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ANABELIAN RESULTS FOR MIXED CHARACTERISTIC COMPLETE DISCRETE VALUATION FIELDS

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ABSTRACT. Let p be a prime number. In the present paper, as a direct application of Fontaine's works, we first observe that any isomorphism of Hodge-Tate type between the absolute Galois groups of mixed characteristic complete discrete valuation fields of residue characteristic p arises from an isomorphism of fields. This observation may be regarded as a generalization of the corresponding result for p-adic local fields due to Mochizuki. Moreover, by combining with an author's previous joint work with Minamide, we prove that any isomorphism between the absolute Galois groups of mixed characteristic complete discrete valuation fields with *nondegenerate* perfect residues that preserves the respective ramification filtrations arises from an isomorphism of fields. Here, we shall say that a field of characteristic p is nondegenerate if a Sylow-p subgroup of its absolute Galois group is nontrivial. While our proof is based on the joint work, this result may be regarded as a much stronger generalization of the corresponding result for p-adic local fields due to Mochizuki than the generalization obtained in the joint work.

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CONTENTS

INTRODUCTION

Throughout the present paper, let p be a prime number. For each field F of characteristic 0, we shall write $\overline{F}$ for the algebraic closure [determined up to isomorphisms] of F; $G_F \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{F}/F)$. The purpose of the present paper is to show that some main anabelian results for p-adic local fields due to Mochizuki may be strengthened in a greater generality via a Fontaine's highly nontrivial work and an author's recent joint work with Minamide.

Let K_1, K_2 be mixed characteristic complete discrete valuation fields of residue characteristic p;

$$\phi : G_{K_1} \overset{\sim}{\to} G_{K_2}$$

an isomorphism. Write $\mathbf{C}_i$ for the p-adic completion of $\overline{K}_i$. Then we shall say that ϕ is of *Hodge-Tate type* if the topological G_{K_1}-module obtained by composing ϕ with the natural action of G_{K_2} on $\mathbf{C}_2$ is isomorphic to the topological G_{K_1}-module $\mathbf{C}_1$. As a direct application of the Fontaine's computation of a certain type of Galois centralizer [cf. [3], Proposition 6.2], we first observe the following [cf. Proposition 1.3]:

Theorem A. *If ϕ is of Hodge-Tate type, then ϕ arises from a commutative diagram*

$$\begin{array}{ccc} \overline{K}_2 & \xrightarrow{\sim} & \overline{K}_1 \\ \uparrow & & \uparrow \\ K_2 & \xrightarrow{\sim} & K_1 \end{array}$$

of fields, where the vertical arrows denote the natural inclusions; the horizontal arrows denote isomorphisms of fields.

Theorem A may be regarded as a generalization of [9], Theorem 3.5, (ii), which was based on more classical p-adic representation theory [cf. [10]].

Next, for each discrete valuation field K of residue characteristic p, we shall say that K is *nondegenerate* if a Sylow-p subgroup of the absolute Galois group of the residue field of K is nontrivial. Then, by combining Theorem A with several results obtained in author's joint works with Minamide, we obtain the following result [cf. Corollary 2.7]:

Theorem B. *Suppose that K_1 and K_2 are nondegenerate mixed characteristic complete discrete valuation fields with perfect residues. Write*

$$\mathrm{OutIsom}^{\mathrm{filt}}(G_{K_1}, G_{K_2})$$

for the set of outer isomorphisms $G_{K_1} \overset{\sim}{\to} G_{K_2}$ of profinite groups that preserve the respective ramification filtrations. Then the natural map

$$\mathrm{Isom}(K_2, K_1) \longrightarrow \mathrm{OutIsom}^{\mathrm{filt}}(G_{K_1}, G_{K_2})$$

is bijective.

While we apply the discussions/results that appear in [7], the above result may be regarded as a much stronger generalization of [8], Theorem, than [7], Theorem G. Several further remarks on this result will be given immediately after Corollary 2.7.

1. Field-theoreticity of isomorphisms of Hodge-Tate type

In the present section, as a direct application of a Fontaine's fundamental work, we generalize an anabelian result concerning isomorphisms of Hodge-Tate-type between the absolute Galois groups of p-adic local fields due to Mochizuki.

Definition 1.1. For $i = 1, 2$, let K_i be a mixed characteristic complete discrete valuation field of residue characteristic p. Let

$$\phi : G_{K_1} \overset{\sim}{\to} G_{K_2}$$

be an isomorphism. Write $\mathbf{C}_i$ for the p-adic completion of $\overline{K}_i$. Then we shall say that ϕ is of *Hodge-Tate type* if the topological G_{K_1}-module obtained by composing ϕ with the natural action of G_{K_2} on $\mathbf{C}_2$ is isomorphic to the topological G_{K_1}-module $\mathbf{C}_1$.

