2 *to be* (k,λ)]. *Moreover, if these two conditions are satisfied, then the closed subscheme of X determined by the polynomial $f_E(t)$ of condition* (2) *coincides with the supersingular divisor* [*cf.* [10, Chapter II, Proposition 2.6, (3)]] *associated to a nilpotent* [*cf.* [10, Chapter II, Definition 2.4]] *ordinary* [*cf.* [10, Chapter II, Definition 3.1]] *indigenous bundle* [*cf.* [10, Chapter I, Definition 2.2]] *on the tetrapod (X,D).*

Proof. — Observe that since [we have assumed that] the polynomial $f_E(t)$ in the statement of Lemma 3.2 is monic, it is immediate that the polynomial $f_E(t)$ satisfies condition (1) of [6, Proposition 4.1]. Thus, Lemma 3.2 is a formal consequence of [6, Proposition 4.1] [cf. also [6, Proposition 3.2], [6, Proposition 3.3]]. □

Theorem 3.3. — *Let p be an odd prime number, and let S be a scheme in characteristic p. Then every tetrapod over S that has an ordinary Legendre parameter is hyperbolically ordinary.*

Proof. — Let (X, D) be a tetrapod over S that has an ordinary Legendre parameter. First, observe that it follows from [10, Chapter II, Proposition 3.4] that, to verify Theorem 3.3, we may assume without loss of generality, by replacing (X, D) by a geometric fiber of (X, D) over S, that S is the spectrum of an algebraically closed field. Then Theorem 3.3 follows from Lemma 1.11, (iii), Lemma 2.13, and Lemma 3.2 [cf. also Remark 2.5.1]. This completes the proof of Theorem 3.3. $\square$

Remark 3.3.1. — In the proof of Theorem 3.3, we obtain certain nilpotent indigenous bundles from certain solutions of Heun's differential equation

$$\left(\frac{d^2}{dt^2} + \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{t-\lambda}\right)\frac{d}{dt} + \frac{t-\beta}{t(t-1)(t-\lambda)}\right)u(t) = 0.$$

On the other hand, the strategy of obtaining suitable nilpotent indigenous bundles from suitable solutions of Heun's differential equations was initiated by I. I. Bouw and S. Wewers [cf. [1], [2]]. Bouw and Wewers first construct, from suitable solutions of Heun's differential equations, deformation data, i.e., suitable pairs each of which consists of a finite Galois covering of curves and a logarithmic differential form on the covering; they then construct nilpotent indigenous bundles from these deformation data.

References

[1] I. I. Bouw, *The accessory parameter problem in positive characteristic*, Preprint arXiv:0705.0458 (2007).

[2] I. I. Bouw and S. Wewers, Indigenous bundles with nilpotent p-curvature, *Int. Math. Res. Not.* (2006), Art. ID 89254, 37 pp.

[3] T. S. Chihara, *An introduction to orthogonal polynomials*, Math. Appl., Vol. **13** Gordon and Breach Science Publishers, New York-London-Paris, 1978.

[4] L. R. A. Finotti, Minimal degree liftings of hyperelliptic curves, *J. Math. Sci. Univ. Tokyo* **11** (2004), no. 1, 1–47.

[5] Y. Hoshi, Nilpotent admissible indigenous bundles via Cartier operators in characteristic three, *Kodai Math. J.* **38** (2015), no. 3, 690–731.

[6] Y. Hoshi, On the supersingular divisors of nilpotent admissible indigenous bundles, *Kodai Math. J.* **42** (2019), no. 1, 1–47.

[7] Y. Hoshi, Hyperbolic ordinariness of hyperelliptic curves of lower genus in characteristic three, *Kyushu J. Math.* **73** (2019), no. 2, 317–335.

[8] Y. Hoshi, Canonical liftings of level two of tetrapods in characteristic three, *Kyushu J. Math.* **79** (2025), no. 2, 209–228.

[9] Y. Hoshi, *On indigenous bundles in characteristic three*, RIMS Preprint **1918** (June 2020).

[10] S. Mochizuki, A theory of ordinary p-adic curves, *Publ. Res. Inst. Math. Sci.* **32** (1996), no. 6, 957–1152.

[11] S. Mochizuki, *Foundations of p-adic Teichmüller theory*, AMS/IP Stud. Adv. Math. **11**, Amer. Math. Soc./Int. Press, 1999.

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