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Abstract

In the present paper, we revisit the result on congruence topologies in the multiplicative groups of finitely generated infinite fields due to Grunewald and Segal from the viewpoint of finite partitions of the sets of closed points of normal models and discuss an abstract principle behind it. As a consequence, we generalize the result of Grunewald and Segal to groups of rational points of semi-abelian varieties over finitely generated infinite fields.

Contents

Introduction

Throughout the present paper, for each connected Noetherian scheme S and each point $s \in S$, we shall write $|S|$ for the set of closed points of S; $k(s)$ for the residue field of S at s; Π_S for the étale fundamental group of S, relative to a suitable choice of basepoint. For each field F, we shall write G_F for the absolute Galois group of F, relative to a choice of a separable closure of F. For each profinite group G, and each topological G-module M, we shall write $H^1(G, M)$ for the first continuous cochain cohomology group. We shall say that a field F is a *generalized global field* if F is a finitely generated infinite field. If Γ is a finitely generated subgroup of the group of F-valued points of a semi-abelian variety over a field F of characteristic 0 (respectively, positive characteristic p), then we shall write

$$\widehat{\Gamma}$$

for the profinite (respectively, pro-prime-to-p) completion of Γ.

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In [10], Grunewald and Segal proved the following type of result concerning congruence topologies in the multiplicative groups of generalized global fields under finite partitions of closed points of models [cf. [10], Theorems A, B; the formulation below differs slightly from the original in that the exceptional set Z is also allowed in the higher dimensional case and is required only to have Dirichlet density 0], which may be regarded as a refinement of the works of Schmidt, Chevalley, and Wehrfritz [cf. [19], [2], [23]]:

Theorem 0.1. *Let K be a generalized global field; X an irreducible normal scheme of finite type over $\mathbb{Z}$ whose function field is K; $Z \subseteq |X|$ a subset of Dirichlet density 0 [cf. Definition 1.8];*

$$|X| \;=\; \coprod_{1\leq i\leq r} S_i$$

a finite partition. Then there exists a positive integer $j \leq r$ satisfying the following condition: For each finitely generated subgroup

$$\Gamma \;\subseteq\; K^{\times}$$

and each nonempty open subscheme $U \subseteq X$ such that every element of Γ is a unit on U, i.e., $\Gamma \subseteq \mathcal{O}_X(U)^{\times}$, the natural homomorphism

$$\widehat{\Gamma} \;\longrightarrow\; \prod_{v\in|U|\cap(S_j\setminus Z)} k(v)^{\times}$$

is injective.

In [22], the author of the present paper and Tachihara applied this result [in the case where K is a number field] to establish a *purely mono-anabelian reconstruction algorithm* of a number field whose input data is the sole underlying topological group of the absolute Galois group or its solvably closed quotient variant of the number field. It appears to the author that this type of phenomenon has the potential for further applications in anabelian geometry. In light of this motivation, in the present paper, to acquire a deeper understanding of this phenomenon concerning finite partitions of closed points of models of generalized global fields, we revisit and generalize the above theorem to the groups of rational points of semi-abelian varieties.

In order to state our main results, we introduce some notations and notions on semi-abelian schemes. For each semi-abelian variety G over a generalized global field K, each irreducible normal scheme U of finite type over $\mathbb{Z}$ whose function field is K, and each prime number l invertible in U, we shall write T_lG for the l-adic Tate module associated to G and say that U is *G-admissible* if G extends [uniquely] to a semi-abelian scheme $\mathcal{G}$ over U that fits into the following exact sequence of semi-abelian schemes

$$0 \longrightarrow \mathcal{T} \longrightarrow \mathcal{G} \longrightarrow \mathcal{A} \longrightarrow 0,$$

where $\mathcal{T}$ denotes a torus over U [with constant rank]; $\mathcal{A}$ denotes an abelian scheme over U. It is not difficult to see that:

- One may obtain a G-admissible open subscheme U after shrinking an irreducible normal scheme, and
- There exists a natural monodromy representation $\Pi_U \to \mathrm{Aut}(T_lG)$ [cf. Lemma 3.2].

On the other hand, we note that, for each topological Π_U-module M and each $v \in |U|$, there exists a natural restriction homomorphism $H^1(\Pi_U, M) \to H^1(G_{k(v)}, M)$ *determined up to isomorphisms* that arise from inner automorphism indeterminacy of the natural homomorphism $G_{k(v)} \to \Pi_U$. In particular, the vanishing of the image of a given class is independent of the choice of basepoints. Thus, this indeterminacy does not cause any problem in our discussion throughout the present paper.

