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Abstract

In birational anabelian geometry, we often discuss the fully-faithfulness of the functor from a certain category of fields to a category of profinite groups with inner automorphism indeterminacy obtained by forming the absolute Galois groups of fields. In the present paper, we obtain various generalizations of earlier results concerning the birational anabelian Grothendieck conjecture for curves in *arbitrary characteristic*. Let Σ be a nonempty set of prime numbers. Then we prove a geometrically pro-Σ version of the relative birational anabelian Grothendieck conjecture for curves over Σ-TKND-AVKF-fields, which may be regarded as a pro-Σ variant of TKND-AVKF-fields [concerning the divisible subgroups of the groups of rational points of semi-abelian varieties] introduced in an author's recent joint work with Hoshi and Mochizuki. For instance, by applying recent results on the finiteness of torsion points of abelian varieties obtained by Murotani-Ozeki or Lombardo-Szamuely, one may observe that any subfield of the maximal Kummer extension field of a finitely generated field of characteristic 0 is Σ-TKND-AVKF. One may also observe that, for each prime number p, any mixed characteristic higher local field with finite finial residues of characteristic p is $\{p\}$-TKND-AVKF. The main difficulty of the verification in the pro-Σ setting compared to the verification in the full profinite setting lies in the fact that we need to consider certain *group-theoretic perfections* of fields that do not carry natural ring structures. Moreover, we prove the [geometrically pro-Σ] semi-absolute birational anabelian Grothendieck conjecture for curves over Σ-TKND-AVKF-fields, which may be regarded as a complete generalization of both the author's earlier result in the corresponding relative setting and Mochizuki's corresponding result in the case of Kummer-faithful base fields. Furthermore, in the case where the base fields are subfields of the maximal Kummer extension fields of number fields, as an application of the above result for curves, together with highly nontrivial techniques due to Koenigsmann and Pop, we prove the absolute birational anabelian Grothendieck conjecture-type/Neukirch-Uchida-type result for algebraic varieties of *arbitrary positive dimension*, which may be regarded as a generalization of Pop's corresponding famous result for finitely generated fields in characteristic 0. In particular, any finitely generated transcendental extension field of $\mathbb{Q}^{\mathrm{ab}}$ may be regarded as an anabelian field. In light of Kronecker dimension and the vanishing of cyclotomic characters, this result may be regarded as the characteristic 0 analogue of the anabelian result for the function fields over $\overline{\mathbb{F}}_p$ of dimension ≥ 2 due to Pop.

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Contents

Introduction

In birational anabelian geometry, we consider the "reconstructibility" of the function fields of algebraic varieties from their [suitable quotients of] absolute Galois groups. One of the formulations of "reconstructibility" [so called the birational anabelian Grothendieck conjecture] is given by the fully-faithfulness of the functor from a certain category of fields to a category of profinite groups with inner automorphism indeterminacy obtained by forming the absolute Galois groups of fields. In this research area, the birational anabelian Grothendieck conjecture has been studied intensively and proved affirmatively in many situations [cf. for instance, see [4], [7], [30], [35], [39], [42], [44], [45], [46], [58]].

In the present paper, we generalize various results concerning the birational anabelian Grothendieck conjecture. In order to state earlier results and our main results, we first introduce various notations. For any field F and any algebraic variety [i.e., a separated, of finite type, and geometrically integral scheme] Z over F, we shall fix an algebraic closure $\overline{F}$ of F and write F^{sep} for the separable closure of F in $\overline{F}$; F^{pf} for the perfection of F in $\overline{F}$ [where if F is of characteristic 0, then $F^{\mathrm{pf}} \stackrel{\mathrm{def}}{=} F$]; $G_F \stackrel{\mathrm{def}}{=} \mathrm{Gal}(F^{\mathrm{sep}}/F)$; $K(Z)$ for the function field of Z;

$$\Delta_{K(Z)}$$

for the kernel of the natural surjection $G_{K(Z)} \twoheadrightarrow G_F$. For each algebraic variety Z over a field F and each nonempty set of prime numbers Σ, we shall write

$$G^{(\Sigma)}_{K(Z)} \stackrel{\mathrm{def}}{=} G_{K(Z)}/\mathrm{Ker}(\Delta_{K(Z)} \twoheadrightarrow \Delta^{\Sigma}_{K(Z)}),$$

where $(-)^{\Sigma}$ denotes the maximal pro-Σ quotient of $(-)$. In particular, we have a natural exact sequence of profinite groups

$$1 \longrightarrow \Delta^{\Sigma}_{K(Z)} \longrightarrow G^{(\Sigma)}_{K(Z)} \longrightarrow G_F \longrightarrow 1.$$

For each pair of fields F_1, F_2 and each pair of profinite groups G_1, G_2, we shall write

$$\mathrm{Isom}(F_1, F_2)$$

for the set of isomorphisms $F_1 \xrightarrow{\sim} F_2$ of fields;

$$\mathrm{Isom}(F_1^{\mathrm{pf}}, F_2^{\mathrm{pf}})/\mathrm{Fr}^{\mathbb{Z}}$$

for the set of isomorphisms $F_1^{\mathrm{pf}} \xrightarrow{\sim} F_2^{\mathrm{pf}}$ of fields, considered up to compositions with the integral powers of the absolute Frobenius automorphism in the case where F_1 and F_2 are of positive characteristic;

$$\mathrm{OutIsom}(G_1, G_2)$$

for the set of outer isomorphisms $G_1 \xrightarrow{\sim} G_2$ of profinite groups. For any algebraic varieties Z_1, Z_2 over a field F and any nonempty set of prime numbers Σ, we shall write

$$\mathrm{Isom}_F(K(Z_1)^{\mathrm{pf}}, K(Z_2)^{\mathrm{pf}})$$

for the set of F-isomorphisms $K(Z_1)^{\mathrm{pf}} \xrightarrow{\sim} K(Z_2)^{\mathrm{pf}}$ of fields;

$$\mathrm{Isom}_{G_F}(G_{K(Z_1)}^{(\Sigma)}, G_{K(Z_2)}^{(\Sigma)})/\mathrm{Inn}(\Delta_{K(Z_2)}^{\Sigma})$$

for the set of isomorphisms $G_{K(Z_1)}^{(\Sigma)} \xrightarrow{\sim} G_{K(Z_2)}^{(\Sigma)}$ of profinite groups that lie over the identity automorphism of G_F with respect to the natural surjective homomorphisms $G_{K(Z_1)}^{(\Sigma)} \twoheadrightarrow G_F$ and $G_{K(Z_2)}^{(\Sigma)} \twoheadrightarrow G_F$, considered up to compositions with inner automorphisms arising from elements of $\in \Delta_{K(Z_2)}^{\Sigma}$. For any algebraic varieties Z_1, Z_2 over fields F_1, F_2, respectively, and any nonempty set of prime numbers Σ, we shall write

$$\mathrm{Isom}(K(Z_1)/F_1, K(Z_2)/F_2)$$

for the set of isomorphisms $K(Z_1) \xrightarrow{\sim} K(Z_2)$ of fields that induce isomorphisms $F_1 \xrightarrow{\sim} F_2$ via the natural embeddings $F_1 \hookrightarrow K(Z_1)$, $F_2 \hookrightarrow K(Z_2)$;

$$\mathrm{OutIsom}(G_{K(Z_1)}^{(\Sigma)}/G_{F_1}, G_{K(Z_2)}^{(\Sigma)}/G_{F_2})$$

for the set of outer isomorphisms $G_{K(Z_1)}^{(\Sigma)} \xrightarrow{\sim} G_{K(Z_2)}^{(\Sigma)}$ of profinite groups that induce outer isomorphisms $G_{F_1} \xrightarrow{\sim} G_{F_2}$ via the natural surjective homomorphisms $G_{K(Z_1)}^{(\Sigma)} \twoheadrightarrow G_{F_1}$ and $G_{K(Z_2)}^{(\Sigma)} \twoheadrightarrow G_{F_2}$.

After the pioneering and celebrated work of K. Uchida for the function fields of algebraic curves [i.e., 1-dimensional algebraic varieties] over finite fields [cf. [58], Theorem], there are many progresses in birational anabelian geometry. For instance, the following results are known:

Theorem 0.1 (Mochizuki, a consequence of [31], Theorem 4.12)**.** *Let p be a prime number; Σ a set of prime numbers that contains p; K a generalized sub-p-adic field, i.e., a field isomorphic to a subfield of a finitely generated extension field of the completion of the maximal unramified extension of the field of p-adic numbers $\mathbb{Q}_p$ [cf. [31], Definition 4.11]; X, Y algebraic curves over K. Then the natural map*

$$\mathrm{Isom}_K(K(Y), K(X)) \longrightarrow \mathrm{Isom}_{G_K}(G_{K(X)}^{(\Sigma)}, G_{K(Y)}^{(\Sigma)})/\mathrm{Inn}(\Delta_{K(Y)}^{\Sigma})$$

is bijective.

Recall that a field F of characteristic 0 is *Kummer-faithful* if, for each semi-abelian variety A over a finite extension field E of F, every divisible element of $A(E)$ is trivial [or equivalently, the Kummer map $A(E) \to H^1(G_E, \pi_1(A \times_E \overline{E}))$ is injective]. Then:

Theorem 0.2 (Mochizuki, [35], Theorem 1.11)**.** *Let K, L be Kummer-faithful fields of characteristic 0; X, Y algebraic curves over K, L, respectively. Then the natural map*

$$\mathrm{Isom}(K(Y)/L, K(X)/K) \longrightarrow \mathrm{OutIsom}(G_{K(X)}/G_K, G_{K(Y)}/G_L)$$

is bijective.

Recall that a field F is

- *TKND* [i.e., "torally Kummer-nondegenerate"] if $F_{\mathrm{div}} \subseteq \overline{F}$ is an infinite field extension, where F_{div} ($\subseteq \overline{F}$) denotes the field obtained by adjoining the divisible elements of the multiplicative groups of finite extension fields of F to the prime field $\subseteq F$;
- *AVKF* [i.e., "abelian variety Kummer-faithful"] if, for each abelian variety A over a finite extension field E of F, every divisible element of $A(E)$ is trivial [or equivalently, the Kummer map $A(E) \to H^1(G_E, \pi_1(A \times_E \overline{E}))$ is injective];
- *TKND-AVKF* if F is both TKND and AVKF.

In particular, the notion of TKND-AVKF-field may be regarded as a generalized notion of Kummer-faithful-field. Then:

Theorem 0.3 (T., [53], Theorem A; [53], Corollary B)**.**

(i) Let K be a TKND-AVKF-field of characteristic 0; X, Y algebraic curves over K. Suppose that there exists a surjective homomorphism $K^\times \twoheadrightarrow \mathbb{Z}$. Then the natural map

$$\mathrm{Isom}_K(K(Y), K(X)) \longrightarrow \mathrm{Isom}_{G_K}(G_{K(X)}, G_{K(Y)})/\mathrm{Inn}(\Delta_{K(Y)})$$

is bijective.

(ii) Let p be a prime number. Write L ($\subseteq \overline{\mathbb{Q}}$) for the field obtained by adjoining all roots of p to $\mathbb{Q}$. Let K be a subfield of a finitely generated extension field of L; X, Y algebraic curves over K. Then the natural map

$$\mathrm{Isom}_K(K(Y), K(X)) \longrightarrow \mathrm{Isom}_{G_K}(G_{K(X)}, G_{K(Y)})/\mathrm{Inn}(\Delta_{K(Y)})$$

is bijective.

Theorem 0.4 (Neukirch-Uchida-Pop, [57], Theorem; [58], Theorem; the discussions in [39], [42]; [41], Theorem 1.3)**.** *Let K, L be finitely generated infinite fields. Then the natural map*

$$\mathrm{Isom}(L^{\mathrm{pf}}, K^{\mathrm{pf}})/\mathrm{Fr}^{\mathbb{Z}} \longrightarrow \mathrm{OutIsom}(G_K, G_L)$$

is bijective.

Before stating our main results, we add various remarks concerning the above results.

Remarks

(a) With regard to Theorem 0.1, if K is a sub-p-adic field, i.e., a field isomorphic to a subfield of a finitely generated extension field of $\mathbb{Q}_p$, then the stronger relative Hom-version for the function fields of algebraic varieties over K of arbitrary dimension is also obtained by S. Mochizuki [cf. [30], Theorem 17.1]. On the other hand, we still do not know

whether or not the relative Hom/semi-absolute/absolute analogue of Theorem 0.1 holds true.

(b) With regard to relations between Theorems 0.1, 0.2, we note that every sub-p-adic field is generalized sub-p-adic and Kummer-faithful [cf. [35], Remark 1.5.4, (i)]. On the other hand, there exists a generalized sub-p-adic field (respectively, Kummer-faithful field) that is not a Kummer-faithful field (respectively, generalized sub-p-adic field) [cf. [53], Proposition 1.7, (ii)]. Furthermore, Theorem 0.2 is a result in the semi-absolute setting, but we do not know whether or not the semi-absolute analogue of Theorem 0.1 holds true, as stated in (a). Interestingly, one may observe that the relative analogue of Theorem 0.2 does not hold in general [cf. see (c) below].

(c) With regard to Theorem 0.3, we note that there are many examples of TKND-AVKF-fields of characteristic 0 that are not Kummer-faithful or generalized sub-p-adic. For instance, any finitely generated extension field of the maximal abelian extension field of the field of rational numbers $\mathbb{Q}$ is TKND-AVKF [cf. [53], Theorem 1.8]. In the situation where the proofs of many anabelian results for curves depend on weight arguments based on the Weil conjecture or p-adic Hodge theory, hence on the existence of highly nontrivial cyclotomic characters, the author's motivation was to obtain the strong form of an anabelian Grothendieck conjecture-type result for curves over fields whose cyclotomic characters totally vanish. Here, we note that many results concerning the weak form of anabelian phenomena [i.e., reconstructions of *isomorphism classes* of geometric objects under considerations from their fundamental groups] for hyperbolic curves over fields whose associated cyclotomic characters vanish have already been obtained by various researchers so far [cf. for instance, see [24], [47], [48], [49], [50], [51], [52], [55]]. Moreover, in a joint work with Y. Hoshi and S. Mochizuki, we proved an anabelian Grothendieck conjecture-type result for higher dimensional configuration spaces associated to hyperbolic curves over TKND-AVKF-fields $\subseteq \overline{\mathbb{Q}}$. In particular, as stated in the Introduction of [53], it is natural to ask

> *whether or not the various anabelian geometric results that have been obtained so far may be generalized to the results in the case where the base fields are TKND-AVKF-fields.*

This was the background of Theorem 0.3. On the other hand, at the time of writing of [53], the author could not prove the semi-absolute version of Theorem 0.3 as opposed to the success of verification of the relative version as in Theorem 0.3. Moreover, in the relative situation, the author imposed a mild assumption on the base field to control the cyclotomic rigidity phenomena. At the time of writing of [53], the author did not know whether or not this assumption may be dropped even if we assume that the base field is Kummer-faithful. However, due to the construction of Kummer-faithful fields of characteristic 0 whose absolute Galois groups are abelian by T. Asayama [cf. [2], Corollary 1.5], one may observe that this type of assumption may not be dropped. Indeed, one may prove that the relative anabelian Grothendieck conjecture-type results imply that the absolute Galois groups of base fields are center-free. Finally, we remark that Theorem 0.3, (ii) follows from Theorem 0.3, (i), and the highly nontrivial finiteness theorem concerning the torsion points of abelian varieties valued in large arithmetic fields [cf. [22], Theorem 1; [54], Theorem A], together with the easily verified properties concerning the multiplicative groups of such large arithmetic fields [cf. [27], Theorem 2; [53], Proposition 4.9].

(d) We discuss briefly the role of TKND-AVKF-property of the base fields in the proof of Theorem 0.3. To reconstruct the field structure of function fields, after synchronizing various cyclotomes, we first construct [a certain subquotient of] the multiplicative group of function field via Kummer theory associated to the étale fundamental groups of curves that arise [as quotients] from the absolute Galois groups of function fields. In this procedure, to capture the information of principal divisors in the group of Weil divisors, we need the AVKF-property of base fields. Indeed, one may observe that the obstruction lies in the divisible group of points [valued in a finite extension field of a base field] of the Jacobian of a smooth, proper curve under consideration. Moreover, to decomplete [in the sense of Kummer completion] the base field portion, we need a technique of Galois evaluations. This technique works efficiently under the assumption that the base field satisfies the TKND-property. The TKND-property is also related to the reconstruction of additive structure of function fields from the viewpoint of quasi-rational functions developed in [53], §3, i.e., certain set-theoretic functions on the set of closed points of a smooth curve over an algebraically closed field associated to a subfield of the base algebraically closed field that are "not so far" from the rational functions.

(e) With regard to Theorem 0.4, the 1-dimensional case [in the sense of Kronecker dimension] is proved by K. Uchida after significant contributions by J. Neukirch, K. Iwasawa, and M. Ikeda. The higher dimensional case is proved by F. Pop. Although the author could not access the original preprint that announces Theorem 0.4 in this form beyond the curve case [cf. [39], Theorem 1], the strategy of proof is summarized in the survey paper [41] by Pop. Moreover, it appears to the author that much stronger results are already proved. For instance, in the positive characteristic case, Pop proved that anabelian results for the function fields of algebraic varieties over $\overline{\mathbb{F}}_p$ of dimension ≥ 2 [cf. [44], Theorem 1, (3)]. Another way to prove this result in characteristic 0 is combining *Observation* in [39], p. 146, together with a Mochizuki's relative anabelian result [cf. [30], Theorem 17.1]. In the higher dimensional case in characteristic 0, we generalize Theorem 0.4 in the case of more general base fields in the final section of the present paper.

Next, in order to state our main results concerning generalizations of the above results, we introduce further several notations and notions on fields. For each nonempty set of prime numbers Σ and each field F, we shall write

$$\Sigma'$$

for the set of prime numbers $\notin \Sigma$;

$$F^{\times \Sigma^\infty} \stackrel{\text{def}}{=} \bigcap_m (F^\times)^m,$$

where m ranges over the positive integers whose prime factors are contained in Σ;

$$F_{\Sigma\text{-div}} \stackrel{\text{def}}{=} \bigcup_{F \subseteq E} F_{\text{prm}}(E^{\times \Sigma^\infty})(\subseteq \overline{F}),$$

where $F_{\text{prm}} \subseteq F$ denotes the prime subfield; $F \subseteq E$ ranges over the finite extension fields $\subseteq \overline{F}$. Then one may introduce a pro-Σ-variant of the notion of TKND-AVKF-field as follows:

- If $F_{\Sigma\text{-div}} \subseteq \overline{F}$ is an infinite field extension, then we shall say that F is a Σ-*TKND-field*.
- If F satisfies the following condition, then we shall say that F is a Σ-*AVKF-field*:

  Let A be an abelian variety over a finite extension field E of F. Then every Σ-divisible element of $A(E)$ is a Σ'-torsion element.

- If F is both a Σ-TKND-field and a Σ-AVKF-field, then we shall say that F is a Σ-*TKND-AVKF-field*.

Here, we note that if Σ is the set of all prime numbers, then the notion of Σ-TKND-AVKF-field coincides with the notion of TKND-AVKF-field. Note also that one may construct many examples of Σ-TKND-AVKF-field as in the case of TKND-AVKF-field. Indeed:

- For each prime number p, any subfield of a mixed/positive characteristic higher local field with finite final residues of characteristic p is a $\{p\}$-TKND-AVKF-field [cf. Proposition 1.12, (i)]. In particular, since a finitely generated extension field of a mixed/positive characteristic higher local field with finite final residues of characteristic p is a subfield of a mixed/positive characteristic higher local field with finite final residues of characteristic p [cf. Remark 1.11.1], one may observe that any sub-p-adic field is a $\{p\}$-TKND-AVKF-field [cf. Proposition 1.12, (ii)].
- For each set of prime numbers Σ, any subfield of the maximal Kummer extension of a finitely generated field of characteristic 0 is a Σ-TKND-AVKF-field [cf. Corollary 1.20, (i); Remark 1.20.1 for a generalization].
- Let l be a prime number; F an l-quasi-finite field [i.e., a perfect field such that $G_F^l \cong \mathbb{Z}_l$] of positive characteristic $\neq l$ that is algebraic over the prime field; K a subfield of the maximal prime-to-l extension field of the perfection of a finitely generated extension field of F. Suppose that K is not algebraic over the prime field. Then K is an $\{l\}$-TKND-AVKF-field [cf. Corollary 1.20, (ii)].

Then our first main result is the following [cf. Theorem 3.7; Corollary 3.13]:

Theorem A. *The following hold:*

(i) Let Σ be a nonempty set of prime numbers; K a perfect Σ-TKND-AVKF-field; $\Sigma^\dagger$ a set of prime numbers that contains Σ; X, Y algebraic curves over K. Suppose that:

> *There exists a nontrivial homomorphism $K^\times \to \mathbb{Z}_{(\Sigma')}$, where $\mathbb{Z}_{(\Sigma')}$ denotes the localization of $\mathbb{Z}$ by the multiplicatively closed subset determined by Σ'.*

Then the natural map

$$\mathrm{Isom}_K(K(Y)^{\mathrm{pf}}, K(X)^{\mathrm{pf}}) \longrightarrow \mathrm{Isom}_{G_K}(G_{K(X)}^{(\Sigma^\dagger)}, G_{K(Y)}^{(\Sigma^\dagger)})/\mathrm{Inn}(\Delta_{K(Y)}^{\Sigma^\dagger})$$

is bijective.

(ii) Let K be a field satisfying one of the following conditions:

(a) There exists a finitely generated field K_0 of characteristic 0 such that K is isomorphic to a subfield of the maximal Kummer extension field of K_0.

(b) There exists a mixed characteristic higher local field K_0 with finite final residues of characteristic $\in \Sigma$ such that K is isomorphic to a subfield of K_0.

(c) K is not algebraic over the prime field. Moreover, there exist a prime number $l \in \Sigma$ and an l-quasi-finite field F of positive characteristic $\neq l$ that is algebraic over the prime field such that K is isomorphic to a perfect subfield of the maximal pro-prime-to-l extension field of the perfection of a finitely generated extension field of F.

Let X, Y be algebraic curves over K. Then the natural map

$$\mathrm{Isom}_K(K(Y)^{\mathrm{pf}}, K(X)^{\mathrm{pf}}) \longrightarrow \mathrm{Isom}_{G_K}(G_{K(X)}^{(\Sigma)}, G_{K(Y)}^{(\Sigma)})/\mathrm{Inn}(\Delta_{K(Y)}^{\Sigma})$$

is bijective.

It is clear that:

- Theorem A, (i), may be regarded as a geometrically pro-Σ variant of Theorem 0.3, (i).
- Theorem A, (ii), under condition (a), may be regarded as a generalization of Theorem 0.3, (ii) from the viewpoints of both base fields and Σ.

With regard to Theorem A, (i), the fundamental difficulty lies in the evident fact that we may only resort to the Σ-portions of groups of points of Jacobians. In particular, there indeed exist inevitable obstructions for being principal divisors, and one may not access directly to the group of Kummer classes as opposed to the full profinite situation discussed in Remarks, (d). On the other hand, the Σ-AVKF-property assures that the group of obstructions is a Σ'-torsion group. Therefore, one may construct a certain saturated version of the group of Kummer classes. Here, it would be important to mention that this saturated object may be regarded as a certain group-theoretic analogue of the usual ring-theoretic operation of perfection of a positive characteristic scheme. Indeed, in the case where the characteristic $\mathrm{char}(K)$ is positive, and Σ is the set of prime numbers $\neq \mathrm{char}(K)$, this saturated object coincides with the group of Kummer classes of the usual perfection of function field. In other words, one may regard this saturated object as a Kummer-theoretic perfection of the multiplicative group of a function field that is not necessarily of positive characteristic. However, since our goal is to reconstruct field structures, we need to reduce this indeterminacy arising from saturations ultimately [especially in the case of characteristic 0]. This is implemented by considering subtle tensions with the additive structures. Here, we recall the definition of quasi-rational function:

> Let K be an algebraically closed field; $k \subseteq K$ a subfield; X a smooth proper curve over K;
>
> $$f \in \mathrm{Fn}(X(K), K \cup \{\infty\})$$
>
> a set-theoretic function on the set of K-valued points of X whose values in $K \cup \{\infty\}$. Then we shall say that f is a *quasi-rational function* on X associated to k if there exist elements
>
> $$\phi_f \in \mathrm{Fn}(X(K), k^\times), \quad g \in K(X)$$
>
> such that $f = \phi_f \cdot g$.

In fact, the language of quasi-rational function introduced in [53], §3, is suitable to discuss our saturated Kummer classes. Indeed, one may observe that roots of quasi-rational functions are also quasi-rational functions in suitable coverings. In light of this observation, together with the discussions in [53], §3, one may prove the following algebraicity criterion [cf. Proposition 3.5]:

> Let K be an algebraically closed field; $k \subseteq K$ a subfield such that the extension is of infinite degree; X a smooth proper curve over K;
>
> $$f \in \mathrm{Fn}(X(K), K \cup \{\infty\})$$
>
> a set-theoretic function. Suppose that:
>
> - A positive integer power of f is a quasi-rational function on X associated to k such that the rational function determined [up to multiplication by a constant function] by the positive integer power of f is nonconstant.
> - For each $a \in K$, there exists a positive integer n_a such that the set-theoretic function $(f+a)^{n_a}$ is a quasi-rational function on X associated to k.
>
> Then f is a rational function on a finite purely inseparable covering of X, where we note that $K(X)^{\mathrm{pf}}$ may be regarded as a subset of $\mathrm{Fn}(X(K), K \cup \{\infty\})$ in a natural way.

Then, by applying this algebraicity criterion, one may deduce that a natural set-theoretic bijection between the respective set-theoretic function spaces obtained by considering cuspidal inertia subgroups [that is compatible with an isomorphism between the respective saturated Kummer classes in a suitable sense] maps the [usual perfection of] group of rational functions $\neq 0$ to the [usual perfection of] group of rational functions $\neq 0$. In this procedure, we also need to apply our displayed assumption in the statement of Theorem A, (i), to discuss a certain weak synchronization of cyclotomes [cf. Proposition 2.14, (i)], which is used to assure a suitable compatibility of the set-theoretic bijection between the respective set-theoretic function spaces with an isomorphism between the respective groups of saturated Kummer classes mentioned above. As discussed in Remarks, (c) above, one may not drop this assumption.

With regard to Theorem A, (ii), under condition (a), the assertion is proved by combining Theorem A, (i), with the following results [cf. Theorems 1.17, 3.11]:

Theorem B. *Let K be a subfield of the maximal Kummer extension of a finitely generated field of characteristic 0; A an abelian variety over K; $K \subseteq L$ $(\subseteq \overline{K})$ a finite field extension. Then $A(L)_{\mathrm{tor}}$ is finite.*

Theorem C. *Let F be a finitely generated field of characteristic 0. Write E $(\subseteq \overline{\mathbb{Q}})$ for the maximal Kummer extension field of F; $R_F \subseteq E^\times$ for the subgroup consisting of [divisible] elements of $E^\times$ whose some positive integer power is contained in F [so $\mu(\overline{F}) \subseteq R_F \subseteq E^{\times\infty}$]. Then, for any subfield $M \subseteq E$, it holds that $M^\times / M \cap R_F$ is a free abelian group.*

Theorem B is an immediate consequence of a recent result on the finiteness of torsion points of abelian varieties obtained by D. Lombardo and T. Szamuely [cf. [25], Theorem 1.1]. On the other hand, if we recall that the divisible elements of the multiplicative groups of abelian extensions of number fields are roots of unity, then Theorem C may be regarded as a certain generalization of W. May's result on the freeness of multiplicative groups up to torsion of abelian extensions of number fields, where we note that the proof of Theorem C depends on May's techniques.

It is also clear that:

- Theorem A, (ii), under condition (b), may be regarded as a certain type of a generalization of Theorem 0.1. [However, this does not contain Theorem 0.1 as a special case.]
- In light of [44], Theorem 1, (3), mentioned above, one may prove a result similar to Theorem A, (ii), under condition

  (c') K is not algebraic over the prime field. Moreover, there exist a prime number $l \in \Sigma$ and a field F of positive characteristic $\neq l$ that is algebraic over the prime field such that K is isomorphic to a perfect subfield of the perfection of a finitely generated extension field of F.

In light of conditions (c), (c'), it may be natural to ask

> whether or not our assumption that F is l-quasi-finite in Theorem A, (ii), under condition (c) may be dropped,

or more generally,

> to which extent pro-l anabelian Grothendieck conjecture-type results hold for certain infinite large algebraic extension fields [such as the maximal pro-prime-to-l extension fields] of function fields over $\overline{\mathbb{F}}_p$ of dimension ≥ 2, where p is a prime number $\neq l$.

However, at the time of writing of the present paper, the author does not know any partial answers to these questions. On the other hand, in a joint paper with A. Minamide and K. Sawada in preparation [cf. [29]], we discuss a closely related anabelian phenomenon for a certain large class of infinite algebraic extension fields of $\mathbb{Q}$.

With regard to Theorem A, (ii), under condition (b), we recall that the proof of Theorem 0.1 depends heavily on p-adic Hodge theoretic techniques. On the other hand, it would be interesting to observe that a geometrically pro-p anabelian result for curves over p-adic fields is proved by a Kummer-theoretic technique. In a subsequent paper [cf. [56]], we introduce a certain new class of fields that contains generalized sub-p-adic fields and prove a full generalization of Theorem 0.1.

Next, by applying Theorem A, together with Theorem 0.4 [that is applied to the base fields], one may obtain the following absolute anabelian result [cf. Theorem 3.15]:

Theorem D. *Let p be a prime number; X, Y algebraic curves over the perfections of finitely generated transcendental fields of characteristic p; Σ a nonempty set of prime numbers such that $\Sigma \neq \{p\}$, and Σ' is also nonempty. Then the natural map*

$$\mathrm{Isom}(K(Y), K(X)) \longrightarrow \mathrm{OutIsom}(G^{(\Sigma)}_{K(X)}, G^{(\Sigma)}_{K(Y)})$$

[cf. Remark 3.15.1] is bijective.

Interestingly, one may not drop the assumption that Σ' is nonempty. Indeed, Pop's theorem implies that, up to compositions with iterates of the absolute Frobenius automorphism, any outer isomorphism $G_{K(X)} \xrightarrow{\sim} G_{K(Y)}$ arises from a unique isomorphism $K(Y)^{\mathrm{pf}} \xrightarrow{\sim} K(X)^{\mathrm{pf}}$. On the other hand, one may observe that, in general, there exists an isomorphism $K(Y)^{\mathrm{pf}} \xrightarrow{\sim} K(X)^{\mathrm{pf}}$ that does not preserve the respective base fields of X, Y. In particular, this isomorphism does not descend to an isomorphism $K(Y) \xrightarrow{\sim} K(X)$. With regard to the assumption that $\Sigma \neq \{p\}$, it would be important to note that there exists no affirmative geometrically pro-p anabelian result for geometric objects over fields of characteristic p. In particular, at the time of writing of the present paper, the author does not know how to eliminate the assumption that $\Sigma \neq \{p\}$.

