RIMS-1999

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July 2026

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July 30, 2026

Abstract

Let p be a prime number; F a nonabelian free pro-p group of finite rank. In the present paper, as an application of results in pro-p combinatorial group theory and classical representation theory, we prove that the subgroup $F \subseteq \mathrm{Aut}(F)$ may be characterized as a unique normal closed subgroup of $\mathrm{Aut}(F)$ isomorphic to F, where we regard F as a subgroup of $\mathrm{Aut}(F)$ via the natural isomorphism between F and the inner automorphism group of F. In particular, F is characteristic in $\mathrm{Aut}(F)$, and $\mathrm{Out}(\mathrm{Aut}(F))$ is trivial. These results may be regarded as pro-p analogues of the corresponding original results in the discrete setting due to Dyer-Formanek/Formanek.

Contents

Introduction

Throughout the present paper, we regard a center-free topological group G as a subgroup of the group $\mathrm{Aut}(G)$ of *continuous* automorphisms of G via the natural isomorphism $G \xrightarrow{\sim} \mathrm{Inn}(G)$ from G to the group of inner automorphisms of G. The present paper is motivated by the following interesting results due to L. Dyer and E. Formanek [cf. [3], [4]]:

Theorem 0.1 (Dyer-Formanek)**.** *Let F^{disc} be a nonabelian free group of finite rank. Then $F^{\mathrm{disc}} \subseteq \mathrm{Aut}(F^{\mathrm{disc}})$ is a characteristic subgroup. In particular, in light of Burnside's criterion, it holds that $\mathrm{Out}(\mathrm{Aut}(F^{\mathrm{disc}})) = \{1\}$.*

Theorem 0.2 (Formanek)**.** *Let F^{disc} be a nonabelian free group of finite rank. Then the subgroup $F^{\mathrm{disc}} \subseteq \mathrm{Aut}(F^{\mathrm{disc}})$ may be characterized as a unique normal subgroup of $\mathrm{Aut}(F^{\mathrm{disc}})$ isomorphic to F^{disc}.*

2020 *Mathematics Subject Classification*: Primary 20F28; Secondary 20E18.

Key words and phrases: free pro-p group; automorphism group; Andreadakis-Johnson filtration.

Note that Theorem 0.1 is an immediate consequence of Theorem 0.2, and the proofs of Theorems 0.1, 0.2 depend on techniques in combinatorial group theory concerning the Andreadakis-Johnson filtration on the automorphism group of a finitely generated free group. Note also that this type of property is closely related to the automorphism tower problem.

On the other hand, it appears to the author that it is natural to ask whether or not there exists a profinite analogue of the above theorems. Let p be a prime number. Since the pro-p setting is much closer than the full profinite setting, in the present paper, as the first step toward this direction, we consider the pro-p analogue of the above theorems. Let G be a profinite group; $H \subseteq G$ a closed subgroup. If any *continuous* automorphism of G preserves H, then we shall say that H is *characteristic* in G. Here, we note that:

- The automorphism group of a topologically finitely generated profinite group admits a structure of profinite group with respect to characteristic open subgroups [cf. [11], Proposition 4.4.3].
- A nonabelian free pro-p group is center-free [cf., e.g., [8], Corollary 3.4; [10], Proposition 1.4].

Then our main results are the following [cf. Theorem 3.1, Corollary 3.2]:

Theorem A. *Let F be a nonabelian free pro-p group of finite rank. Then the subgroup $F \subseteq \mathrm{Aut}(F)$ may be characterized as a unique normal closed subgroup of $\mathrm{Aut}(F)$ isomorphic to F.*

Corollary B. *Let F be a nonabelian free pro-p group of finite rank. Then the subgroup $F \subseteq \mathrm{Aut}(F)$ is characteristic. In particular, in light of Burnside's criterion, it holds that $\mathrm{Out}(\mathrm{Aut}(F)) = \{1\}$.*

The proof of Theorem A resorts to the pro-p combinatorial group theory and some classical representation-theoretic results. On the other hand, the author's original strategy was to apply results in anabelian geometry to imitate Formanek's proof. However, the author could not compute a certain centralizer in the case where $n = 2$, which was needed to imitate Formanek's proof. As a result, our discussion becomes a totally different one from the Formanek's strategy even though we apply the essentially same representation-theoretic result [cf. Lemma 1.4]. For future researches, we explicitly record the following open questions:

> **Q.1** Apart from the nilpotent situations, for a general set of prime numbers Σ, is a nonabelian free pro-Σ group characteristic in its automorphism group?
>
> **Q.2** Can we obtain similar results for other types of highly nonabelian profinite groups? [For instance, is a nonabelian Demushkin group or a profinite surface group characteristic in its automorphism group?]

Here, we note that, in the arithmetic side, the famous Neukirch-Uchida theorem asserts that: Any outer isomorphism of the absolute Galois groups of number fields arises from a unique field isomorphism [cf. [12]]. In particular, the absolute Galois group of a number field satisfies a similar property to the property in Theorem A, hence, in particular, is characteristic in its automorphism group. Thus, in light of Q.2, it is natural to ask whether or not the absolute Galois group of a p-adic local field satisfies a similar property. At the time of writing of the present paper, the author does not have any partial answers to the above questions, which reflects our situation that a conceptual understanding of this phenomenon is still lacking. In this context, obtaining a conceptual understanding of this phenomenon remains an important task for future work.

The present paper is organized as follows. In §1, we discuss preliminary results on pro-p combinatorial group theory and representation theory surrounding the pro-p Andreadakis-Johnson filtration. In §2, we discuss some elementary properties on matrix groups, which will be of use in the proof of our main results. In §3, by applying various results in the previous sections, we prove Theorem A.

Notations and conventions

Numbers: Let p be a prime number. Then we shall write $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{F}_p$ for the ring of p-adic integers, the field of p-adic numbers, the field of cardinality p, respectively.