Then our main results are the following [cf. Theorems 3.9, 3.10]:

Theorem A. *Let n be a positive integer; K a generalized global field. For each positive integer $i \leq n$, let G_i be a semi-abelian variety over K; l_i a prime number. Let U be an irreducible normal scheme of finite type over $\mathbb{Z}$ whose function field is K such that l_i is invertible in U, and U is G_i-admissible for each $i \leq n$. For each positive integer $i \leq n$, let*

$$\xi_i \ \in \ H^1(\Pi_U, T_{l_i}G_i)$$

be a nonzero element. Write

$$T \ \subseteq \ |U|$$

for the subset of closed points $v \in U$ such that the image of ξ_i via the natural homomorphism

$$H^1(\Pi_U, T_{l_i}G_i) \ \longrightarrow \ H^1(G_{k(v)}, T_{l_i}G_i)$$

is still nonzero for any positive integer $i \leq n$. Then T contains a set with positive Dirichlet density.

Theorem B. *Let K be a generalized global field; X an irreducible normal scheme of finite type over $\mathbb{Z}$ whose function field is K; $Z \subseteq |X|$ a subset of Dirichlet density 0;*

$$|X| \ = \ \coprod_{1 \leq i \leq r} S_i$$

a finite partition. Then there exists a positive integer $j \leq r$ such that, for each semi-abelian variety G over K, each G-admissible open subscheme $U \subseteq X$, and each prime number l invertible in U, the natural homomorphism

$$H^1(\Pi_U, T_lG) \ \longrightarrow \prod_{v \in |U| \cap (S_j \setminus Z)} H^1(G_{k(v)}, T_lG)$$

is injective.

Theorem B may be regarded as a direct consequence of Theorem A. One of the interesting points of Theorem B lies in the fact that the choice of j is universal for all the semi-abelian varieties over K [and for all U and l]. This phenomenon may not be observed in the situation of Theorem 0.1 due to the evident fact that Theorem 0.1 only discusses the $\mathbb{G}_{\mathrm{m}}$-case. On the other hand, one may ask whether or not there exist analogues of Theorems A, B for l-adic Galois representations concerning higher l-adic cohomology groups associated to arbitrary smooth, proper varieties. In fact, our proof depends on the Weil conjecture for abelian varieties and the semisimplicity results for l-adic Galois representations associated to abelian varieties due to Faltings, Zarhin, and Mori [cf. [6], [7], [16]]. While the Weil conjecture has already been proved in this generality by Deligne [cf. [3]], the corresponding generalizations of the semisimplicity results are still open [i.e., the Grothendieck-Serre semisimplicity conjecture]. Thus, at the time of writing of the present paper, the author does not know an answer to the question on such a generalization.

Finally, one may apply Theorem B to obtain the following generalization of Theorem 0.1 [cf. Corollary 3.13]:

Corollary C. *Let K be a generalized global field; X an irreducible normal scheme of finite type over $\mathbb{Z}$ whose function field is K; $Z \subseteq |X|$ a subset of Dirichlet density 0;*

$$|X| \ = \ \coprod_{1 \leq i \leq r} S_i$$

a finite partition. Fix a positive integer $j \leq r$ as in Theorem B. Let G be a semi-abelian variety over K;

$$\Gamma \ \subseteq \ G(K)$$

a finitely generated subgroup. Note that there exists a G-admissible open subscheme $U \subseteq X$ such that any element of Γ extends to a U-valued point of the semi-abelian scheme $\mathcal{G}$ over U that lifts G. Then, for each such open subscheme U, the natural homomorphism

$$\widehat{\Gamma} \longrightarrow \prod_{v \in |U| \cap (S_j \setminus Z)} \mathcal{G}_{k(v)}(k(v))'$$

is injective, where $\mathcal{G}_{k(v)} \stackrel{\mathrm{def}}{=} \mathcal{G} \times_U \mathrm{Spec}\ k(v)$; $\mathcal{G}_{k(v)}(k(v))' \stackrel{\mathrm{def}}{=} \mathcal{G}_{k(v)}(k(v))$ (respectively, $\mathcal{G}_{k(v)}(k(v))'$ denotes the prime-to-p part of $\mathcal{G}_{k(v)}(k(v))$) if K is of characteristic 0 (respectively, positive characteristic p).