The following result may be regarded as a key result in our discussion, which is easily derived from the corresponding highly nontrivial result in the perfect residue case due to Fontaine.

Proposition 1.2. *Let K be a mixed characteristic complete discrete valuation field of residue characteristic p. Write $\mathbf{C}$ for the p-adic completion of $\overline{K}$; $\mathrm{End}^{\mathrm{cont}}_{\mathbb{Q}_p[G_K]}(\mathbf{C})$ for the set of continuous G_K-equivariant endomorphisms of $\mathbb{Q}_p$-vector spaces. Then it holds that*

$$\mathrm{End}^{\mathrm{cont}}_{\mathbb{Q}_p[G_K]}(\mathbf{C}) \;=\; K.$$

Proof. In the perfect residue case, the equality follows from [3], Proposition 6.2. In the general case, let $\sigma \in \mathrm{End}^{\mathrm{cont}}_{\mathbb{Q}_p[G_K]}(\mathbf{C})$. Let $K \subseteq L$ be a weakly unramified algebraic field extension such that L is a Henselian discrete valuation field with perfect residues. [For instance, such an L is constructed by adjoining suitable p-power roots of lifts of a p-basis of the residue field.] Write $\widehat{L}$ for the p-adic completion of L. Then since $\sigma \in \mathrm{End}^{\mathrm{cont}}_{\mathbb{Q}_p[G_{\widehat{L}}]}(\mathbf{C})$, it follows immediately from the perfect residue case that σ is multiplication by $\sigma(1) \in \widehat{L}$. However, since σ is G_K-equivariant, it holds that $\sigma(1) \in \mathbf{C}^{G_K} = K$ [cf. [4], Proposition 3.8], as desired. This completes the proof of Proposition 1.2. $\square$

As a corollary of Proposition 1.2, we obtain the following:

Proposition 1.3. *In the notation of Definition 1.1, suppose that ϕ is of Hodge-Tate type. Then ϕ arises from a commutative diagram*

$$\begin{array}{ccc} \overline{K}_2 & \xrightarrow{\sim} & \overline{K}_1 \\ \uparrow & & \uparrow \\ K_2 & \xrightarrow{\sim} & K_1 \end{array}$$

of fields, where the vertical arrows denote the natural inclusions; the horizontal arrows denote isomorphisms of fields.

Proof. In light of the definition of a Hodge-Tate type isomorphism, there exists a continuous isomorphism

$$\mathbf{C}_1 \ \xrightarrow{\sim}\ \mathbf{C}_2$$

of $\mathbb{Q}_p$-vector spaces compatible with ϕ and the respective natural Galois actions. Then it follows immediately from the various definitions involved that ϕ induces an isomorphism of the respective *endomorphism rings*

$$\mathrm{End}^{\mathrm{cont}}_{\mathbb{Q}_p[G_{K_1}]}(\mathbf{C}_1) \ \xrightarrow{\sim}\ \mathrm{End}^{\mathrm{cont}}_{\mathbb{Q}_p[G_{K_2}]}(\mathbf{C}_2).$$

Then, in light of Proposition 1.2, we observe that ϕ induces an isomorphism $K_1 \xrightarrow{\sim} K_2$ of fields. Thus, by applying the above discussions to the restrictions of ϕ to the open subgroups of G_{K_1}, we conclude that there exists an isomorphism $\overline{K}_1 \xrightarrow{\sim} \overline{K}_2$ of fields compatible with ϕ and the respective natural Galois actions. This completes the proof of Proposition 1.3. □

Remark 1.3.1. To obtain various variants of Proposition 1.3, we need to generalize Proposition 1.2. For instance, one may ask whether or not a similar result holds for mixed characteristic rank 1 complete valuation fields of residue characteristic p whose value groups have no nontrivial p-divisible elements. This class includes the completion of the maximal tame extension field of a mixed characteristic complete discrete valuation of residue characteristic p. The author expects that this class of valuation fields is anabelian if we take into account an additional structure such as a topological Galois module $\mathbf{C}$ in our context [cf. also the data of geometric fundamental groups in the context of [11]] . However, at the time of writing of the present paper, the author does not obtain a proof of the analogue of Proposition 1.2 for this class.