Topological groups: Let G be a topological group; $I, H \subseteq G$ closed subgroups of G. Then we shall write $[I, H] \subseteq G$ for the *closure* of the usual commutator subgroup defined by I, H; $Z_G(H)$ for the *centralizer* of $H \subseteq G$; $Z(G) \overset{\text{def}}{=} Z_G(G)$; $G^{\text{ab}} \overset{\text{def}}{=} G/[G,G]$, i.e., the maximal abelian quotient of G; $\text{Aut}(G)$ for the group of *continuous* automorphisms of the topological group G; $\text{Inn}(G) \subseteq \text{Aut}(G)$ for the group of inner automorphisms of G; $\text{Out}(G) \overset{\text{def}}{=} \text{Aut}(G)/\text{Inn}(G)$. Now suppose that G is *center-free* [i.e., $Z(G) = \{1\}$]. Then we have a natural exact sequence of groups

$$1 \longrightarrow G \ \ (\overset{\sim}{\to} \text{Inn}(G)) \longrightarrow \text{Aut}(G) \longrightarrow \text{Out}(G) \longrightarrow 1.$$

We regard G as a subgroup of $\text{Aut}(G)$ via the composite $G \overset{\sim}{\to} \text{Inn}(G) \subseteq \text{Aut}(G)$. Suppose further that G is *profinite* and *topologically finitely generated*. Then one verifies immediately that the topology of G admits a basis of *characteristic open subgroups*, which thus induces a *profinite topology* on the groups $\text{Aut}(G)$ and $\text{Out}(G)$.

1 Preliminaries on pro-p combinatorial group theory and representation theory

Let p be a prime number; n a positive integer ≥ 2; F^{disc} a free group of rank n; F a free pro-p group of rank n. In the present section, we introduce and prove some basic properties concerning the pro-p Andreadakis-Johnson filtrations of the automorphism groups of free pro-p groups, which may be regarded as pro-p combinatorial group-theoretic and representation theoretic preliminaries of the proof of our main theorem.

First, we begin by introducing some notations concerning the lower central series of a topological group. Note that, in the present paper, the symbol $[-,-]$ is always assumed to be the closure of a commutator subgroup [cf. Notations and conventions].

Definition 1.1. Let G be a topological group. Then:

(i) For each positive integer i, we shall write G_i for the i-th term of the lower central series of G, which is defined inductively as follows:

- $G_1 = G$;
- $G_{i+1} = [G, G_i]$.

Note that G_i is a characteristic subgroup of G for each positive integer i.

(ii) For each positive integer i, we shall write

$$K_i(G)$$

for the kernel of the natural homomorphism $\text{Aut}(G) \to \text{Aut}(G/G_{i+1})$.

Next, we discuss some basic properties concerning pro-p Andreadakis-Johnson filtrations.

Proposition 1.2. *The following hold:*

(i) For each positive integer i, the natural homomorphism $\mathrm{Aut}(F) \to \mathrm{Aut}(F/F_i)$ is surjective.

(ii) For each positive integer i, it holds that $K_i(F) \cap F = F_i$. In particular, it holds that $K_i(F) \cap FK_{i+1}(F) = F_i K_{i+1}(F)$.

(iii) For each positive integer i, it holds that $(K_i(F)/K_i(F) \cap FK_{i+1}(F) \xrightarrow{\sim})\ FK_i(F)/FK_{i+1}(F)$ is a torsion-free $\mathbb{Z}_p$-module.

(iv) For each pair of positive integers i, j, it holds that $[K_i(F), K_j(F)] \subseteq K_{i+j}(F)$. In particular, for each positive integer i, it holds that $K_1(F)_i \subseteq K_i(F)$.

(v) For each positive integer i, it holds that $K_i(F)/K_{i+1}(F)$ is a torsion-free abelian group. In particular, it holds that $K_1(F)$ is torsion-free.

(vi) It holds that $\cap_{i\geq 1} K_i(F) = \{1\}$, where i ranges over the positive integers.

(vii) It holds that $\cap_{i\geq 1} FK_i(F) = F$, where i ranges over the positive integers.

Proof. Assertion (i) follows immediately from [8], Proposition 5.2, together with the fact that the subgroup F_i is a characteristic subgroup of F.

Next, we verify assertion (ii). Observe that the equality is equivalent to the equality $Z(F/F_{i+1}) = F_i/F_{i+1}$. Since $F_{i+1} = [F, F_i]$, the inclusion $F_i/F_{i+1} \subseteq Z(F/F_{i+1})$ is immediate. Let $x \in Z(F/F_{i+1})$. Suppose that $x \notin F_i/F_{i+1}$. Write $d < i$ for the maximal positive integer among the positive integers m satisfying $x \in F_m/F_{i+1}$; $(0 \neq)\ \overline{x} \in F_d/F_{d+1}$ for the image of x via the natural surjective homomorphism $F_d/F_{i+1} \twoheadrightarrow F_d/F_{d+1}$; $\mathrm{Lie}(F) \overset{\mathrm{def}}{=} \oplus_{m\geq 1} F_m/F_{m+1}$ for the free Lie algebra over $\mathbb{Z}_p$ associated to F, where m ranges over the positive integers. Then since $x \in Z(F/F_{i+1})$, and $i+1 \geq d+2$, for each $y \in F_1/F_2 \subseteq \mathrm{Lie}(F)$, it holds that $[\overline{x}, y] = 0 \in \mathrm{Lie}(F)$. In particular, by considering the natural embedding $\mathrm{Lie}(F) \hookrightarrow \mathbb{Z}_p\langle X_1, \cdots, X_n\rangle$, where $\mathbb{Z}_p\langle X_1, \cdots, X_n\rangle$ denotes the noncommutative free algebra of n variables over $\mathbb{Z}_p$, we observe that the image of $\overline{x}$ lies in the center of $\mathbb{Z}_p\langle X_1, \cdots, X_n\rangle$. Then since $n \geq 2$, it holds that $\overline{x} = 0$. This is a contradiction. Thus, we conclude that $x \in F_i/F_{i+1}$. This completes the proof of assertion (ii).