The present paper is organized as follows. In §1, we investigate a general abstract principle behind the injectivity results under finite partitions of closed points of models of generalized global fields. In particular, in an abstract setting, we reduce our injectivity problem to the problem of constructing a certain type of element in the image of a representation of G_K, which we shall refer to as *central regular elements*. Here, the only nontrivial arithmetic result we apply is the Chebotarev density theorem. In §2, we construct central regular elements for representations that arise from finite sets of Galois representations associated to abelian varieties or tori. In fact, if this finite set consists of the Galois representations associated to tori only, then the construction is almost trivial. In particular, by applying the result in §1 only, one may give a simpler alternative proof of Theorem 0.1. Thus, most of the discussion in this section is devoted to the case of abelian varieties. We resort to numerous highly nontrivial results on Galois representations associated to abelian varieties such as the semisimplicity theorem due to Faltings, Zarhin, and Mori or the Weil conjecture. In §3, by combining the results obtained in §1, §2, together with basic generalities on semi-abelian schemes, we prove Theorems A, B, and Corollary C.

Notations and conventions

Numbers: The notation $\mathfrak{Primes}$ will be used to denote the set of prime numbers. The notation $\mathbb{Z}$ will be used to denote the ring of integers. The notation $\widehat{\mathbb{Z}}$ will be used to denote the profinite completion of $\mathbb{Z}$. Let p be a prime number. Then $\mathbb{Z}_p$ will be used to denote the ring of p-adic integers; $\mathbb{Q}_p$ will be used to denote the field of p-adic numbers.

Fields: Let F be a field. Then we shall write $\overline{F}$ for the algebraic closure [determined up to isomorphisms] of F; F^{sep} ($\subseteq \overline{F}$) for the separable closure of F; $G_F \stackrel{\mathrm{def}}{=} \mathrm{Gal}(F^{\mathrm{sep}}/F)$.

Topological groups: Let G be a profinite group. Then we shall write $Z(G)$ for the center of G; $\mathrm{Aut}(G)$ for the group of *continuous* automorphisms of G. If G is *topologically finitely generated*, then one verifies immediately that the topology of G admits a basis of *characteristic open subgroups*, which thus induces a *profinite topology* on the group $\mathrm{Aut}(G)$. Let M be a topological G-module; $g \in G$. Then we shall write $g - 1 : M \to M$ for the endomorphism of $\mathbb{Z}$-modules that maps $m \mapsto gm - m$. It is immediate that, if $g \in Z(G)$, then $g - 1$ is an endomorphism of G-modules. For each nonnegative integer i, we shall write $H^i(G, M)$ for the i-th continuous cochain cohomology group defined by the standard continuous cochain complex [cf. [18], Definition 2.7.1].

Schemes: Let S be a scheme; $s \in S$ a point. Then we shall write $|S|$ for the set of closed points of S; $k(s)$ for the residue field of S at s.

Fundamental groups: For a connected Noetherian scheme S, we shall write Π_S for the étale fundamental group of S, relative to a suitable choice of basepoint [cf. [9]].

1 Abstract principle of profinite injectivity under finite partitions

In the present section, we discuss an abstract principle behind the injectivity under finite partitions of closed points of normal models of finitely generated infinite fields in question. The discussion consists of the following observations:

- The first cohomology groups of the absolute Galois groups of finite fields $\cong \widehat{\mathbb{Z}}$ are explicitly computed. This explicit computation is closely related to an elementary property of 1-cocycles concerning the action of central elements in an abstract setting [cf. Lemma 1.3].

- In a finite quotient of the absolute Galois group of a finitely generated infinite field, it follows from the Chebotarev density theorem that the conjugacy class of any element is represented by Frobenius elements associated to a set of closed points of positive Dirichlet density.

- In light of purely algebraic considerations on finite sets of 1-cocycles concerning topological modules of profinite groups [cf. Proposition 1.6], together with the above observations, one may prove that finitely many nonzero elements of a first cohomology group in the global setting are still nonzero in the corresponding first cohomology group at the residue fields associated to a set of closed points of positive Dirichlet density [cf. Proposition 1.9], under the existence of a certain type of central element [cf. Definition 1.5].