Next, in order to state our result concerning the semi-absolute version of birational anabelian Grothendieck conjecture, we introduce the following condition on Σ:

> Let k be a field. Then we shall say that the pair (k, Σ) is of *finite torsion type* if, for each abelian variety A over a finite extension field $k^{\dagger}$ of k, the cardinality of group of Σ'-torsion elements of $A(k^{\dagger})$ is *finite*.

Note that one may regard the above condition as a condition concerning an "ampleness" of Σ in some sense. Then our second main theorem is the following [cf. Theorem 4.15; Corollary 4.16]:

Theorem E. *The following hold:*

(i) Let X, Y be smooth, proper curves over Σ-TKND-AVKF infinite perfect fields k_X, k_Y of characteristic $\notin \Sigma$; $\Sigma^{\dagger}$ a set of prime numbers that contains Σ. Suppose that the pairs (k_X, Σ) and (k_Y, Σ) are of finite torsion type. Then the natural map

$$\mathrm{Isom}(K(Y)/k_Y, K(X)/k_X) \longrightarrow \mathrm{OutIsom}(G_{K(X)}^{(\Sigma^{\dagger})}/G_{k_X}, G_{K(Y)}^{(\Sigma^{\dagger})}/G_{k_Y})$$

is bijective.

(ii) Let k_X, k_Y be fields that belong to one of the following types of fields:

(a) A field isomorphic to a subfield of the maximal Kummer extension field of a finitely generated field of characteristic 0.

(b) A field isomorphic to a subfield of a mixed characteristic higher local field with finite final residues of characteristic $\in \Sigma$.

(c) A perfect field isomorphic to a subfield of the perfection of a finitely generated extension field of a quasi-finite field of positive characteristic that is algebraic over the prime field.

Let X, Y be algebraic curves over k_X, k_Y. If k_X or k_Y is a field of type (c), then we suppose that Σ' is finite. Then the natural map

$$\mathrm{Isom}(K(Y)/k_Y, K(X)/k_X) \longrightarrow \mathrm{OutIsom}(G_{K(X)}^{(\Sigma)}/G_{k_X}, G_{K(Y)}^{(\Sigma)}/G_{k_Y})$$

is bijective.

Note that Theorem E, (i), may be regarded as a complete generalization of Theorem 0.2 and an affirmative resolution of Question 1 posed in [53], Remark 4.7.2. With regard to the proof of Theorem E, (i), one fundamental difficulty lies in the evident fact that the field structures of base fields may not be regarded as input datum. Recall that, in the proof of Theorem 0.3 or Theorem A, we apply algebraicity criteria for quasi-rational functions. However, these criteria can be used only after reconstructing the field structures of base fields. To overcome this difficulty, we combine Saïdi-Tamagawa's lemmas in [46] [or alternatively, the proof of Uchida's lemma in the case where $\Sigma = \mathfrak{Primes}$] with the fundamental theorem of projective geometry to reconstruct the field structures of base fields [cf. Propositions 4.11; 4.12, (ii)]. On the other hand, at the time of writing of the present paper, the author does not know

whether or not our assumption on Σ may be eliminated.

This is due to an essentially similar reason to the reason why condition (ϵ_X) may not be eliminated in [46], Theorem C. Here, we also note that:

- The sophisticated version of an algebraicity criterion for quasi-rational functions mentioned above is also applied in the proof of Theorem E, (i) [cf. Proposition 4.12, (iii)].
- The problem of cyclotomic rigidity does not appear in the semi-absolute setting. Indeed, since the base fields are different, there is no need to synchronize/identify respective cyclotomes of the domain and codomain function fields from the scratch.

As an application of Theorem E, (ii), one may obtain the following Neukirch-Uchida-type absolute anabelian results [cf. Corollary 4.17]:

Corollary F. *Let X, Y be smooth proper curves over fields. Then the following hold:*

(i) Suppose that the base fields of X and Y are isomorphic to subfields of the maximal Kummer extension fields of number fields, and $\Sigma \subsetneq \mathfrak{Primes}$. Then the natural map

$$\mathrm{Isom}(K(Y), K(X)) \longrightarrow \mathrm{OutIsom}(G_{K(X)}^{(\Sigma)}, G_{K(Y)}^{(\Sigma)})$$

is bijective.

(ii) Suppose that the base fields of X and Y are positive characteristic quasi-finite fields that are algebraic over the respective prime fields, and Σ' is finite. Then the natural map

$$\mathrm{Isom}(K(Y), K(X)) \longrightarrow \mathrm{OutIsom}(G_{K(X)}^{(\Sigma)}, G_{K(Y)}^{(\Sigma)})$$

is bijective.

It appears to the author that Corollary F, (i), may be regarded as a strong evidence of the existence of anabelian phenomena for objects over sufficiently large fields such as $\mathbb{Q}^{\mathrm{ab}}$. It would be interesting to pursue the hyperbolic curve analogues of Theorem E, (ii), and Corollary F. Note that Corollary F, (ii), may be regarded as a certain generalization of [46], Theorem C from the viewpoint of base field. With regard to Corollary F, (i), in fact, the assumption that $\Sigma \subsetneq \mathfrak{Primes}$ may be eliminated [cf. Theorem G below]. For achieving this, we need to reconstruct the absolute Galois groups of base fields in the case where $\Sigma = \mathfrak{Primes}$, which is much harder to prove compared to the case where $\Sigma \subsetneq \mathfrak{Primes}$. This is proved during the verification of Theorem G below. On the other hand, in light of Theorem E, (ii), under condition (b), it is natural to ask whether or not a p-adic local analogue of Corollary F holds. In fact, one may prove such an assertion affirmatively, and its proof will be included in an ongoing joint work with E. Lepage in a more general p-adic context.

Finally, we introduce our third main theorem concerning the absolute version of birational anabelian Grothendieck conjecture or a Neukirch-Uchida-Pop-type result [cf. Theorem 5.21]:

Theorem G. *Let Σ be a nonempty set of prime numbers; X, Y algebraic varieties of positive dimension over subfields of the maximal Kummer extensions of number fields. Then the natural map*

$$\mathrm{Isom}(K(Y), K(X)) \longrightarrow \mathrm{OutIsom}(G_{K(X)}^{(\Sigma)}, G_{K(Y)}^{(\Sigma)})$$

is bijective.

Note that Theorem G may be regarded as a complete generalization of Theorem 0.4 in characteristic 0. With regard to the proof of Theorem G, by resorting to a Koenigsmann's result on constructing valuations, together with some group-theoretic considerations, we first reconstruct the absolute Galois groups of base fields. At this point, one may eliminate the assumption that $\Sigma \subsetneq \mathfrak{Primes}$ in the statement of Corollary

F, (i). Next, by using techniques of Engler-Koenigsmann-Nogueira and Pop, we reconstruct geometric divisorial subgroups. After that, in light of the already established 2-dimensional case [in the sense of Kronecker dimension] as above, together with a highly nontrivial global theory due to Pop, we proceed by induction on the dimension. To complete induction, we use the fundamental theorem of projective geometry again. It would be worth mentioning that the 1-dimensional analogue [again, in the sense of Kronecker dimension] of Theorem G should not hold. Indeed, in light of Shafarevich's conjecture, the open subgroups of the absolute Galois group of the maximal abelian extension of $\mathbb{Q}$ can be isomorphic, hence, may not hold the information of their field structures. It appears to the author that this situation is very similar to the situation of Pop's birational anabelian result for algebraic varieties of dimension ≥ 2 over algebraic closures of the prime fields in positive characteristic. Indeed, it is well-known that the absolute Galois groups of 1-dimensional function fields over algebraically closed fields are free profinite groups [cf. [16], Theorem 4.4; [39], Corollary of Theorem A].

The present paper is organized as follows. In §1, we first introduce the notion of pro-Σ variant of TKND-AVKF-field as above. Then we investigate basic properties of these fields with respect to various fundamental field-theoretic operations and give some examples and counter-examples. Examples we discuss include mixed/positive characteristic higher local fields and subfields of the maximal Kummer extensions of finitely generated fields of characteristic 0. In particular, Theorem B is proved in this section. In §2, we reconstruct, from the data of the geometrically pro-Σ quotient of the absolute Galois group of function field of a smooth curve over a perfect Σ-TKND-AVKF-field, together with the natural surjective homomorphism onto the absolute Galois group of the base field, a certain group-theoretic perfection of the image of the multiplicative group of function field via the Kummer map. The discussion that appears in this section may be regarded as a geometrically pro-Σ variant of the discussion in [53], §2. In §3, after preparing the function space associated to cuspidal inertia subgroups/Galois sections and a certain sophisticated version of an algebraicity criterion for quasi-rational functions on smooth curves over algebraically closed fields, by combining with the reconstruction results obtained in the previous sections, we prove Theorem A, (i), (ii). Theorem C is also proved in this section. Moreover, in light of a suitable consideration on Frobenius twists, by applying Theorem A, together with Theorem 0.4, we prove Theorem D. In §4, by applying the semi-absolute reconstruction of perfectized group of Kummer classes in §2, together with various techniques to reconstruct additive structures mentioned above, we prove Theorem E and Corollary F. In §5, by resorting to the techniques due to Koenigsmann and Pop, together with several group-theoretic arguments and the fundamental theorem of projective geometry, we apply Theorem E to prove Theorem G by induction on the Kronecker dimension.

Notations and Conventions

Sets: Let A, B be sets. Then we shall write $\mathrm{Fn}(A, B)$ for the set of maps from A to B.

Numbers: The notation $\mathfrak{Primes}$ will be used to denote the set of prime numbers. For each subset $\Sigma \subseteq \mathfrak{Primes}$, we shall write $\Sigma' \stackrel{\text{def}}{=} \mathfrak{Primes} \setminus \Sigma$ and refer to an integer whose prime factors are elements $\in \Sigma$ as a Σ-*integer*. We shall refer to a finite extension field of $\mathbb{Q}$ as a *number field*. The notation $\widehat{\mathbb{Z}}$ will be used to denote the profinite completion of $\mathbb{Z}$. If p is a prime number, then $\mathbb{Q}_p$ will be used to denote the field of p-adic numbers; $\mathbb{Q}_p^{\mathrm{ur}}$ will be used to denote the maximal unramified extension field of $\mathbb{Q}_p$. We shall refer to a finite extension field of $\mathbb{Q}_p$ as a *p-adic local field*.

Groups: Let M be an abelian group; $\Sigma \subseteq \mathfrak{Primes}$ a nonempty subset. Then, for each $x \in M$, we shall say that x is a Σ-*torsion* element if there exists a Σ-integer n such that $nx = 0$. For each $x \in M$, we shall say that x is Σ-*divisible* if, for each prime number $l \in \Sigma$, it holds that x is an l-divisible element in M.

For each subgroup $N \subseteq M$, we shall refer to the subgroup of M consisting of the elements $y \in M$ that admits a Σ-integer n such that $ny \in N$ as the *Σ-saturation* of N in M.

Fields: Let F be a field; n a positive integer; p a prime number. Then we shall write $\mathrm{char}(F)$ for the characteristic of F; $F_{\mathrm{prm}} \subseteq F$ for the prime subfield; $\overline{F}$ for the algebraic closure [determined up to isomorphisms] of F; F^{pf} ($\subseteq \overline{F}$) for the perfection of F [so $F = F^{\mathrm{pf}}$ in the case where $\mathrm{char}(F) = 0$]; F^{sep} ($\subseteq \overline{F}$) for the separable closure of F; $G_F \stackrel{\mathrm{def}}{=} \mathrm{Gal}(F^{\mathrm{sep}}/F)$;

$$\mu_n(F) \stackrel{\mathrm{def}}{=} \{x \in F^\times \mid x^n = 1\}; \quad \mu(F) \stackrel{\mathrm{def}}{=} \bigcup_m \mu_m(F);$$

$$F^{\times\infty} \stackrel{\mathrm{def}}{=} \bigcap_m (F^\times)^m; \quad F^{\times p^\infty} \stackrel{\mathrm{def}}{=} \bigcap_m (F^\times)^{p^m},$$

where m ranges over the positive integers;

$$F_{\mathrm{div}} \stackrel{\mathrm{def}}{=} \bigcup_{F \subseteq E} F_{\mathrm{prm}}(E^{\times\infty}) \ (\subseteq \overline{F}),$$

where $F \subseteq E$ ranges over the finite extension fields $\subseteq \overline{F}$. If Σ is a nonempty set of prime numbers, then we shall write

$$F^{\times\Sigma^\infty} \stackrel{\mathrm{def}}{=} \bigcap_m (F^\times)^m,$$

where m ranges over the positive Σ-integers. Here, we note that $F^{\times\Sigma^\infty} \subseteq F^\times$ is Σ'-saturated. Indeed, let m be a positive Σ-integer; $x \in F^{\times\Sigma^\infty}$; $x^{\frac{1}{m'}} \in F^\times$ an m'-th root of x, where m' is a positive Σ'-integer. Fix a positive integer a such that m' divides $am-1$ and an m-th root $x^{\frac{1}{m}} \in F^\times$ of x. Write $b \stackrel{\mathrm{def}}{=} -\frac{am-1}{m'}$. Then it is immediate to see that $((x^{\frac{1}{m'}})^a \cdot (x^{\frac{1}{m}})^b)^m = x^{\frac{1}{m'}}$, hence that $x^{\frac{1}{m'}} \in (F^\times)^m$. Thus, by varying m, we observe that $x^{\frac{1}{m'}} \in F^{\times\Sigma^\infty}$. We shall write

$$F_{\Sigma\text{-div}} \stackrel{\mathrm{def}}{=} \bigcup_{F \subseteq E} F_{\mathrm{prm}}(E^{\times\Sigma^\infty}) \ (\subseteq \overline{F}),$$

where $F \subseteq E$ ranges over the finite extension fields $\subseteq \overline{F}$. In particular, it holds that $F^{\times\infty} = F^{\times\mathfrak{Primes}^\infty}$, and $F_{\mathrm{div}} = F_{\mathfrak{Primes}\text{-div}}$.

Topological groups: Let G be a profinite group; $H \subseteq G$ a subgroup of G. Then we shall write $\overline{H} \subseteq G$ for the closure of H in G; $Z_G(H)$ for the *centralizer* of $H \subseteq G$, i.e.,

$$Z_G(H) \stackrel{\mathrm{def}}{=} \{g \in G \mid ghg^{-1} = h \text{ for any } h \in H\}.$$

Note that since G is Hausdorff, the centralizer $Z_G(H)$ is automatically closed in G. For each nonempty set of prime numbers Σ, we shall write G^Σ for the maximal pro-Σ quotient of G. For each prime number l, we shall write $G^l \stackrel{\mathrm{def}}{=} G^{\{l\}}$. [Similar abbreviations appear throughout the present paper.] We shall write G^{ab} for the quotient of G by $\overline{[G,G]} \subseteq G$; $\mathrm{Aut}(G)$ for the group of *continuous* automorphisms of G; $\mathrm{Inn}(G) \subseteq \mathrm{Aut}(G)$ for the group of inner automorphisms of G; $\mathrm{Out}(G) \stackrel{\mathrm{def}}{=} \mathrm{Aut}(G)/\mathrm{Inn}(G)$.

Let G_1, G_2 be profinite groups. Then an *outer surjective homomorphism* $G_1 \twoheadrightarrow G_2$ is defined to be an equivalence class of continuous surjective homomorphisms $G_1 \twoheadrightarrow G_2$, where we say that two surjective homomorphisms $f, g : G_1 \twoheadrightarrow G_2$ are equivalent if f coincides with the composite of g with an inner automorphism of G_2.

Schemes: Let K be a field; $K \subseteq L$ a field extension; X an algebraic variety [i.e., a separated, of finite type, and geometrically integral scheme] over K. Then we shall write $X(L)$ for the set of L-valued points of X; $X_L \stackrel{\text{def}}{=} X \times_K L$.

Fundamental groups: For a connected Noetherian scheme S, we shall write Π_S for the étale fundamental group of S, relative to a suitable choice of basepoint. Let K be a field; X an algebraic variety over K. Then we shall write

$$\Delta_X \stackrel{\text{def}}{=} \Pi_{X_{K^{\text{sep}}}}.$$

In particular, we have a natural exact sequence of profinite groups

$$1 \longrightarrow \Delta_X \longrightarrow \Pi_X \longrightarrow G_K \longrightarrow 1$$

[cf. [13], Exposé IX, Théorème 6.1]. We shall write

$$\Pi_X^{(\text{ab})} \stackrel{\text{def}}{=} \Pi_X/\text{Ker}(\Delta_X \twoheadrightarrow \Delta_X^{\text{ab}}).$$

For each nonempty set of prime numbers Σ, we shall write

$$\Pi_X^{(\Sigma)} \stackrel{\text{def}}{=} \Pi_X/\text{Ker}(\Delta_X \twoheadrightarrow \Delta_X^{\Sigma}).$$

In particular, we have a natural exact sequence of profinite groups

$$1 \longrightarrow \Delta_X^{\Sigma} \longrightarrow \Pi_X^{(\Sigma)} \longrightarrow G_K \longrightarrow 1.$$

We shall write

$$K(X)$$

for the function field of X;

$$\Delta_{K(X)} \stackrel{\text{def}}{=} G_{K(X)\otimes_K K^{\text{sep}}}.$$

In particular, we have a natural exact sequence of profinite groups

$$1 \longrightarrow \Delta_{K(X)} \longrightarrow G_{K(X)} \longrightarrow G_K \longrightarrow 1.$$

For each nonempty set of prime numbers Σ, we shall write

$$G_{K(X)}^{(\Sigma)} \stackrel{\text{def}}{=} G_{K(X)}/\text{Ker}(\Delta_{K(X)} \twoheadrightarrow \Delta_{K(X)}^{\Sigma}).$$

In particular, we have a natural exact sequence of profinite groups

$$1 \longrightarrow \Delta_{K(X)}^{\Sigma} \longrightarrow G_{K(X)}^{(\Sigma)} \longrightarrow G_K \longrightarrow 1.$$

Curves: Let K be a field; X a smooth curve [i.e., a smooth and 1-dimensional algebraic variety] over K. Then we shall write $\overline{X}$ for the smooth compactification of X over K. We shall refer to an element of $\overline{X}(\overline{K}) \setminus X(\overline{K})$ as a *cusp* of X. Then we shall say that X is a *hyperbolic curve* if $2g - 2 + r > 0$, where g denotes the genus of X; r denotes the cardinality of the set of cusps of X.

Next, suppose that X is a hyperbolic curve over K. Let Σ be a nonempty set of prime numbers. Then we shall refer to the stabilizer subgroup of Δ_X^{Σ} associated to some pro-cusp of the pro-Σ universal étale covering of $X_{\overline{K}}$ that lies over a cusp $x \in \overline{X}(\overline{K}) \setminus X(\overline{K})$ as a *cuspidal inertia subgroup* of $\Pi_X^{(\Sigma)}$ [or Δ_X^{Σ}]

associated to x. Write $\{U_i\}_{i\in I}$ for the family of open subschemes of $X_{\overline{K}}$. Then it follows immediately from the various definitions involved that there exists a natural isomorphism of profinite groups

$$\Delta^{\Sigma}_{K(X)} \xrightarrow{\sim} \varprojlim_{i\in I} \Delta^{\Sigma}_{U_i},$$

where the transition maps are the [outer] surjective homomorphisms induced by the natural open immersions. We shall refer to an inverse limit of cuspidal inertia subgroups of some cofinal collection of $\Delta^{\Sigma}_{U_i}$'s as a *cuspidal inertia subgroup* of $G^{(\Sigma)}_{K(X)}$ [or $\Delta^{\Sigma}_{K(X)}$].

1 Pro-Σ variant of the notion of TKND-AVKF-fields

Let Σ be a nonempty set of prime numbers. In the present section, we discuss a certain pro-Σ variant of the notion of *TKND-AVKF-fields*. Here, we recall that, if a field F satisfies the following condition, then we shall say that F is an *AVKF-field* [i.e., "abelian variety Kummer-faithful field" — cf. [20], Definition 6.1, (iii)]:

> Let A be an abelian variety over a finite extension field E of F. Then every divisible element of $A(E)$ is trivial.

If we approach Grothendieck conjecture-type questions via Kummer theory, this condition arises naturally during the step to reconstruct Kummer classes of function fields of curves. Moreover, the TKND-property on fields, which is also closely related to the Kummer theory for $\mathbb{G}_{\mathrm{m}}$, plays essential roles in other places [cf. Introduction, Remarks, (d)]. Our motivation to introduce a new class of fields arises from the following trivial observation:

> If we consider the [geometrically] pro-Σ variant of Grothendieck conjecture-type questions from the viewpoint of Kummer theory, then one may only access the Σ-portion of Kummer theory.

In the case where we consider [geometrically] pro-Σ variant of Grothendieck conjecture-type questions, we see the problem of the notion of TKND-AVKF-field. Indeed, the definition of TKND-AVKF-field is based on divisible elements but Σ-divisible elements. Thus, we need some modifications of this notion. In the present section, we introduce the notion of Σ-TKND-AVKF-field, which is a new class of fields that has compatibility with this situation, and give some examples and counter-examples of these fields.

Definition 1.1. Let F be a field. Then:

(i) If $F_{\Sigma\text{-div}} \subseteq \overline{F}$ is an infinite field extension, then we shall say that F is a Σ-*TKND-field* [i.e., "torally Kummer-nondegenerate field"]. Here, we recall that F is TKND [cf. [20], Definition 6.6, (i), (ii)] if $F_{\mathrm{div}} \subseteq \overline{F}$ is an infinite field extension.

(ii) If F satisfies the following condition, then we shall say that F is a Σ-*KF-field* [i.e., "Kummer-faithful field"]:

> Let A be a semi-abelian variety over a finite extension field E of F. Then every Σ-divisible element of $A(E)$ is a Σ'-torsion element.

(iii) If F satisfies the following condition, then we shall say that F is a Σ-*AVKF-field*:

> Let A be an abelian variety over a finite extension field E of F. Then every Σ-divisible element of $A(E)$ is a Σ'-torsion element.

(iv) If F is both a Σ-TKND-field and a Σ-AVKF-field, then we shall say that F is a Σ-*TKND-AVKF-field*.

Remark 1.1.1. It follows immediately from the various definitions involved that:

- Let $\Sigma^\dagger \subseteq \Sigma$ be a nonempty subset. Then every $\Sigma^\dagger$-TKND-field is a Σ-TKND-field.
- Let $\Sigma_1 \subseteq \mathfrak{Primes}$, $\Sigma_2 \subseteq \mathfrak{Primes}$ be nonempty subsets. Then, if a field is both Σ_1-AVKF and Σ_2-AVKF, then the field is $(\Sigma_1 \cup \Sigma_2)$-AVKF.
- The notion of Σ-TKND-field (respectively, Σ-AVKF-field; Σ-TKND-AVKF-field) may be regarded as a natural Σ-variant of the notion of TKND-field (respectively, AVKF-field; TKND-AVKF-field).
- Every Σ-KF-field that is not isomorphic to a subfield of $\overline{\mathbb{F}}_p$ for any prime number p is a Σ-TKND-field.
- Every Σ-KF-field is a Σ-AVKF-field.
- Every subfield of a Σ-AVKF-field is a Σ-AVKF-field. On the other hand, we note that a similar statement for TKND-field does not hold [cf. [54], Remark 1.1.1].

Lemma 1.2. *Let F be a field whose characteristic $\notin \Sigma$. Then the following hold:*

(i) Let A be a semi-abelian variety over F. Then any Σ-divisible element of $A(F^{\mathrm{pf}})$ is a Σ-divisible element of $A(F^\dagger)$ for some finite purely inseparable extension field $F^\dagger$ of F.

(ii) Suppose that F is a Σ-KF-field (respectively, Σ-TKND-field; Σ-AVKF-field; Σ-TKND-AVKF-field). Then F^{pf} is also a Σ-KF-field (respectively, Σ-TKND-field; Σ-AVKF-field; Σ-TKND-AVKF-field).

Proof. First, we verify assertion (i). Let $x \in A(F^{\mathrm{pf}})$ be a Σ-divisible element; $F \subseteq F^\dagger$ a finite purely inseparable field extension such that $x \in A(F^\dagger)$. To verify assertion (i), it suffices to verify that, for each $l \in \Sigma$ and each element $y \in A(F^{\mathrm{pf}})$ such that $y^l = x$, it holds that $y \in A(F^\dagger)$. However, this follows immediately from the fact that the morphism $[l] : A \to A$ obtained by multiplication by l is a finite étale covering [cf. our assumption that $l \neq \mathrm{char}(F)$]. This completes the proof of assertion (i). Assertion (ii) follows immediately from assertion (i), together with the various definitions involved. This completes the proof of Lemma 1.2. $\square$

Here, we recall that:

Theorem 1.3. *Let k be a field; $k \subseteq K$ a finitely generated, regular field extension; A an abelian variety over K. Write*

$$(\mathrm{Tr}_{K/k}(A),\ \tau : \mathrm{Tr}_{K/k}(A)_K \longrightarrow A)$$

for the K/k-trace associated to A [cf. [6], Theorem 6.2]. Then $A(K)/\tau(\mathrm{Tr}_{K/k}(A)(k))$ is finitely generated [cf. [6], Theorem 7.1; [23], Theorem 1].

Next, we prove several permanent properties concerning Σ-TKND-field and Σ-AVKF-field whose statement is similar to [53], Proposition 1.2:

Lemma 1.4. *Let*

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3$$

be an exact sequence of abelian groups. Suppose that:

(a) For any Σ-integer m, the group of m-torsions of M_2 is finite.

(b) The Σ-divisible elements of M_1 are Σ'-torsion elements.

(c) The group of Σ-divisible elements of M_3 is finite.

Then the Σ-divisible elements of M_2 are Σ'-torsion elements.

Proof. Observe that condition (a) implies that the Σ-divisible subgroup $M_2^{\times\Sigma\infty}$ of M_2 is also Σ-divisible. On the other hand, it follows from condition (c) that there exists a positive Σ'-integer n such that $nM_2^{\times\Sigma\infty} \subseteq M_1$. Then since $nM_2^{\times\Sigma\infty}$ is Σ-divisible, it follows from condition (b) that $nM_2^{\times\Sigma\infty}$ consists of Σ'-torsion elements. Thus, since n is a positive Σ'-integer, we conclude that the Σ-divisible elements of M_2 are Σ'-torsion elements. This completes the proof of Lemma 1.4. $\square$

Proposition 1.5. *Let K be a field whose characteristic $\notin \Sigma$; L a finitely generated extension field of K. Then the following hold:*

(i) Suppose that K is a Σ-TKND-field. Then L^{pf} is also a Σ-TKND-field.

(ii) Suppose that K is a Σ-AVKF-field. Then L^{pf} is also a Σ-AVKF-field.

(iii) Suppose that K is a Σ-KF-field such that, for any abelian variety over a finite extension field $K^\dagger$ of K, the group of Σ-divisible elements of $A(K^\dagger)$ is finite. Then L^{pf} is also a Σ-KF-field.

(iv) Suppose that K is a Σ-TKND-AVKF-field. Then L^{pf} is also a Σ-TKND-AVKF-field.

Proof. First, in light of Lemma 1.2, (ii), assertion (i) follows immediately from a similar argument to the argument applied in the proof of [53], Proposition 1.2, (i).

Next, we verify assertion (ii). In light of Lemma 1.2, (ii), it suffices to verify that L is a Σ-AVKF-field. Let $L \subseteq L^\dagger$ be a finite field extension; A an abelian variety over $L^\dagger$. Then our goal is to prove that every Σ-divisible element of $A(L^\dagger)$ is a Σ'-torsion element. Note that, if the field extension $K \subseteq L$ is algebraic, then since $K \subseteq L^\dagger$ is finite, the desired assertion follows immediately from our assumption that K is a Σ-AVKF-field. Then, in light of the various definitions involved, by replacing K, L by suitable finite field extensions, respectively, we may assume without loss of generality that $L = L^\dagger$, and the field extension $K \subseteq L$ is transcendental and regular. Thus, since the Σ-divisible subgroup of a finitely generated abelian group is a finite [pro-]Σ' group, we conclude from Theorem 1.3, together with Lemma 1.4, that every Σ-divisible element of $A(L)$ is a Σ'-torsion element. This completes the proof of assertion (ii).

Next, we verify assertion (iii). In light of Lemma 1.2, (ii), it suffices to verify that L is a Σ-KF-field. Observe that it follows from the proof of assertion (ii), together with our assumption on K, that L is a Σ-AVKF-field such that, for any abelian variety over a finite extension field $L^\dagger$ of L, the group of Σ-divisible elements of $A(L^\dagger)$ is finite. On the other hand, since K is a Σ-KF-field, it follows immediately from a similar argument to the argument applied in the proof of [53], Proposition 1.2, (i), that, for each finite extension field $L^\dagger$ of L, the Σ-divisible elements of the $L^\dagger$-valued points of a torus are Σ'-torsion elements. Here, we note that a semi-abelian variety is an extension of an abelian variety by a torus. Thus, by applying Lemma 1.4, we conclude that L is a Σ-KF-field, as desired. This completes the proof of assertion (iii).

Finally, assertion (iv) follows immediately from assertions (i), (ii). This completes the proof of Proposition 1.5. $\square$

Proposition 1.6. *Let K be a field whose characteristic $\notin \Sigma$; $K = K_0 \subsetneq K_1 \subsetneq \cdots \subsetneq K_i \subsetneq K_{i+1} \subsetneq \cdots$ an infinite countable chain of finitely generated regular transcendental extensions. Write $L \stackrel{\mathrm{def}}{=} \cup_{i \geq 0} K_i$. Then, if K is a Σ-TKND-field (respectively, Σ-AVKF-field; Σ-TKND-AVKF-field), then L^{pf} is also a Σ-TKND-field (respectively, Σ-AVKF-field; Σ-TKND-AVKF-field).*

Proof. In light of Lemma 1.2, (ii); Proposition 1.5, (i), (ii), (iv), Proposition 1.6 follows from a similar argument to the argument applied in [35], Remark 1.5.4, (ii). $\square$

Proposition 1.7. *Let K be a field whose characteristic $\notin \Sigma$. Write L for the maximal pro-Σ' extension field of K. Suppose that Σ is finite. Then, if K is a Σ-TKND-field (respectively, Σ-AVKF-field; Σ-TKND-AVKF-field), then L^{pf} is also a Σ-TKND-field (respectively, Σ-AVKF-field; Σ-TKND-AVKF-field).*

Proof. In light of Lemma 1.2, (ii), it suffices to verify the respective properties for L. Note that any finite extension field of L is the composite of L with a finite extension field of K. Then, by replacing K, L by finite extension fields, respectively, we observe that it suffices to verify that, for each semi-abelian variety A over K, there exists a finite extension field $K^{\dagger}$ ($\subseteq L$) of K such that the group of Σ-divisible elements of $A(L)$ is contained in $A(K^{\dagger})$ [cf. also the observation in the proof of Lemma 1.4 concerning condition (a)]. Note that since Σ is finite, by replacing K by a finite extension field of K, if necessary, we may assume that, for each $l \in \Sigma$, the morphism $[l] : A \to A$ obtained by multiplication by l is a finite étale Galois covering [cf. our assumption that $l \neq \mathrm{char}(K)$]. In this situation, since $K \subseteq L$ is a pro-Σ' field extension, it is immediate to see that the group of Σ-divisible elements of $A(L)$ is contained in $A(K^{\dagger})$ for some finite extension field $K^{\dagger}$ of K. This completes the proof of Proposition 1.7. $\square$

Definition 1.8. Let F be a perfect field. Then:

- We shall say that F is *quasi-finite* if $G_F \cong \widehat{\mathbb{Z}}$.
- Let l be a prime number. Then we shall say that F is *l-quasi-finite* if $G_F^l \cong \mathbb{Z}_l$.