Next, we verify assertion (iii). In light of assertion (ii), it suffices to verify that $I_i \overset{\mathrm{def}}{=} F_i K_{i+1}(F)/K_{i+1}(F)$ is saturated in $K_i(F)/K_{i+1}(F)$. Write

$$J_i : K_i(F)/K_{i+1}(F) \ \hookrightarrow\ \mathrm{Hom}_{\mathbb{Z}_p}(F^{\mathrm{ab}}, F_{i+1}/F_{i+2})$$

for the natural injective [Andreadakis-Johnson] homomorphism of $\mathbb{Z}_p$-modules determined by the assignment $\phi \mapsto (x \mapsto \phi(x)x^{-1})$. Note that the restriction $J_i|_{F_i/F_{i+1}}$ of J_i is determined by forming the commutator. If the image of $J_i|_{F_i/F_{i+1}}$ is not saturated in $\mathrm{Hom}_{\mathbb{Z}_p}(F^{\mathrm{ab}}, F_{i+1}/F_{i+2})$, then the natural homomorphism

$$F_i/F_{i+1} \otimes_{\mathbb{Z}_p} \mathbb{F}_p \ \longrightarrow\ \mathrm{Hom}_{\mathbb{F}_p}(F^{\mathrm{ab}} \otimes_{\mathbb{Z}_p} \mathbb{F}_p, F_{i+1}/F_{i+2} \otimes_{\mathbb{Z}_p} \mathbb{F}_p)$$

is not injective. However, this contradicts the center-freeness of $\mathrm{Lie}(F) \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ [cf. the discussion in the proof of (ii)]. Therefore, the image of $J_i|_{F_i/F_{i+1}}$ is saturated in $\mathrm{Hom}_{\mathbb{Z}_p}(F^{\mathrm{ab}}, F_{i+1}/F_{i+2})$. Thus, since J_I is injective, we conclude that I_i is saturated in $K_i(F)/K_{i+1}(F)$. This completes the proof of assertion (iii).

Assertion (iv) follows immediately from [8], Lemma 5.3. Assertion (v) follows immediately from the existence of the above displayed injective homomorphism or [8], Lemma 5.7. Assertion (vi) follows from [8], Theorem 5.8, (b). Assertion (vii) follows immediately from assertion (vi), together with Lemma 1.3 below. This completes the proof of Proposition 1.2.

∎

Lemma 1.3. *Let G be a profinite group; H a normal closed subgroup of G; $\{K_i\}_{i\in I}$ a family of normal closed subgroups of G such that $\cap_{i\in I} K_i = \{1\}$. Then $\cap_{i\in I} HK_i = H$.*

Proof. Lemma 1.3 follows immediately from [11], Proposition 2.1.4, (a). $\square$

Next, we prove a pro-p analogue of [4], Theorem 2 [cf. Proposition 1.5] whose proof is similar to the proof of [4], Theorem 2. The key lemma is the following representation-theoretic result. [The details are omitted in [4].]

Lemma 1.4. *Let n be a positive integer ≥ 2; K a field of characteristic 0. Write V ($\stackrel{\text{def}}{=} K^n$) for the standard $\mathrm{SL}_n(K)$-representation;*

$$W \stackrel{\text{def}}{=} \mathrm{Hom}(V, \wedge_2 V) \xrightarrow{\sim} \wedge_{n-1}V \otimes \wedge_2 V.$$

Then the following hold:

(i) Suppose that $n = 2$. Then W is irreducible.

(ii) Suppose that $n \geq 3$. Then there exists an irreducible $\mathrm{SL}_n(K)$-subrepresentation of W of rank $\frac{n(n+1)(n-2)}{2}$ ($= \dim_K W - n$). [In this case, we note that $\frac{n(n+1)(n-2)}{2} > n$.]

Proof. First, we recall that since K is of characteristic 0, any irreducible polynomial representation of $\mathrm{GL}_n(K)$ is obtained from a Schur module associated to a partition [cf. [6], Theorem (3.5a)]. Then since the irreducible polynomial representations of $\mathrm{GL}_n(K)$ are absolutely irreducible, it follows immediately from the equality $\mathrm{GL}_n(\overline{K}) = Z(\mathrm{GL}_n(\overline{K}))\cdot\mathrm{SL}_n(\overline{K})$, together with Schur's lemma, that the restriction to $\mathrm{SL}_n(K)$ of the irreducible polynomial representations of $\mathrm{GL}_n(K)$ are still irreducible. Write $\mathbb{S}_{(-)}$ [i.e., the Schur functor] for the irreducible polynomial representation of $\mathrm{GL}_n(K)$, hence also of $\mathrm{SL}_n(K)$, associated to a partition $(-)$. Here, we recall that any polynomial representation of $\mathrm{GL}_n(K)$ is completely reducible [cf. [6], Corollary (2.6e)]. Then, in light of the isomorphism discussed in Remarks, (ii) immediately after [6], Theorem (3.5a), by using Littlewood-Richardson rule concerning the product of two Schur polynomials, we observe that:

- If $n = 2$, then $W \cong \mathbb{S}_{(2,1)}$.
- If $n \geq 3$, then $W \cong \mathbb{S}_{(2,2,1,\ldots,1)} \oplus \mathbb{S}_{(2,1,\ldots,1)}$, where the cardinalities of rows of the partitions $(2, 2, 1, \ldots, 1)$, $(2, 1, \ldots, 1)$ are $n-1$, n, respectively.

Therefore, it suffices to verify that $\dim_K \mathbb{S}_{(2,2,1,\ldots,1)} = \frac{n(n+1)(n-2)}{2}$ in the case where $n \geq 3$. However, this follows from Weyl's formula [cf. e.g., [7], §24.3, Corollary; [5], Theorem 6.3, (i) [over $\mathbb{C}$]]. This completes the proof of Lemma 1.4. $\square$

Proposition 1.5. *Consider the natural action of $\mathrm{Aut}(F)$ on $K_1(F)/K_2(F)$ by conjugation. Then the $\mathbb{Z}_p$-submodule*

$$FK_2(F)/K_2(F) \subseteq K_1(F)/K_2(F)$$

may be characterized as the unique $\mathrm{Aut}(F)$-invariant $\mathbb{Z}_p$-submodule of rank n that is a direct summand of $K_1(F)/K_2(F)$.

Proof. First, we note that $K_1(F^{\mathrm{disc}})/K_2(F^{\mathrm{disc}}) \otimes \mathbb{Z}_p \xrightarrow{\sim} K_1(F)/K_2(F)$ [cf. [1], Theorem 3.1; [1], §4]. Then it follows immediately from [4], Theorem 2, that the $\mathbb{Z}_p$-submodule $FK_2(F)/K_2(F) \subseteq K_1(F)/K_2(F)$ is an $\mathrm{Aut}(F)$ invariant $\mathbb{Z}_p$-submodule of rank n that is a direct summand of $K_1(F)/K_2(F)$.