2. Field-theoreticity of isomorphisms between the filtered absolute Galois groups of nondegenerate mixed characteristic complete discrete valuation fields with perfect residues

In the present section, by applying Proposition 1.3, together with some results obtained in an author's recent joint work with Minamide, we prove that any isomorphism between the absolute Galois groups of nondegenerate mixed characteristic complete discrete valuation fields with perfect residues that preserves the respective ramification filtrations arises from an isomorphism of fields.

First, we give some definitions concerning nondegeneracy, as follows.

Definition 2.1. Let k be a field of characteristic p. Then we shall say that k is *nondegenerate* if a Sylow-p subgroup of G_k is nontrivial.

Definition 2.2. Let K be a discrete valuation field of residue characteristic p. Then we shall say that K is *nondegenerate* if the residue field of K is nondegenerate.

Next, we introduce a basic setup.

Definition 2.3. For $i = 1, 2$, let K_i be a mixed characteristic complete discrete valuation field with a perfect residue field k_i of characteristic p. Let

$$\phi : G_{K_1} \overset{\sim}{\to} G_{K_2}$$

be an isomorphism that preserves the respective ramification filtrations.

Definition 2.4. Let k be a perfect field of characteristic p. Then we shall say that k is *strongly p-quasi-finite* if, for any finite extension field $k^\dagger$ of k, it holds that $G^p_{k^\dagger} \overset{\sim}{\to} \mathbb{Z}_p$.

Lemma 2.5. *In the notation of Definition 2.3, suppose that k_1 is strongly p-quasi-finite. Then ϕ arises from a commutative diagram*

$$\begin{tikzcd} \overline{K}_2 \arrow[r, "\sim"] & \overline{K}_1 \\ K_2 \arrow[u] \arrow[r, "\sim"] & K_1 \arrow[u] \end{tikzcd}$$

of fields, where the vertical arrows denote the natural inclusions; the horizontal arrows denote isomorphisms of fields. Moreover, the isomorphisms that appear in the diagram are unique.

Proof. First, we note that since ϕ preserves the respective inertia subgroups, it is immediate that k_2 is also strongly p-quasi-finite. Next, we recall that, in the strongly p-quasi-finite residue case, there exists a well-behaved p-local class field theory [cf. [7], §2]. As an application, together with [7], Theorem 7.2, (iii), one may observe that ϕ induces an isomorphism

$$\phi_f : \overline{K}_1 \overset{\sim}{\to} \overline{K}_2$$

of $\mathbb{Q}_p$-vector spaces compatible with ϕ and the respective natural Galois actions. Moreover, since ϕ preserves the respective ramification filtrations, it follows from a similar argument to the argument applied in the proof of [7], Theorem 8.11, that ϕ_f is, in fact, a homeomorphism. Thus, we conclude from Proposition 1.3 that ϕ arises from a commutative diagram of field embeddings as in the statement. Finally, the uniqueness follows immediately from [5], Theorem. This completes the proof of Lemma 2.5. $\square$

Theorem 2.6. *In the notation of Definition 2.3, suppose that K_1 is nondegenerate. Then ϕ arises from a commutative diagram*

$$\begin{array}{ccc} \overline{K}_2 & \xrightarrow{\sim} & \overline{K}_1 \\ \uparrow & & \uparrow \\ K_2 & \xrightarrow{\sim} & K_1 \end{array}$$

of fields, where the vertical arrows denote the natural inclusions; the horizontal arrows denote isomorphisms of fields. Moreover, the isomorphisms that appear in the diagram are unique.

Proof. Since K_1 is nondegenerate, it is immediate that there exists an unramified field extension $K_1 \subseteq L_1$ such that the residue field of L_1 is strongly p-quasi-finite. Write $K_2 \subseteq L_2$ for the unramified field extension that corresponds to the field extension $K_1 \subseteq L_1$ via ϕ. Note that since ϕ preserves the respective ramification filtrations, the isomorphism $\phi|_{G_{L_1}} : G_{L_1} \xrightarrow{\sim} G_{L_2}$ induced by ϕ also preserves the respective ramification filtrations. Then it follows from Lemma 2.5 that $\phi|_{G_{L_1}}$ arises from a unique commutative diagram

$$\begin{array}{ccc} \overline{K}_2 & \xrightarrow{\sim} & \overline{K}_1 \\ \uparrow & & \uparrow \\ L_2 & \xrightarrow{\sim} & L_1 \end{array}$$

of field embeddings, where the horizontal arrows are isomorphisms. In particular, since ϕ preserves the respective inertia subgroups I_1, I_2, we observe that the outer isomorphism $I_1 \xrightarrow{\sim} I_2$ determined by ϕ arises from a unique field isomorphism of the respective maximal unramified extension fields of K_1, K_2. Thus, since the inertia subgroups are normal, by Galois descent, together with the internal indecomposability of G_{K_1} [cf. [6], Theorem A, (i); [6], Lemma 1.9], we conclude that ϕ arises from a unique commutative diagram

$$\begin{array}{ccc} \overline{K}_2 & \xrightarrow{\sim} & \overline{K}_1 \\ \uparrow & & \uparrow \\ K_2 & \xrightarrow{\sim} & K_1 \end{array}$$

of field embeddings, as desired. This completes the proof of Theorem 2.6. $\square$