First, we begin by discussing three elementary group-theoretic lemmas. The third one is a property of 1-cocycles related to centers mentioned above.

Lemma 1.1. *Let l be a prime number; M a finitely generated free $\mathbb{Z}_l$-module. Then the following hold:*

(i) Let $f : M \to M$ be an endomorphism of $\mathbb{Z}_l$-modules; $u \in M \setminus \mathrm{Im}(f)$. Then there exists a positive integer n such that $u \notin \mathrm{Im}(f) + l^n M$.

(ii) Let $N \subseteq M$ be a nontrivial $\mathbb{Z}_l$-submodule. Then there exists a positive integer n such that, for each positive integer $m \geq n$, it holds that $[N : N \cap l^m M] \geq l^{m-n}$.

Proof. Assertion (i) follows immediately from the facts that $\mathrm{Im}(f) \subseteq M$ is a closed subgroup, and $\cap_{m\geq 1} l^m M = \{0\}$, where m ranges over the positive integers. Assertion (ii) follows immediately by considering the saturation of N in M. This completes the proof of Lemma 1.1. $\square$

Lemma 1.2. *Let G be a group; $H_1, \ldots, H_n \subseteq G$ subgroups of finite index such that $\sum_{1\leq i\leq n} \frac{1}{[G:H_i]} < 1$. For each positive integer $i \leq n$, let $C_i \subseteq G$ be a coset of H_i. Then it holds that $\cup_{1\leq i\leq n} C_i \subsetneq G$.*

Proof. Lemma 1.2 follows immediately from the consideration on cardinalities in the finite quotient of G by a normal subgroup of finite index contained in $\cap_{1\leq i\leq n} H_i$. $\square$

Lemma 1.3. *Let G be a profinite group; M a topological G-module; $c : G \to M$ a continuous 1-cocycle; $z \in Z(G)$. Then the following hold:*

(i) For each $g \in G$, it holds that $(z - 1)c(g) = (g - 1)c(z)$.

(ii) Suppose that the endomorphism $z - 1 : M \to M$ is injective, and the element $[c] \in H^1(G, M)$ determined by c is nonzero. Then $c(z) \notin \mathrm{Im}(z - 1)$.

Proof. First, we verify assertion (i). Since c is a 1-cocycle, it holds that

$$c(gz) \; = \; gc(z) + c(g); \quad c(zg) \; = \; zc(g) + c(z).$$

On the other hand, since $z \in Z(G)$, it holds that $c(gz) = c(zg)$. Thus, we obtain the desired conclusion. This completes the proof of assertion (i).

Next, we verify assertion (ii). Suppose that there exists an element $m \in M$ such that $c(z) = (z-1)m$. Note that since $z \in Z(G)$, for any $g \in G$, it holds that $z - 1$ and $g - 1$ commute as elements of $\mathrm{End}(M)$. Then, by applying assertion (i), we observe that

$$(z-1)(c(g) - (g-1)m) \; = \; 0$$

for any $g \in G$. Therefore, it follows from the injectivity of $z - 1$ that $c(g) - (g-1)m = 0$ for any $g \in G$. This contradicts our assumption that $[c] \neq 0$. Thus, we conclude that $c(z) \notin \mathrm{Im}(z-1)$, as desired. This completes the proof of assertion (ii), hence of Lemma 1.3. $\square$

Remark 1.3.1. Lemma 1.3 may be regarded as a cocycle-level variant of Sah's lemma [cf. [15], Lemma 6.21].

Next, we introduce a basic setup to discuss a general principle behind the injectivity phenomena under finite partitions of closed points in question.

Definition 1.4. Let

$$\Gamma$$

be a profinite group;

$$S \; \subseteq \; \mathfrak{Primes}$$

a nonempty finite subset. For each $l \in S$, let

$$M_l$$

be a finitely generated free $\mathbb{Z}_l$-module, which may be regarded as a profinite group in a natural way;

$$\rho_l : \Gamma \; \longrightarrow \; \mathrm{Aut}(M_l)$$

a continuous representation. Write

$$I_\Gamma \; \subseteq \; \prod_{l \in S} \mathrm{Aut}(M_l)$$

for the image of Γ via the natural homomorphism induced by $\{\rho_l\}_{l \in S}$.