Next, we verify the uniqueness part by a similar discussion to the proof of [4], Theorem 2. Fix an isomorphism $F^{\mathrm{ab}} \xrightarrow{\sim} \mathbb{Z}_p^n$. Then the conjugate action of $\mathrm{Aut}(F)$ on

$$K_1(F)/K_2(F) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}_p}(F^{\mathrm{ab}}, F_2/F_3) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p^n, \wedge_2 \mathbb{Z}_p^n)$$

factors through the natural action of $\mathrm{GL}_n(\mathbb{Z}_p)$ on $\mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p^n, \wedge_2\mathbb{Z}_p^n)$. Let $M \subseteq \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p^n, \wedge_2\mathbb{Z}_p^n)$ be a $\mathrm{GL}_n(\mathbb{Z}_p)$-submodule of rank n that is a direct summand. Then it follows immediately from the Zariski density of $\mathrm{SL}_n(\mathbb{Z}_p) \subseteq \mathrm{SL}_n(\mathbb{Q}_p)$ that $M \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \subseteq \mathrm{Hom}_{\mathbb{Q}_p}(\mathbb{Q}_p^n, \wedge_2\mathbb{Q}_p^n)$ is an $\mathrm{SL}_n(\mathbb{Q}_p)$-subrepresentation of rank n. On the other hand, it follows immediately from Lemma 1.4 that an irreducible $\mathrm{SL}_n(\mathbb{Q}_p)$-subrepresentation of $\mathrm{Hom}_{\mathbb{Q}_p}(\mathbb{Q}_p^n, \wedge_2\mathbb{Q}_p^n)$ of rank n is unique. Thus, since M is direct summand, we conclude that such an M is also unique. This completes the proof of Proposition 1.5. $\square$

Definition 1.6. For each pro-p group G, write $\Phi(G) \subseteq G$ for the Frattini subgroup.

Lemma 1.7. *Let d be a positive integer; $G \subseteq \mathrm{Aut}(F)$ a closed subgroup; $Q \subseteq K_d(F)$ a closed subgroup that is not contained in $K_{d+1}(F)$ and normalized by the conjugation action by G. Suppose that Q^{ab} is a finitely generated free $\mathbb{Z}_p$-module, and the G-space $Q/\Phi(Q)$ is irreducible. Then the natural homomorphism*

$$Q^{\mathrm{ab}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \longrightarrow K_d(F)/K_{d+1}(F) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

is injective.

Proof. Since the homomorphism is nonzero, it suffices to verify that the G-space $Q^{\mathrm{ab}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is irreducible. Suppose that there exists a proper G-subspace $L \subsetneq Q^{\mathrm{ab}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Then it is immediate that $L \cap Q^{\mathrm{ab}}$ is a saturated proper G-submodule of Q^{ab}. Therefore, the natural homomorphism $(L \cap Q^{\mathrm{ab}})/p(L \cap Q^{\mathrm{ab}}) \to Q/\Phi(Q) = Q^{\mathrm{ab}}/pQ^{\mathrm{ab}}$ is injective. On the other hand, since Q^{ab} is a finitely generated free $\mathbb{Z}_p$-module, $L \cap Q^{\mathrm{ab}}$ is also a finitely generated free $\mathbb{Z}_p$-module whose rank is less than the rank of Q^{ab}. This contradicts our assumption that $Q/\Phi(Q)$ is an irreducible G-space. This completes the proof of Lemma 1.7. $\square$

2 Basic properties and computations concerning matrix groups

Let p be a prime number; m, n positive integers. In this section, we discuss some well-known or computational properties on matrix groups, which will be of use in the proof of our main theorem in the next section.

Lemma 2.1. *The following hold:*

(i) Any normal p-subgroup of $\mathrm{GL}_n(\mathbb{F}_p)$ is trivial.

(ii) The kernel of the natural surjective homomorphism

$$\mathrm{GL}_n(\mathbb{Z}/p^{m+1}\mathbb{Z}) \twoheadrightarrow \mathrm{GL}_n(\mathbb{Z}/p^m\mathbb{Z})$$

may be naturally identified with $\mathrm{M}_n(\mathbb{F}_p)$, which is an elementary abelian p-group.

(iii) The natural action of the kernel of the natural surjective homomorphism

$$\mathrm{GL}_n(\mathbb{Z}/p^{m+1}\mathbb{Z}) \twoheadrightarrow \mathrm{GL}_n(\mathbb{F}_p)$$

on the kernel of the natural surjective homomorphism

$$\mathrm{GL}_n(\mathbb{Z}/p^{m+1}\mathbb{Z}) \twoheadrightarrow \mathrm{GL}_n(\mathbb{Z}/p^m\mathbb{Z})$$

via conjugation is trivial. Moreover, the induced action of $\mathrm{GL}_n(\mathbb{F}_p)$ *on* $\mathrm{M}_n(\mathbb{F}_p)$ *[cf. (ii)] coincides with the natural conjugation action.*

(iv) Let r, s *be positive integers;* $A, B \in \mathrm{M}_n(\mathbb{Z}_p)$. *Then*

$$[1 + p^r A, 1 + p^s B] \equiv 1 + p^{r+s}(AB - BA) \pmod{\Gamma_{r+s+\min\{r,s\}}},$$

where $\Gamma_i \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\mathrm{GL}_n(\mathbb{Z}_p) \twoheadrightarrow \mathrm{GL}_n(\mathbb{Z}/p^i\mathbb{Z}))$ *for each positive integer* i. *In particular, it holds that* $[\Gamma_r, \Gamma_s] \subseteq \Gamma_{r+s}$.

(v) Suppose that p *is odd (respectively,* $p = 2$*). Then* Γ_1 *(respectively,* Γ_2*) is torsion-free.*

Proof. First, we verify assertion (i). Let $M \subseteq \mathrm{GL}_n(\mathbb{F}_p)$ be a normal p-subgroup of $\mathrm{GL}_n(\mathbb{F}_p)$. Write U (respectively, L) for the subgroup of $\mathrm{GL}_n(\mathbb{F}_p)$ consisting of the upper (respectively, lower) triangle matrices whose diagonal components are equal to 1. Since U and L are Sylow p-subgroups of $\mathrm{GL}_n(\mathbb{F}_p)$, it follows immediately from Sylow's theorem, together with our choice of M, that $M \subseteq U \cap L = \{1\}$. This completes the proof of assertion (i). Assertions (ii), (iii), (iv) follow immediately from direct calculations. Assertion (v) is a well-known property. This completes the proof of Lemma 2.1. $\square$