In particular, quasi-finite fields are l-quasi-finite fields for any prime number l.

Proposition 1.9. *Let l be a prime number; F an l-quasi-finite field of positive characteristic $\neq l$ that is algebraic over the prime field. Then any subfield of the perfection of the maximal pro-prime-to-l extension field of a finitely generated extension field of F is l-AVKF.*

Proof. Note that since F is algebraic over the prime field, any finite extension field of F is also l-quasi-finite. In particular, in light of Remark 1.1.1 and Propositions 1.5, (ii); 1.7, it suffices to verify that, for each abelian variety A over F, it holds that $T_l A^{G_F} = \{0\}$, where $T_l A$ denotes the l-adic Tate module associated to A. On the other hand, for each finitely generated $\mathbb{Z}_l$-module M, we recall that $\mathrm{Aut}_{\mathbb{Z}_l}(M)$ admits a pro-l open subgroup. Then, after replacing F by a finite extension field of F, we observe that there exists a finite subfield $F_0 \subseteq F$ such that A descends to an abelian variety over F_0, and if we write $\rho : G_{F_0} \to \mathrm{Aut}(T_l A)$ for the natural Galois representation, then $\rho(G_F) = \rho(G_{F_0})$. Thus, we conclude from the easily verified fact that $T_l A^{G_{F_0}} = \{0\}$ that $T_l A^{G_F} = \{0\}$. This completes the proof of Proposition 1.9. $\square$

Definition 1.10. Let p be a prime number; F a field. Then we shall say that:

- F is a *sub-p-adic field* if F is isomorphic to a subfield of a finitely generated extension field of $\mathbb{Q}_p$ [cf. [30], Definition 15.4, (i)].
- F is a *generalized sub-p-adic field* if F is isomorphic to a subfield of a finitely generated extension field of the completion of $\mathbb{Q}_p^{\mathrm{ur}}$ [cf. [31], Definition 4.11].

In particular, sub-p-adic fields are generalized sub-p-adic fields.

Definition 1.11. Let F be a field; $F^{(0)}$ a perfect field; d a positive integer. Then a structure of *local field of dimension* d over $F^{(0)}$ on F is a sequence of complete discrete valuation fields $F^{(d)} \stackrel{\mathrm{def}}{=} F, F^{(d-1)}, \ldots, F^{(0)}$ such that for each integer $0 \leq i \leq d-1$, $F^{(i)}$ is the residue field of the complete discrete valuation field $F^{(i+1)}$.

Remark 1.11.1. Let K be a mixed characteristic/positive characteristic complete discrete valuation field with a perfect residue field. Write $\mathcal{O}_K$ for the ring of integers of K; k for the residue field of $\mathcal{O}_K$; $K\{\{t\}\}$ for the field of fractions of the completion of the localization of $\mathcal{O}_K[[t]]$ at the prime ideal (π_K), where t denotes an indeterminate; π_K denotes a uniformizer of $\mathcal{O}_K$. Then it is immediate to see that $K(t) \subseteq K\{\{t\}\}$, and $K\{\{t\}\}$ is a mixed characteristic/positive characteristic complete discrete valuation field whose residue field is $k((t))$. Thus, we observe that $K\{\{t\}\}$ admits a structure of local field of dimension 2 whose final residue field is k. In particular, any finitely generated extension field of a mixed characteristic/positive characteristic higher local field is a subfield of a mixed characteristic/positive characteristic higher local field.

Proposition 1.12. *Let p be a prime number. Then the following hold:*

(i) Let k be a finite field of characteristic p; K a field that admits a structure of a mixed characteristic or positive characteristic higher local field over k. Let M be a subfield of K. Then M is p-KF. In particular, any sub-p-adic field is p-KF [cf. Remark 1.11.1].

(ii) Any generalized sub-p-adic field is p-TKND. In particular, any sub-p-adic field is p-TKND-AVKF [cf. (i)].

(iii) There exists a generalized sub-p-adic field that is not p-AVKF.

Proof. First, we verify assertion (i). In light of Remark 1.1.1, it suffices to verify that K is p-KF. By induction on the dimension of K as a local field, one may observe that it follows immediately from the finiteness of k that the group of p-divisible elements of the multiplicative group of a finite extension field of K is finite, hence consists of prime-to-p torsion elements. Then, in light of Lemma 1.4, together with the definitions of p-KF-field or semi-abelian variety, our goal is to verify the following assertion:

> Claim 1.12.A: Let $K \subseteq K^\dagger$ be a finite field extension; A an abelian variety over $K^\dagger$. Then the group of p-divisible elements of $A(K^\dagger)$ is finite, hence consists of prime-to-p torsion elements.

By replacing K by a finite extension field of K, we may assume without loss of generality that $K = K^\dagger$, and A has semi-stable reduction over K [cf. [15], Exposé IX, Théorème 3.6]. Write R for the ring of integers of K; $K^{(d-1)}$ for the residue field of R, where d denotes the dimension of K; $\mathcal{A}$ for the semi-abelian scheme over R that extends A; $\mathcal{A}_s$ for the special fiber of $\mathcal{A}$; $\widehat{\mathcal{A}}$ for the formal completion of $\mathcal{A}$ at the origin. Then the reduction map induces a natural exact sequence

$$0 \longrightarrow \widehat{\mathcal{A}}(R) \longrightarrow A(K) = \mathcal{A}(R) \longrightarrow \mathcal{A}_s(K^{(d-1)}).$$

Recall that any p-divisible element of $\widehat{\mathcal{A}}(R)$ is trivial [cf. e.g., [53], Lemma 1.3]. Therefore, to verify that any p-divisible element of $A(K)$ is a [finite] prime-to-p-torsion element, it follows from Lemma 1.4 that it suffices to verify that the group of p-divisible elements of $\mathcal{A}_s(K^{(d-1)})$ is finite. In light of induction on the dimension d as a local field, it suffices to consider the case where $d = 1$, hence $k = K^{(0)}$. However, since k is finite, it is immediate to see that $\mathcal{A}_s(k)$ is finite. This completes the proof of Claim 1.12.A, hence of assertion (i).

Assertion (ii) follows immediately from [52], Lemma 3.4, (iii), together with assertion (i). Assertion (iii) follows immediately from [53], Proposition 1.7, (ii). This completes the proof of Proposition 1.12. □

Remark 1.12.1. With regard to Proposition 1.12, (ii), much more general assertion may be proved by applying Lemma 1.19 below.

Definition 1.13. Let K be a field. Then we shall refer to the extension field of K [in $\overline{K}$] obtained by adjoining all the roots of elements in K to K as the *maximal Kummer extension* of K, which we shall denote by K^{Kum}.

Remark 1.13.1. Let K be a field. Then it follows immediately from the various definitions involved that each finitely generated extension field of K^{Kum} is contained in the maximal Kummer extension field of a finitely generated extension field of K.

Next, we recall the following recent theorems concerning the finiteness of torsion points of abelian varieties:

Theorem 1.14 (Murotani-Ozeki)**.** *Let p be a prime number; K_0 a subfield of the maximal cyclotomic extension field of a finitely generated field of characteristic* 0 *whose associated p-adic cyclotomic character is open; $K \subseteq K_0^{\mathrm{Kum}}$ a subfield; A an abelian variety over K; $K \subseteq L$ $(\subseteq \overline{K})$ a finite field extension. Then the p-power torsion subgroup $A(L)_{p\text{-tor}} \subseteq A(L)$ is finite.*

Proof. In light of Ribet's theorem [cf. [22], Appendix, Theorem 1], together with Theorem 1.3, Theorem 1.14 follows immediately from [36], Proposition 3.15, (i). □

Remark 1.14.1. In fact, by applying [36], Proposition 3.15, (i), one may construct a more general class of fields that satisfies a similar property [cf., e.g., the statement of [36], Corollary 3.16].

Theorem 1.15 (Lombardo-Szamuely)**.** *Let K be a subfield of the maximal Kummer extension of a number field; A an abelian variety over K; $K \subseteq L$ $(\subseteq \overline{K})$ a finite field extension. Then the torsion subgroup $A(L)_{\mathrm{tor}} \subseteq A(L)$ is finite.*

Proof. Theorem 1.15 follows immediately from [25], Theorem 1.1. □

Remark 1.15.1. [25], Theorem 1.1 may be regarded as an ideal generalization of [36], Theorem 1.1 obtained by T. Murotani and Y. Ozeki, which was already a much stronger result than the author's earlier observation in [54]. On the other hand, we note that Theorem 1.14 does not follow from Theorem 1.15.

Here, in light of the discussions below, one may generalize Theorem 1.15 to the case of the maximal Kummer extension of a finitely generated field of characteristic 0 [cf. Theorem 1.17].

Lemma 1.16. *The following hold:*

(i) Let F be a complete discrete valuation field of equicharacteristic 0; $F \subseteq E$ an algebraic field extension contained in F^{Kum} $(\subseteq \overline{F})$. Write $k_{(-)}$ for the residue field of $(-)$. Then the residue field extension $k_F \subseteq k_E$ is contained in k_F^{Kum}.

(ii) Let k be a field of characteristic 0; S a normal variety over k; $K(S) \subseteq K(T)$ an algebraic field extension contained in $K(S)^{\mathrm{Kum}}$ $(\subseteq \overline{K(S)})$. Write T for the normalization of S in $K(T)$ [so T is not necessarily Noetherian]. Then the residue field extensions of the natural morphism $T \to S$ over a point s of S of codimension 1 are contained in the maximal Kummer extension fields of the residue field of S at s.

Proof. First, we verify assertion (i). Note that since F is a complete discrete valuation field of equicharacteristic 0, it holds that F is isomorphic to $k_F((t))$, where t denotes an indeterminate. Then it holds that the principal unit of F is divisible. In particular, one may observe that F^{Kum} is a Henselian valuation field whose completion is isomorphic to the completion of $\cup_{n \geq 1} k_F^{\mathrm{Kum}}((t^{\frac{1}{n}}))$. Thus, we conclude that $k_E \subseteq k_F^{\mathrm{Kum}}$. This completes the proof of assertion (i). Assertion (ii) follows immediately from assertion (i), together with Krasner's lemma. This completes the proof of Lemma 1.16. □

Theorem 1.17. *Let K be a subfield of the maximal Kummer extension of a finitely generated field of characteristic 0; A an abelian variety over K; $K \subseteq L$ $(\subseteq \overline{K})$ a finite field extension. Then $A(L)_{\mathrm{tor}}$ is finite.*

Proof. In light of induction on the transcendental degree of the finitely generated field over the prime field, by applying the theory of specialization to suitable abelian schemes [over non-Noetherian bases], together with Lemma 1.16, (ii), one may observe that Theorem 1.17 follows immediately from Theorem 1.15. □

Next, we record a certain variant of an observation by Murotani-Ozeki concerning a criterion on TKND-field.

Definition 1.18. Let l be a prime number; F a field. Then we shall say that F is *stably l-$\times\mu$-indivisible* if, for each finite field extension $F \subseteq E$, it holds that $E^{\times l^{\infty}} \subseteq \mu(E)$.

Lemma 1.19. *Let l be a prime number; F a stably l-$\times\mu$-indivisible field of characteristic $\neq l$ such that $\mu_{l^\infty}(\overline{F}) \subseteq F$; $F \subseteq E$ $(\subseteq \overline{F})$ a Galois extension. Write E' $(\subseteq \overline{F})$ for the field obtained by adjoining all the prime-to-l-roots of elements in E to E. Then $E_{\{l\}\text{-div}} \subseteq E'$. In particular, if $\mu(\overline{F}) \subseteq F$, then the field extension $E \subseteq E \cdot E_{\{l\}\text{-div}}$ is abelian.*

Proof. To verify Lemma 1.19, it follows immediately from the definition of $E_{\{l\}\text{-div}}$ that it suffices to verify that, for each finite extension $E \subseteq E^{\dagger}$ and each $x \in (E^{\dagger})^{\times l^{\infty}}$, there exists an $\{l\}'$-integer d such that $x^d \in E^{\times}$. However, this follows from a similar argument to the argument applied in the proof of [36], Proposition 3.3. This completes the proof of Lemma 1.19. □

Corollary 1.20. *The following hold:*

(i) Let K be a subfield of the maximal Kummer extension of a finitely generated field of characteristic 0. Then K is a Σ-TKND-AVKF-field [cf. Remark 1.20.1]. In particular, any subfield of a finitely generated extension field of the maximal Kummer extension field of a number field is also a Σ-TKND-AVKF-field [cf. Remark 1.13.1].

(ii) Let l be a prime number; F an l-quasi-finite field of positive characteristic $\neq l$ that is algebraic over the prime field; K a field such that:

- *K is not algebraic over the prime field.*
- *K is a subfield of the maximal pro-prime-to-l extension field of the perfection of a finitely generated extension field of F.*

Then K is an l-TKND-AVKF-field.

Proof. First, we consider the Σ-AVKF portions. In the case of assertion (i), in light of Theorem 1.17, it follows from a similar argument to the argument applied in the proof of [52], Theorem 3.1, that the maximal Kummer extension of a finitely generated field of characteristic 0 is a Σ-AVKF-field. On the other hand, in the case of assertion (ii), it follows from Proposition 1.9 that K is l-AVKF. This completes the proof of the respective Σ-AVKF portions of assertions (i), (ii).

Next, we verify the Σ-TKND portions. In light of Remark 1.1.1, we may assume without loss of generality that $\Sigma = \{l\}$. In the case of assertion (i), we note that any abelian extension of a finitely generated field of characteristic 0 is stably l-$\times\mu$-indivisible [cf. [52], Lemma 3.4, (iii), (iv)]. Note also that the absolute Galois group of a finite step solvable extension field of a finitely generated field of characteristic 0 is not finite. Thus, by applying Lemma 1.19, we observe that the maximal Kummer extension field of a finitely generated field of characteristic 0 is l-TKND. On the other hand, we *observe* that:

- Any subfield of the maximal pro-prime-to-l extension field of the perfection of a stably l-$\times\mu$-indivisible field of characteristic $\neq l$ is also a stably l-$\times\mu$-indivisible field.
- Any l-divisible element of the multiplicative group a finitely generated extension field of a base field is a l-divisible element of the multiplicative group of a finite extension field of the base field [cf. the discussion in the proof of [53], Proposition 1.2, (i)].

Then, in the case of assertion (ii), by applying the above *observations*, one may deduce that K is stably l-$\times\mu$-indivisible. Thus, since K is not algebraic over the prime field, we conclude that K is l-TKND. This completes the proof of the respective Σ-TKND portions of assertions (i), (ii), hence of Corollary 1.20. □

Remark 1.20.1. In the proof of Corollary 1.20, (i), we applied the generalized version of Lombardo-Szamuely's result [cf. Theorem 1.17]. On the other hand, by applying the Murotani-Ozeki's result [cf. Theorem 1.14], together with the lemma below [cf. Lemma 1.21], one may prove a generalization of Corollary 1.20, (i):

> Let K_0 be a subfield of the maximal cyclotomic extension field of a finitely generated field of characteristic 0 such that, for each $p \in \Sigma$, the p-adic cyclotomic character $G_{K_0} \to \mathbb{Z}_p^\times$ is open. Then any subfield of K_0^{Kum} is a Σ-TKND-AVKF-field.

However, since the condition depends on Σ, in the remainder of the present paper, for simplicity, we restrict our attention to the situation of Corollary 1.20, (i), when applying the result.

Remark 1.20.2. Let p be a prime number such that $\Sigma \neq \{p\}$; A an abelian variety over $\overline{\mathbb{F}}_p$. Then all the torsion points of the abelian variety $A_{\overline{\mathbb{F}}_p(t)}$ over $\overline{\mathbb{F}}_p(t)$ [where t denotes an indeterminate] are divisible. In particular, it holds that $\overline{\mathbb{F}}_p(t)$ is not Σ-AVKF.

Remark 1.20.3. Let k be a finite field. Recall that the notion of k-largeness concerning sets of prime numbers played an important role in Saïdi-Tamagawa's paper [46] [cf. [46], Definition 3.2]. In fact, it follows immediately from the various definitions involved that Σ is k-large if and only if k is Σ-TKND [cf. [46], Definition/Proposition 3.1].

Lemma 1.21. *Let K be a field; $K \subseteq L$ an algebraic field extension; A an abelian variety over K. Consider the following conditions:*

(a) The Σ-divisible elements of $A(L)$ are Σ'-torsion.

(b) The Σ-divisible elements of the torsion subgroup $A(L)_{\mathrm{tor}} \subseteq A(L)$ are Σ'-torsion.

(c) For each $l \in \Sigma$, the l-power torsion subgroup $A(L)_{l\text{-tor}} \subseteq A(L)$ is finite.

Then (a) $\Rightarrow$ (b) $\Leftrightarrow$ (c). Moreover, if K is Σ-AVKF, and the field extension $K \subseteq L$ is Galois, then (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c).

Proof. Lemma 1.21 follows from a similar argument to the argument applied in the proof of [38], Proposition 2.4. $\square$

2 Reconstruction of certain perfections of the group of Kummer classes of rational functions

Let Σ be a nonempty set of prime numbers; K a perfect field of characteristic $\notin \Sigma$; X a proper hyperbolic curve over K. In the present section, in the case where K is a Σ-TKND-AVKF-field, by discussing the geometrically pro-Σ analogue of [53], §2, we establish the semi-absolute reconstruction of the group-theoretic perfections of the group of Kummer classes of $K(X_{\overline{K}})^\times$ from the data of the natural surjection $G^{(\Sigma)}_{K(X)} \twoheadrightarrow G_K$, together with the data of cuspidal inertia subgroups [cf. Definition 2.5; Proposition 2.8]. Moreover, in several arithmetic settings, we observe that any isomorphism between the data $G^{(\Sigma)}_{K(X)} \twoheadrightarrow G_K$ and a similar data "$G^{(\Sigma)}_{K(Y)} \twoheadrightarrow G_L$" maps the cuspidal inertia subgroups of $G^{(\Sigma)}_{K(X)}$ to the cuspidal inertia subgroups of $G^{(\Sigma)}_{K(Y)}$. This implies that any such isomorphism induces an isomorphism between the respective group-theoretic perfections of Kummer classes [cf. Corollary 2.10]. Finally, we also discuss a phenomenon of certain very weak cyclotomic rigidity in the relative anabelian geometric situation [cf. Proposition 2.14], which will be applied to obtain a corollary of the geometrically pro-Σ relative birational version of the Grothendieck Conjecture for smooth curves over Σ-TKND-AVKF-fields in §3.

First, we begin by discussing a pro-Σ variant of the synchronization of geometric cyclotomes in a similar style to [53], Proposition 2.3.

Definition 2.1.

(i) We shall write

$$\Lambda_X$$

for the dual $\widehat{\mathbb{Z}}^{\Sigma}$-module of the second cohomology group $H^2(\Delta_X^{\Sigma}, \widehat{\mathbb{Z}}^{\Sigma})$. Note that:

- Since the kernel of the natural surjective homomorphism $\Delta_X \twoheadrightarrow \Delta_X^{\Sigma}$ has l-cohomological dimension ≤ 1 for each prime number $l \in \Sigma$, one may observe that the natural inflation map $H^2(\Delta_X^{\Sigma}, \widehat{\mathbb{Z}}^{\Sigma}) \to H^2(\Delta_X, \widehat{\mathbb{Z}}^{\Sigma})$ is an isomorphism.
- Since X is a smooth proper curve of genus ≥ 2, it holds that $H^2(\Delta_X, \widehat{\mathbb{Z}}^{\Sigma}) \xrightarrow{\sim} H^2_{\text{ét}}(X_{\overline{K}}, \widehat{\mathbb{Z}}^{\Sigma})$.

In particular, it follows immediately from Poincaré duality that Λ_X is isomorphic to $\widehat{\mathbb{Z}}^{\Sigma}(1)$. Here, we recall that $\widehat{\mathbb{Z}}^{\Sigma}(1) \stackrel{\text{def}}{=} \varprojlim_n \mu_n(\overline{K})$, where n ranges over the positive Σ-integers.

(ii) Let $x \in X(K)$ be a closed point. Then we shall write $X_x \stackrel{\text{def}}{=} X \setminus \{x\}$;

$$\Delta_{X_x}^{\Sigma-\text{cn}}$$

for the maximal cuspidally central quotient associated to the outer surjection $\Delta_{X_x}^{\Sigma} \twoheadrightarrow \Delta_X^{\Sigma}$ induced by the natural open immersion $X_x \hookrightarrow X$ [cf. [33], Definition 1.1, (i); [53], Definition 2.2, (i)].

Proposition 2.2. *Let $x \in X(K)$ be a closed point; $I_x \subseteq \Delta_{K(X)}^{\Sigma}$ a cuspidal inertia subgroup associated to x. Then we have a natural exact sequence of profinite groups*

$$1 \longrightarrow I_x \longrightarrow \Delta_{X_x}^{\Sigma-\text{cn}} \longrightarrow \Delta_X^{\Sigma} \longrightarrow 1.$$

Moreover, one may reconstruct, in a purely group-theoretic way, from the natural outer surjection $\Delta_{X_x}^{\Sigma} \twoheadrightarrow \Delta_X^{\Sigma}$, the natural scheme-theoretic identification

$$\Lambda_X \xrightarrow{\sim} I_x \ (\xrightarrow{\sim} \widehat{\mathbb{Z}}^{\Sigma}(1))$$

— where we identify I_x with its image in $\Delta_{X_x}^{\Sigma-\text{cn}}$ — as follows: The Leray-Serre spectral sequence

$$E_2^{i,j} = H^i(\Delta_X^{\Sigma}, H^j(I_x, I_x)) \ \Rightarrow \ H^{i+j}(\Delta_{X_x}^{\Sigma-\text{cn}}, I_x)$$

associated to the above exact sequence induces a differential

$$H^1(I_x, I_x) = H^0(\Delta_X^{\Sigma}, H^1(I_x, I_x)) = E_2^{0,1} \longrightarrow E_2^{2,0} = H^2(\Delta_X^{\Sigma}, H^0(I_x, I_x)) = \operatorname{Hom}(\Lambda_X, I_x).$$

Then the image of the identity automorphism $\in \operatorname{Hom}(I_x, I_x) = H^1(I_x, I_x)$ via the above differential gives us the isomorphism $\Lambda_X \xrightarrow{\sim} I_x$, as desired.

Proof. Proposition 2.2 follows from a similar argument to the argument applied in the proof of [53], Proposition 2.3. □

Next, by applying a similar argument to the argument applied in [53], Definition 2.5, under a certain assumption on K, we shall construct, in a purely group-theoretic way, from the data of the natural surjection $G_{K(X)}^{(\Sigma)} \twoheadrightarrow G_K$, together with the data of cuspidal inertia subgroups of $G_{K(X)}^{(\Sigma)}$, a subset

$$\overline{K}(X)^{\kappa\text{-sat}} \ \subseteq \ \varinjlim_{K \subseteq K^\dagger} \ H^1(G_{K(X) \otimes_K K^\dagger}^{(\Sigma)}, \Lambda_X),$$

where $K \subseteq K^\dagger$ ($\subseteq \overline{K}$) ranges over the finite field extensions [cf. Definition 2.5].

Definition 2.3. Let F be a field. Then we shall say that F is Σ^∞-*AV-tor-finite* [cf. [20], Definition 6.1, (iv)]:

> Let A be an abelian variety over a finite extension field $F^\dagger$ of F. Then, for each prime number $l \in \Sigma$, the group of l-power torsion points $\in A(F^\dagger)$ is *finite*.

In light of the discussion in the proof of [53], Lemma 2.4 [i.e., the duality theory on abelian varieties], if $\mathrm{char}(F) \notin \Sigma$, then one may observe that this condition is equivalent to the following condition:

> Let A be an abelian variety over a finite extension field $F^\dagger$ of F. Then it holds that $H^1(\Delta_A, \widehat{\mathbb{Z}}^\Sigma(1))^{G_{F^\dagger}} = \{0\}$.

Remark 2.3.1. It follows immediately from the various definitions involved that:

- Any subfield of a Σ^∞-AV-tor-finite field is also Σ^∞-AV-tor-finite.
- Any Σ-AVKF-field is Σ^∞-AV-tor-finite.

Proposition 2.4. *The perfection of a finitely generated extension field of a* Σ^∞*-AV-tor-finite field of characteristic* $\notin \Sigma$ *is also* Σ^∞*-AV-tor-finite.*

Proof. Since the perfection does not change the absolute Galois group, it follows immediately from the various definitions involved that it suffices to verify that any finitely generated transcendental, regular extension field of a Σ^∞-AV-tor-finite field is also Σ^∞-AV-tor-finite. Then, by considering a suitable model of a finitely generated transcendental, regular extension and applying the theory of specialization, one may obtain the desired assertion. This completes the proof of Proposition 2.4. $\square$

Until the end of Definition 2.5, suppose that

$$K \text{ is } \Sigma^\infty\text{-}\textit{AV-tor-finite}.$$

Let $K \subseteq M$ ($\subseteq \overline{K}$) be a finite separable field extension; $S \subseteq X(M) = X_M(M)$ ($\subseteq X(\overline{K})$) a nonempty finite subset of M-valued points of X. Write $U \overset{\mathrm{def}}{=} X_M \setminus S \subseteq X_M$. Observe that the natural exact sequence

$$1 \longrightarrow \Delta_U^\Sigma \longrightarrow \Pi_U^{(\Sigma)} \longrightarrow G_M \longrightarrow 1$$

determines an exact sequence

$$0 \longrightarrow H^1(G_M, \Lambda_X) \longrightarrow H^1(\Pi_U^{(\Sigma)}, \Lambda_X) \xrightarrow{r} H^1(\Delta_U^\Sigma, \Lambda_X)^{G_M}.$$

Thus, by allowing the [sufficiently large] finite extension fields $K \subseteq K^\dagger$ to vary, we obtain an exact sequence

$$0 \longrightarrow \varinjlim_{K \subseteq K^\dagger} H^1(G_{K^\dagger}, \Lambda_X) \longrightarrow \varinjlim_{K \subseteq K^\dagger} H^1(\Pi_{U_{K^\dagger}}^{(\Sigma)}, \Lambda_X) \longrightarrow \varinjlim_{K \subseteq K^\dagger} H^1(\Delta_U^\Sigma, \Lambda_X)^{G_{K^\dagger}}.$$

Here, we observe that, for any finite field extension $K \subseteq K^\dagger$,

$$H^1(\Delta_U^\Sigma, \Lambda_X)^{G_{K^\dagger}} = H^1((\Delta_U^\Sigma)^{\mathrm{ab}}, \Lambda_X)^{G_{K^\dagger}}.$$

Next, for each $x \in S$, let I_x be a cuspidal inertia subgroup of $\Delta^{\Sigma}_{K(X)}$ $(\subseteq G^{(\Sigma)}_{K(X)})$ associated to x. Then we have an exact sequence of G_M-modules

$$\bigoplus_{x\in S} I_x \longrightarrow (\Delta^{\Sigma}_U)^{\mathrm{ab}} \longrightarrow (\Delta^{\Sigma}_X)^{\mathrm{ab}} \longrightarrow 0,$$

which determines an exact sequence of modules

$$0 \longrightarrow H^1((\Delta^{\Sigma}_X)^{\mathrm{ab}}, \Lambda_X)^{G_M} \longrightarrow H^1((\Delta^{\Sigma}_U)^{\mathrm{ab}}, \Lambda_X)^{G_M} \longrightarrow \bigoplus_{x\in S} H^1(I_x, \Lambda_X).$$

Note that since K is Σ^{∞}-AV-tor-finite, it follows immediately from [34], Proposition A.6, (iv), that, for any finite field extension $K \subseteq K^{\dagger}$,

$$H^1(\Delta^{\Sigma}_X, \Lambda_X)^{G_{K^{\dagger}}} = \{0\}.$$

Thus, we obtain a natural injection

$$i : H^1((\Delta^{\Sigma}_U)^{\mathrm{ab}}, \Lambda_X)^{G_M} \hookrightarrow \bigoplus_{x\in S} H^1(I_x, \Lambda_X).$$

Write

$$1_x \in H^1(I_x, \Lambda_X) = \mathrm{Hom}(I_x, \Lambda_X)$$

for the isomorphism $I_x \overset{\sim}{\to} \Lambda_X$ of Proposition 2.2;

$$\mathbb{Z}_x \subseteq H^1(I_x, \Lambda_X)$$

for the subgroup generated by 1_x;

$$i_x : G_M \hookrightarrow \Pi^{(\Sigma)}_{X_M}$$

for the section of the natural surjection $\Pi^{(\Sigma)}_{X_M} \twoheadrightarrow G_M$ determined by the image of $N_{G^{(\Sigma)}_{K(X)\otimes_K M}}(I_x)$ via the natural surjection $G^{(\Sigma)}_{K(X)\otimes_K M} \twoheadrightarrow \Pi^{(\Sigma)}_{X_M}$. Next, we fix $x_0 \in S$. Write

$$D_x \in H^1(G_M, (\Delta^{\Sigma}_X)^{\mathrm{ab}})$$

for the element obtained by forming the difference between i_{x_0} and i_x;

$$\mathcal{P}^{\mathrm{sat}}_{S,M} \subseteq \bigoplus_{x\in S} \mathbb{Z}_x \quad \Big(\subseteq \bigoplus_{x\in S} H^1(I_x, \Lambda_X) \Big)$$

for the Σ'-saturation [in $\oplus_{x\in S} \mathbb{Z}_x$] of the subgroup consisting of $(n_x)_{x\in S} \in \bigoplus_{x\in S} \mathbb{Z}_x$ such that

$$\sum_{x\in S} n_x = 0, \qquad \sum_{x\in S} n_x \cdot D_x = 0 \quad \big(\in H^1(G_M, (\Delta^{\Sigma}_X)^{\mathrm{ab}}) \big).$$

Here, we identify $\mathbb{Z}_x$ with $\mathbb{Z}$ via the unique isomorphism $\mathbb{Z}_x \overset{\sim}{\to} \mathbb{Z}$ that maps 1_x to 1, and note that one verifies immediately that these conditions on $(n_x)_{x\in S}$ are *independent* of the choice of $x_0 \in S$. Write

$$\mathcal{P}^{\kappa\text{-sat}}_{S,M}$$

for the image of $(i \circ r)^{-1}(\mathcal{P}^{\mathrm{sat}}_{S,M})$ via the natural homomorphism

$$H^1(\Pi^{(\Sigma)}_{U_M}, \Lambda_X) \longrightarrow \varinjlim_{K\subseteq K^{\dagger}} H^1(\Pi^{(\Sigma)}_{U_{K^{\dagger}}}, \Lambda_X).$$

Let $y \in U(K^{\mathrm{sep}})$ be an element; $I_y \subseteq \Delta^{\Sigma}_{K(X)} \subseteq G^{(\Sigma)}_{K(X)}$ a cuspidal inertia subgroup associated to y. Fix a finite field extension $M \subseteq K_y$ such that $y \in U(K_y)$. Write

$$E_y : \varinjlim_{K \subseteq K^{\dagger}} H^1(\Pi^{(\Sigma)}_{U_{K^{\dagger}}}, \Lambda_X) \longrightarrow \varinjlim_{K \subseteq K^{\dagger}} H^1(G_{K^{\dagger}}, \Lambda_X)$$

for the natural restriction homomorphism induced by the section of the natural surjection $\Pi^{(\Sigma)}_{U_{K_y}} \twoheadrightarrow G_{K_y}$ determined by the image of $N_{G^{(\Sigma)}_{K(X)\otimes_K K_y}}(I_y)$ via the natural surjection $G^{(\Sigma)}_{K(X)\otimes_K K_y} \twoheadrightarrow \Pi^{(\Sigma)}_{U_{K_y}}$. Then, in light of the inductive limit of inflation maps

$$\varinjlim_{K \subseteq K^{\dagger}} H^1(\Pi^{(\Sigma)}_{U_{K^{\dagger}}}, \Lambda_X) \hookrightarrow \varinjlim_{K \subseteq K^{\dagger}} H^1(G^{(\Sigma)}_{K(X)\otimes_K K^{\dagger}}, \Lambda_X),$$

by allowing the finite field extension $K \subseteq M$ and the nonempty finite subset $S \subseteq X(M)$ to vary, we obtain

$$\mathcal{P}^{\kappa\text{-sat}} \stackrel{\text{def}}{=} \bigcup_{S,\ M} \mathcal{P}^{\kappa\text{-sat}}_{S,M} \subseteq \varinjlim_{K \subseteq K^{\dagger}} H^1(G^{(\Sigma)}_{K(X)\otimes_K K^{\dagger}}, \Lambda_X).$$

Definition 2.5. We define

$$\overline{K}(X)^{\kappa\text{-sat}} \subseteq \mathcal{P}^{\kappa\text{-sat}} \quad \left(\subseteq \varinjlim_{K \subseteq K^{\dagger}} H^1(G^{(\Sigma)}_{K(X)\otimes_K K^{\dagger}}, \Lambda_X) \right)$$

as a subset consisting of elements $f \in \mathcal{P}^{\kappa\text{-sat}}$ satisfying one of the following conditions:

(i) There exist a finite separable field extension $K \subseteq M_f$ $(\subseteq \overline{K})$, a nonempty finite subset $S_f \subseteq X(M_f)$ $(\subseteq X(\overline{K}))$, and $z_1, z_2 \in X(\overline{K}) \setminus S_f$ such that

$$f \in \mathcal{P}^{\kappa\text{-sat}}_{S_f,M_f}, \quad E_{z_1}(f) = 0, \quad E_{z_2}(f) \neq 0.$$

(ii) There exist an element $g \in \mathcal{P}^{\kappa\text{-sat}}_{S_g,M_g} \subseteq \mathcal{P}^{\kappa\text{-sat}}$ satisfying condition (i) and $w \in X(\overline{K}) \setminus S_g$ such that the image of $E_w(g)$ via the inductive limit of inflation maps

$$\varinjlim_{K \subseteq K^{\dagger}} H^1(G_{K^{\dagger}}, \Lambda_X) \hookrightarrow \varinjlim_{K \subseteq K^{\dagger}} H^1(G^{(\Sigma)}_{K(X)\otimes_K K^{\dagger}}, \Lambda_X)$$

coincides with f.