Definition 1.5. In the notation of Definition 1.4, we shall say that $\{M_l, \rho_l\}_{l \in S}$ satisfies the ECR-*property* [i.e., "Existence of Central Regular elements"] if, for each positive integer m, there exists an element

$$z \; = \; (z_l)_{l \in S} \; \in \; Z(I_\Gamma)$$

such that, for each $l \in S$,

- the endomorphism $z_l - 1 : M_l \to M_l$ is injective, and
- $\mathrm{Im}(z_l - 1) \subseteq l^m M_l$,

where we regard M_l as a topological $\mathrm{Aut}(M_l)$-module in a natural way.

Proposition 1.6. *In the notation of Definition 1.4, suppose that $\{M_l, \rho_l\}_{l\in S}$ satisfies the* ECR*-property. For each $l \in S$, let C_l be a finite set of nonzero elements of $H^1(\Gamma, M_l)$. Then there exists an element $\gamma \in \Gamma$ such that*

$$c(\gamma) \;\notin\; \mathrm{Im}(\gamma - 1)$$

for each $[c] \in \coprod_{l\in S} C_l$, where $c : \Gamma \to M_l$ is a continuous 1-cocycle that represents $[c]$ if $[c] \in C_l$.

Proof. Write $H \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\Gamma \twoheadrightarrow I_\Gamma)$. In particular, for each $[c] \in C_l$, the restriction $c|_H : H \to M_l$ is a homomorphism. For each $[c] \in C_l$, write $N_c \stackrel{\mathrm{def}}{=} \mathrm{Im}(c|_H) \subseteq M_l$. In light of Lemma 1.1, (ii), let m be a positive integer such that

$$\sum_{[c]\in\coprod_{l\in S} C_l,\ N_c\neq\{0\}} \frac{1}{[N_c : N_c \cap l^m M_l]} \;<\; 1.$$

Then since $\{M_l, \rho_l\}_{l\in S}$ satisfies the ECR-property, one may fix an element $z = (z_l)_{l\in S} \in Z(I_\Gamma)$ such that, for each $l \in S$,

- the homomorphism $z_l - 1 : M_l \to M_l$ is injective, and
- $\mathrm{Im}(z_l - 1) \subseteq l^m M_l$.

Next, for each $[c] \in \coprod_{l\in S} C_l$ such that $N_c = \{0\}$, it follows immediately from the inflation-restriction sequence that, if $[c] \in C_l$, then c determines a continuous 1-cocycle $\tilde{c} : I_\Gamma \to M_l$ such that $[\tilde{c}] \in H^1(I_\Gamma, M_l)$ is nonzero. Then, in this case, by applying Lemma 1.3, (ii), together with the various definitions involved, we observe that

$$\tilde{c}(z) \;\notin\; \mathrm{Im}(z_l - 1).$$

Next, let $\tilde{z} \in \Gamma$ be a lifting of $z \in Z(I_\Gamma) \subseteq I_\Gamma$ via the natural surjective homomorphism $\Gamma \twoheadrightarrow I_\Gamma$. For each $[c] \in C_l$ such that $N_c \neq \{0\}$, write

$$B_c \;\stackrel{\mathrm{def}}{=}\; \{h \in H \mid c(h\tilde{z}) \in \mathrm{Im}(z_l - 1)\}.$$

Here, we note that since H acts trivially on M_l, it holds that $c(h\tilde{z}) = c(\tilde{z}) + c(h)$, and B_c is independent of the choice of a representative of $[c]$. In particular,

$$B_c \;=\; c|_H^{-1}\Big(N_c \cap \big(\mathrm{Im}(z_l - 1) - c(\tilde{z})\big)\Big).$$

Since $z_l - 1$ is an injective endomorphism on the finitely generated free $\mathbb{Z}_l$-module M_l, it is immediate that $\mathrm{Im}(z_l - 1) \subseteq M_l$ is an open subgroup. In particular, the subgroup $L_c \stackrel{\mathrm{def}}{=} N_c \cap \mathrm{Im}(z_l - 1) \subseteq N_c$ is also open. On the other hand, if $N_c \cap (\mathrm{Im}(z_l - 1) - c(\tilde{z})) \neq \emptyset$, then $N_c \cap (\mathrm{Im}(z_l - 1) - c(\tilde{z})) = t + L_c$ for some $t \in N_c \cap (\mathrm{Im}(z_l - 1) - c(\tilde{z}))$. Therefore, if $B_c \neq \emptyset$, then it holds that $B_c \subseteq H$ is a coset of a subgroup of finite index $[N_c : L_c]$. On the other hand, it follows from our choices of z, m that

$$\sum_{[c]\in\coprod_{l\in S} C_l,\ N_c\neq\{0\}} \frac{1}{[N_c : L_c]} \;\leq\; \sum_{[c]\in\coprod_{l\in S} C_l,\ N_c\neq\{0\}} \frac{1}{[N_c : N_c \cap l^m M_l]} \;<\; 1.$$