Lemma 2.2. *Suppose that* $n \geq 3$. *Then* $\mathrm{Out}(\mathrm{GL}_n(\mathbb{F}_2))$ *is of order* 2 *and generated by the outer automorphism* $A \mapsto (A^{-1})^{\mathrm{t}}$. *In particular, the pull-back of the standard representation of* $\mathrm{GL}_n(\mathbb{F}_2)$ *via an automorphism of* $\mathrm{GL}_n(\mathbb{F}_2)$ *is isomorphic to the standard representation or its dual.*

Proof. Lemma 2.2 follows immediately from the classification of automorphisms of $\mathrm{GL}_n(\mathbb{F}_p)$ [cf. [2], Chapitre IV, §1]. $\square$

Lemma 2.3. *Suppose that* $n \geq 3$. *Write* V *for the standard representation of* $G \stackrel{\mathrm{def}}{=} \mathrm{GL}_n(\mathbb{F}_2)$; $V^\vee$ *for the dual* G*-representation of* V. *Then*

$$\mathrm{Hom}_G(V, \mathrm{End}(V)) = \{0\}; \quad \mathrm{Hom}_G(V^\vee, \mathrm{End}(V)) = \{0\}.$$

Proof. For each positive integer i, write $I_i \in \mathrm{GL}_i(\mathbb{F}_2)$ for the identity matrix. Let $f : V \to \mathrm{End}(V) = \mathrm{M}_n(\mathbb{F}_2)$ be a G-equivariant homomorphism; $e \in V$ a nonzero element. Write $S_e \subseteq G$ for the stabilizer subgroup of e. Then since f is G-equivariant, it holds that $f(e) \in Z_{\mathrm{M}_n(\mathbb{F}_2)}(S_e)$. Observe that $Z_{\mathrm{M}_n(\mathbb{F}_2)}(S_e) = \mathbb{F}_2 \cdot I_n$. Indeed, one may replace e by the element of standard basis associated to the first coordinate. Then this equality follows immediately from the well-known equality $Z_{\mathrm{M}_{n-1}(\mathbb{F}_2)}(\mathrm{GL}_{n-1}(\mathbb{F}_2)) = \mathbb{F}_2 \cdot I_{n-1}$, together with a direct computation under the assumption that $n \geq 3$. In particular, the image of f consists of diagonal matrices. Note that the action of $\mathrm{GL}_n(\mathbb{F}_2)$ on $V \setminus \{0\}$ is transitive. Therefore, the set $f(V \setminus \{0\})$ consists of a single diagonal matrix, which must be zero due to the fact that f is a homomorphism. Thus, we conclude that $\mathrm{Hom}_G(V, \mathrm{End}(V)) = \{0\}$. The equality $\mathrm{Hom}_G(V^\vee, \mathrm{End}(V)) = \{0\}$ follows from a totally similar argument to the above argument. This completes the proof of Lemma 2.3. $\square$

Lemma 2.4. *We regard* $\mathrm{M}_2(\mathbb{F}_2)$ *as a* $\mathrm{GL}_2(\mathbb{F}_2)$*-module with respect to the natural conjugation action. Write*

$$A_1 \stackrel{\mathrm{def}}{=} I_2 + E_{12} \in \mathrm{M}_2(\mathbb{F}_2); \quad A_2 \stackrel{\mathrm{def}}{=} I_2 + E_{21} \in \mathrm{M}_2(\mathbb{F}_2); \quad A_3 \stackrel{\mathrm{def}}{=} A_1 + A_2 \in \mathrm{M}_2(\mathbb{F}_2),$$

where $I_2 \in \mathrm{GL}_2(\mathbb{F}_2)$ *denotes the identity matrix;* $E_{ij} \in \mathrm{M}_2(\mathbb{F}_2)$ *denotes the matrix unit associated to* (i,j)*-component;*

$$S \subseteq \mathrm{M}_2(\mathbb{F}_2)$$

for the subgroup of order 4 *generated by* A_1, A_2, A_3. *Then the following hold:*

(i) The group $\mathrm{GL}_2(\mathbb{F}_2)$ *is isomorphic to the symmetric group* $\mathfrak{S}_3$ *on* 3*-letters. In particular,* $\mathrm{Out}(\mathrm{GL}_2(\mathbb{F}_2)) = \{1\}$.

(ii) The subgroup $S \subseteq \mathrm{M}_2(\mathbb{F}_2)$ *is a unique irreducible* $\mathrm{GL}_2(\mathbb{F}_2)$*-submodule of dimension* 2.

(iii) It holds that $[S,S]_{\mathrm{L}} = \mathbb{F}_2 \cdot I_2$, *where* $[-,-]_{\mathrm{L}}$ *denotes the Lie bracket.*

(iv) It holds that $A_i + A_i^2 \not\subseteq \mathbb{F}_2 \cdot I_2$ *for each* $i = 1,2,3$.

Proof. Assertion (i) follows immediately from the easily verified fact that $\mathrm{GL}_2(\mathbb{F}_2)$ is a nonabelian group of order 6. Next, we verify assertion (ii). Note that $\mathrm{GL}_2(\mathbb{F}_2) = \{I_2, A_1, A_2, A_3, A_1A_2, A_2A_1\}$, and $A_1^2 = A_2^2 = A_3^2 = I_2$. This observation, together with Sylow's theorem, implies that $S \subseteq \mathrm{M}_2(\mathbb{F}_2)$ is a $\mathrm{GL}_2(\mathbb{F}_2)$-submodule of dimension 2, and any two nonzero elements of S are conjugate in $\mathrm{GL}_2(\mathbb{F}_2)$, which shows that S is irreducible. Let $W \subseteq \mathrm{M}_2(\mathbb{F}_2)$ be an irreducible $\mathrm{GL}_2(\mathbb{F}_2)$-submodule of dimension 2. Then the irreducibility of W implies that the trace of any element of W is 0. Therefore, it holds that $W \subseteq \mathbb{F}_2 \cdot I_2 \oplus S$. Thus, we conclude from the irreducibility of W again, together with the dimension of W, that $W = S$. This completes the proof of assertion (ii). Assertions (iii), (iv) follow immediately from direct computations. This completes the proof of Lemma 2.4. □

Remark 2.4.1. Lemma 2.4, (i), will not be applied in the present paper. This is stated just for the comparison with Lemma 2.2.

3 Characterization of a nonabelian free pro-p group in its automorphism group

In this section, by applying the results obtained in the previous sections, we prove our main theorem [cf. Theorem 3.1]. Throughout the present section, let p be a prime number; n a positive integer ≥ 2; F a free pro-p group of rank n.