Proposition 2.6. *We maintain the above notations. Then the following hold:*

(i) Suppose that K is Σ-AVKF [cf. Remark 2.3.1]. Then the subgroup $\mathcal{P}^{\mathrm{sat}}_{S,M} \subseteq \bigoplus_{x\in S} \mathbb{Z}_x$ $(\xrightarrow{\sim} \bigoplus_{x\in S} \mathbb{Z})$ coincides with the Σ'-saturation of the group of principal divisors associated to the rational functions on X_M whose support are contained in S.

(ii) Suppose that K is Σ-AVKF [cf. Remark 2.3.1]. Let $h \in \mathcal{P}^{\kappa\text{-sat}}_{S,M}$ be an element. Then there exist a positive Σ'-integer m, a rational function h_f on U and an element $c \in H^1(G_M, \Lambda_X)$, such that

$$h^m = \overline{c} \cdot h^{\kappa}_f,$$

where $\overline{c}$ denotes the image of c via the natural composite homomorphism

$$H^1(G_M, \Lambda_X) \to \varinjlim_{K \subseteq K^{\dagger}} H^1(G_{K^{\dagger}}, \Lambda_X) \hookrightarrow \varinjlim_{K \subseteq K^{\dagger}} H^1(\Pi^{(\Sigma)}_{U_{K^{\dagger}}}, \Lambda_X);$$

h^{κ}_f denotes the image of h_f via the Kummer map.

(iii) Suppose that K is Σ-TKND. Let $K \subseteq K^\dagger$ be a finite field extension; f a nonconstant rational function on $X_{K^\dagger}$. Then there exist points $z_1, z_2 \in X_{K^\dagger}(\overline{K})$ such that z_1 and z_2 are neither zeros nor poles of f, and the image of $f(z_1)$ (respectively, $f(z_2)$) via the inductive limit of Kummer maps

$$\overline{K}^\times = \varinjlim_{K \subseteq K^\dagger} (K^\dagger)^\times \longrightarrow \varinjlim_{K \subseteq K^\dagger} H^1(G_{K^\dagger}, \Lambda_X)$$

is the trivial element (respectively, a nontrivial [or equivalently, non-Σ'-torsion] element).

(iv) Suppose that K is a Σ-TKND-AVKF-field [cf. Remark 2.3.1]. Let $h \in \mathcal{P}^{\kappa\text{-sat}}$ be a nonconstant element, i.e., the rational function h_f [cf. (ii)] is nonconstant [where we note that this criterion of nonconstancy is well-defined due to (iii)]. Then it holds that $h \in \overline{K}(X)^{\kappa\text{-sat}}$ if and only if some positive Σ'-integer power of h coincides with the Kummer class of a nonconstant rational function on $X_{K^\dagger}$ for some finite field extension $K \subseteq K^\dagger$.

(v) Let $c \in \overline{K}^\times$ be an element; g a nonconstant rational function on X. Then there exists a point $z \in X(\overline{K})$ such that $g(z)$ coincides with c [cf. [53], Proposition 2.6, (v)].

Proof. First, we verify assertion (i). As in the proof of [53], Proposition 2.6, (i), write $J(X_M)$ for the Jacobian of X_M;

$$\iota : X_M \hookrightarrow J(X_M)$$

for the Albanese embedding relative to $x_0 \in X(M) = X_M(M)$;

$$\chi : J(X_M)(M) \longrightarrow H^1(G_M, \Delta^{\Sigma}_{J(X_M)})$$

for the natural homomorphism obtained by assigning to $z \in J(X_M)(M)$ the difference of the Galois sections $G_M \to \Pi^{(\Sigma)}_{J(X_M)}$ associated to z and the origin. Then, by a similar argument to the argument applied in the proof of [53], Proposition 2.6, (i), together with our assumption that K is Σ-AVKF, one may observe that:

- $\mathrm{Ker}(\chi)$ is a Σ'-torsion group.
- For each

  $$(n_x)_{x \in S} \in \bigoplus_{x \in S} \mathbb{Z}_x$$

  such that $\sum_{x \in S} n_x = 0$, it holds that

  $$(n_x)_{x \in S} \in \mathcal{P}^{\mathrm{sat}}_{S,M} \Leftrightarrow \sum_{x \in S} m n_x \iota(x) \in \mathrm{Ker}(\chi) \ (\subseteq J(X_M)(M)) \text{ for some positive } \Sigma'\text{-integer } m.$$

Thus, we obtain a desired conclusion. This completes the proof of assertion (i). The remainder of the proof is totally similar to the argument applied in the proof of [53], Proposition 2.6, (ii), (iii), (iv). This completes the proof of Proposition 2.6. $\square$

Definition 2.7. In the notation of immediately preceding Definition 2.5, write

$${}_\infty H^1(G_{\overline{K}(X)}, \Lambda_X) \stackrel{\text{def}}{=} \bigcup_{S,M} \varinjlim_{K \subseteq K^\dagger} H^1(\Pi^{(\Sigma)}_{U_{K^\dagger}}, \Lambda_X) \subseteq \varinjlim_{K \subseteq K^\dagger} H^1(G^{(\Sigma)}_{K(X) \otimes_K K^\dagger}, \Lambda_X),$$

where we identify $H^1(\Pi^{(\Sigma)}_{U_{K^\dagger}}, \Lambda_X)$ as a subgroup of $H^1(G^{(\Sigma)}_{K(X)\otimes_K K^\dagger}, \Lambda_X)$ via the natural inflation map. Here, we note that

$$\overline{K}(X)^{\kappa\text{-sat}} \subseteq {}_\infty H^1(G_{\overline{K}(X)}, \Lambda_X)$$

[cf. Definition 2.5], and the inductive limit of Kummer maps

$$K(X_{\overline{K}})^\times = \varinjlim_{K\subseteq K^\dagger} (K(X)\otimes_K K^\dagger)^\times \longrightarrow \varinjlim_{K\subseteq K^\dagger} H^1(G^{(\Sigma)}_{K(X)\otimes_K K^\dagger}, \Lambda_X)$$

factors through a homomorphism

$$\kappa_X : K(X_{\overline{K}})^\times \longrightarrow {}_\infty H^1(G_{\overline{K}(X)}, \Lambda_X).$$

Proposition 2.8. *Suppose that K is a Σ-TKND-AVKF-field. Then the Σ'-saturation of* $\mathrm{Im}(\kappa_X)$ *in* ${}_\infty H^1(G_{\overline{K}(X)}, \Lambda_X)$ *coincides with the Σ'-saturation of $\overline{K}(X)^{\kappa\text{-sat}}$ in ${}_\infty H^1(G_{\overline{K}(X)}, \Lambda_X)$. In particular, in light of the group-theoreticity of $\overline{K}(X)^{\kappa\text{-sat}}$, the group of the Σ'-saturation of Kummer classes of $K(X_{\overline{K}})^\times$ may be reconstructed, in a purely group-theoretic way, from the data of the natural surjection $G^{(\Sigma)}_{K(X)} \twoheadrightarrow G_K$, together with the data of cuspidal inertia subgroups of $G^{(\Sigma)}_{K(X)}$.*

Proof. Proposition 2.8 follows immediately from Proposition 2.6, (iii), (iv), (v), together with the various constructions involved. □

Next, by applying a similar argument to the argument applied in the proof of [53], Proposition 2.8, one may prove the following assertion concerning the preservation of cuspidal inertia subgroups.

Proposition 2.9. *Suppose that K is Σ^∞-AV-tor-finite. Let L be a perfect Σ^∞-AV-tor-finite-field of characteristic $\notin \Sigma$; Y a proper hyperbolic curve over L;*

$$\sigma : G^{(\Sigma)}_{K(X)} \xrightarrow{\sim} G^{(\Sigma)}_{K(Y)}$$

an isomorphism of profinite groups that induces an isomorphism $G_K \xrightarrow{\sim} G_L$ via the natural surjections $G^{(\Sigma)}_{K(X)} \twoheadrightarrow G_K$ and $G^{(\Sigma)}_{K(Y)} \twoheadrightarrow G_L$. Then σ maps the cuspidal inertia subgroups of $G^{(\Sigma)}_{K(X)}$ to the cuspidal inertia subgroups of $G^{(\Sigma)}_{K(Y)}$.

Corollary 2.10. *Suppose that K is a Σ-TKND-AVKF-field. Let L be a perfect Σ-TKND-AVKF-field of characteristic $\notin \Sigma$; Y a proper hyperbolic curve over L;*

$$\sigma : G^{(\Sigma)}_{K(X)} \xrightarrow{\sim} G^{(\Sigma)}_{K(Y)}$$

an isomorphism of profinite groups that induces an isomorphism $G_K \xrightarrow{\sim} G_L$ via the natural surjections $G^{(\Sigma)}_{K(X)} \twoheadrightarrow G_K$ and $G^{(\Sigma)}_{K(Y)} \twoheadrightarrow G_L$. For each finite field extension $K \subseteq K^\dagger$ ($\subseteq \overline{K}$), write $L \subseteq L^\dagger$ ($\subseteq \overline{L}$) for the corresponding finite field extension via the isomorphism $G_K \xrightarrow{\sim} G_L$. Then σ induces naturally commutative diagrams of groups:

$$\begin{array}{ccc}
\left(\varinjlim_{K\subseteq K^\dagger} (K^\dagger)^\times/(K^\dagger)^{\times\Sigma^\infty}\right)^{\mathrm{sat}} & \longrightarrow & \varinjlim_{K\subseteq K^\dagger} H^1(G_{K^\dagger}, \Lambda_X) \\
\downarrow\wr & & \downarrow\wr \\
\left(\varinjlim_{L\subseteq L^\dagger} (L^\dagger)^\times/(L^\dagger)^{\times\Sigma^\infty}\right)^{\mathrm{sat}} & \longrightarrow & \varinjlim_{L\subseteq L^\dagger} H^1(G_{L^\dagger}, \Lambda_Y),
\end{array}$$

$$\begin{array}{ccc}
\left(\varinjlim_{K\subseteq K^\dagger} (K(X)\otimes_K K^\dagger)^\times/(K^\dagger)^{\times\Sigma\infty}\right)^{\mathrm{sat}} & \longrightarrow & {}_\infty H^1(G_{\overline{K(X)}},\Lambda_X) \\
\downarrow\wr & & \downarrow\wr \\
\left(\varinjlim_{L\subseteq L^\dagger} (K(Y)\otimes_L L^\dagger)^\times/(L^\dagger)^{\times\Sigma\infty}\right)^{\mathrm{sat}} & \longrightarrow & {}_\infty H^1(G_{\overline{K(Y)}},\Lambda_Y),
\end{array}$$

where $K \subseteq K^\dagger$ $(\subseteq \overline{K})$ ranges over the finite field extensions; $(-)^{\mathrm{sat}}$ denotes the Σ'-saturation of $(-)$ in the respective codomains of the natural injections induced by the inductive limits of Kummer maps; the horizontal arrows denote the natural injections.

Proof. In light of Proposition 2.2, Corollary 2.10 follows immediately from Propositions 2.8, 2.9. $\square$

Next, we discuss a phenomenon of certain very weak cyclotomic rigidity in the relative anabelian geometric situation.

Lemma 2.11. *The image of the natural injective homomorphism $\mathbb{Q} \hookrightarrow \mathbb{Q} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^{\Sigma}$ is saturated.*

Proof. Lemma 2.11 follows immediately from the fact that the injective homomorphism is a homomorphism of $\mathbb{Q}$-algebras. $\square$

Definition 2.12. For each [possibly empty] subset $S \subseteq \mathfrak{Primes}$, we shall write $\mathbb{Z}_{(S)}$ for the localization of $\mathbb{Z}$ by the multiplicatively closed subset determined by S.

Lemma 2.13. *Write*

$$\kappa : \overline{K}^\times = \varinjlim_{K\subseteq K^\dagger} (K^\dagger)^\times \longrightarrow \varinjlim_{K\subseteq K^\dagger} H^1(G_{K^\dagger}, \widehat{\mathbb{Z}}^{\Sigma}(1))$$

— where $K \subseteq K^\dagger$ $(\subseteq \overline{K})$ ranges over the finite field extensions — for the inductive limit of Kummer maps. Let $\sigma \in \mathrm{Aut}(\widehat{\mathbb{Z}}^{\Sigma}(1)) = (\widehat{\mathbb{Z}}^{\Sigma})^\times$ be an element. Write

$$\tau \in \mathrm{Aut}\left(\varinjlim_{K\subseteq K^\dagger} H^1(G_{K^\dagger}, \widehat{\mathbb{Z}}^{\Sigma}(1))\right)$$

for the automorphism induced by σ. Suppose that

- *$\tau(\mathrm{Im}(\kappa)^{\mathrm{sat}}) = \mathrm{Im}(\kappa)^{\mathrm{sat}}$ [where $\mathrm{Im}(\kappa)^{\mathrm{sat}}$ denotes the Σ'-saturation of $\mathrm{Im}(\kappa)$], and*
- *there exists a nontrivial homomorphism $K^\times \to \mathbb{Z}_{(\Sigma')}$.*

Then it holds that $\sigma \in \mathbb{Q}^\times$ in $\mathbb{Q} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^{\Sigma}$.

Proof. Fix a nontrivial homomorphism $\phi : K^\times \to \mathbb{Z}_{(\Sigma')}$ and write $A \overset{\mathrm{def}}{=} \mathrm{Im}(\phi)$. As in the proof of [53], Lemma 2.11, write $\tilde{\phi} : \overline{K}^\times \twoheadrightarrow \mathbb{Q}$ for the surjective homomorphism obtained by assigning

$$\overline{K}^\times \ni x \ \mapsto\ \frac{\phi(\mathrm{Nm}_{K^\dagger/K}(x))}{[K^\dagger : K]} \in \mathbb{Q},$$

where $K^\dagger$ denotes a finite extension field of K that contains x; $\mathrm{Nm}_{K^\dagger/K} : (K^\dagger)^\times \to K^\times$ denotes the norm map. Write $A^\Sigma \stackrel{\mathrm{def}}{=} \varprojlim_n A/nA$, where n ranges over the positive Σ-integers. Then we have the following commutative diagram

$$\begin{array}{ccc} K^\times & \xrightarrow{\phi} & A \\ \downarrow & & \downarrow \\ \overline{K}^\times & \xrightarrow{\tilde{\phi}} & \mathbb{Q} \\ {\scriptstyle\kappa}\downarrow & & \downarrow \\ \varinjlim_{K \subseteq K^\dagger} H^1(G_{K^\dagger}, \widehat{\mathbb{Z}}^\Sigma(1)) & \xrightarrow{\hat{\phi}} & \mathbb{Q} \otimes_{\mathbb{Z}} A^\Sigma, \end{array}$$

where the right-hand vertical arrows denote the natural injections; the lower horizontal arrow denotes the homomorphism induced by $\tilde{\phi}$. Here, we note that since $\{0\} \neq A \subseteq \mathbb{Z}_{(\Sigma')}$, it follows immediately that $A^\Sigma \cong \widehat{\mathbb{Z}}^\Sigma$. Observe that the natural actions of σ on the domain and the codomain of $\hat{\phi}$ is compatible with $\hat{\phi}$. In particular, since $\tau((\mathrm{Im}(\kappa))^{\mathrm{sat}}) = \mathrm{Im}(\kappa)^{\mathrm{sat}}$, it follows from Lemma 2.11 that the automorphism of $\mathbb{Q} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^\Sigma$ determined by $\sigma \in (\widehat{\mathbb{Z}}^\Sigma)^\times$ maps $1 \in \mathbb{Q} \subseteq \mathbb{Q} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^\Sigma$ to some nonzero rational number $\in \mathbb{Q} \subseteq \mathbb{Q} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^\Sigma$. Thus, we conclude that $\sigma \in \mathbb{Q}^\times$, as desired. This completes the proof of Lemma 2.13. ☐

Proposition 2.14. *In the notation and assumption of Corollary 2.10, suppose that*

- $K = L$, *and*
- *the isomorphism σ lies over the identity automorphism of G_K.*

Then the following hold:

(i) Suppose, moreover, that there exists a nontrivial homomorphism $K^\times \to \mathbb{Z}_{(\Sigma')}$. Then the natural automorphism $\widehat{\mathbb{Z}}^\Sigma(1) \xrightarrow{\sim} \Lambda_X \xrightarrow{\sim} \Lambda_Y \xleftarrow{\sim} \widehat{\mathbb{Z}}^\Sigma(1)$ induced by σ [cf. Proposition 2.2] is multiplication by a nonzero rational number.

(ii) Let l be a prime number. Suppose that $\Sigma = \{l\}$, and K is a positive characteristic l-quasi-finite field of characteristic $\neq l$ that is algebraic over the prime field. Then, after replacing σ by the composite of σ by an inner automorphism of $G^{(l)}_{K(Y)}$, the natural automorphism $\mathbb{Z}_l(1) \xrightarrow{\sim} \Lambda_X \xrightarrow{\sim} \Lambda_Y \xleftarrow{\sim} \mathbb{Z}_l(1)$ induced by σ [cf. Proposition 2.2] is multiplication by a nonzero rational number.

Proof. In light of the first commutative diagram of Corollary 2.10, assertion (i) follows immediately from Lemma 2.13. On the other hand, in light of our assumption on K, assertion (ii) follows immediately from the easily verified fact that G_K is abelian, and the l-adic cyclotomic character $G_K \to \mathbb{Z}_l^\times$ is open. This completes the proof of Proposition 2.14. ☐

3 Geometrically pro-Σ relative birational anabelian Grothendieck Conjecture-type results for curves over Σ-TKND-AVKF-fields

Throughout the present section, we fix a nonempty subset

$$\Sigma \subseteq \mathfrak{Primes}$$

of prime numbers. In the present section, by combining the proof of algebraicity criterion concerning quasi-rational functions developed in [53], §3, with the results obtained in the previous sections, we prove several geometrically pro-Σ birational anabelian Grothendieck Conjecture-type results for smooth curves. For instance, the results include:

- Geometrically pro-Σ relative birational anabelian Grothendieck Conjecture for curves over subfields of the maximal Kummer extension fields of finitely generated fields of characteristic 0.
- Geometrically pro-p relative birational anabelian Grothendieck Conjecture for curves over subfields of mixed characteristic higher local fields with finite final residues of characteristic p.

Definition 3.1. Let K be a perfect field of characteristic $\notin \Sigma$; X an algebraic variety over K. Then:

(i) We shall write

$$\mathrm{Sect}^{\mathrm{Gal}}(X)$$

for the set of equivalence classes of sections of the natural surjective homomorphism $\Pi_X^{(\Sigma)} \twoheadrightarrow G_K$ that arise from the K-valued points of X, where we consider two such sections to be equivalent if they differ by composition with an inner automorphism induced by an element of Δ_X^{Σ}. In particular, we have a natural surjection

$$X(K) \twoheadrightarrow \mathrm{Sect}^{\mathrm{Gal}}(X).$$

(ii) We shall write

$$\mathrm{Sect}^{\mathrm{Op\text{-}Gal}}(X) \stackrel{\mathrm{def}}{=} \varinjlim_{K \subseteq K^{\dagger}} \mathrm{Sect}^{\mathrm{Gal}}(X_{K^{\dagger}}),$$

where $K \subseteq K^{\dagger}$ $(\subseteq \overline{K})$ ranges over the finite field extensions; the transition maps [that appear in the inductive limit] are the natural maps obtained by restricting the domains of sections. In particular, we have a natural surjection

$$X_{\overline{K}}(\overline{K}) \twoheadrightarrow \mathrm{Sect}^{\mathrm{Op\text{-}Gal}}(X).$$

Proposition 3.2. *Let K be a perfect field of characteristic $\notin \Sigma$; X a smooth curve over K. Write*

$$\mathcal{I}_X$$

for the set of conjugacy classes of cuspidal inertia subgroups of $\Delta_{K(X)}^{\Sigma}$ that are not associated to cusps of $X_{\overline{K}}$. Then the natural surjection

$$X_{\overline{K}}(\overline{K}) \twoheadrightarrow \mathcal{I}_X$$

is bijective.

Proof. Proposition 3.2 follows immediately from [32], Proposition 1.2, (i), together with the various definitions involved. $\square$

Proposition 3.3. *We maintain the notation of Proposition 3.2. Suppose that X is a hyperbolic curve over K. Then the composite*

$$\mathcal{I}_X \stackrel{\sim}{\leftarrow} X_{\overline{K}}(\overline{K}) \twoheadrightarrow \mathrm{Sect}^{\mathrm{Op\text{-}Gal}}(X)$$

of the natural maps coincides with the map obtained by forming the images of the normalizers in $G_{K(X)}^{(\Sigma)}$ of cuspidal inertia subgroups of $\Delta_{K(X)}^{\Sigma}$ via the natural [Δ_X^{Σ}-outer] surjective homomorphism $G_{K(X)}^{(\Sigma)} \twoheadrightarrow \Pi_X^{(\Sigma)}$ that lies over the identity automorphism of G_K.

Proof. Proposition 3.3 follows immediately from various definitions involved. ☐

Next, as an application of [53], §3, we prove a key algebraicity result. Before proceeding, we recall the notion of quasi-rational function defined in [53], §3.

Definition 3.4. Let K be an algebraically closed field; $k \subseteq K$ a subfield; X a smooth proper curve over K;

$$f \in \mathrm{Fn}(X(K), K \cup \{\infty\})$$

a set-theoretic function. Then we shall say that f is a *quasi-rational function* on X associated to k if there exist elements

$$\phi_f \in \mathrm{Fn}(X(K), k^{\times}), \quad g \in K(X)$$

such that $f = \phi_f \cdot g$.

Remark 3.4.1. In the above notation, it is immediate to see that the rational functions on X are quasi-rational functions associated to k, and g is determined by f up to multiplication by a constant function.

Proposition 3.5. *Let K be an algebraically closed field; $k \subseteq K$ a subfield such that the extension is of infinite degree; X a smooth proper curve over K;*

$$f \in \mathrm{Fn}(X(K), K \cup \{\infty\})$$

a set-theoretic function. Suppose that:

- *A positive integer power of f is a quasi-rational function on X associated to k such that the rational function determined [up to multiplication by a constant function] by the positive integer power of f is nonconstant.*
- *For each $a \in K$, there exists a positive integer n_a such that the set-theoretic function $(f + a)^{n_a}$ is a quasi-rational function on X associated to k.*

Then f is a rational function on a finite purely inseparable covering of X, where we note that $K(X)^{\mathrm{pf}}$ may be regarded as a subset of $\mathrm{Fn}(X(K), K \cup \{\infty\})$ *in a natural way.*

Proof. First, since the field extension $k \subseteq K$ is of infinite degree, there exists an intermediate extension field $k \subseteq M \subseteq K$ such that the field extension $M \subseteq K$ is an infinite algebraic field extension [cf. [53], Lemma 4.5]. In particular, by replacing k by M, if necessary, we may assume without loss of generality that $k \subseteq K$ is algebraic.

Next, in light of Artin-Schreier theorem [that implies that if the order of the absolute Galois group of a field is finite, then the order is equal to 1 or 2], one may *observe* that, for each positive integer n, if we write k_n $(\subseteq K)$ for the algebraic extension field of k obtained by adjoining k to the n-th roots of elements of k, then the extension $k_n \subseteq K$ is still infinite.

Next, let $b \in K \setminus k$ be an element. Note that since any positive integer power of a quasi-rational function on X is also a quasi-rational function on X, it follows immediately from our assumption that there exist a positive integer n, set-theoretic functions c_1, c_2, c_3, c_4 on $X(K)$ valued in k, and rational functions g_1, g_2, g_3, g_4 on X such that

$$f^n = c_1 g_1, \quad (1-f)^n = (-1)^n \cdot (f-1)^n = c_2 g_2,$$

$$(f+b)^n \;=\; c_3 g_3, \quad (1-f-b)^n \;=\; (-1)^n \cdot (f+b-1)^n \;=\; c_4 g_4.$$

Let $Y \to X$ be a finite ramified normal [not necessary Galois!] covering over K such that, for each $i \in \{1,2,3,4\}$, the rational function g_i admits an n-th root $g_i^{\frac{1}{n}}$ in $K(Y)$. Then, for each $i \in \{1,2,3,4\}$, there exists an n-th root $c_i^{\frac{1}{n}}$ of c_i [valued in k_n] such that

$$f = c_1^{\frac{1}{n}} g_1^{\frac{1}{n}}, \quad 1-f = c_2^{\frac{1}{n}} g_2^{\frac{1}{n}}, \quad f+b = c_3^{\frac{1}{n}} g_3^{\frac{1}{n}}, \quad 1-f-b = c_4^{\frac{1}{n}} g_4^{\frac{1}{n}},$$

where the equalities are considered in $\mathrm{Fn}(Y(K), K \cup \{\infty\})$. On the other hand, in light of our assumption on nonconstancy, we note that f is a quasi-rational function on Y such that the rational function determined [up to multiplication by a constant function] by f is nonconstant. Then since the extension $k_n \subseteq K$ is algebraic and of infinite degree, it follows immediately from [53], Lemma 3.5, that f is rational function on Y, hence on the purely inseparable closure of X in Y. This completes the proof of Proposition 3.5. $\square$

Lemma 3.6. *Let K be a perfect field of characteristic $\notin \Sigma$; X a smooth, proper curve of genus ≥ 2 over K. Then the natural homomorphism*

$$\mathrm{Aut}_K(K(X)^{\mathrm{pf}}) \;\longrightarrow\; \mathrm{Out}(\Delta^{\Sigma}_{K(X)})$$

is injective.