Corollary 2.7. *For $i = 1, 2$, let K_i be a nondegenerate mixed characteristic complete discrete valuation field with perfect residues. Write*

$$\mathrm{OutIsom}^{\mathrm{filt}}(G_{K_1}, G_{K_2})$$

for the set of outer isomorphisms $G_{K_1} \xrightarrow{\sim} G_{K_2}$ of profinite groups that preserve the respective ramification filtrations. Then the natural map

$$\mathrm{Isom}(K_2, K_1) \longrightarrow \mathrm{OutIsom}^{\mathrm{filt}}(G_{K_1}, G_{K_2})$$

is bijective.

Proof. The injectivity follows immediately from [5], Theorem. The surjectivity follows from Theorem 2.6. This completes the proof of Corollary 2.7. □

Remark 2.7.1. Since we do not need to assume that the base fields are algebraic over the prime fields, Corollary 2.7 may be regarded as a much stronger generalization of [8], Theorem, than [7], Theorem G. On the other hand, our proof is based on various discussions/results in [7].

Remark 2.7.2. It is not clear to the author that the nondegeneracy assumption may be eliminated. Since anabelian phenomena do not occur in the situation of cohomological dimension 1, it appears to the author that our assumption is intrinsic. It would be also interesting to compare with the situation of local class field theory. Indeed, there exists a general p-local class field theory under this type of nondegeneracy assumption due to Fesenko [cf. [2]]. On the other hand, one may not expect that there exists a usual local class field theory in the algebraically closed residue case.

Remark 2.7.3. For another direction, one may ask whether or not there exists a similar result including imperfect residue cases. At the time of writing of the present paper, it is not even clear to the author which ramification theory among the various existing ramification theory is suitable to be considered in anabelian geometry. This is a remaining important task.

Remark 2.7.4. At this point, there is a big difference between the mixed characteristic situation and the positive characteristic situation. Indeed, in the case of positive characteristic, the only known anabelian result is the Abrashkin's result for positive characteristic local fields, i.e., the case of finite residues [cf. [1]]. It is natural to ask whether or not a similar result holds in the positive characteristic case.

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References

[1] V. Abrashkin, Modified proof of a local analogue of the Grothendieck conjecture, *Journal Theorie des Nombres de Bordeaux* **22** (2010), pp. 1–50.

[2] I. Fesenko, Local class field theory: perfect residue case, *Russian Acad. Sci. Izv. Math.* **43** (1994), pp. 65–81.

[3] J.-M. Fontaine, Arithmétique des représentations galoisiennes p-adiques, *Astérisque* **295** (2004), pp. 1-115.

[4] J.-M. Fontaine and Y. Ouyang, *Theory of p-adic Galois representations.*

[5] Y. Hoshi and S. Tsujimura, On the injectivity of the homomorphisms from the automorphism groups of fields to the outer automorphism groups of the absolute Galois groups, *Res. Number Theory* **9**, Paper No. 44 (2023).

[6] A. Minamide and S. Tsujimura, Internal indecomposability of profinite groups, *Adv. Math.* **409** (2022), Paper No. 108689.

[7] A. Minamide and S. Tsujimura, Anabelian geometry for Henselian discrete valuation fields with quasi-finite residues, to appear in *Amer. J. Math.*

[8] S. Mochizuki, A version of the Grothendieck conjecture for p-adic local fields, *Internat. J. Math.* **8** (1997), pp. 499–506.

[9] S. Mochizuki, Topics in absolute anabelian geometry I: Generalities, *J. Math. Sci. Univ. Tokyo* **19** (2012), pp. 139–242.

[10] J.-P. Serre, *Abelian l-adic Representations and Elliptic Curves*, W. A. Benjamin, Inc.,(1968).

[11] S. Tsujimura, *On pro-p anabelian geometry for hyperbolic curves of genus* 0 *over p-adic fields*, *RIMS preprint* **1983** (April 2024).

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