Our goal is to prove the following result:

Theorem 3.1. *The closed subgroup* $F \subseteq \mathrm{Aut}(F)$ *may be characterized as a unique normal closed subgroup of* $\mathrm{Aut}(F)$ *isomorphic to* F.

Remark 3.1.1. Theorem 3.1 may be regarded as a pro-p analogue of the Formanek's corresponding result in the discrete setting.

Remark 3.1.2. At the time of writing of the present paper, the author does not whether or not a similar assertion to Theorem 3.1 for nonabelian free profinite groups holds.

The following is an immediate consequence of Theorem 3.1.

Corollary 3.2. *It holds that F is a characteristic subgroup of* $\mathrm{Aut}(F)$*. In particular, one may observe that* $\mathrm{Out}(\mathrm{Aut}(F)) = \{1\}$*.*

Remark 3.2.1. The equality $\mathrm{Out}(\mathrm{Aut}(F)) = \{1\}$ is equivalent to the assertion that the natural injective homomorphism $\mathrm{Aut}(F) \hookrightarrow \mathrm{Aut}(\mathrm{Aut}(F))$, via conjugation, is an isomorphism. Thus, the automorphism tower terminates in a very early stage. This phenomenon also can be seen in the arithmetic side. Indeed, if we write G_K for the absolute Galois group of a number field, then it follows immediately from the Neukirch-Uchida theorem [cf. [12]] that the natural injective homomorphism $\mathrm{Aut}(G_K) \hookrightarrow \mathrm{Aut}(\mathrm{Aut}(G_K))$ is an isomorphism, hence that $\mathrm{Out}(\mathrm{Aut}(G_K)) = \{1\}$. At the time of writing of the present paper, the author does not have a clear understanding why this phenomenon occurs for highly nonabelian profinite groups that appear in anabelian geometry. With regard to the arithmetic side, one may ask whether or not a similar property holds for the absolute Galois groups of p-adic local fields.

In the remainder, to verify Theorem 3.1, let

$$N \subseteq \mathrm{Aut}(F)$$

be a free pro-p normal closed subgroup of rank n. Write

$$\phi : \mathrm{Aut}(F) \longrightarrow \mathrm{Aut}(N)$$

for the natural homomorphism determined by conjugation;

$$\phi_\theta : \mathrm{Aut}(F) \longrightarrow \mathrm{Aut}(F)$$

for the composite of ϕ with the automorphism $\mathrm{Aut}(N) \overset{\sim}{\to} \mathrm{Aut}(F)$ induced by an isomorphism $\theta : N \overset{\sim}{\to} F$. In particular, $\phi_\theta(N) = F$.

Lemma 3.3. *It holds that ϕ, hence also ϕ_θ, is injective.*

Proof. Observe that the [tautological] outer representation associated to the natural exact sequence

$$1 \longrightarrow F \longrightarrow \mathrm{Aut}(F) \longrightarrow \mathrm{Out}(F) \longrightarrow 1$$

is the identity automorphism. On the other hand, if $N \cap F = \{1\}$, then the normality of N and F imply that N commutes with F. This contradicts the above observation. Therefore, $N \cap F \neq \{1\}$. In particular, any automorphism $\in \mathrm{Ker}(\phi)$ of F induces the identity automorphism of a nontrivial normal closed subgroup of F. Here, we note that it follows from [11], Proposition 8.7.8, that F is internally indecomposable [i.e., a property that every nontrivial normal closed subgroup has the trivial centralizer — cf. [9], Definition 1.1, (v); [9], Proposition 1.2]. Thus, we conclude from [9], Lemma 1.9, that $\mathrm{Ker}(\phi) = \{1\}$. This completes the proof of Lemma 3.3. □

Definition 3.4. Write

$$\pi : \mathrm{Aut}(F) \twoheadrightarrow \mathrm{GL}(F^{\mathrm{ab}}) \; (\cong \mathrm{GL}_n(\mathbb{Z}_p))$$

for the natural surjective homomorphism [cf. Proposition 1.2, (i)];

$$\overline{\pi} : \mathrm{Aut}(F) \twoheadrightarrow \mathrm{GL}(F/\Phi(F))$$

for the natural surjective homomorphism induced by π;

$$P \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\overline{\pi})$$

$$\overline{\phi} : \mathrm{Aut}(F) \longrightarrow \mathrm{GL}(N/\Phi(N))$$

for the natural homomorphism induced by ϕ.

Lemma 3.5. *It holds that $\overline{\phi}$ is surjective, and $\mathrm{Ker}(\overline{\phi}) = P$. In particular, we obtain the following commutative diagram:*

$$\begin{array}{ccc} \mathrm{Aut}(F) & \xrightarrow{\phi} & \mathrm{Aut}(N) \\ {\scriptstyle\overline{\pi}}\big\downarrow & & \big\downarrow \\ \mathrm{GL}(F/\Phi(F)) & \xrightarrow{\sim} & \mathrm{GL}(N/\Phi(N)), \end{array}$$

where the right-hand vertical arrow denotes the natural surjective homomorphism [cf. Proposition 1.2, (i)].

Proof. Recall that the kernel of the natural surjective homomorphism $\mathrm{Aut}(N) \twoheadrightarrow \mathrm{GL}(N/\Phi(N))$ is a pro-p group. Then since ϕ is injective [cf. Lemma 3.3], it holds that $\mathrm{Ker}(\overline{\phi})$ is a pro-p normal subgroup of $\mathrm{Aut}(F)$. On the other hand, there exists no nontrivial pro-p normal subgroup of $\mathrm{GL}(F/\Phi(F))$ [cf. Lemma 2.1, (i)]. Therefore, it holds that $\mathrm{Ker}(\overline{\phi}) \subseteq P$. However, since the cardinality of $\mathrm{GL}(F/\Phi(F))$ and $\mathrm{GL}(N/\Phi(N))$ are same, we conclude that $\mathrm{Ker}(\overline{\phi}) = P$, and $\overline{\phi}$ is surjective. This completes the proof of Lemma 3.5. ☐

Definition 3.6. For each positive integer r, write

$$\Gamma_m \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\mathrm{GL}(F^{\mathrm{ab}}) \twoheadrightarrow \mathrm{GL}(F^{\mathrm{ab}}/p^r F^{\mathrm{ab}})).$$

Write

$$H \stackrel{\mathrm{def}}{=} \pi(N) \subseteq \mathrm{GL}(F^{\mathrm{ab}}).$$

In particular, it follows immediately from Lemma 3.5 that $H \subseteq \Gamma_1$.