Proof. First, since K is perfect, it holds that $K(X)^{\mathrm{pf}} \otimes_K \overline{K}$ is a perfect field and may be identified with $K(X_{\overline{K}})^{\mathrm{pf}}$. In particular, we may assume without loss of generality that K is algebraically closed. Then, in the case where $\mathrm{char}(K) = 0$, the injectivity of the natural map

$$\mathrm{Aut}_K(K(X)) \;=\; \mathrm{Aut}_K(X) \;\longrightarrow\; \mathrm{Out}(\Delta^{\Sigma}_X)$$

follows from a similar argument to the argument applied in the proof of [8], Lemma 1.14. Next, suppose that $\mathrm{char}(K) = p > 0$. Let $\sigma \in \mathrm{Aut}_K(K(X)^{\mathrm{pf}})$ be an automorphism that induces the identity outer automorphism of $\Delta^{\Sigma}_{K(X)}$. Write $\mathrm{Fr} : K(X)^{\mathrm{pf}} \xrightarrow{\sim} K(X)^{\mathrm{pf}}$ for the absolute Frobenius morphism. Then since X is a smooth curve, there exists an integer m that fits into the following commutative diagram of fields:

$$\begin{array}{ccc}
K(X)^{\mathrm{pf}} & \xrightarrow[\sigma\circ\mathrm{Fr}^m]{\sim} & K(X)^{\mathrm{pf}} \\
\uparrow & & \uparrow \\
K(X) & \xrightarrow[f]{} & K(X),
\end{array}$$

where the vertical arrows denote the natural inclusions; f denotes a finite, purely inseparable field extension that induces the p^m-th Frobenius automorphism on K. Here, we note that $\sigma \circ \mathrm{Fr}^m$ still induces the identity outer automorphism on $\Delta^{\Sigma}_{K(X)}$, hence the identity automorphism on Λ_X. On the other hand, since X is a smooth curve over K, the field extension f factors as the composite morphism $K(X) \xrightarrow{\sim} K(X^{(p^{m'})}) \hookrightarrow K(X)$, where $X^{(p^{m'})}$ denotes the $p^{m'}$-th relative Frobenius twist; the second arrow denotes the field extension associated to the relative Frobenius morphism of degree $p^{m'}$. Therefore, since $\sigma \circ \mathrm{Fr}^m$ induces the identity automorphism on Λ_X, the commutativity of the above diagram implies that $m = m'$, hence that f coincides with the restriction of Fr^m on $K(X)$ up to composition with an automorphism of $K(X)$ over K. Thus, we conclude that σ induces an automorphism on $K(X)$, hence from the discussion in the case where $\mathrm{char}(K) = 0$ that σ is the identity automorphism. This completes the proof of Lemma 3.6. $\square$

Next, we prove our first main theorem [cf. Theorem A]:

Theorem 3.7. *Let K be a perfect Σ-TKND-AVKF-field; $\Sigma^\dagger$ a set of prime numbers that contains Σ; X, Y smooth proper curves over K. Write*

$$\mathrm{Isom}_K(K(Y)^{\mathrm{pf}}, K(X)^{\mathrm{pf}})$$

for the set of K-isomorphisms between $K(Y)^{\mathrm{pf}}$ and $K(X)^{\mathrm{pf}}$;

$$\mathrm{Isom}_{G_K}(G_{K(X)}^{(\Sigma^\dagger)}, G_{K(Y)}^{(\Sigma^\dagger)})/\mathrm{Inn}(\Delta_{K(Y)}^{\Sigma^\dagger})$$

for the set of isomorphisms $G_{K(X)}^{(\Sigma^\dagger)} \xrightarrow{\sim} G_{K(Y)}^{(\Sigma^\dagger)}$ of profinite groups that lie over the identity automorphism of G_K, considered up to compositions with inner automorphisms that arise from elements of $\Delta_{K(Y)}^{\Sigma^\dagger}$. Suppose that:

There exists a nontrivial homomorphism $K^\times \to \mathbb{Z}_{(\Sigma')}$

[cf. Definition 2.12]. Then the natural map

$$\mathrm{Isom}_K(K(Y)^{\mathrm{pf}}, K(X)^{\mathrm{pf}}) \longrightarrow \mathrm{Isom}_{G_K}(G_{K(X)}^{(\Sigma^\dagger)}, G_{K(Y)}^{(\Sigma^\dagger)})/\mathrm{Inn}(\Delta_{K(Y)}^{\Sigma^\dagger})$$

is bijective.

Proof. First, it follows from a routine discussion concerning Galois decent, together with Lemma 3.6, we may assume without loss of generality that

$$\Sigma^\dagger = \Sigma.$$

Next, we verify the injectivity. It follows immediately from the various definitions involved that we may assume without loss of generality that $X = Y$. Moreover, in light of [21], Lemma 1.2, by replacing X by a geometrically pro-Σ finite ramified Galois covering of X, we may assume without loss of generality that X is a smooth proper curve of genus ≥ 2. Then the desired injectivity follows immediately from Lemma 3.6.

Next, we verify the surjectivity. Recall that $\Delta_{K(X)}^{\Sigma}$ is isomorphic to the inverse limit of a system of free pro-Σ profinite groups whose transition maps are surjective. In particular, for every open subgroup $H \subseteq \Delta_{K(X)}^{\Sigma}$, it holds that $Z_{\Delta_{K(X)}^{\Sigma}}(H) = \{1\}$. Thus, to verify the surjectivity, it follows immediately from Galois descent, together with [53], Lemma 4.6, (i), (ii), that we may assume without loss of generality that X and Y have genus ≥ 2. In particular, one may apply the constructions in §2. Let

$$\sigma \in \mathrm{Isom}_{G_K}(G_{K(X)}^{(\Sigma)}, G_{K(Y)}^{(\Sigma)})/\mathrm{Inn}(\Delta_{K(Y)}^{\Sigma})$$

be an element; $\tilde{\sigma} \in \mathrm{Isom}_{G_K}(G_{K(X)}^{(\Sigma)}, G_{K(Y)}^{(\Sigma)})$ a lifting of σ. Then, in light of Propositions 2.9, 3.3, the isomorphism $\tilde{\sigma}$ determines a commutative diagram

$$\begin{array}{ccccc}
X_{\overline{K}}(\overline{K}) & \xrightarrow{\sim} & \mathcal{I}_X & \longrightarrow & \mathrm{Sect}^{\text{Op-Gal}}(X) \\
\downarrow \wr & & \downarrow \wr & & \downarrow \wr \\
Y_{\overline{K}}(\overline{K}) & \xrightarrow{\sim} & \mathcal{I}_Y & \longrightarrow & \mathrm{Sect}^{\text{Op-Gal}}(Y).
\end{array}$$

Write

$$p_{\tilde{\sigma}}^{\kappa} : {}_{\infty}H^1(G_{\overline{K(Y)}}, \Lambda_Y) \xrightarrow{\sim} {}_{\infty}H^1(G_{\overline{K(X)}}, \Lambda_X)$$

[cf. Corollary 2.10] for the isomorphism induced by $\tilde{\sigma}$;

$$p_{\tilde{\sigma}} : \mathrm{Fn}(Y_{\overline{K}}(\overline{K}), \overline{K} \cup \{\infty\}) \xrightarrow{\sim} \mathrm{Fn}(X_{\overline{K}}(\overline{K}), \overline{K} \cup \{\infty\})$$

for the bijection induced by the bijection $X_{\overline{K}}(\overline{K}) \xrightarrow{\sim} Y_{\overline{K}}(\overline{K})$ that appears in the above commutative diagram. Then it follows immediately from Corollary 2.10 that the isomorphism $p^{\kappa}_{\tilde{\sigma}}$ induces an isomorphism

$$f_{\tilde{\sigma}} : \left(\varinjlim_{K \subseteq K^{\dagger}} (K(Y) \otimes_K K^{\dagger})^{\times}/(K^{\dagger})^{\times \Sigma^{\infty}} \right)^{\mathrm{sat}} \xrightarrow{\sim} \left(\varinjlim_{K \subseteq K^{\dagger}} (K(X) \otimes_K K^{\dagger})^{\times}/(K^{\dagger})^{\times \Sigma^{\infty}} \right)^{\mathrm{sat}}$$

via the inductive limits of Kummer maps. Write

$$D_{K,\Sigma} \stackrel{\mathrm{def}}{=} \varinjlim_{K \subseteq K^{\dagger}} (K^{\dagger})^{\times \Sigma^{\infty}}.$$

Next, we make the following observation:

Claim 3.7.A: The bijection $f_{\tilde{\sigma}}$ induces an isomorphism

$$f^{D}_{\tilde{\sigma}} : \left((K(Y) \otimes_K \overline{K})^{\times}/D_{K,\Sigma}\right)^{\mathrm{sat}} \xrightarrow{\sim} \left((K(X) \otimes_K \overline{K})^{\times}/D_{K,\Sigma}\right)^{\mathrm{sat}}.$$

Moreover, there exists an element $c \in \mathbb{Q}^{\times} \cap (\widehat{\mathbb{Z}}^{\Sigma})^{\times}$ such that the diagram

$$\begin{array}{ccc} \left((K(Y) \otimes_K \overline{K})^{\times}/D_{K,\Sigma}\right)^{\mathrm{sat}} & \xrightarrow[f^{D}_{\tilde{\sigma}}]{\sim} & \left((K(X) \otimes_K \overline{K})^{\times}/D_{K,\Sigma}\right)^{\mathrm{sat}} \\ \downarrow & & \downarrow \\ \mathrm{Fn}(Y_{\overline{K}}(\overline{K}), (\overline{K}^{\times}/D_{K,\Sigma})^{\mathrm{sat}} \cup \{0,\infty\}) & \xrightarrow{\sim} & \mathrm{Fn}(X_{\overline{K}}(\overline{K}), (\overline{K}^{\times}/D_{K,\Sigma})^{\mathrm{sat}} \cup \{0,\infty\}) \end{array}$$

— where the lower horizontal arrow denotes the bijection induced by the bijection $X_{\overline{K}}(\overline{K}) \xrightarrow{\sim} Y_{\overline{K}}(\overline{K})$ that appears in the above commutative diagram; the vertical arrows denote the Galois evaluation maps — commutes up to the c-power automorphism of $\mathrm{Fn}(X_{\overline{K}}(\overline{K}), (\overline{K}^{\times}/D_{K,\Sigma})^{\mathrm{sat}} \cup \{0,\infty\})$ [cf. the first commutative diagram in the statement of Corollary 2.10; Proposition 2.14, (i), together with the displayed assumption in the statement of Theorem 3.7].

Indeed, Claim 3.7.A follows immediately from the various definitions involved.

Here, we note that, for each $* \in \{X, Y\}$, there exists a commutative diagram

$$\begin{array}{ccc} (K(*) \otimes_K \overline{K})^{\times}/D_{K,\Sigma} & \longrightarrow & \left((K(*) \otimes_K \overline{K})^{\times}/D_{K,\Sigma}\right)^{\mathrm{sat}} \\ \downarrow & & \downarrow \\ \mathrm{Fn}((*)_{\overline{K}}(\overline{K}), (\overline{K}^{\times}/D_{K,\Sigma}) \cup \{0,\infty\}) & \longrightarrow & \mathrm{Fn}((*)_{\overline{K}}(\overline{K}), (\overline{K}^{\times}/D_{K,\Sigma})^{\mathrm{sat}} \cup \{0,\infty\}), \end{array}$$

where the horizontal arrows denote the natural injections; the left-hand vertical arrow denotes the natural map; the right-hand vertical arrow denotes the Galois evaluation map. Next, we verify the following assertion:

Claim 3.7.B: The bijection $p_{\tilde{\sigma}}$ induces a field isomorphism

$$(K(Y) \otimes_K \overline{K})^{\mathrm{pf}} \xrightarrow{\sim} (K(X) \otimes_K \overline{K})^{\mathrm{pf}}$$

over $\overline{K}$. Moreover, this field isomorphism is G_K-equivariant, hence, in particular, induces a field isomorphism

$$K(Y)^{\mathrm{pf}} \xrightarrow{\sim} K(X)^{\mathrm{pf}}$$

over K.

Indeed, to verify the first assertion of Claim 3.7.B, it suffices to prove that $p_{\tilde{\sigma}}$ maps any rational function on $Y_{\overline{K}}$ to a rational function on the perfection of $X_{\overline{K}}$, where $K(X_{\overline{K}})^{\mathrm{pf}}$ may be regarded as a subset of $\mathrm{Fn}(X(\overline{K}), \overline{K} \cup \{\infty\})$ in a natural way. Note that it follows immediately from the various definitions involved that $p_{\tilde{\sigma}}$ maps any constant rational function on $Y_{\overline{K}}$ to a constant rational function on $X_{\overline{K}}$. Recall that K is a Σ-TKND-field. Then it holds that $K_{\Sigma\text{-div}} \subsetneq \overline{K}$ is an infinite field extension. Note that $D_{K,\Sigma} \subseteq K_{\Sigma\text{-div}}$. Then, by applying Claim 3.7.A, one may observe that $p_{\tilde{\sigma}}$ maps any rational function f_Y on $Y_{\overline{K}}$ to a set-theoretic function f_X on $X_{\overline{K}}$ such that f_X^m is a quasi-rational function on $X_{\overline{K}}$ associated to $K_{\Sigma\text{-div}}$ for some positive integer m. Indeed, it follows immediately from Claim 3.7.A that there exists a positive Σ'-integer n' such that the $c' \overset{\mathrm{def}}{=} cn'$-power of the image of f_X in $\mathrm{Fn}(X_{\overline{K}}(\overline{K}), (\overline{K}^\times / D_{K,\Sigma})^{\mathrm{sat}} \cup \{0, \infty\})$ coincides with the image of a rational function on $X_{\overline{K}}$. Then since c' is a rational number, by considering an integer power of $f_X^{c'}$, we obtain a desired positive integer m. Here, we note that, for each $a \in \overline{K}$, it follows from the construction that the bijection $p_{\tilde{\sigma}}$ maps the rational function $f_Y + a$ on $Y_{\overline{K}}$ to the set-theoretic function $f_X + a$ on $X_{\overline{K}}$. Thus, in light of the above discussion, by applying Proposition 3.5, we conclude that f_X is a rational function in a finite purely inseparable covering, hence that $p_{\tilde{\sigma}}$ maps any rational function on $Y_{\overline{K}}$ to a rational function on the perfection of $X_{\overline{K}}$, as desired. This completes the proof of the first assertion of Claim 3.7.B. Since $\tilde{\sigma}$ lies over the identity automorphism of G_K, the second assertion follows immediately from the first assertion, together with the various constructions involved. This completes the proof of Claim 3.7.B.

Finally, observe that the construction of the bijection $p_{\tilde{\sigma}}$ is functorial with respect to the restrictions of $\tilde{\sigma}$ to the open subgroups of $G_{K(X)}^{(\Sigma)}$. Thus, by applying Claim 3.7.B to the isomorphisms between the open subgroups of $G_{K(X)}^{(\Sigma)}$ and the open subgroups of $G_{K(Y)}^{(\Sigma)}$ induced by $\tilde{\sigma}$, we conclude that $\tilde{\sigma}$ arises from a unique field isomorphism $K(Y)^{\mathrm{pf}} \xrightarrow{\sim} K(X)^{\mathrm{pf}}$ over K. This completes the proof of Theorem 3.7. $\square$

Remark 3.7.1. Note that Theorem 3.7 may be regarded as a desired affirmative answer to the question posed in [53], Remark 4.7.3.

Remark 3.7.2. One may pose further questions concerning the semi-absolute or the curve versions of Theorem 3.7 as in [53], Remarks 4.7.1, 4.7.2. An answer to the former question will be given in the next section. Interestingly, in the semi-absolute case, we do not need to assume the existence of a certain homomorphism from $K^\times$ that appears in the statement of Theorem 3.7 but need to impose a finiteness assumption on torsion points of Jacobians.

Remark 3.7.3. Note that one may not eliminate the displayed assumption in the statement of Theorem 3.7 even if we assume that $\Sigma = \mathfrak{Primes}$, and the base field is a Kummer-faithful field of characteristic 0. Indeed, suppose that Theorem 3.7 holds true without the displayed assumption. Then, by considering the assertion for the function field of smooth curves of genus 0, one may observe that the absolute Galois group of the base field is center-free. On the other hand, T. Asayama recently proved that there exist abundant Kummer-faithful fields of characteristic 0 whose absolute Galois groups are abelian [cf. [2], Corollary 1.5]. This is a contradiction.

Next, we discuss a generalization of [26], Proposition 1, due to W. May:

Proposition 3.8. *Let F be a finitely generated field of characteristic* 0*; $F \subseteq E$ a finite field extension. Then $E^\times / F^\times$ is the product of a finite group and a free abelian group.*

Proof. Write $k_{(-)}$ for the algebraic closure of $\mathbb{Q}$ in $(-)$. Let X be a proper, smooth model of F over k_F. Write $\phi : Y \to X$ for the normalization of X in E; $D_{(-)}$ for the group of Weil divisors on $(-)$; $P_{(-)}$ for the group of principal divisors on $(-)$. Then we have the following exact sequences

$$\mathrm{Ker}(\phi^* : \mathrm{Pic}(X) \to \mathrm{Pic}(Y)) \longrightarrow P_Y/P_X \longrightarrow D_Y/D_X,$$

$$1 \longrightarrow k_E^\times/k_F^\times \longrightarrow E^\times/F^\times \longrightarrow P_Y/P_X \longrightarrow 1.$$

Note that the finite morphism $\phi : Y \to X$ is unramified over all but finite points of X of codimension 1. This immediately implies that D_Y/D_X, hence also the image of P_Y/P_X in D_Y/D_X, is the product of a finite group and a free abelian group. On the other hand, by considering the norm map on the Picard groups, one may observe that $\mathrm{Ker}(\phi^* : \mathrm{Pic}(X) \to \mathrm{Pic}(Y))$ is a torsion subgroup of $\mathrm{Pic}(X)$. Here, we recall that the Neron-Severi group of $X_{\overline{k}_F}$ is a finitely generated abelian group [cf., e.g., [5], §8.4, Theorem 7]. Then since there exists a natural injective homomorphism $\mathrm{Pic}(X) \hookrightarrow \mathrm{Pic}_{X/k_F}(k_F)$ [cf., e.g., [5], §8.1, Proposition 4], it follows immediately from Mordell-Weil theorem that $\mathrm{Ker}(\phi^* : \mathrm{Pic}(X) \to \mathrm{Pic}(Y))$ is finite. Therefore, it follows from Lemma 3.9 below that P_Y/P_X is also the product of a finite group and a free abelian group. Recall that $k_E^\times/k_F^\times$ is the product of a finite group and a free abelian group [cf. [26], Proposition 1 — where, in light of Dirichlet's unit theorem, the proof is totally similar to the above discussion]. Thus, again, by applying Lemma 3.9 below, we conclude that $E^\times/F^\times$ is the product of a finite group and a free abelian group. This completes the proof of Proposition 3.8. □

Lemma 3.9. *Let*

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

be an exact sequence of abelian groups. Suppose that, for $i = 1, 3$, M_i is the product of a finite group and a free abelian group. Then M_2 is also the product of a finite group and a free abelian group.

Proof. For $i = 1, 2, 3$, write $T_i \subseteq M_i$ for the torsion subgroup. Then since T_1 and T_3 are finite, it holds that T_2 is also finite. Note that M_2/T_2 is torsion-free. Moreover, since the free abelian groups are projective, it follows immediately from our assumptions on M_1 and M_3 that there exists a finite index subgroup of M_2/T_2, which is free. Therefore, since M_2/T_2 is torsion-free, it is immediate to see that M_2/T_2 is a free abelian group. Thus, by applying the fact that the free abelian groups are projective again, we conclude that M_2 is the product of a finite group and a free abelian group. This completes the proof of Lemma 3.9. □

Here, we recall the following W. May's observation [cf. [26], Proposition 2]:

Proposition 3.10 (W. May)**.** *Let p be a prime number; F a field characteristic $\neq p$; $F \subseteq E$ a field extension of degree p obtained by adjoining a p-th root $a \in E$ of an element of F. Suppose that $E \neq F(\sqrt{-1})$ in the case where $p = 2$. Then the torsion subgroup of $E^\times/F^\times$ is generated by the images of a and $\mu(E)$.*

Then one may apply the above two propositions to prove the following result, which may be regarded as a certain generalization of W. May's corresponding result on the abelian extensions of number fields [cf. [27], Theorem 2, where we note that the divisible elements of multiplicative groups of abelian extensions of number fields are roots of unity]:

Theorem 3.11. *Let F be a finitely generated field of characteristic 0. Write E $(\subseteq \overline{F})$ for the maximal Kummer extension field of F; $R_F \subseteq E^\times$ for the subgroup consisting of [divisible] elements of $E^\times$ whose some positive integer power is contained in F [so $\mu(\overline{F}) \subseteq R_F \subseteq E^{\times\infty}$]. Then, for any subfield $M \subseteq E$, it holds that $M^\times/M \cap R_F$ is a free abelian group.*

Proof. First, we note that any subgroup of a free abelian group is also a free abelian group. Then it follows immediately from the various definitions involved that we may assume without loss of generality that $M = E$. Let

$$F_0 \overset{\text{def}}{=} F \ \subseteq \ F_1 \overset{\text{def}}{=} F(\sqrt{-1}) \ \subseteq \ F_2 \ \subseteq \ \cdots \ \subseteq \ F_i \ \subseteq \ \cdots$$

a countable sequence of field extensions such that:

- For each positive integer i, the field extension $F_{i-1} \subseteq F_i$ is obtained by adjoining a p_i-th root of a power root of an element of F for some prime number p_i.
- It holds that $E = \bigcup_{i\geq 0} F_i$, where i ranges over the nonnegative integers.

For each nonnegative integer i, write $R_i \overset{\text{def}}{=} F_i \cap R_F$ [so $R_0 = F^\times$]. In particular, we observe that $E^\times/R_F = \varinjlim_{i\geq 0} F_i^\times/R_i$, where the transition maps are the natural injective homomorphisms. Note that $F_1^\times/R_1$ is the quotient of $F_1^\times/F^\times$ by the torsion subgroup, hence a free abelian group [cf. Proposition 3.8]. Then since the free abelian groups are projective, it suffices to verify that $F_{i+1}^\times/R_{i+1}F_i^\times$ is a free abelian group for any positive integer i. To see this, in light of Proposition 3.8 again, it suffices to verify that the subgroup $R_{i+1}F_i^\times/F_i^\times \subseteq F_{i+1}^\times/F_i^\times$ coincides with the torsion subgroup of $F_{i+1}^\times/F_i^\times$. However, in light of the definition of E, this follows immediately from Proposition 3.10, together with the condition on the above countable sequence of field extensions. This completes the proof of Theorem 3.11. $\square$

Lemma 3.12. *The following hold:*

(i) Let K be a mixed characteristic discrete valuation field; L a subfield of a finitely generated extension field of K. Then there exists a surjective homomorphism $L^\times \twoheadrightarrow \mathbb{Z}$.

(ii) Let l, p be distinct prime numbers; K a field of characteristic p; L a subfield of the maximal pro-prime-to-l extension field E of the perfection of a finitely generated extension field of K. Suppose that L is not contained in the algebraic closure of K in E. Then there exists a nontrivial homomorphism $L^\times \to \mathbb{Z}_{(\mathfrak{Primes}\setminus\{l\})}$.

Proof. First, we verify assertion (i). Note that $\mathbb{Q}^\times$ is not contained in the unit group of any mixed characteristic discrete valuation field. In particular, we may assume without loss of generality that L is not contained in any finite extension field of K. Then it follows immediately from [53], Proposition 4.8, that there exists a surjective homomorphism $L^\times \twoheadrightarrow \mathbb{Z}$. This completes the proof of assertion (i).

Next, we verify assertion (ii). Note that any discrete valuation on the finitely generated extension field of K whose restriction on K is trivial extends to a valuation on E whose value group is isomorphic to $\mathbb{Z}_{(\mathfrak{Primes}\setminus\{l\})}$. Then, in light of our assumption on L, it is immediate to see that there exists a nontrivial homomorphism $L^\times \to \mathbb{Z}_{(\mathfrak{Primes}\setminus\{l\})}$. This completes the proof of assertion (ii), hence of Lemma 3.12. $\square$

Corollary 3.13. *Let K be a field satisfying one of the following conditions:*

(a) There exists a finitely generated field K_0 of characteristic 0 such that K is isomorphic to a subfield of the maximal Kummer extension field of K_0.

(b) *There exists a mixed characteristic higher local field K_0 with finite final residues of characteristic $\in \Sigma$ such that K is isomorphic to a subfield of K_0.*

(c) *K is not algebraic over the prime field. Moreover, there exist a prime number $l \in \Sigma$ and a l-quasi-finite field F of positive characteristic $\neq l$ that is algebraic over the prime field such that K is isomorphic to a perfect subfield of the maximal pro-prime-to-l extension field of the perfection of a finitely generated extension field of F.*

Let X, Y be smooth proper curves over K. Then the natural map

$$\mathrm{Isom}_K(K(Y)^{\mathrm{pf}}, K(X)^{\mathrm{pf}}) \longrightarrow \mathrm{Isom}_{G_K}(G^{(\Sigma)}_{K(X)}, G^{(\Sigma)}_{K(Y)})/\mathrm{Inn}(\Delta^{\Sigma}_{K(Y)})$$

is bijective.

Proof. The bijectivity in question follows immediately from Theorem 3.7, together with Proposition 1.12, (i); Corollary 1.20, (i), (ii); Remark 1.1.1, and Theorem 3.11 [in the case of condition (a)]; Lemma 3.12, (i) [in the case of condition (b)]; Lemma 3.12, (ii) [in the case of condition (c)]. This completes the proof of Corollary 3.13. □

Remark 3.13.1. With regard to condition (b) in the statement of Corollary 3.13, Mochizuki proved that one may take K to be any generalized sub-p-adic field [i.e., an immediate consequence of [31], Theorem 4.12]. While one may not obtain this generalization by our method in the present paper [cf. Proposition 1.12, (iii)], our result may be applied to much wider class of fields including fields whose associated cyclotomic characters totally vanish [cf. condition (a)]. In contrast, we recall that the proof obtained by Mochizuki heavily depends on the reconstruction of differentials via p-adic Hodge theory that requires the nontriviality of p-adic cyclotomic characters. Anyway, we need additional works to unify the situations. In our subsequent paper [cf. [56]], we introduce a new class of fields that contain generalized sub-p-adic fields and prove a generalization of Mochizuki's result.

Remark 3.13.2. Recall that, in [44], Theorem 1, (3), F. Pop proved the pro-l birational anabelian Grothendieck conjecture for function fields over $\overline{\mathbb{F}}_p$ of dimension ≥ 2, where p is a prime number $\neq l$. Note that, for each field F, a perfect subfield of the perfection of a finitely generated extension field of F that contains F is also the perfection of a finitely generated extension field of F. Then, in light of suitable Galois descent, together with the various definitions involved, by applying [44], Theorem 1, (3) to our relative setting, one may prove a result similar to Theorem A, (ii), under the following condition:

(c') K is not algebraic over the prime field. Moreover, there exist a prime number $l \in \Sigma$ and a field F of positive characteristic $\neq l$ that is algebraic over the prime field such that K is isomorphic to a perfect subfield of the perfection of a finitely generated extension field of F.

In particular, one may take F to be $\overline{\mathbb{F}}_p$ in condition (c'). This situation may not be realized in condition (c). On the other hand, in condition (c'), one may not take K to be the perfection of the maximal pro-prime-to-l extension field of a finitely generated transcendental field. Thus, in light of conditions (c), (c'), it may be natural to ask

> whether or not our assumption that F is l-quasi-finite in condition (c) may be dropped,

or more generally,

> to which extent pro-l anabelian Grothendieck conjecture-type results hold for certain infinite large algebraic extension fields [such as the maximal pro-prime-to-l extension fields] of function fields over $\overline{\mathbb{F}}_p$ of dimension ≥ 2.

However, at the time of writing of the present paper, the author does not know any partial answers to these questions. On the other hand, in a joint paper with A. Minamide and K. Sawada in preparation [cf. [29]], we discuss a closely related anabelian phenomenon for a certain large class of infinite algebraic extension fields of $\mathbb{Q}$.

Remark 3.13.3. With regard to condition (c) in the statement of Corollary 3.13, one may observe that this includes positive characteristic finitely generated transcendental fields. In particular, we obtain a geometrically pro-l anabelian Grothendieck Conjecture-type result in the positive characteristic setting. On the other hand, it is worth mentioning that we still do not know whether or not the geometrically pro-l birational anabelian Grothendieck Conjecture for curves over finite fields is affirmative. [For sufficiently large Σ, there indeed exists such a result obtained by Saïdi-Tamagawa — cf. [46], Theorem C.]

Remark 3.13.4. With regard to Remark 3.13.3, the author hopes to be able to address a hyperbolic curve analogue of the geometrically pro-l birational anabelian Grothendieck Conjecture-type result for curves over positive characteristic finitely generated transcendental fields in a future paper. However, we recall that, in many situations, an anabelian result over a higher dimensional base field is proved by applying a corresponding anabelian result over a lower dimensional base field via reduction. On the other hand, we only have very limited results for the geometrically pro-l anabelian Grothendieck Conjecture for hyperbolic curves over finite fields. With regard to this direction, it would be worth mentioning that Hoshi proved a partial but highly nontrivial geometrically pro-l anabelian result for the tripod over a finite field [cf. [19], Theorem B]. Anyway, we need a substantially new technique to solve the above problem affirmatively.

Finally, by applying Pop's highly nontrivial result, we strengthen Corollary 3.13 in the case where the base field is the perfection of a positive characteristic finitely generated transcendental field.

Lemma 3.14. *Let K be a Hilbertian field; $\Sigma \subsetneq \mathfrak{Primes}$ a nonempty proper subset. Then G_K admits no nontrivial pro-Σ normal closed subgroup.*

Proof. Suppose not. Let $N \subseteq G_K$ be a nontrivial pro-Σ normal closed subgroup. Recall that every open subgroup of G_K [that is the absolute Galois group of a Hilbertian field again] is center-free, which is an immediate consequence of Weissauer's theorem [cf. [12], Theorem 15.3.1, (b)]. In particular, in light of the conjugation action of G_K on N, it holds that N is not finite. Let $H \subseteq N$ be a proper open subgroup. Then it follows from Weissauer's theorem again that H is the absolute Galois group of a Hilbertian field. In particular, since $\Sigma \subsetneq \mathfrak{Primes}$, it holds that H is not a pro-Σ group [cf. e.g., [12], Corollary 18.3.6]. This contradicts our assumption that N is a pro-Σ group. This completes the proof of Lemma 3.14. $\square$

Theorem 3.15. *Let p be a prime number; X, Y smooth, proper curves over the perfections of finitely generated transcendental fields of characteristic p; $\Sigma \subsetneq \mathfrak{Primes}$ a nonempty proper subset that contains a prime number $\neq p$. Write*

$$\mathrm{Isom}(K(Y), K(X))$$

for the set of isomorphisms between $K(Y)$ and $K(X)$;

$$\mathrm{OutIsom}(G_{K(X)}^{(\Sigma)}, G_{K(Y)}^{(\Sigma)})$$

for the set of outer isomorphisms $G_{K(X)}^{(\Sigma)} \xrightarrow{\sim} G_{K(Y)}^{(\Sigma)}$ of profinite groups. Then the natural map

$$\mathrm{Isom}(K(Y), K(X)) \longrightarrow \mathrm{OutIsom}(G_{K(X)}^{(\Sigma)}, G_{K(Y)}^{(\Sigma)})$$

is bijective [cf. Remark 3.15.1 below].