Thus, by applying Lemma 1.2, one may take $h \in H \setminus \cup_{[c]\in\coprod_{l\in S} C_l,\ N_c\neq\{0\}} B_c$. Write $\gamma \stackrel{\mathrm{def}}{=} h\tilde{z}$. Note that $\rho_l(\gamma) = \rho_l(h\tilde{z}) = \rho_l(\tilde{z}) = z_l$. Then it follows immediately from the above discussions that

$$c(\gamma) \;\notin\; \mathrm{Im}(\gamma - 1)$$

for each $[c] \in \coprod_{l\in S} C_l$. This completes the proof of Proposition 1.6. $\square$

Next, we apply the above purely algebraic result to an arithmetic setting. For this, we prepare some notions [cf. [11], [20]].

Definition 1.7. A *generalized global field* is a finitely generated infinite field. In particular, a global field is a generalized global field.

Definition 1.8. Let U be a finite-type integral scheme over $\mathbb{Z}$ of dimension $n \geq 1$; $A \subseteq |U|$ a subset. For each $v \in |U|$, write $N(v)$ for the cardinality of the residue field $k(v)$. Then we shall say that A has Dirichlet density $\delta(A)$ if

$$\delta(A) \ = \ \lim_{s \to n^+} \left(\sum_{v \in A} \frac{1}{N(v)^s} \right) / \log\left(\frac{1}{s-n} \right),$$

where $s \to n^+$ means that s approaches n on the real axis from the right.

Remark 1.8.1. In the notation of Definition 1.8, suppose that A has Dirichlet density. Let $B \subseteq |U|$ be a subset such that $A \subseteq B$, and B has Dirichlet density. Then it is immediate that $\delta(A) \leq \delta(B)$ [cf., e.g., [11], Lemma 2.3, (b)].

The following result may be regarded as a key nonvanishing result in this abstract setting. The only nontrivial arithmetic result we need to apply is the Chebotarev density theorem.

Proposition 1.9. *Let K be a generalized global field; U an irreducible normal scheme of finite type over $\mathbb{Z}$ whose function field is K. In the notation of Definition 1.4, suppose that*

$$\Gamma \ = \ \Pi_U,$$

and there exists a continuous homomorphism

$$\psi : \Gamma^\dagger \ \longrightarrow \ \Gamma$$

of profinite groups such that:

- *For each $l \in S$, the natural homomorphism $H^1(\Gamma, M_l) \to H^1(\Gamma^\dagger, M_l)$ induced by ψ is injective.*
- *$\{M_l, \rho_l^\dagger\}_{l \in S}$ satisfies the* ECR*-property, where $\rho_l^\dagger \stackrel{\mathrm{def}}{=} \rho_l \circ \psi : \Gamma^\dagger \to \mathrm{Aut}(M_l)$.*

For each $l \in S$, let

$$C_l \ \subseteq \ H^1(\Gamma, M_l)$$

be a finite subset of nonzero elements of $H^1(\Gamma, M_l)$. Write

$$T \ \subseteq \ |U|$$

for the subset of closed points $v \in U$ such that the image of $[c] \in C_l$ via the natural homomorphism

$$H^1(\Gamma, M_l) \ \longrightarrow \ H^1(G_{k(v)}, M_l)$$

is still nonzero for any $[c] \in \coprod_{l \in S} C_l$. Then T contains a set with positive Dirichlet density.