Lemma 3.7. *Suppose that $H \neq 1$. Write*

$$m \stackrel{\mathrm{def}}{=} \max\{r \geq 1 \mid H \subseteq \Gamma_r\};$$

$$W \stackrel{\mathrm{def}}{=} H \cdot \Gamma_{m+1}/\Gamma_{m+1} \subseteq \Gamma_m/\Gamma_{m+1} \cong \mathrm{M}_n(\mathbb{F}_p).$$

Then the composite

$$N/\Phi(N) \twoheadrightarrow H/\Phi(H) \twoheadrightarrow W$$

of the natural surjective homomorphisms is an isomorphism.

Proof. Note that the composite is $\mathrm{Aut}(F)$-equivariant. On the other hand, it follows immediately from Lemma 3.5, together with the irreducibility of the standard representation of $\mathrm{GL}_n(\mathbb{F}_p)$, that the $\mathrm{Aut}(F)$-module $N/\Phi(N)$ is irreducible. Thus, we conclude that the composite is injective, hence an isomorphism. This completes the proof of Lemma 3.7. □

Proposition 3.8. *It holds that $H = \{1\}$, hence that $N \subseteq K_1(F)$.*

Proof. Suppose that $H \neq \{1\}$ to derive a contradiction. We maintain the notation of Lemma 3.7.

First, we consider the case where p is odd. Let c be a nontrivial element of $Z(\mathrm{GL}(F/\Phi(F)))$. Then the action of c on $N/\Phi(N)$ is a nontrivial scalar multiplication [cf. Lemma 3.5], and the action of c on W is trivial. This contradicts Lemma 3.7.

Next, we consider the case where $p = 2$, and $n \geq 3$. It follows immediately from Lemmas 2.2; 2.3; 3.5, together with Lemma 2.1, (iii), that $W = \{0\}$. This contradicts Lemma 3.7.

Finally, we consider the case where $p = 2$, and $n = 2$. Write

$$\psi : N^{\mathrm{ab}} \twoheadrightarrow H^{\mathrm{ab}}$$

for the natural $\mathrm{Aut}(F)$-equivariant surjective homomorphism induced by π. Then it follows from Lemma 3.7 that $\mathrm{Ker}(\psi) \subseteq 2N^{\mathrm{ab}}$. In the remainder, we use the notations of Lemma 2.4. Note that, by applying Lemma 2.4, (ii), together with Lemma 3.5, we observe that $W = S$ under the natural identification $\Gamma_m/\Gamma_{m+1} \xrightarrow{\sim} \mathrm{M}_2(\mathbb{F}_2)$. Let $A \in S \setminus \{0\}$; $h \in H$ a lifting of A. Then, by a slight abuse of notation, one may write $h \equiv I_2 + 2^m A \pmod{\Gamma_{m+1}}$. If $m \geq 2$, then $h^2 \equiv I_2 + 2^{m+1}A \pmod{\Gamma_{m+2}}$. Therefore, since $[H, H] \subseteq [\Gamma_m, \Gamma_m] \subseteq \Gamma_{2m}$ [cf. Lemma 2.1, (iv)], it holds that $h^2 \notin [H, H]$. On the other hand, if $m = 1$, then:

- $h^2 \equiv I_2 + 4(A + A^2) \pmod{\Gamma_3}$.
- $[H, H] \equiv \{[I_2 + 2A_1, I_2 + 2A_2] = I_2 + 4[A_1, A_2]_{\mathrm{L}} \mid A_1, A_2 \in S\} \pmod{\Gamma_3}$ [cf. Lemma 2.1, (iv)], where we note that the right-hand side coincides with $\{I_2, 5I_2\} \pmod{\Gamma_3}$ [cf. Lemma 2.4, (iii)].

Then it follows immediately from Lemma 2.4, (iv), that $h^2 \notin [H, H]$. In either case, we observe that $\mathrm{Ker}(\psi) \subsetneq 2N^{\mathrm{ab}}$. In light of the irreducibility of $2N^{\mathrm{ab}}/4N^{\mathrm{ab}}$, together with Nakayama's lemma, that

$$\mathrm{Ker}(\psi) \subseteq 4N^{\mathrm{ab}}.$$

Fix a basis $\{x_1, x_2\}$ of a free group of rank 2, and identify F with the pro-2 completion of the free group. Write

$$g \in \mathrm{Aut}(F)$$

for the element of order 2 determined by the assignments $x_1 \mapsto x_1^{-1}$, $x_2 \mapsto x_2^{-1}$. Then since $\pi(g)$ lies in the center of $\mathrm{GL}(F^{\mathrm{ab}})$, it holds that g acts trivially on H. Thus, in light of the above displayed inclusion $\mathrm{Ker}(\psi) \subseteq 4N^{\mathrm{ab}}$, together with the $\mathrm{Aut}(F)$-equivariance of ψ, we observe that $\phi(g)$ lies in the kernel of the natural surjective homoorphism $\mathrm{Aut}(N) \twoheadrightarrow \mathrm{GL}(N^{\mathrm{ab}}/4N^{\mathrm{ab}})$. However, since $g^2 = 1$, and the kernel is torsion-free [cf. Proposition 1.2, (v); Lemma 2.1, (v)], we deduce that $\phi(g) = 1$. This contradicts the injectivity of ϕ [cf. Lemma 3.3].