Proof. First, the injectivity follows immediately from [21], Theorem 1.5. Next, we verify the surjectivity. Let $\sigma : G^{(\Sigma)}_{K(X)} \xrightarrow{\sim} G^{(\Sigma)}_{K(Y)}$ be an isomorphism of profinite groups. Write k_X, k_Y for the respective base fields of X, Y. Then since the operation of perfection does not change the absolute Galois group, it follows immediately from Lemma 3.14, together with the fact that finitely generated transcendental fields are Hilbertian, that σ induces an isomorphism $\sigma_k : G_{k_X} \xrightarrow{\sim} G_{k_Y}$ of profinite groups via the natural surjective homomorphisms $G^{(\Sigma)}_{K(X)} \twoheadrightarrow G_{k_X}$, $G^{(\Sigma)}_{K(Y)} \twoheadrightarrow G_{k_Y}$. Thus, by applying Pop's theorem, to verify that σ arises from a field isomorphism $K(Y) \xrightarrow{\sim} K(X)$, we may assume without loss of generality that $k \stackrel{\mathrm{def}}{=} k_X = k_Y$, and σ_k is the identity automorphism. In this situation, it follows from Corollary 3.13 that σ arises from the following commutative diagram of fields:

$$\begin{array}{ccc} \overline{K(Y)} & \xrightarrow{\sim} & \overline{K(X)} \\ \uparrow & & \uparrow \\ K(Y)^{\mathrm{pf}} & \xrightarrow[\sigma_f]{\sim} & K(X)^{\mathrm{pf}}, \end{array}$$

where the vertical arrows denote the natural inclusions; the upper horizontal arrow denotes an isomorphism that lies over the identity automorphism on $\overline{k}$; the lower horizontal arrow denotes an isomorphism that lies over the identity automorphism on k. Then there exists a positive integer m such that $\sigma_f \circ \mathrm{Fr}^m : K(Y)^{\mathrm{pf}} \xrightarrow{\sim} K(X)^{\mathrm{pf}}$ induces a finite purely inseparable field extension $K(Y) \hookrightarrow K(X)$ [via the natural injections] that factors as the composite $K(Y) \xrightarrow{\sim} K(Y^{(p^m)}) \hookrightarrow K(X)$, where $(-)^{(p^*)}$ denotes the p^*-th Frobenius twist of $(-)$; the first arrow denotes the natural isomorphism; the second arrow denotes a finite purely inseparable extension over k. In particular, there exists a positive integer m' such that the extension $K(Y^{(p^m)}) \hookrightarrow K(X)$ factors as the composite of an isomorphism $K(Y^{(p^m)}) \xrightarrow{\sim} K(X^{(p^{m'})})$ over k and the natural injection $K(X^{(p^{m'})}) \hookrightarrow K(X)$ [determined by the $p^{m'}$-th relative Frobenius morphism]. Thus, it follows immediately from the various definitions involved that the composite

$$K(Y) \xrightarrow{\sim} K(Y^{(p^m)}) \xrightarrow{\sim} K(X^{(p^{m'})}) \xleftarrow{\sim} K(X)$$

[where the final arrow denotes the natural isomorphism] induces the outer isomorphism determines by σ. This completes the proof of Theorem 3.15. □

Remark 3.15.1. Note that since the base fields of X and Y are perfect, by considering the p-divisible elements of their function fields, one may observe that any isomorphism $K(Y) \xrightarrow{\sim} K(X)$ induces an isomorphism between their base fields. This observation is applied to construct the natural map in the statement of Theorem 3.15.

Remark 3.15.2. One may not drop the assumption that $\Sigma \subsetneq \mathfrak{Primes}$. Indeed, Pop's theorem implies that, up to compositions with iterates of the absolute Frobenius automorphism, any outer isomorphism $G_{K(X)} \xrightarrow{\sim} G_{K(Y)}$ arises from a unique isomorphism $K(Y)^{\mathrm{pf}} \xrightarrow{\sim} K(X)^{\mathrm{pf}}$. On the other hand, one may observe that, in general, there exists an isomorphism $K(Y)^{\mathrm{pf}} \xrightarrow{\sim} K(X)^{\mathrm{pf}}$ that does not preserve the respective base fields of X, Y. In particular, this isomorphism does not descend to an isomorphism $K(Y) \xrightarrow{\sim} K(X)$. Thus, we conclude that the natural map

$$\mathrm{Isom}(K(Y), K(X)) \longrightarrow \mathrm{OutIsom}(G_{K(X)}, G_{K(Y)})$$

is not surjective in general.

4 Geometrically pro-Σ semi-absolute birational anabelian Grothendieck Conjecture-type results for curves over Σ-TKND-AVKF-fields

Throughout the present section, we fix a nonempty subset

$$\Sigma \subseteq \mathfrak{Primes}$$

of prime numbers. In the present section, we discuss the semi-absolute analogue of Theorem 3.7 under a certain assumption concerning an ampleness of $\Sigma \subseteq \mathfrak{Primes}$:

Definition 4.1. Let k be a field. Then we shall say that the pair (k, Σ) is of *finite torsion type* if, for each abelian variety A over a finite extension field $k^\dagger$ of k, the cardinality of group of Σ'-torsion elements of $A(k^\dagger)$ is *finite*.

Remark 4.1.1. In the present section, the finite torsion type assumption plays a similar role to condition (ϵ_X) [that depends on a smooth, proper curve X over a finite field] in [46]. In fact, one may weaken the above condition to a similar condition to (ϵ_X) that depends on a curve under consideration. However, as a matter of taste, the author chose to discuss anabelian results under a condition that is independent of the choice of a curve.

Remark 4.1.2. In the semi-absolute setting, in addition to the Σ-TKND-AVKF assumption on base fields, we assume that the pair (k, Σ) is of finite torsion type. On the other hand, since this additional condition can also be confirmed for many fields, this is not so harmful to obtain interesting applications.

For instance, the results in the present section include:

- Geometrically pro-Σ semi-absolute birational anabelian Grothendieck Conjecture for curves over subfields of the maximal Kummer extension fields of finitely generated fields of characteristic 0.
- Geometrically pro-p semi-absolute birational anabelian Grothendieck Conjecture for curves over subfields of mixed characteristic higher local fields with finite final residues of characteristic p.
- Geometrically pro-Σ semi-absolute birational anabelian Grothendieck Conjecture for curves over perfect subfields of the perfections of finitely generated extension fields of quasi-finite fields of positive characteristic that are algebraic over the prime fields under the assumption that Σ' is finite.
- Geometrically pro-Σ absolute birational anabelian Grothendieck Conjecture for curves over subfields of the maximal Kummer extension fields of number fields under the assumption that $\Sigma' \neq \emptyset$. [In fact, this assumption may be eliminated in the next section.]
- Geometrically pro-Σ absolute birational anabelian Grothendieck Conjecture for curves over quasi-finite fields of positive characteristic that are algebraic over the prime fields under the assumption that Σ' is finite.

To reduce a semi-absolute setting to a relative setting, we need to reconstruct the field structures of base fields. For this, we combine the proof of certain lemmas in Saïdi-Tamagawa's paper [46] with the fundamental theorem of projective geometry [that is applied to base fields!].

First, we observe that the finite torsion type assumption enables us to prove that the groups of principal divisors are preserved up to forming suitable finite index subgroups under isomorphisms in question in the semi-absolute settings. For possible use in the future, we discuss several propositions in a setting that includes higher dimensional varieties.

Proposition 4.2. *Let k be a perfect Σ-AVKF-field such that the pair (k, Σ) is of finite torsion type; X a proper, normal variety over k. Then the group of Σ-divisible elements of the Picard group* $\mathrm{Pic}(X)$ *of X is a finite [pro-]Σ' group.*

Proof. First, since X is proper over k, it follows from [5], §8.2, Theorem 3, that $\mathrm{Pic}_{X/k}$ is representable by a scheme. Note that there exists a natural injective homomorphism $\mathrm{Pic}(X) \hookrightarrow \mathrm{Pic}_{X/k}(k) = \mathrm{Pic}^{\mathrm{red}}_{X/k}(k)$ [cf., e.g., [5], §8.1, Proposition 4]. Then it suffices to verify that the group of Σ-divisible elements of $\mathrm{Pic}^{\mathrm{red}}_{X/k}(k)$ is a finite [pro-]Σ' group. On the other hand, we recall that the Neron-Severi group of $X_{\overline{k}}$ is a finitely generated abelian group [cf., e.g., [5], §8.4, Theorem 7]. Therefore, it suffices to verify that the group of Σ-divisible elements of $(\mathrm{Pic}^0_{X/k})^{\mathrm{red}}(k)$ is a finite [pro-]Σ' group. Since X is a proper, normal variety over a perfect field k, one may observe that $\mathrm{Pic}^0_{X/k}$ is proper over k [cf. e.g., [34], Theorem A.4]. In particular, the group scheme $(\mathrm{Pic}^0_{X/k})^{\mathrm{red}}$ is an abelian variety over k. Thus, the desired property follows from the definition of Σ-AVKF-field and our finiteness assumption on the pair (k, Σ). This completes the proof of Proposition 4.2. ☐

Definition 4.3. Let k be a field; K a finitely generated transcendental regular extension field of k; v a geometric valuation on K, i.e., a valuation on K whose restriction to k is trivial. Then we shall say that v is *divisorial* if there exists a proper normal variety X over k whose function field coincides with K such that v arises from the generic point of an integral closed subscheme of X of codimension 1.

In the remainder of the present section, for each abelian group M, we shall write

$$\widehat{M} \overset{\mathrm{def}}{=} \varprojlim_{n} M/nM,$$

where n ranges over the positive Σ-integers.

Definition 4.4. Let k be a field of characteristic $\notin \Sigma$; K a finitely generated transcendental regular extension field of k. Note that any [normalized] divisorial valuation $v : (K^\times \twoheadrightarrow)\ K^\times/k^{\times\Sigma\infty} \twoheadrightarrow \mathbb{Z}$ on K extends to a surjective homomorphism $(K^\times/k^{\times\Sigma\infty} \subseteq)\ H^1(G_K, \widehat{\mathbb{Z}}^\Sigma(1)) = \widehat{K}^\times \twoheadrightarrow \widehat{\mathbb{Z}}^\Sigma$, which is also written as v by a slight abuse of notation. Then:

(i) For each set of [normalized] divisorial valuations $\mathbb{V}$ on K, write

$$E_{\mathbb{V}} : K^\times/k^{\times\Sigma\infty} \longrightarrow \prod_{\mathbb{V}} \mathbb{Z},$$

$$\widehat{E}_{\mathbb{V}} : \widehat{K}^\times \longrightarrow \prod_{\mathbb{V}} \widehat{\mathbb{Z}}^\Sigma$$

for the natural homomorphisms induced by elements in $\mathbb{V}$.

(ii) Let $\mathbb{V}$ be a set of [normalized] divisorial valuations on K. Then we shall say that $\mathbb{V}$ is *arithmetic-like* if $E_{\mathbb{V}}$ induces an injection $K^\times/k^\times \hookrightarrow \prod_{\mathbb{V}} \mathbb{Z}$,

$$\mathrm{Im}(E_{\mathbb{V}}) \subseteq \bigoplus_{\mathbb{V}} \mathbb{Z}\ (\subseteq \prod_{\mathbb{V}} \mathbb{Z}),$$

and the group of Σ-divisible elements of the cokernel of this inclusion is a finite [pro-]Σ' group.

Lemma 4.5. *Let M be an abelian group. Then the following hold:*

(i) Let n be a positive Σ-integer. Then the natural homomorphism $\iota : M/nM \to \widehat{M}/n\widehat{M}$ is an isomorphism.

(ii) Let $N \subseteq M$ be a subgroup such that the group of Σ-divisible elements of M/N is a finite [pro-]Σ' group. Suppose that the natural homomorphism $M \to \widehat{M}$ is injective. Then the cokernel of the inclusion

$$N \subseteq M \bigcap \mathrm{Im}(\widehat{N}) \quad (\subseteq \widehat{M})$$

is a finite [pro-]Σ' group. In particular, if $\Sigma = \mathfrak{Primes}$, then it holds that

$$N = M \bigcap \mathrm{Im}(\widehat{N}) \quad (\subseteq \widehat{M}).$$

Proof. First, we verify assertion (i). Write $q : \widehat{M}/n\widehat{M} \twoheadrightarrow M/nM$ for the natural homomorphism induced by the projection morphism $\widehat{M} \twoheadrightarrow M/nM$. Then since $q \circ \iota$ is the identity automorphism, it holds that ι is injective. To verify the surjectivity of ι, it suffices to verify that q is injective. Let $x = (x_m)_{\geq 1} \in \varprojlim_{m\geq 1} nM/nmM \subseteq \varprojlim_{m\geq 1} M/nmM = \widehat{M}$, where m ranges over the positive Σ-integers. To see the injectivity of q, our goal is to prove that $x \in n\widehat{M}$. For each positive Σ-integer m, let $y_m \in M/nmM$ be such that $ny_m = x_m$ and write $C_m \subseteq M/nmM$ for the image of $y_m + M[n] \subseteq M$ via the natural surjection $M \twoheadrightarrow M/nmM$. Then it is immediate to see that $\{C_m\}$ forms a countable inverse system whose transition morphisms are surjective. In particular, its inverse limit is nonempty. Thus, if we replace y_m by a suitable element $\in y_m + M[n]$, then $y \stackrel{\mathrm{def}}{=} (y_m) \in \widehat{M}$, and $x = ny$. This completes the proof of assertion (i).

Next, we verify assertion (ii). In light of our assumption on M/N, we observe that the cokernel of the inclusion $N \subseteq \cap_{n\geq 1} N + nM$ [where n ranges over the positive Σ-integers] is a finite [pro-]Σ' group. On the other hand, for each positive Σ-integer n, it follows immediately from assertion (i) that $M \cap \mathrm{Im}(\widehat{N}) \subseteq N + nM$. Thus, we obtain the desired property. This completes the proof of assertion (ii), hence of Lemma 4.5. $\square$

Proposition 4.6. *In the notation of Definition 4.4, the following hold:*

(i) Suppose that k is Σ-AVKF, and the pair (k, Σ) is of finite torsion type. Then there exists an arithmetic-like set of divisorial valuations on K. Moreover, if K is the function field of a smooth, proper curve over k, then the set of all divisorial valuations on K is arithmetic-like.

(ii) Let $\mathbb{V}$ be an arithmetic-like set of divisorial valuations on K. Then the cokernel of the inclusion

$$\mathrm{Im}(E_{\mathbb{V}}) \subseteq \left(\bigoplus_{\mathbb{V}} \mathbb{Z}\right) \bigcap \mathrm{Im}(\widehat{E}_{\mathbb{V}}) \quad \left(\subseteq \prod_{\mathbb{V}} \widehat{\mathbb{Z}}^{\Sigma}\right)$$

is a finite [pro-]Σ' group. In particular, if $\Sigma = \mathfrak{Primes}$, then it holds that

$$\mathrm{Im}(E_{\mathbb{V}}) = \left(\bigoplus_{\mathbb{V}} \mathbb{Z}\right) \bigcap \mathrm{Im}(\widehat{E}_{\mathbb{V}}) \quad \left(\subseteq \prod_{\mathbb{V}} \widehat{\mathbb{Z}}\right).$$

Proof. First, we verify assertion (i). Let X be a proper, normal model of K over k. Then it follows immediately from Proposition 4.2, together with elementary properties on the discrete valuations on proper, normal models, that the set of divisorial valuations on X is arithmetic-like. This completes the proof of assertion (i).

Next, we verify assertion (ii). Observe that

$$\bigoplus_{\mathbb{V}} \mathbb{Z} \subseteq \widehat{\bigoplus_{\mathbb{V}} \mathbb{Z}} \subseteq \widehat{\prod_{\mathbb{V}} \mathbb{Z}} = \prod_{\mathbb{V}} \widehat{\mathbb{Z}}^{\Sigma}.$$

Then, if we take "M", "N", to be $\bigoplus_{\mathbb{V}} \mathbb{Z}$, $\mathrm{Im}(E_{\mathbb{V}})$, in Lemma 4.5, (ii), respectively, then the desired property follows from Lemma 4.5, (ii). This completes the proof of assertion (ii), hence of Proposition 4.6. $\square$

Definition 4.7. Let k be a field of characteristic $\notin \Sigma$. Then we shall write

$$\tilde{k} \subseteq \hat{k}^{\times}$$

for the subgroup of elements $x \in \hat{k}^{\times}$ satisfying the following condition:

> There exists a finite separable extension field $k^{\dagger}$ of k such that the image of x in $\widehat{k^{\dagger}}^{\times}$ via the natural homomorphism $\hat{k}^{\times} \to \widehat{k^{\dagger}}^{\times}$ is contained in the image of the natural homomorphism $(k^{\dagger})^{\times} \to \widehat{k^{\dagger}}^{\times}$.

Proposition 4.8. *In the notation of Corollary 2.10, suppose that the pairs (K, Σ) and (L, Σ) are of finite torsion type. Then the following hold:*

(i) The isomorphism σ induces an isomorphism

$$\sigma_D : \mathrm{Div}_X \xrightarrow{\sim} \mathrm{Div}_Y$$

between the respective groups of Weil divisors on X, Y.

(ii) Write $\mathrm{PDiv}_{(-)} \subseteq \mathrm{Div}_{(-)}$ for the subgroup of principal divisors on $(-)$. Then there exist respective finite index subgroups $U_X \subseteq \mathrm{PDiv}_X$, $U_Y \subseteq \mathrm{PDiv}_Y$ such that $\sigma_D(U_X) = U_Y$.

(iii) Fix U_X, U_Y as above. Write $\mathcal{U}_X \subseteq K(X)^{\times}/K^{\times\Sigma^{\infty}}$, $\mathcal{U}_Y \subseteq K(Y)^{\times}/L^{\times\Sigma^{\infty}}$ for the respective finite index subgroups obtained by pulling-back U_X, U_Y, via the natural surjective homomorphisms. Then, in light of the Kummer maps, the isomorphism σ induces isomorphisms

$$(H^1(G^{(\Sigma)}_{K(X)}, \Lambda_X) \supseteq) \; \tilde{K} \cdot \mathcal{U}_X \xrightarrow{\sim} \tilde{L} \cdot \mathcal{U}_Y \; (\subseteq H^1(G^{(\Sigma)}_{K(Y)}, \Lambda_Y)),$$

$$(H^1(G_K, \Lambda_X) \supseteq) \; \tilde{K} \xrightarrow{\sim} \tilde{L} \; (\subseteq H^1(G_L, \Lambda_Y)).$$

(iv) In light of the Kummer maps, the isomorphism σ induces an isomorphism

$$\left(\varinjlim_{K \subseteq K^{\dagger}} H^1(G_{K^{\dagger}}, \Lambda_X) \supseteq \right) \varinjlim_{K \subseteq K^{\dagger}} (K^{\dagger})^{\times}/(K^{\dagger})^{\times\Sigma^{\infty}} \xrightarrow{\sim} \varinjlim_{L \subseteq L^{\dagger}} (L^{\dagger})^{\times}/(L^{\dagger})^{\times\Sigma^{\infty}} \left(\subseteq \varinjlim_{L \subseteq L^{\dagger}} H^1(G_{L^{\dagger}}, \Lambda_Y) \right),$$

where $K \subseteq K^{\dagger}$, $L \subseteq L^{\dagger}$ ranges over the respective finite separable field extensions.

(v) Suppose that K and L are infinite. Then there exist a finite separable extension field $K^{\dagger}$ of K and a nonconstant rational function $f \in K(X_{K^{\dagger}})$ such that:

- *The supports of f and $1+f$ are $K^{\dagger}$-rational, and* $\deg(f)$ *[i.e., the degree of pole divisor of f] coincides with the gonality of $X_{\overline{K}}$.*
- *The images of f, $1+f$ in $K(X_{K^{\dagger}})^{\times}/(K^{\dagger})^{\times\Sigma^{\infty}}$ lie in the image of $\mathcal{U}_X$ via the natural homomorphism $K(X)^{\times}/K^{\times\Sigma^{\infty}} \to K(X_{K^{\dagger}})^{\times}/(K^{\dagger})^{\times\Sigma^{\infty}}$.*
- *Write $L \subseteq L^{\dagger}$ for the finite separable field extension that corresponds to the finite separable field extension $K \subseteq K^{\dagger}$ with respect to σ. Then the images of f, $1+f$ via the composite $K(X_{K^{\dagger}})^{\times} \to \widehat{K(X_{K^{\dagger}})}^{\times} \xrightarrow{\sim} \widehat{K(Y_{L^{\dagger}})}^{\times}$ [induced by σ] lie in the image of $K(Y_{L^{\dagger}})^{\times}$.*

Moreover, the gonalities of $X_{\overline{K}}$ and $Y_{\overline{L}}$ coincide.

Proof. Assertion (i) follows immediately from the construction of "$\mathbb{Z}_x$" immediately preceding Definition 2.5, together with a suitable procedure forming Galois invariants with respect to the base field extensions. Assertion (ii) follows immediately from Proposition 4.6, (i), (ii), together with our assumption that the pairs (K, Σ) and (L, Σ) are of finite torsion type. Assertion (iii) follows from a similar argument to the argument applied in the proof of Proposition 2.6, (iv) [i.e., a technique via Galois evaluation]. Next, by applying assertion (iii) to the finite base extensions of X, Y, one may observe from the definition of $\widetilde{(-)}$ that assertion (iv) holds true. Finally, in light of assertion (iii), assertion (v) follows from a similar argument to the argument applied in the proof of [46], Lemma 4.8 and the former portion of the proof of [46], Lemma 4.9. This completes the proof of Proposition 4.8. $\square$

Next, we prepare a basic setup.

Definition 4.9. Let k be a perfect field of characteristic $\notin \Sigma$; X a smooth, proper [hyperbolic] curve over k. Write

$$\widetilde{\mathcal{K}}(X)^{\times} \overset{\text{def}}{=} \left(\varinjlim_{k \subseteq k^{\dagger}} (K(X) \otimes_k k^{\dagger})^{\times}/(k^{\dagger})^{\times\Sigma^{\infty}} \right)^{\text{sat}},$$

where $k \subseteq k^{\dagger}$ ($\subseteq \overline{k}$) ranges over the finite field extensions; $(-)^{\text{sat}}$ denotes the Σ'-saturation of $(-)$ in a sense similar to Corollary 2.10;

$$\mathcal{K}(X)^{\times} \subseteq \widetilde{\mathcal{K}}(X)^{\times}$$

for the subgroup of elements $f \in \widetilde{\mathcal{K}}(X)^{\times}$ whose [class of] values [at all the points $\in X(\overline{k})$ that f has values in the sense of Galois evaluations] in $\varinjlim_{k \subseteq k^{\dagger}} H^1(G_{k^{\dagger}}, \Lambda_X)$ are contained in the image of $\overline{k}^{\times}$ via the natural Kummer map. In particular, there exists a natural homomorphism

$$(K(X_{\overline{k}})^{\text{pf}})^{\times} \longrightarrow \mathcal{K}(X)^{\times}.$$

Definition 4.10. Let k_X, k_Y be perfect fields of characteristic $\notin \Sigma$; X, Y smooth, proper [hyperbolic] curves over k_X, k_Y. Write

$$D_X \subseteq \overline{k}_X^{\times}; \quad D_Y \subseteq \overline{k}_Y^{\times}$$

for the respective kernels of the natural homomorphisms $\overline{k}_X^{\times} \to \mathcal{K}(X)^{\times}$, $\overline{k}_Y^{\times} \to \mathcal{K}(Y)^{\times}$. Let

$$\phi : X(\overline{k}_X) \xrightarrow{\sim} Y(\overline{k}_Y)$$

be a set-theoretic bijection;

$$\iota : \mathcal{K}(X)^{\times} \xrightarrow{\sim} \mathcal{K}(Y)^{\times}$$

an isomorphism satisfying the following conditions:

(a) For each $x \in X(\overline{k}_X)$, there exists the following commutative diagram:

$$\begin{array}{ccc} \mathcal{K}(X)^{\times} & \xrightarrow[\iota]{\sim} & \mathcal{K}(Y)^{\times} \\ {\scriptstyle \mathrm{ord}_x}\downarrow & & {\scriptstyle \mathrm{ord}_{\phi(x)}}\downarrow \\ \mathbb{Z}_{(\Sigma')} & = & \mathbb{Z}_{(\Sigma')}, \end{array}$$

where $\mathrm{ord}_{(-)}$ denotes the function induced by the order function at $(-)$ on the group of rational functions.

(b) Let $x \in X(\overline{k}_X)$, $f \in \mathcal{K}(X)^{\times}$ be such that $\mathrm{ord}_x(f) = 0$. Write

$$\iota_k : \overline{k}_X^{\times}/D_X \overset{\sim}{\to} \overline{k}_Y^{\times}/D_Y$$

for the natural isomorphism induced by τ via the consideration on divisible elements. Then $\mathrm{ord}_{\phi(x)}(\iota(f)) = 0$ [cf. condition (a)], and the respective [class of] values of f, $\iota(f)$ at x, $\phi(x)$ are compatible with ι_k.

(c) There exists a nonconstant rational function $f \in K(X)$ such that:

- The supports of f and $1 + f$ are k_X-rational, and $\deg(f)$ coincides with the gonality of $X_{\overline{k}_X}$.
- The images of f, $1 + f$ via the composite $K(X)^{\times} \to \mathcal{K}(X)^{\times} \overset{\sim}{\to} \mathcal{K}(Y)^{\times}$ [induced by ι] lie in the image of $K(Y)^{\times}$.

Moreover, the gonalities of $X_{\overline{k}_X}$ and $Y_{\overline{k}_Y}$ coincide.

Let

$$\tilde{\iota} : \overline{k}_X^{\times} \longrightarrow \overline{k}_Y^{\times}$$

be a [set-theoretic] map that lifts ι_k with respect to the natural surjective homomorphisms $\overline{k}_X^{\times} \twoheadrightarrow \overline{k}_X^{\times}/D_X$, and $\overline{k}_Y^{\times} \twoheadrightarrow \overline{k}_Y^{\times}/D_Y$.

Next, with regard to the above basic setup, we discuss certain properties on preservations of field structures.

Proposition 4.11. *In the notation of Definition 4.10, let c_1, c_2 be elements in $\overline{k}_X^{\times}$ such that $c_1 + c_2 \neq 0$. Then there exist elements a, b in D_Y such that*

$$\tilde{\iota}(c_1 + c_2) \;=\; a\tilde{\iota}(c_1) + b\tilde{\iota}(c_2) \;\in\; \overline{k}_Y^{\times}.$$

Proof. In light of conditions (a), (b), (c), Proposition 4.11 follows from a similar argument to the argument applied in the latter portion of the proof of [46], Lemma 4.9. [Here, we note that the discussion on the preservation of -1 in the proof of Claim in the proof of [46], Lemma 4.9 is not essential.] □

Remark 4.11.1. In the case where $\Sigma = \mathfrak{Primes}$, Proposition 4.11 follows immediately from a similar argument to the argument applied in the proof of [35], Proposition 1.2, (ii), (iii), i.e., the proof of Uchida's lemma. In particular, in this case, we do not need to resort to the much more complicated proofs of lemmas in [46].

Proposition 4.12. *In the notation of Definition 4.10, the following hold:*

(i) The isomorphism ι_k induces an isomorphism

$$\iota_f : \overline{k}_X^\times/((k_X)_{\mathrm{prm}}(D_X))^\times \ \overset{\sim}{\to}\ \overline{k}_Y^\times/((k_Y)_{\mathrm{prm}}(D_Y))^\times$$

that preserves collineations.

(ii) Suppose that the field extensions $(k_X)_{\mathrm{prm}}(D_X) \subseteq \overline{k}_X$, $(k_Y)_{\mathrm{prm}}(D_Y) \subseteq \overline{k}_Y$ are of infinite degree. Then ι_f [cf. (i)] arises from a unique field isomorphism

$$\overline{k}_X \ \overset{\sim}{\to}\ \overline{k}_Y$$

that lies over a field isomorphism $(k_X)_{\mathrm{prm}}(D_X) \overset{\sim}{\to} (k_Y)_{\mathrm{prm}}(D_Y)$.

(iii) Suppose that the field extensions $(k_X)_{\mathrm{prm}}(D_X) \subseteq \overline{k}_X$, $(k_Y)_{\mathrm{prm}}(D_Y) \subseteq \overline{k}_Y$ are of infinite degree. Write

$$\iota_{\mathrm{Fn}} : \mathrm{Fn}(Y(\overline{k}_Y), \overline{k}_Y \cup \{\infty\}) \ \overset{\sim}{\to}\ \mathrm{Fn}(X(\overline{k}_X), \overline{k}_X \cup \{\infty\})$$

for the bijection induced by the bijection $\phi : X(\overline{k}_X) \overset{\sim}{\to} Y(\overline{k}_Y)$ and the isomorphism $\overline{k}_X \overset{\sim}{\to} \overline{k}_Y$ [cf. (ii)]. Then $\iota_{\mathrm{Fn}}(K(Y_{\overline{k}_Y})^{\mathrm{pf}}) = K(X_{\overline{k}_X})^{\mathrm{pf}}$, where we regard $K(-)^{\mathrm{pf}}$ as a subset of $\mathrm{Fn}((-)(\overline{k}_{(-)}), \overline{k}_{(-)} \cup \{\infty\})$ in a natural way. In particular, the isomorphism $\iota : \mathcal{K}(X)^\times \overset{\sim}{\to} \mathcal{K}(Y)^\times$ induces an isomorphism $(K(X_{\overline{k}_X})^{\mathrm{pf}})^\times/D_X \overset{\sim}{\to} (K(Y_{\overline{k}_Y})^{\mathrm{pf}})^\times/D_Y$ that arises from a field isomorphism $K(X_{\overline{k}_X})^{\mathrm{pf}} \overset{\sim}{\to} K(Y_{\overline{k}_Y})^{\mathrm{pf}}$.

Proof. Assertion (i) follows immediately from Proposition 4.11. Since ι_f preserves collineations, assertion (ii) follows immediately from the fundamental theorem of projective geometry, together with our assumption that the field extensions $(k_X)_{\mathrm{prm}}(D_X) \subseteq \overline{k}_X$, $(k_Y)_{\mathrm{prm}}(D_Y) \subseteq \overline{k}_Y$ are of infinite degree [cf. [1]]. Assertion (iii) follows immediately from Proposition 3.5. This completes the proof of Proposition 4.12. ☐

Definition 4.13. For each function field F of a smooth curve over a field, write $(F^{\mathrm{sep}})^{(\Sigma)}$ ($\subseteq F^{\mathrm{sep}}$) for the maximal geometrically pro-Σ separable extension field of F.

Lemma 4.14. *Let k be an algebraically closed field; X a smooth, proper curve over k; $\sigma : G^{(\Sigma)}_{K(X)} \overset{\sim}{\to} G^{(\Sigma)}_{K(X)}$ an automorphism of profinite groups. Suppose that σ is compatible with the natural action of $G^{(\Sigma)}_{K(X)}$ on $((K(X)^{\mathrm{sep}})^{(\Sigma)})^\times/k^\times$. Then σ is the identity automorphism.*

Proof. Let $s \in G^{(\Sigma)}_{K(X)}$ be an element. Then it follows from our assumption that $s' \overset{\mathrm{def}}{=} \sigma(s)s^{-1}$ acts trivially on $((K(X)^{\mathrm{sep}})^{(\Sigma)})^\times/k^\times$. It suffices to verify that s' is the identity element. Let $f \in ((K(X)^{\mathrm{sep}})^{(\Sigma)})^\times \backslash k^\times$ be an element. Then there exist elements $a, b \in k^\times$ such that $s'(f) = af$, and $(1 - s'(f) =) s'(1-f) = b(1-f)$. Since $f \in ((K(X)^{\mathrm{sep}})^{(\Sigma)})^\times \setminus k^\times$, these equalities immediately imply that $a = b = 1$. In particular, it holds that s' fixes the nonconstant rational functions. Thus, since any constant function may be written as the difference of nonconstant rational functions, we conclude that s' is the identity element, as desired. This completes the proof of Lemma 4.14. ☐

Theorem 4.15. *Let X, Y be smooth, proper curves over Σ-TKND-AVKF infinite perfect fields k_X, k_Y of characteristic $\notin \Sigma$; $\Sigma^\dagger$ a set of prime numbers that contains Σ. Suppose that the pairs (k_X, Σ) and (k_Y, Σ) are of finite torsion type. Write*

$$\mathrm{Isom}(K(Y)/k_Y, K(X)/k_X)$$

for the set of isomorphisms $K(Y) \xrightarrow{\sim} K(X)$ of fields that induce isomorphisms $k_Y \xrightarrow{\sim} k_X$ via the natural inclusions $k_Y \subseteq K(Y)$, $k_X \subseteq K(X)$;

$$\mathrm{OutIsom}(G^{(\Sigma^\dagger)}_{K(X)}/G_{k_X}, G^{(\Sigma^\dagger)}_{K(Y)}/G_{k_Y})$$

for the set of outer isomorphisms $G^{(\Sigma^\dagger)}_{K(X)} \xrightarrow{\sim} G^{(\Sigma^\dagger)}_{K(Y)}$ of profinite groups that induce outer isomorphisms $G_{k_X} \xrightarrow{\sim} G_{k_Y}$ via the natural surjective homomorphisms $G^{(\Sigma^\dagger)}_{K(X)} \twoheadrightarrow G_{k_X}$, $G^{(\Sigma^\dagger)}_{K(Y)} \twoheadrightarrow G_{k_Y}$. Then the natural map

$$\mathrm{Isom}(K(Y)/k_Y, K(X)/k_X) \longrightarrow \mathrm{OutIsom}(G^{(\Sigma^\dagger)}_{K(X)}/G_{k_X}, G^{(\Sigma^\dagger)}_{K(Y)}/G_{k_Y})$$

is bijective.

Proof. First, the injectivity follows immediately from [21], Theorem 1.5. Then, in light of Galois descent, by replacing X, Y by their suitable respective finite [possibly, ramified] coverings, we may assume without loss of generality that:

$$\Sigma \;=\; \Sigma^\dagger; \;\; X \text{ and } Y \text{ have genus} \geq 2.$$

Next, we verify the surjectivity. Let

$$\sigma : G^{(\Sigma)}_{K(X)} \xrightarrow{\sim} G^{(\Sigma)}_{K(Y)}$$

be an isomorphism of profinite groups that induces an isomorphism $G_{k_X} \xrightarrow{\sim} G_{k_Y}$ of profinite groups. Write

$$D_X \stackrel{\mathrm{def}}{=} \varinjlim_{k_X \subseteq k_X^\dagger} (k_X^\dagger)^{\times \Sigma\infty}, \quad D_Y \stackrel{\mathrm{def}}{=} \varinjlim_{k_Y \subseteq k_Y^\dagger} (k_Y^\dagger)^{\times \Sigma\infty},$$

where $k_X \subseteq k_X^\dagger$ $(\subseteq \overline{k}_X)$, $k_Y \subseteq k_Y^\dagger$ $(\subseteq \overline{k}_Y)$ range over the respective finite field extensions. Then, in light of Corollary 2.10; Proposition 4.8, (iv), by applying a similar argument to the argument applied in the proof of Theorem 3.7, one may observe that σ induces

- a set-theoretic bijection

  $$\phi : X(\overline{k}_X) \xrightarrow{\sim} Y(\overline{k}_Y)$$

  and

- an isomorphism

  $$\iota : \mathcal{K}(X)^\times \xrightarrow{\sim} \mathcal{K}(Y)^\times$$

  [that induces an isomorphism $\iota_k : \overline{k}_X^\times/D_X \xrightarrow{\sim} \overline{k}_Y^\times/D_Y$] compatible with the respective valuation maps $\mathrm{ord}_{(-)}$ on $K(X_{\overline{k}_X})^\times/D_X$, $K(Y_{\overline{k}_Y})^\times/D_Y$ with respect to ϕ

such that the set-theoretic bijection

$$\mathrm{Fn}(Y(\overline{k}_Y), (\overline{k}_Y^\times/D_Y) \cup \{0, \infty\}) \xrightarrow{\sim} \mathrm{Fn}(X(\overline{k}_X), (\overline{k}_X^\times/D_X) \cup \{0, \infty\})$$

induced by ϕ and ι_k is compatible with ι with respect to the natural embeddings

$$\mathcal{K}(X)^\times \hookrightarrow \mathrm{Fn}(X(\overline{k}_X), (\overline{k}_X^\times/D_X) \cup \{0, \infty\}),$$

$$\mathcal{K}(Y)^{\times} \hookrightarrow \mathrm{Fn}(Y(\overline{k}_Y), (\overline{k}_Y^{\times}/D_Y) \cup \{0, \infty\})$$

that arise from Galois evaluations. Moreover, since k_X and k_Y are infinite, it follows immediately from Proposition 4.8, (v), that, after replacing k_X, k_Y by their suitable finite extension fields, if necessary, we may assume without loss of generality that:

- There exists a nonconstant rational function $f \in K(X)$ such that:
  - The supports of f and $1+f$ are k_X-rational, and $\deg(f)$ coincides with the gonality of $X_{\overline{k}_X}$.
  - The images of f, $1+f$ via the composite $K(X)^{\times} \to \mathcal{K}(X)^{\times} \xrightarrow{\sim} \mathcal{K}(Y)^{\times}$ [induced by ι] lie in the image of $K(Y)^{\times}$.
- The gonalities of $X_{\overline{k}_X}$ and $Y_{\overline{k}_Y}$ coincide.

Here, we note that since k_X and k_Y are Σ-TKND-fields, the field extensions $(k_X)_{\mathrm{prm}}(D_X) \subseteq \overline{k}_X$, $(k_Y)_{\mathrm{prm}}(D_Y) \subseteq \overline{k}_Y$ are of infinite degree. Then it follows from Proposition 4.12, (ii), (iii), together with Proposition 4.8, (iii) [for the deperfection], that there exist

- a field isomorphism
$$\overline{k}_Y \xrightarrow{\sim} \overline{k}_X$$
compatible with ι_k^{-1} with respect to the natural surjective homomorphisms $\overline{k}_X^{\times} \twoheadrightarrow \overline{k}_X^{\times}/D_X$, $\overline{k}_Y^{\times} \twoheadrightarrow \overline{k}_Y^{\times}/D_Y$. [Note that this isomorphism is compatible with the isomorphism $G_{k_X} \xrightarrow{\sim} G_{k_Y}$ induced by σ due to the uniqueness in the statement of Proposition 4.12, (ii).]
- a field isomorphism
$$K(Y_{\overline{k}_Y}) \xrightarrow{\sim} K(X_{\overline{k}_X})$$
compatible with the isomorphism $G_{k_X} \xrightarrow{\sim} G_{k_Y}$ and a set-theoretic bijection
$$\mathrm{Fn}(Y(\overline{k}_Y), \overline{k}_Y \cup \{\infty\}) \xrightarrow{\sim} \mathrm{Fn}(X(\overline{k}_X), \overline{k}_X \cup \{\infty\})$$
[induced by ϕ and the above field isomorphism $\overline{k}_Y \xrightarrow{\sim} \overline{k}_X$] with respect to the natural embeddings
$$K(X_{\overline{k}_X}) \hookrightarrow \mathrm{Fn}(X(\overline{k}_X), \overline{k}_X \cup \{\infty\}),$$
$$K(Y_{\overline{k}_Y}) \hookrightarrow \mathrm{Fn}(Y(\overline{k}_Y), \overline{k}_Y \cup \{\infty\})$$
that arise from usual evaluations.

Thus, in light of the uniqueness in Proposition 4.12, (ii), by applying the above constructions to the isomorphisms obtained by restricting σ to open subgroups of $G^{(\Sigma)}_{K(X)}$, we obtain a field isomorphism

$$\psi : (K(Y_{\overline{k}_Y})^{\mathrm{sep}})^{(\Sigma)} \xrightarrow{\sim} (K(X_{\overline{k}_X})^{\mathrm{sep}})^{(\Sigma)}$$

[that lies over the field isomorphism $K(Y_{\overline{k}_Y}) \xrightarrow{\sim} K(X_{\overline{k}_X})$] such that the isomorphism $((K(Y_{\overline{k}_Y})^{\mathrm{sep}})^{(\Sigma)})^{\times}/\overline{k}_Y^{\times} \xrightarrow{\sim} ((K(X_{\overline{k}_X})^{\mathrm{sep}})^{(\Sigma)})^{\times}/\overline{k}_X^{\times}$ induced by ψ is compatible with the isomorphism $\Delta^{\Sigma}_{K(X)} \xrightarrow{\sim} \Delta^{\Sigma}_{K(Y)}$ induced by σ with respect to the natural actions. Then it follows immediately from Lemma 4.14 that one may observe that the isomorphism $\Delta^{\Sigma}_{K(X)} \xrightarrow{\sim} \Delta^{\Sigma}_{K(Y)}$ arises from ψ. Thus, since the field isomorphism $K(Y_{\overline{k}_Y}) \xrightarrow{\sim} K(X_{\overline{k}_X})$ is compatible with the isomorphism $G_{k_X} \xrightarrow{\sim} G_{k_Y}$, we conclude from the center-freeness of $\Delta^{\Sigma}_{K(X)}$ that σ arises from ψ that lies over a field isomorphism $K(Y) \xrightarrow{\sim} K(X)$. This completes the proof of Theorem 4.15. $\square$

Remark 4.15.1. Theorem 4.15 may be regarded as an affirmative resolution of Question 1 in [53], Remark 4.7.2 and a substantial generalization of the Mochizuki's corresponding result for function fields of curves over Kummer-faithful fields [cf. [35], Theorem 1.11] even if we restrict our attention to the case of characteristic 0.

Remark 4.15.2. At the time of writing of the present paper, the author does not know whether or not the finite torsion type assumption in the statement of Theorem 4.15 may be eliminated.

Corollary 4.16. *Let k_X, k_Y be fields that belong to one of the following types of fields:*

(a) A field isomorphic to a subfield of the maximal Kummer extension field of a finitely generated field of characteristic 0.

(b) A field isomorphic to a subfield of a mixed characteristic higher local field with finite final residues of characteristic $\in \Sigma$.

(c) A perfect field isomorphic to a subfield of the perfection of a finitely generated extension field of a quasi-finite field of positive characteristic that is algebraic over the prime field.

Let X, Y be smooth proper curves over k_X, k_Y. If k_X or k_Y is a field of type (c), then we suppose that $\Sigma \subseteq \mathfrak{Primes}$ is a cofinite subset. Then the natural map

$$\mathrm{Isom}(K(Y)/k_Y, K(X)/k_X) \longrightarrow \mathrm{OutIsom}(G^{(\Sigma)}_{K(X)}/G_{k_X}, G^{(\Sigma)}_{K(Y)}/G_{k_Y})$$

is bijective.

Proof. First, in light of Galois descent, by shrinking Σ, if necessary, we may assume without loss of generality that Σ does not contain the characteristics of k_X, k_Y. Next, we observe that, if $\Sigma \subseteq \mathfrak{Primes}$ is a cofinite subset, then it is immediate to see that any quasi-finite field of positive characteristic $\notin \Sigma$ that is algebraic over the prime field is Σ-TKND. Then it follows immediately from Propositions 1.9; 1.12, (i); Corollary 1.20, (i), (ii); Remark 1.1.1, that the fields of types (a), (c) are Σ-TKND-AVKF, and the fields of type (b) are $\{p\}$-TKND-AVKF for some prime number $p \in \Sigma$. Moreover, it follows immediately from the proofs of Propositions 1.9; 1.12, (i); Theorem 1.17, that the pairs (k_X, Σ) and (k_Y, Σ) are of finite torsion type. On the other hand, we note that if k_X or k_Y is finite, then, by replacing k_X, k_Y, by suitable infinite algebraic extension fields of k_X, k_Y that are quasi-finite, we may assume without loss of generality that both k_X and k_Y are infinite. Thus, the bijectivity in question follows from Theorem 4.15. This completes the proof of Corollary 4.16. ☐

Corollary 4.17. *Let X, Y be smooth proper curves over fields. Write*

$$\mathrm{Isom}(K(Y), K(X))$$

for the set of isomorphisms $K(Y) \xrightarrow{\sim} K(X)$ of fields;

$$\mathrm{OutIsom}(G^{(\Sigma)}_{K(X)}, G^{(\Sigma)}_{K(Y)})$$

for the set of outer isomorphisms $G^{(\Sigma)}_{K(X)} \xrightarrow{\sim} G^{(\Sigma)}_{K(Y)}$ of profinite groups. Then the following hold:

(i) Suppose that the base fields of X and Y are isomorphic to subfields of the maximal Kummer extension fields of number fields, and $\Sigma \subsetneq \mathfrak{Primes}$. Then the natural map

$$\mathrm{Isom}(K(Y), K(X)) \ \longrightarrow \ \mathrm{OutIsom}(G^{(\Sigma)}_{K(X)}, G^{(\Sigma)}_{K(Y)})$$

is bijective.

(ii) Suppose that the base fields of X and Y are positive characteristic quasi-finite fields that are algebraic over the prime field, and $\Sigma \subseteq \mathfrak{Primes}$ is a cofinite subset. Then the natural map

$$\mathrm{Isom}(K(Y), K(X)) \ \longrightarrow \ \mathrm{OutIsom}(G^{(\Sigma)}_{K(X)}, G^{(\Sigma)}_{K(Y)})$$

is bijective.

Proof. Write k_X, k_Y for the respective base fields of X, Y. Then, in light of Corollary 4.16, it suffices to verify the equalities

$$\mathrm{Isom}(K(Y), K(X)) \ = \ \mathrm{Isom}(K(Y)/k_Y, K(X)/k_X),$$

$$\mathrm{OutIsom}(G^{(\Sigma)}_{K(X)}, G^{(\Sigma)}_{K(Y)}) \ = \ \mathrm{OutIsom}(G^{(\Sigma)}_{K(X)}/G_{k_X}, G^{(\Sigma)}_{K(Y)}/G_{k_Y}).$$

The first equality follows immediately from the fact that k_X and k_Y are algebraic over the respective prime fields.

Next, we verify the second equality in the case of assertion (i). It is immediate to see that it suffices to verify that the absolute Galois group of a subfield F of the maximal Kummer extension field E of a number field admits no nontrivial pro-Σ normal closed subgroup. Let $N \subseteq G_F$ be a pro-Σ normal closed subgroup. Note that since E is a metabelian extension field of a number field, it follows immediately from [12], Theorem 19.2.3, that E is Hilbertian. Then it follows from Lemma 3.14 that $N \cap G_E = \{1\}$, hence that $N \subseteq Z_{G_F}(G_E)$. On the other hand, since $G_{\mathbb{Q}}$ is sn-internally indecomposable [cf. [28], Theorem B], one may observe that $Z_{G_F}(G_E) \subseteq Z_{G_{\mathbb{Q}}}(G_E) = \{1\}$, hence that $N = \{1\}$. This completes the proof of the second equality in the case of assertion (i).

Finally, we verify the second equality in the case of assertion (ii). Note that if $\mathrm{char}(k_X) \notin \Sigma$, then since k_X is quasi-finite and algebraic over the prime field, it follows immediately from a weight argument for each prime number $l \in \Sigma$ that any surjective homomorphism $\phi : G^{(\Sigma)}_{K(X)} \twoheadrightarrow \widehat{\mathbb{Z}}$ factors through the natural quotient $G^{(\Sigma)}_{K(X)} \twoheadrightarrow G_{k_X}$. Moreover, since $G_{k_X} \cong \widehat{\mathbb{Z}}$, the surjective homomorphism $G_{k_X} \twoheadrightarrow \widehat{\mathbb{Z}}$ induced by ϕ is an isomorphism. Therefore, we complete the proof of the equality in this case. Thus, by symmetry, we may assume without loss of generality that $\mathrm{char}(k_X) \in \Sigma$, and $\mathrm{char}(k_Y) \in \Sigma$. In fact, this case follows from a similar [but slightly different] discussion to the proof of [46], Lemma 2.4.3. Indeed, since k_X and k_Y are quasi-finite and algebraic over the prime fields, the preservation of decomposition subgroups associated to closed points is proved by a totally similar technique using Brauer groups [cf. the proof of [46], Proposition 1.3]. Moreover, in light of the structure of decomposition subgroups [cf. [46], Proposition 1.2], together with our assumption that $\mathrm{char}(k_{(-)}) \in \Sigma$, one may observe that $\mathrm{char}(k_{(-)})$ may be characterized as the unique prime number l such that there exists a topologically non-finitely generated pro-l normal closed subgroup of a decomposition subgroup of $G^{(\Sigma)}_{K(-)}$. Write $p \overset{\mathrm{def}}{=} \mathrm{char}(k_X) = \mathrm{char}(k_Y)$. Here, we note that since $k_{(-)}$ is a positive characteristic quasi-finite field algebraic over the prime field, and $\Sigma \subseteq \mathfrak{Primes}$ is a cofinite subset,

- the pro-$(\Sigma \setminus \{p\})$ cyclotomic character associated to $k_{(-)}$ is injective, and
- the pro-l cyclotomic character associated to $k_{(-)}$ is open for each prime number $l \neq p$.

Then, in light of the first bullet, together with the structure of tame quotient, the wild inertia subgroup of a decomposition subgroup may be characterized as the maximal pro-p normal closed subgroup of the decomposition subgroup. Furthermore, in light of the second bullet, the inertia subgroup of a decomposition subgroup may be characterized as the pull-back of the maximal abelian normal closed subgroup of the tame quotient of the decomposition subgroup. If the inertia subgroups of decomposition subgroups may be reconstructed, then one may reduce our problem to a similar problem for the geometrically pro-Σ fundamental groups of X, Y, i.e., the smooth, *proper* curves. Then the desired equality follows from a similar weight argument that appears in the above discussion in the case where $\mathrm{char}(k_X) \notin \Sigma$. This completes the proof of the second equality in the case of assertion (ii), hence of Corollary 4.17. $\square$

Remark 4.17.1. In the statement of Corollary 4.17, (i), we suppose that $\Sigma \subsetneq \mathfrak{Primes}$. In fact, this assumption may be dropped in the next section in a more general higher dimensional setting [cf. Theorem 5.21].

Remark 4.17.2. Corollary 4.17, (ii), may be regarded as a generalization of the classical Uchida's theorem to the function fields of curves over quasi-finite bases. At the time of writing of the present paper, the author does not know whether or not our assumption that the base fields are algebraic over the prime fields may be eliminated.

Remark 4.17.3. Let p be a prime number. In light of Corollary 4.16, under condition (b), it is natural to ask whether or not a p-adic local analogue of Corollary 4.17 holds. In fact, one may prove the absolute birational version of geometrically pro-Σ [where $p \in \Sigma$] Grothendieck Conjecture for curves over p-adic local fields. The proof of this assertion will be included in an ongoing joint work with E. Lepage in a more general p-adic anabelian context.

5 An analogue of Neukirch-Uchida-Pop theorem for finitely generated transcendental fields over arbitrary Kummer extensions of number fields

In the present section, as an application of Theorem 4.15, together with strong reconstruction results due to J. Koenigsmann and F. Pop, by induction on dimension, we prove an analogue of Neukirch-Uchida-Pop theorem for finitely generated transcendental fields over arbitrary subfields of the maximal Kummer extension fields of number fields [cf. Theorem 5.21]. In light of vanishing of cyclotomic characters, this result may be regarded as a certain characteristic 0 analogue of Pop's birational anabelian result for algebraic varieties of dimension ≥ 2 over algebraic closures of the prime fields in positive characteristic [cf. Remark 5.21.2].

First, we discuss a certain semi-absoluteness of isomorphisms of the absolute Galois groups of function fields of algebraic varieties over arbitrary subfields of finite step solvable extension fields of number fields [cf. Proposition 5.7].

Definition 5.1. Let K be a field. Then:

(i) We shall write $\dim K$ for the Kronecker dimension of K, i.e., the transcendental degree of K over the prime field when $\mathrm{char}(K) > 0$; 1+ the transcendental degree of K over the prime field when $\mathrm{char}(K) = 0$.

(ii) For each valuation v on K, we shall write K_v for the residue field of the valuation ring $\mathcal{O}_v \subseteq K$ of v; rr_v for the rational rank of the value group Γ_v of v; $D_v \subseteq G_K$ for the decomposition subgroup [determined up to conjugacy — cf. [11], Theorem 3.2.15] associated to v; $I_v \subseteq D_v$ for the inertia subgroup. In the situation where K is a function field of an algebraic variety, for each prime number $l \neq \mathrm{char}(K)$, we write $D_v^{(l)}$, $I_v^{(l)}$ for the respective images of D_v, I_v, in the geometrically pro-l quotient of G_K.

(iii) Let v be a valuation on K. Then it follows immediately from [11], Theorem 3.4.3, together with [11], Theorem 2.1.4, (a), that $\mathrm{rr}_v + \dim K_v \leq \dim K$. If the equality $\mathrm{rr}_v + \dim K_v = \dim K$ holds, then we shall say that v is *defectless*.

Remark 5.1.1. Let K be a field; v_1, v_2 valuations on K such that $v_1 \leq v_2$, i.e., $\mathcal{O}_{v_2} \subseteq \mathcal{O}_{v_1}$. Then it follows immediately from [11], Lemma 2.3.1, that D_{v_2} is isomorphic to a closed subgroup of D_{v_1} [cf. e.g., [39], 1.1, (3), (a)].

Definition 5.2. Let k be a field; K a finitely generated transcendental extension field over k; v a valuation on K. Then we shall say that v is *geometric* if v lies over the trivial valuation on k.

Remark 5.2.1. In the notation of Definition 5.2, suppose that k is of characteristic 0, and v is a geometric discrete valuation. Recall that any geometric discrete valuation on the residue field of v determines a geometric valuation w on K such that $\mathcal{O}_w \subseteq \mathcal{O}_v$. Write $n \overset{\mathrm{def}}{=} \mathrm{trdeg}_{K/k} \geq 1$. By repeating this procedure [cf. [11], Theorem 3.4.3], we obtain a geometric valuation v_0 on K such that $\mathcal{O}_{v_0} \subseteq \mathcal{O}_v$, and $\Gamma_{v_0} \cong \mathbb{Z}^{\oplus n}$. In this situation, under suitable choices of respective extensions of v, v_0, to $\overline{K}$ [cf. also [11], Lemma 3.1.5], we note that $\widehat{\mathbb{Z}}(1) \cong I_v \subseteq I_{v_0} \cong \widehat{\mathbb{Z}}(1)^{\oplus n}$ [cf. e.g., [39], 1.1, (3), (a); [39], 1.6, (1)], and $D_{v_0} \cong I_{v_0} \rtimes G_{k'}$, where k' is a finite extension field of k [cf. [39], 1.6, (1)].

Definition 5.3. Let $K \subseteq \overline{\mathbb{Q}}$ be a subfield. Then we shall say that K is of *finite solvable type* if K is a subfield of a finite step solvable extension field of a number field $\subseteq \overline{\mathbb{Q}}$.

Remark 5.3.1. In fact, it follows from Bary-Spoker-Fehm-Wiese's theorem [cf. [3], Theorem 1.1] that any field of finite solvable type is Hilbertian.

Lemma 5.4. *Let n be a positive integer; $K \subseteq \overline{\mathbb{Q}}$ a subfield of finite solvable type; X an algebraic variety over K of dimension n; $H, I \subseteq G_{K(X)}$ closed subgroups. Suppose that:*

- *H is isomorphic to the absolute Galois group of a field of finite solvable type, and I is an abelian profinite group that contains a profinite group isomorphic to $\widehat{\mathbb{Z}}^{\oplus n}$.*
- *I commutes with H.*

Then $I \subseteq \Delta_{K(X)}$.

Proof. Write $C \subseteq G_{K(X)}$ for the closed subgroup topologically generated by I and H. Note that H contains a nonabelian free pro-p group for each prime number p. Then, in light of Koenigsmann's theorem [cf. Proposition 5.12], for each prime number p, there exists a nontrivial Henselian valuation v_p on $K(X)$ such that $C \subseteq D_{v_p}$ whose residue characteristic $\neq p$. Observe that there exists a prime number p such that the restriction of v_p to K is trivial. Indeed, suppose not. Then it follows from the well-known classification of valuations on $\mathbb{Q}$, together with [37], Corollary 12.1.3, and easily verified fact that $C \subsetneq \Delta_{K(X)}$ [cf. the fact that the p-cohomological dimension of $\Delta_{K(X)}$ is n], that the nontrivial restrictions of $\{v_p\}_{p \in \mathfrak{Primes}}$ on K coincide. However, since the residue characteristic of v_p is $\neq p$, this is a contradiction. In particular, one may fix a geometric valuation v on $K(X)$ such that $C \subseteq D_v \cong I_v \rtimes G_{K_v}$. Let l be a prime number. Note that there exists an injective homomorphism $J \stackrel{\text{def}}{=} \mathbb{Z}_l^{\oplus n+1} \hookrightarrow C$. If we write $m \stackrel{\text{def}}{=} \text{trdeg}_{K_v/K}$, then it follows from Lemma 5.5 that the $\mathbb{Z}_l$-rank of the image of J in G_{K_v} is $\leq m+1$. Therefore, the $\mathbb{Z}_l$-rank of $I_v \cap J$ is $\geq n-m$. On the other hand, it follows from the fundamental inequality that $\text{rr}_v \leq n-m$. Then it follows immediately from [39], 1.6, (1), that the $\mathbb{Z}_l$-rank of I_v is $\leq n-m$. Thus, since $I_v \cap J \subseteq I_v$, we observe that the $\mathbb{Z}_l$-rank of I_v is $n-m$, hence that $\text{rr}_v = n-m$. This implies that v is defectless. In this case, it follows from [11], Theorem 3.4.3, that K_v is finitely generated over K. Moreover, we observe that:

- Since v is nontrivial, it holds that $m \leq n-1$.
- Since $I_v \cong \widehat{\mathbb{Z}}^{\oplus n-m}$, the image of I in G_{K_v} via the composite $C \subseteq D_v \twoheadrightarrow G_{K_v}$ contains a profinite group isomorphic to $\widehat{\mathbb{Z}}^{\oplus m}$.
- Since H admits no nontrivial abelian normal closed subgroup, the closed subgroup H injects into G_{K_v} via the composite $C \subseteq D_v \twoheadrightarrow G_{K_v}$.

Thus, by induction on n, we may assume without loss of generality that $n = 1$, and H injects into G_K via the natural surjection $G_{K(X)} \twoheadrightarrow G_K$. On the other hand, it follows immediately from Geyer's theorem that any abelian closed subgroup of G_K is pro-cyclic [cf. e.g., [11], Exercise 5.5.5, (c)]. Thus, since H commutes with I, and any p-Sylow subgroup of H contains a nonabelian free pro-p group for any prime number p, we conclude that $I \subseteq \Delta_{K(X)}$. This completes the proof of Lemma 5.4. $\square$

Lemma 5.5. *Let $K \subseteq \overline{\mathbb{Q}}$ be a field of finite solvable type; X an algebraic variety over K. Then the dimension of X may be characterized as the maximal nonnegative integer among the nonnegative integers n satisfying the following condition:*

($*$) There exists a closed subgroup $I \subseteq G_{K(X)}$ isomorphic to $\mathbb{Z}_l^{\oplus n+1}$ for some prime number l.

Proof. First, if X is of dimension n, then it follows immediately from our assumption on K, together with Remark 5.2.1, that a suitable closed subgroup of a decomposition subgroup associated to a geometric valuation satisfies condition ($*$). Next, suppose that there exists a closed subgroup $I \subseteq G_{K(X)}$ isomorphic to $\mathbb{Z}_l^{\oplus m}$ for some prime number l and some positive integer $m > n+1$. Then since every abelian closed subgroup of $G_{\mathbb{Q}}$ is pro-cyclic, it follows immediately that there exists a closed subgroup of $\Delta_{K(X)}$ isomorphic to $\mathbb{Z}_l^{\oplus m-1}$. Since $m-1 > n$, this contradicts the well-known computation of cohomological dimension of $\Delta_{K(X)}$. This completes the proof of Lemma 5.5. $\square$

Remark 5.5.1. Let l be a prime number. Then, by applying a similar argument to the argument applied in the proof of Lemma 5.5, one may prove a geometrically pro-l variant of Lemma 5.5, i.e., the characterization of dimension of X via the maximal nonnegative integer among the nonnegative integers n satisfying the following condition:

$$\text{There exists a closed subgroup } I \subseteq G^{(l)}_{K(X)} \text{ isomorphic to } \mathbb{Z}_l^{\oplus n+1}.$$

Indeed, it suffices to observe that the l-cohomological dimension of $\Delta^l_{K(X)}$ is equal to $\dim X$. Note that, by considering an inertia subgroup, we obtain an injective homomorphism $\mathbb{Z}_l^{\oplus \dim X} \hookrightarrow \Delta^l_{K(X)}$. Therefore, since $\Delta^l_{K(X)}$ is a pro-l group, it suffices to verify that $H^{n+1}(\Delta^l_{K(X)}, \mathbb{F}_l) = \{0\}$. On the other hand, in light of the Bloch-Kato conjecture, it holds that $H^i(\mathrm{Ker}(\Delta_{K(X)} \twoheadrightarrow \Delta^l_{K(X)}), \mathbb{F}_l) = \{0\}$ for each nonnegative integer i. Thus, we conclude that $H^{n+1}(\Delta^l_{K(X)}, \mathbb{F}_l) \xrightarrow{\sim} H^{n+1}(\Delta_{K(X)}, \mathbb{F}_l) = \{0\}$.

Lemma 5.6. *Let K be a field of finite solvable type; X a proper, normal variety over an algebraically closed field of characteristic 0; $\phi : \Pi_X \to G_K$ a homomorphism whose image is subnormal in G_K. Then ϕ is trivial.*

Proof. In light of our assumption on K, we observe that K is Hilbertian [cf. Remark 5.3.1]. In particular, $\mathrm{Im}(\phi)$ is isomorphic to a subnormal closed subgroup of the absolute Galois group of a Hilbertian field. On the other hand, it is well-known that $\mathrm{Im}(\phi)$ is topologically finitely generated. Thus, we conclude from the sn-elastic portion of [28], Theorem B, that $\mathrm{Im}(\phi) = \{1\}$. This completes the proof of Lemma 5.6. □

Proposition 5.7. *Let Σ be a nonempty set of prime numbers. For each $i = 1, 2$, let $K_i \subseteq \overline{\mathbb{Q}}$ be a field of finite solvable type; X_i an algebraic variety over K_i. Then any isomorphism*

$$G^{(\Sigma)}_{K(X_1)} \xrightarrow{\sim} G^{(\Sigma)}_{K(X_2)}$$

of profinite groups induces an isomorphism

$$G_{K_1} \xrightarrow{\sim} G_{K_2}$$

of profinite groups via the natural surjective homomorphisms $G^{(\Sigma)}_{K(X_1)} \twoheadrightarrow G_{K_1}$, $G^{(\Sigma)}_{K(X_2)} \twoheadrightarrow G_{K_2}$.