Thus, we conclude that $H = \{1\}$ in any event. This completes the proof of Proposition 3.8. □

Definition 3.9. Write

$$\beta \stackrel{\mathrm{def}}{=} \pi \circ \phi_\theta : \mathrm{Aut}(F) \longrightarrow \mathrm{GL}(F^{\mathrm{ab}});$$

$$H' \stackrel{\mathrm{def}}{=} \beta(F).$$

Proposition 3.10. *It holds that $H' = \{1\}$, hence that $\phi_\theta(F) \subseteq K_1(F)$.*

Proof. Suppose that $H' \neq \{1\}$ to derive a contradiction. Write

$$\psi' : F^{\mathrm{ab}} \twoheadrightarrow (H')^{\mathrm{ab}}$$

for the natural surjective homomorphism, which is not necessarily $\mathrm{Aut}(F)$-equivariant. However, in light of Lemma 3.5, except the case where $p = 2$, and $n = 2$, by applying a similar argument to the argument applied in the proof of Proposition 3.8, we obtain a contradiction. Next, we consider the case where $p = 2$, and $n = 2$. Note that $N \subseteq K_1(F)$ [cf. Proposition 3.8]. Write $r \stackrel{\mathrm{def}}{=} \max\{m \geq 1 \mid N \subseteq K_m(F)\}$ [cf. Proposition 1.2, (vi)]. Then, by considering the natural $\mathrm{Aut}(F)$-equivariant injective homomorphism

$$N^{\mathrm{ab}} \hookrightarrow K_r(F)/K_{r+1}(F)$$

[cf. Lemma 1.7], together with the actions of g [cf. the proof of Proposition 3.8] on the domain and codomain of the injective homomorphism, we observe that $\beta(g) = (-1)^r I_2$. On the other hand, in light of Lemma 3.5, by applying an argument to ψ' similar to the argument applied in the proof of Proposition 3.8, one may observe that $\mathrm{Ker}(\psi') \subseteq 4F^{\mathrm{ab}}$. Note that since $\beta(g)$ lies in the center of $\mathrm{GL}(F^{\mathrm{ab}})$, it holds that $\beta(g)$ acts trivially on $(H')^{\mathrm{ab}}$. Then since $\mathrm{Ker}(\psi') \subseteq 4F^{\mathrm{ab}}$, we observe that g acts trivially on $F^{\mathrm{ab}}/4F^{\mathrm{ab}}$, which is a contradiction.

Thus, we conclude that $H' = \{1\}$ in any event. This completes the proof of Proposition 3.10. □

Lemma 3.11. *In light of Propositions 3.8; 3.10, write*

$$r \stackrel{\mathrm{def}}{=} \max\{m \geq 1 \mid N \subseteq K_m(F)\}; \quad s \stackrel{\mathrm{def}}{=} \max\{m \geq 1 \mid \phi_\theta(F) \subseteq K_m(F)\}$$

[cf. Proposition 1.2, (vi)]. Then it holds that $r = s = 1$.

Proof. First, by applying Lemma 1.7, together with Lemma 3.5, we observe that the natural homomorphisms

$$N^{\mathrm{ab}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \longrightarrow K_r(F)/K_{r+1}(F) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p, \quad \phi_\theta(F)^{\mathrm{ab}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \longrightarrow K_s(F)/K_{s+1}(F) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

are injective. Let $\lambda \in 1 + p^2\mathbb{Z}_p \subseteq \mathbb{Z}_p^\times$; $a_\lambda \in \mathrm{Aut}(F)$ a lifting of $\lambda I \in \mathrm{GL}(F^{\mathrm{ab}})$ with respect to π, where I denotes the identity matrix. Then since the natural action of a_λ on $K_r(F)/K_{r+1}(F)$ is multiplication by λ^r, the injectivity of the first homomorphism implies that the natural action of a_λ on N^{ab}, hence the natural action of $\phi_\theta(a_\lambda)$ on F^{ab}, is multiplication by λ^r. Therefore, it holds that the natural action of $\phi_\theta(a_\lambda)$ on $K_s(F)/K_{s+1}(F)$ is multiplication by λ^{rs}, hence from the injectivity of the second homomorphism that the natural action of $\phi_\theta(a_\lambda)$ on $\phi_\theta(F)^{\mathrm{ab}}$ is multiplication by λ^{rs}. Thus, we conclude from the definition on a_λ that $\lambda = \lambda^{rs}$, hence from the choice of λ that $r = s = 1$. This completes the proof of Lemma 3.11. □

Proposition 3.12. *For each positive integer $i \geq 2$, it holds that $N \subseteq FK_i(F)$.*

Proof. First, we note that $N \subseteq K_1(F)$ [cf. Proposition 3.8]. We prove the inclusion $N \subseteq FK_i(F)$ by induction on i. In the case where $i = 2$, by considering the natural $\mathrm{Aut}(F)$-equivariant injective homomorphism

$$f : N^{\mathrm{ab}} \hookrightarrow K_1(F)/K_2(F),$$

we observe that the inclusion in question follows immediately from Proposition 1.5, together with Lemma 1.4. Next, suppose that $N \subseteq FK_i(F)$. Then we consider the natural $\mathrm{Aut}(F)$-equivariant homomorphism

$$N^{\mathrm{ab}} \longrightarrow FK_i(F)/FK_{i+1}(F).$$

Let $\lambda \in \mathbb{Z}_p^\times$ be such that $\lambda^i - \lambda \neq 0$; $a_\lambda \in \mathrm{Aut}(F)$ a lifting of $\lambda I \in \mathrm{GL}(F^{\mathrm{ab}})$ with respect to π, where I denotes the identity matrix. In light of the $\mathrm{Aut}(F)$-equivariance of f, we observe that the natural action of a_λ on N^{ab} is multiplication by λ. On the other hand, since $FK_i(F)/FK_{i+1}(F)$ is a quotient of $K_i(F)/K_{i+1}(F)$, the natural action of a_λ on $FK_i(F)/FK_{i+1}(F)$ is multiplication by λ^i. Thus, we conclude from the torsion-freeness of $FK_i(F)/FK_{i+1}(F)$ [cf. Proposition 1.2, (iii)], together with our choice of λ, that the above displayed homomorphism must be trivial, hence that $N \subseteq FK_{i+1}(F)$. This completes the proof of Proposition 3.12. $\square$

Proof of Theorem 3.1. It follows immediately from Proposition 3.12, together with Proposition 1.2, (vii), that $N \subseteq F$. On the other hand, it is well-known that any nontrivial topologically finitely generated normal closed subgroup of F is open [cf., e.g., [8], Corollary 3.3; [10], Theorem 1.5]. In particular, $N \subseteq F$ is a normal open subgroup. Thus, since N and F are free pro-p groups of same rank $n \geq 2$, we conclude that $N = F$. This completes the proof of Theorem 3.1. $\square$

Acknowledgements

This research was supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University, as well as the Center for Research in Next Generation Geometry. This work is part of the “Arithmetic and Homotopic Galois Theory” project, supported by the CNRS France-Japan AHGT International Research Network between the RIMS Kyoto University, the LPP of Lille University, and the DMA of ENS PSL.

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