Proof. In the case where $\Sigma \subsetneq \mathfrak{Primes}$, the assertion follows immediately from Lemma 3.14, together with Remark 5.3.1. In the case where $\Sigma = \mathfrak{Primes}$, in light of Remark 5.2.1, the assertion follows immediately from Lemmas 5.4, 5.5, 5.6. This completes the proof of Proposition 5.7. □

Remark 5.7.1. One may observe that the proof of semi-absoluteness in the case where $\Sigma = \mathfrak{Primes}$ is much more complex than the proof of semi-absoluteness in the case where $\Sigma \subsetneq \mathfrak{Primes}$ using Lemma 3.14. The complexity arises from a similarlity of the respective profinite group structures of geometric and arithmetic parts.

Next, we reconstruct the divisorial subgroups of the geometrically pro-l quotients of the absolute Galois groups of higher dimensional function fields following Pop.

Definition 5.8. Let l be a prime number; k a field such that $\mathrm{char}(k) \neq l$; K a finitely generated transcendental regular extension field of k; $G \subseteq G^{(l)}_K$ a closed subgroup. Then:

(i) We shall say that G is *divisorial* if $G = D^{(l)}_v$ for some divisorial valuation v on K [cf. Definition 4.3].

(ii) We shall say that G is *divisorial-like* if G is isomorphic to $D^{(l)}_v$ for some divisorial valuation v on K.

Remark 5.8.1. In the notation of Definition 5.8, let v be a defectless geometric valuation on K of rational rank m. Then it is well-known that:

(i) $\Gamma_v \cong \mathbb{Z}^{\oplus m}$, and K_v is finitely generated over k [cf. [11], Theorem 3.4.3]. In particular, the divisorial valuations on K precisely correspond to the defectless geometric valuations on K of rational rank 1 [cf. [39], Remark 1.13, (4); the discussion in [42]].

(ii) $D_v^{(l)} \cong \mathbb{Z}_l(1)^{\oplus m} \rtimes G_{K_v}^{(l)}$ [cf. [39], 1.5, (1); [39], 1.6, (1)], where we regard K_v as the function field of an algebraic variety over a finite extension field of k in a natural way.

Lemma 5.9. *Let l be a prime number; $k \subseteq \overline{\mathbb{Q}}$ a subfield of finite solvable type; K a finitely generated transcendental extension field over k; v a valuation on K. Then the following hold:*

(i) Suppose that v is a [geometric] divisorial valuation. Then the image of $D_v^{(l)}$ via the natural homomorphism $G_K^{(l)} \to G_k$ is open.

(ii) Suppose that the image of $D_v^{(l)}$ via the natural homomorphism $G_K^{(l)} \to G_k$ is open. Then v is geometric.

Proof. Assertion (i) follows immediately from Remark 5.8.1, (i), (ii). To verify assertion (ii), observe that our assumption on k implies that the decomposition subgroups of G_k associated to nontrivial valuations on k are infinite and do not form nontrivial intersections mutually [cf. [37], Corollary 12.1.3]. Thus, we conclude that, if v is not geometric, then the image of $D_v^{(l)}$ via the natural homomorphism $G_K^{(l)} \to G_k$ is not open. This completes the proof of assertion (ii), hence of Lemma 5.9. □

Lemma 5.10. *Let l be a prime number; k a field; K a finitely generated transcendental extension field over k; $N \subseteq G_K$ a normal closed subgroup. Write $G \stackrel{\text{def}}{=} G_K/\text{Ker}(N \twoheadrightarrow N^l)$. Then any normal closed subgroup of G is center-free. In particular, the profinite group G admits no nontrivial abelian nornal closed subgroup.*

Proof. Recall that K is Hilbertian. Note that G may be written as the inverse limit

$$\varprojlim_{N \subseteq M \subseteq G_K} G_K/\text{Ker}(M \twoheadrightarrow M^l),$$

where M ranges over the normal open subgroups of G_K that contains N; the transition maps are natural surjective homomorphisms. Thus, since the transition maps are surjective, and $G_K/\text{Ker}(M \twoheadrightarrow M^l)$ is an almost pro-l-maximal quotient [cf. [34], Definition 1.1, (iii)], we conclude from [28], Theorem B, that any normal closed subgroup of G is center-free. This completes the proof of Lemma 5.10. □

Lemma 5.11. *Let l be a prime number; $k \subseteq \overline{\mathbb{Q}}$ a subfield of finite solvable type; K a finitely generated regular extension field of k such that* $\text{trdeg}_{K/k} \geq 2$*; $G \subseteq G_K^{(l)}$ a closed subgroup isomorphic to a divisorial subgroup of the geometrically pro-l quotient of the absolute Galois group of a finitely generated regular extension field of a field of finite solvable type with the same transcendental degree* $\text{trdeg}_{K/k}$*; v a nontrivial geometric valuation on K such that $G \subseteq D_v^{(l)}$. Then $D_v^{(l)}$ is divisorial.*

Proof. Note that it follows immediately from Remark 5.8.1, (ii), that G contains a closed subgroup H isomorphic to the geometrically pro-l quotient of the absolute Galois group of a finitely generated extension field of a field of finite solvable type whose transcendental degree coincides with $\mathrm{trdeg}_{K/k} - 1 \geq 1$. Then, in light of Lemma 5.10, together with [39], 1.5, (1), one may observe that H injects into $G^{(l)}_{K_v}$ via the natural composite $H \subseteq G \subseteq D^{(l)}_v \twoheadrightarrow G^{(l)}_{K_v}$. On the other hand, since v is nontrivial, it follows from [11], Theorem 3.4.3, that $\mathrm{trdeg}_{K_v/k} \leq \mathrm{trdeg}_{K/k} - 1$. Then since $H \hookrightarrow G^{(l)}_{K_v}$, it follows from Remark 5.5.1 that $\mathrm{trdeg}_{K_v/k} = \mathrm{trdeg}_{K/k} - 1$, hence that v is defectless of rational rank 1. Thus, we conclude from Remark 5.8.1, (i), that $D^{(l)}_v$ is divisorial. This completes the proof of Lemma 5.11. $\square$

Proposition 5.12. *Let l be a prime number; k a field of characteristic $\neq l$ such that $\zeta_{l^2} \in k$; K a finitely generated transcendental regular extension field of k; $G \subseteq G^{(l)}_K$ a closed subgroup that admits a nontrivial abelian pro-l normal closed subgroup that is of infinite index in an l-Sylow subgroup of G. Then there exists a nontrivial valuation v on K such that $G \subseteq D^{(l)}_v$, and the residue characteristic of $v \neq l$.*

Proof. Proposition 5.12 follows immediately from a well-known result due to Engler-Koenigsmann-Nogueira [cf. [9], [10]]. $\square$

Next, we review the following well-known fact.

Lemma 5.13. *Let l be a prime number K a finitely generated transcendental extension of an algebraically closed field of characteristic $\neq l$; v_1, v_2 inequivalent divisorial valuations on K. Then $D^{(l)}_{v_1} \subsetneq D^{(l)}_{v_2}$.*

Proof. In light of Artin-Whaples approximation theorem, there exists an element $x \in K^\times$ such that $v_1(x) = 1$, and x lies in the principal unit of the valuation ring associated to v_2. In particular, in light of Kummer theory, we obtain an element $\in H^1(G^l_K, \mathbb{F}_l)$ whose image in $H^1(D^l_{v_2}, \mathbb{F}_l)$ is trivial but whose image in $H^1(D^l_{v_1}, \mathbb{F}_l)$ is nontrivial. This immediately implies that $D^{(l)}_{v_1} \subsetneq D^{(l)}_{v_2}$. This completes the proof of Lemma 5.13. $\square$

Theorem 5.14. *Let l be a prime number. For each $i = 1, 2$, let $K_i \subseteq \overline{\mathbb{Q}}$ be a subfield of finite solvable type; X_i an algebraic variety of dimension ≥ 2 over K_i. Let*

$$\phi : G^{(l)}_{K(X_1)} \overset{\sim}{\to} G^{(l)}_{K(X_2)}$$

be an isomorphism of profinite groups. Then ϕ preserves the respective divisorial subgroups and their inertia subgroups.

Proof. First, to verify the preservation of decomposition subgroups, in light of [37], Corollary 12.1.3, together with Proposition 5.7, by replacing K_1, K_2, by respective finite Galois extensions of K_1, K_2, we may assume without loss of generality that $\zeta_{l^2} \in K_1$, and $\zeta_{l^2} \in K_2$. Next, let v_1 be a divisorial valuation on $K(X_1)$ [whose restriction to K_1 is trivial]. In light of [39], 1.6, (2); Remark 5.8.1, (ii), one may observe that any l-Sylow subgroup of the closed subgroup $D^{(l)}_{v_1} \subseteq G^{(l)}_{K(X_1)}$ satisfies the assumption on G in the statement of Proposition 5.12. Then since $\phi(D^{(l)}_{v_1})$ is isomorphic to $D^{(l)}_{v_1}$, it follows immediately from Proposition 5.12 that there exists a nontrivial valuation v_2 on $K(X_2)$ such that $\phi(D^{(l)}_{v_1}) \subseteq D^{(l)}_{v_2}$. Our goal is to verify that $\phi(D^{(l)}_{v_1}) = D^{(l)}_{v_2}$. Note that since v_1 is geometric, it follows immediately from Proposition 5.7, together with Lemma 5.9, (i), (ii), that v_2 is also geometric. Thus, in light of Remark 5.5.1, by applying Lemma 5.11, we observe that v_2 is divisorial. Next, by applying the above discussion to ϕ^{-1} and $D^{(l)}_{v_2}$, we also observe that $\phi^{-1}(D^{(l)}_{v_2})$ is contained in a divisorial subgroup $D^{(l)}_{v'_1} \subseteq G^{(l)}_{K(X_1)}$, hence that $D^{(l)}_{v_1} \subseteq D^{(l)}_{v'_1}$. However, since both v_1 and v'_1 are of rank 1, we conclude from Lemma 5.13 [where we take "K" to be the finite extensions of the Henselization of $K(X_1)$ with respect to v_1] that $D^{(l)}_{v_1} = D^{(l)}_{v'_1}$, hence, in particular, that $\phi(D^{(l)}_{v_1}) = D^{(l)}_{v_2}$. Finally, the preservation of their inertia subgroups follows immediately from Lemma 5.10. This completes the proof of Theorem 5.14. ☐

Lemma 5.15. *Let Σ be a nonempty set of prime numbers; F a complete discrete valuation field of equicharacteristic 0. Write k_F for the residue field of F. Then the Kummer theory induces the following commutative diagram*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & k_F^\times/k_F^{\times\Sigma\infty} & \longrightarrow & F^\times/k_F^{\times\Sigma\infty}(1+\mathfrak{m}_F) & \longrightarrow & \mathbb{Z} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & H^1(G_{k_F}, \widehat{\mathbb{Z}}^\Sigma(1)) & \longrightarrow & H^1(G_F, \widehat{\mathbb{Z}}^\Sigma(1)) & \longrightarrow & \widehat{\mathbb{Z}}^\Sigma & \longrightarrow & 1, \end{array}$$

where $\mathfrak{m}_F$ denotes the maximal ideal of the valuation ring of F; the vertical arrows denote the natural injective homomorphisms; $F^\times/k_F^{\times\Sigma\infty}(1+\mathfrak{m}_F) \to \mathbb{Z}$ denotes the surjective homomorphism induced by the discrete valuation on F.

Proof. Lemma 5.15 follows immediately from the easily verified fact that $1+\mathfrak{m}_F$ is divisible. ☐

The next result is a consequence of a highly nontrivial global theory due to Pop:

Proposition 5.16. *Let l be a prime number. For each $i = 1, 2$, let k_i be an algebraically closed field of characteristic 0; K_i a finitely generated [regular] extension field of k_i such that $\mathrm{trdeg}_{K_i/k_i} \geq 2$. Let*

$$\sigma : G^l_{K_1} \xrightarrow{\sim} G^l_{K_2}$$

be an isomorphism of profinite groups such that σ preserves the decomposition subgroups associated to divisorial valuations, and the induced isomorphisms between the maximal pro-l quotients of the absolute Galois groups of residue fields arise from field isomorphisms. Then there exists an l-adic unit $\epsilon \in \mathbb{Z}_l^\times$ such that the isomorphism

$$H^1(G^l_{K_2}, \mathbb{Z}_l) \longrightarrow H^1(G^l_{K_1}, \mathbb{Z}_l)$$

determined by σ and ϵ induces an isomorphism

$$(K_2^\times/k_2^\times)^{\mathrm{sat}} \xrightarrow{\sim} (K_1^\times/k_1^\times)^{\mathrm{sat}}$$

via the respective Kummer maps, where $(-)^{\mathrm{sat}}$ denotes the $\{l\}'$-saturation of $(-)$ in the cohomology group.

Proof. Proposition 5.16 follows immediately from [43], Propositions 11, 22, 23. ☐

Lemma 5.17. *Let n be a positive integer; k an algebraically closed field of characteristic 0; K a finitely generated [regular] extension field of k such that $\mathrm{trdeg}_{K/k} \geq 2$; $f \in K^\times$. Suppose that, for each divisorial valuation v on K that is not contained in the support of f, the image of f in $K_v^\times$ is contained in $(K_v^\times)^n$. Then it holds that $f \in (K^\times)^n$.*

Proof. Let X be a smooth, projective model of K over k [cf. [17], [18]]. Write $D \stackrel{\mathrm{def}}{=} (f) = \sum_{1 \leq i \leq m} a_i D_i$ for the Weil divisor on X determined by f. Fix a positive integer $i \leq m$. Let $x \in D_i \setminus \cup_{j \neq i} D_j \subseteq X$ be such that x is also a regular point of D_i of codimension 1; $C \subseteq X$ an integral closed subscheme such that $C \not\subseteq \mathrm{Supp}(D)$, C intersects transversally with D_i at x, and x is also a regular point of C of codimension 1. In particular, since $\mathcal{O}_{X,x}$ is regular, hence, UFD, it follows immediately from the various definitions involved that there exist a uniformizer $\pi_i \in K^\times$ associated to D_i and an element $u \in \mathcal{O}_{X,x}^\times$ such that $f = u\pi_i^{a_i}$. Therefore, the valuation of the image of f in $K(C)$ at x is a_i. Then since $C \not\subseteq \mathrm{Supp}(D)$, our assumption implies that $n \mid a_i$. Here, we note that an n-th root of f determines an element $\kappa_f \in \mathrm{Hom}(\Pi_U, \mathbb{Z}/n\mathbb{Z})$, where $U \stackrel{\mathrm{def}}{=} X \setminus \mathrm{Supp}(D)$. Observe that since $n \mid a_i$ for each i, the Kummer class κ_f arises from $\mathrm{Hom}(\Pi_X, \mathbb{Z}/n\mathbb{Z})$. On the other hand, since X is a smooth, projective scheme over k of dimension ≥ 2, it follows from [13], Exposé X, Lemme 2.10 [or [14], Exposé XII, Corollaire 3.5, together with its remarque], together with Bertini's theorem, that there exists a smooth closed subscheme $E \subseteq X$ of codimension 1 such that the natural homomorphism $\Pi_E \to \Pi_X$ is surjective. Note that since E is smooth, the natural homomorphism $G_E \to \Pi_E$ is surjective. In particular, our assumption implies that the image of κ_f via the natural injective homomorphism $\mathrm{Hom}(\Pi_X, \mathbb{Z}/n\mathbb{Z}) \hookrightarrow \mathrm{Hom}(\Pi_E, \mathbb{Z}/n\mathbb{Z})$ is trivial, hence that κ_f is trivial. Thus, we conclude that $f \in (K^\times)^n$. This completes the proof of Lemma 5.17. ☐

Definition 5.18. Let k be an algebraically closed field; K a finitely generated [regular] extension field of k; $\mathbb{V}$ a set of divisorial valuations on K. Then we shall say that $\mathbb{V}$ is *ample* if, for each $f \in K$ and each finite subset $S \subseteq \mathbb{V}$ such that $v(f) = 0$ for any $v \in \mathbb{V} \setminus S$, it holds that

$$\bigcap_{v \in \mathbb{V} \setminus S} k + kf + \mathfrak{m}_v \; = \; k + kf,$$

where $\mathfrak{m}_v$ denotes the maximal ideal of the valuation ring associated to v.

Remark 5.18.1. In the notation of Definition 5.18, suppose that $\mathrm{trdeg}_{K/k} = 1$. Then it is immediate to see that $\mathbb{V}$ is not ample.

Remark 5.18.2. In the notation of Definition 5.18, let $\mathbb{V}^\dagger$ be a set of divisorial valuations on K such that $\mathbb{V} \subseteq \mathbb{V}^\dagger$. Then, if $\mathbb{V}$ is ample, then $\mathbb{V}^\dagger$ is also ample.

Lemma 5.19. *In the notation of Definition 5.18, suppose that* $\mathrm{trdeg}_{K/k} \geq 2$*, and* $\mathbb{V}$ *coincides with the set of divisorial valuations of a normal model of K over k [i.e., a normal algebraic variety over k whose function field coincides with K]. Then* $\mathbb{V}$ *is ample.*

Proof. Let X be a normal model of K over k under consideration; $f \in K$; $S \subseteq \mathbb{V}$ a finite subset such that $v(f) = 0$ for any $v \in \mathbb{V} \setminus S$. To verify the equality $\cap_{v \in \mathbb{V} \setminus S}\, k + kf + \mathfrak{m}_v \; = \; k + kf$, it follows immediately that we may assume without loss of generality that f is nonconstant. Let $U \subseteq X$ be an affine open subscheme such that $S \subseteq X \setminus U$. Then it follows from our assumption on S that f determines a dominant morphism $\psi_f : U \to \mathbb{A}^1_k$ over k. In particular, since $\mathbb{A}^1_k$ is a Dedekind scheme, the morphism ψ_f is flat. Let $g \in \cap_{v \in \mathbb{V} \setminus S} k + kf + \mathfrak{m}_v$. Then, in light of the purity of dimensions of fibers of flat morphisms, we observe that g determines a morphism $\psi_g : U \to \mathbb{A}^1_k$ over k such that $\psi_g(\psi_f^{-1}(x))$ consists of finite closed points for each closed point $\in \mathbb{A}^1_k$. Thus, if we write $\Delta_{f,g} \subseteq \mathbb{A}^1_k \times \mathbb{A}^1_k$ for the scheme-theoretic image of $\psi_f \times \psi_g : U \to \mathbb{A}^1_k \times \mathbb{A}^1_k$, then one may observe that the composite $\Delta_{f,g} \subseteq \mathbb{A}^1_k \times \mathbb{A}^1_k \stackrel{\mathrm{pr}_1}{\to} \mathbb{A}^1_k$ is a quasi-finite morphism, hence that $\Delta_{f,g}$ is an integral scheme of dimension 1. Let v be a point $\in U$ of codimension 1 that lies over the generic point of $\Delta_{f,g}$ [whose existence follows from the fact that $\dim U \geq 2$]. Write t_1, t_2 for the respective standard coordinates of the components of $\mathbb{A}^1_k \times \mathbb{A}^1_k$. Then since $g \in k + kf + \mathfrak{m}_v$, by evaluating f, g at v, we note that there exists an element $a_v, b_v \in k$ such that $t_2 = a_v + b_v t_1$ at the residue field of the generic point of $\Delta_{f,g}$. Thus, since the generic point of U maps to the generic point of $\Delta_{f,g}$ via $\psi_f \times \psi_g$, we conclude that $g \in k + kf$. This completes the proof of Lemma 5.19. $\square$

Proposition 5.20. *For $i = 1, 2$, let k_i be an algebraically closed field; K_i a finitely generated transcendental [regular] extension field of k_i; $\mathbb{V}_i$ an ample set of [normalized] divisorial valuations on K_i such that the support in $\mathbb{V}_i$ of any element of $K_i^\times$ is finite. Let*

$$\sigma : K_1^\times/k_1^\times \ \xrightarrow{\sim}\ K_2^\times/k_2^\times$$

be an isomorphism;

$$\phi : \mathbb{V}_1 \ \xrightarrow{\sim}\ \mathbb{V}_2$$

a bijection. For each $v_1 \in \mathbb{V}_1$, suppose that:

- *$\sigma(\mathcal{O}^\times_{v_1}/k_1^\times) = \mathcal{O}^\times_{\phi(v_1)}/k_2^\times$.*
- *The isomorphism $\mathcal{O}^\times_{v_1}/k_1^\times \xrightarrow{\sim} \mathcal{O}^\times_{\phi(v_1)}/k_2^\times$ determined by σ induces an isomorphism $(K_1)^\times_{v_1}/k_1^\times \xrightarrow{\sim} (K_2)^\times_{\phi(v_1)}/k_2^\times$. Moreover, this induced isomorphism arises from an isomorphism $(K_1)_{v_1} \xrightarrow{\sim} (K_2)_{\phi(v_1)}$ of fields.*

Then σ arises from a uniquely determined isomorphism $K_1 \xrightarrow{\sim} K_2$ of fields.

Proof. In light of the fundamental theorem of projective geometry, it suffices to verify that σ preserves the collineations. Let x_1, y_1 be elements of $K_1^\times$; $a_1 \in k_1^\times$ an element such that $z_1 \stackrel{\text{def}}{=} x_1 + a_1 y_1 \in K_1^\times$; x_2, y_2, z_2 respective liftings of $\sigma(\overline{x}_1)$, $\sigma(\overline{y}_1)$, $\sigma(\overline{z}_1)$ in $K_2^\times$, where $\overline{(-)}_1$ denotes the image of $(-)_1$ in $K_1^\times/k_1^\times$. Then it suffices to verify that $z_2 \in k_2x_2 + k_2y_2 \subseteq K_2$. Write $S_1 \subseteq \mathbb{V}_1$ for the union of the supports of x_1, y_1, and z_1. In particular, it follows immediately from our assumption on $\mathbb{V}_1$ that S_1, hence $S_2 \stackrel{\text{def}}{=} \phi(S_1)$, is finite. Note that it follows immediately from our assumptions on σ, ϕ, that, for each $v_2 \in \mathbb{V}_2 \setminus S_2$, the image of $\frac{z_2}{x_2}$ in $(K_2)^\times_{v_2}/k_2^\times$ is contained in the image of $k_2 + k_2\overline{\frac{y_2}{x_2}}$ in $(K_2)^\times_{v_2}/k_2^\times$, where $\overline{\frac{y_2}{x_2}}$ denotes the image of $\frac{y_2}{x_2}$ in $(K_2)^\times_{v_2}$. In particular, it holds that $\frac{z_2}{x_2} \in k_2 + k_2\frac{y_2}{x_2} + \mathfrak{m}_{v_2}$. Thus, since $\mathbb{V}_2$ is ample, and S_2 is finite, we conclude that $\frac{z_2}{x_2} \in k_2 + k_2\frac{y_2}{x_2} \subseteq K_2$, hence that $z_2 \in k_2x_2 + k_2y_2 \subseteq K_2$. This completes the proof of Proposition 5.20. $\square$

Theorem 5.21. *Let Σ be a nonempty set of prime numbers; X, Y algebraic varieties of positive dimension over subfields over the maximal Kummer extensions of number fields. Write*

$$\mathrm{Isom}(K(Y), K(X))$$

for the set of isomorphisms $K(Y) \xrightarrow{\sim} K(X)$ of fields;

$$\mathrm{OutIsom}(G^{(\Sigma)}_{K(X)}, G^{(\Sigma)}_{K(Y)})$$

for the set of outer isomorphisms $G^{(\Sigma)}_{K(X)} \xrightarrow{\sim} G^{(\Sigma)}_{K(Y)}$ of profinite groups. Then the natural map

$$\mathrm{Isom}(K(Y), K(X)) \ \longrightarrow\ \mathrm{OutIsom}(G^{(\Sigma)}_{K(X)}, G^{(\Sigma)}_{K(Y)})$$

is bijective.

Proof. First, the injectivity follows immediately from [21], Theorem 1.5. Next, we verify the surjectivity by induction on the dimension of X. Here, we may assume without loss of generality that $\mathrm{OutIsom}(G^{(\Sigma)}_{K(X)}, G^{(\Sigma)}_{K(Y)}) \neq \emptyset$. In this case, we note that $\dim X = \dim Y$ [cf. Remark 5.5.1; Proposition 5.7]. Then, in the case where $\dim X = 1$, the surjectivity follows immediately from Theorems 1.20, (i); 4.15, together with Proposition 5.7.

Next, suppose that $\dim X \geq 2$. In light of Galois descent with respect to almost pro-l-maximal quotients for some prime number $l \in \Sigma$ [cf. [34], Definition 1.1, (iii)], we may assume without loss of generality that

$$\Sigma \ = \ \{l\}$$

for some prime number l. Let

$$\phi : G^{(l)}_{K(X)} \ \overset{\sim}{\to} \ G^{(l)}_{K(Y)}$$

be an isomorphism of profinite groups. Note that ϕ preserves the respective divisorial subgroups and their inertia subgroups [cf. Theorem 5.14]. Therefore, in light of the induction hypothesis, we may assume without loss of generality that the base fields of X and Y are equal, and the automorphism between their absolute Galois groups induced by ϕ is the identity automorphism [cf. Proposition 5.7]. In particular, there exists no confusion on "$\mathbb{Z}_l(1)$". Next, write k for the base field of X or Y;

$$\hat{g} : \widehat{K(Y_{\overline{k}})}^{\times} \ \overset{\sim}{\to} \ H^1(\Delta^l_{K(Y)}, \mathbb{Z}_l(1)) \ \overset{\sim}{\to} \ H^1(\Delta^l_{K(X)}, \mathbb{Z}_l(1)) \ \overset{\sim}{\leftarrow} \ \widehat{K(X_{\overline{k}})}^{\times}$$

for the G_k-equivariant isomorphism induced by ϕ and the identity automorphism of $\mathbb{Z}_l(1)$;

$$\mathbb{V}_X$$

for the set of divisorial valuations on a proper, normal model of $K(X_{\overline{k}})$;

$$\mathbb{V}_Y$$

for the set of divisorial valuations on $K(Y_{\overline{k}})$ determined by $\mathbb{V}_X$ and the correspondence of divisorial subgroups induced by ϕ. In particular, there exists a natural bijection

$$\phi_{\mathbb{V}} : \mathbb{V}_X \ \overset{\sim}{\to} \ \mathbb{V}_Y.$$

In light of the induction hypothesis, together with Lemma 5.15, we observe that, for each $v \in \mathbb{V}_X$, the isomorphism $H^1(G^l_{K(Y_{\overline{k}})_{\phi_{\mathbb{V}}(v)}}, \mathbb{Z}_l(1)) \overset{\sim}{\to} H^1(G^l_{K(X_{\overline{k}})_v}, \mathbb{Z}_l(1))$ induced by ϕ and the identity automorphism of $\mathbb{Z}_l(1)$ arises from an isomorphism $K(Y_{\overline{k}})_{\phi_{\mathbb{V}}(v)} \overset{\sim}{\to} K(X_{\overline{k}})_v$ of fields over $\overline{k}$. Then, in light of Proposition 5.16, together with Lemma 5.15, by considering the natural action on $\mathbb{Z}_l(1)$, hence on cohomology groups, of the l-adic unit that appears in Proposition 5.16, we deduce that $\hat{g}$ induces a G_k-equivariant isomorphism

$$g^{\mathrm{sat}} : (K(Y_{\overline{k}})^{\times}/\overline{k}^{\times})^{\mathrm{sat}} \ \overset{\sim}{\to} \ (K(X_{\overline{k}})^{\times}/\overline{k}^{\times})^{\mathrm{sat}},$$

where $(-)^{\mathrm{sat}}$ denotes the $\{l\}'$-saturation of $(-)$ in the cohomology group. Moreover, by applying Lemma 5.17, one may observe that g^{sat} induces a G_k-equivariant isomorphism

$$g : K(Y_{\overline{k}})^{\times}/\overline{k}^{\times} \ \overset{\sim}{\to} \ K(X_{\overline{k}})^{\times}/\overline{k}^{\times}.$$

Write

$$\mathbb{V}^{\dagger}_Y$$

for the union of $\mathbb{V}_Y$ with the set of divisorial valuations on a proper, normal model of $K(Y_{\overline{k}})$;

$$\mathbb{V}^{\dagger}_X$$

for the set of divisorial valuations on $K(X_{\overline{k}})$ that corresponds to $\mathbb{V}_Y^\dagger$ via ϕ. Note that $\mathbb{V}_X \subseteq \mathbb{V}_X^\dagger$. Then, in light of Lemma 5.19; Remark 5.18.2, together with the induction hypothesis, we observe that g and the natural bijection $\mathbb{V}_X^\dagger \overset{\sim}{\to} \mathbb{V}_Y^\dagger$ satisfy similar conditions to the conditions in the statement of Proposition 5.20. Thus, we conclude from Proposition 5.20 that g arises from a *uniquely* determined G_k-equivariant field isomorphism

$$K(Y_{\overline{k}}) \ \overset{\sim}{\to}\ K(X_{\overline{k}}),$$

which induces a field isomorphism

$$K(Y) \ \overset{\sim}{\to}\ K(X).$$

Finally, in light of the above *uniqueness*, by applying the above discussion to the isomorphisms obtained by restricting ϕ to the open subgroups of $G_{K(X)}^{(l)}$, we observe that ϕ arises from the field isomorphism $K(Y) \overset{\sim}{\to} K(X)$. This completes the proof of Theorem 5.21. $\square$

Remark 5.21.1. Theorem 5.21 may be regarded as a generalization of both a Pop's celebrated birational anabelian result in characteristic 0 [cf. [39], Theorem 1; [41], Theorem 1.3] and the author's previous result [cf. [53], Corollary B].

Remark 5.21.2. Theorem 5.21 includes the Neukirch-Uchida-type result for the function fields of curves over the maximal cyclotomic extension fields of number fields. Here, we note that, in light of the Shafarevich conjecture, the Neukirch-Uchida-type result for the maximal cyclotomic extension fields of number fields should not hold. Indeed, the Shafarevich conjecture implies that every open subgroups of $G_{\mathbb{Q}^{\mathrm{ab}}}$ are free profinite groups of countable rank, hence isomorphic. On the other hand, it is immediate to see that any proper finite extension field of $\mathbb{Q}^{\mathrm{ab}}$ is not isomorphic to $\mathbb{Q}^{\mathrm{ab}}$ as a field. In light of Kronecker dimension, together with the vanishing of cyclotomic characters, it appears to the author that this situation is very similar to the situation of Pop's birational anabelian result for algebraic varieties of dimension ≥ 2 over algebraic closures of the prime fields in positive characteristic. Indeed, it is well-known that the absolute Galois groups of 1-dimensional function fields over algebraically closed fields are free profinite groups [cf. [16], Theorem 4.4; [39], Corollary of Theorem A].

Remark 5.21.3. Note that Theorem 5.21 may be regarded as an anabelian result for a certain class of fields including infinite algebraic extensions of finitely generated transcendental extensions of $\mathbb{Q}^{\mathrm{ab}}$. Then, in light of a similarity discussed in Remark 5.21.2, it would be interesting to pursue anabelian phenomena for a class of fields including infinite algebraic extensions of the function fields of dimension ≥ 2 over algebraic closures of the prime fields in positive characteristic. However, at the time of writing of the present paper, the author has no clue for such a generalization.

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