Proof. First, for each $[c] \in \coprod_{l \in S} C_l$, we fix a continuous 1-cocycle c that represents $[c]$. Then, in light of the conditions on ψ, by applying Proposition 1.6 to $\{M_l, \rho_l^\dagger\}_{l \in S}$ and the images of $\{C_l\}_{l \in S}$ in $\{H^1(\Gamma^\dagger, M_l)\}_{l \in S}$, one may obtain an element $\gamma \in \Gamma$ such that $c(\gamma) \notin \mathrm{Im}(\gamma - 1)$ for any $[c] \in \coprod_{l \in S} C_l$. Then it follows immediately from Lemma 1.1, (i), that there exists a positive integer n such that $c(\gamma) \notin \mathrm{Im}(\gamma - 1) + l^n M_l$ for any $[c] \in \coprod_{l \in S} C_l$. Write

$$\phi : \Pi_U \longrightarrow \prod_{[c] \in \coprod_{l \in S} C_l} M_l / l^n M_l \rtimes \mathrm{Aut}(M_l / l^n M_l)$$

for the natural homomorphism, where we note that $[c] \in C_l$ and ρ_l determine a homomorphism $\Pi_U \to M_l / l^n M_l \rtimes \mathrm{Aut}(M_l / l^n M_l)$;

$$Q \overset{\mathrm{def}}{=} \mathrm{Im}(\phi); \quad q \overset{\mathrm{def}}{=} \phi(\gamma) \in Q;$$

$$D \subseteq |U|$$

for the subset of closed points $v \in U$ such that the conjugacy class $\subseteq \Pi_U$ of Frobenius elements associated to v maps to the conjugacy class of q in Q via ϕ. Then, by applying the Chebotarev density theorem [cf., e.g., [20], p.91, Theorem 7; [11], Theorem 2.4] to the finite quotient $\Pi_U \twoheadrightarrow Q$, we observe that D has positive density. On the other hand, it follows immediately from the fact that $c(\gamma) \notin \mathrm{Im}(\gamma - 1) + l^n M_l$ for any $[c] \in \coprod_{l \in S} C_l$, together with the well-known computation of $H^1(G_{k(v)}, M_l)$, i.e., the first cohomology group of $\widehat{\mathbb{Z}}$, that the image of $[c]$ via the natural homomorphism $H^1(\Gamma, M_l) \to H^1(G_{k(v)}, M_l / l^n M_l)$ is still nonzero for any $v \in D$ and any $[c] \in \coprod_{l \in S} C_l$. In particular, the set T contains the set D which has positive Dirichlet density. This completes the proof of Proposition 1.9. □

2 Construction of central regular elements via semisimplicity and purity

In the present section, we construct "central regular elements" as in Definition 1.5 for Galois representations associated to abelian varieties or tori over generalized global fields. The key theorems for our construction are the following:

- The semisimplicity of l-adic Galois representations associated to abelian varieties due to Faltings [in the case of characteristic 0], Zarhin [in the case of positive characteristic $\neq 2$], and Mori [in the case of characteristic 2].
- The Weil conjecture for abelian varieties.

First, we begin by recalling the above theorems:

Definition 2.1. Let F be a field; l a prime number invertible in F; G a semi-abelian variety over F. Then we shall write

$$T_l G$$

for the l-adic Tate module associated to G;

$$V_l G \overset{\mathrm{def}}{=} T_l G \otimes_{\mathbb{Z}_l} \mathbb{Q}_l; \quad \overline{V}_l G \overset{\mathrm{def}}{=} T_l G \otimes_{\mathbb{Z}_l} \overline{\mathbb{Q}}_l.$$

Theorem 2.2. *Let K be a generalized global field; l a prime number invertible in K; A an abelian variety over K. Then the l-adic Galois representation $G_K \to \mathrm{GL}(V_l A)$, hence also $G_K \to \mathrm{GL}(\overline{V}_l A)$, associated to A is semisimple.*

Proof. Since the coefficient field $\mathbb{Q}_l$ is of characteristic 0, it follows immediately from [14], Propositions 5.11, 5.13, that it suffices to verify that $G_K \to \mathrm{GL}(V_l A)$ is semisimple. Thus, Theorem 2.2 follows from [6]; [7], Theorem 1; [16], Chapter XII, Théorème 2.5. $\square$

Theorem 2.3. *Let k be a finite field of cardinality q; l a prime number invertible in k; A an abelian variety over k. Then the absolute values [in $\mathbb{C}$] of Frobenius eigenvalues associated to the l-adic Galois representation $G_k \to \mathrm{GL}(\overline{V}_l A)$ are $\sqrt{q}$ with respect to arbitrary field isomorphisms $\overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$. In particular, for any G_k-stable nonzero subspace $W \subseteq \overline{V}_l A$, the determinant of the Frobenius action on W is not equal to* 1*.*

Proof. Theorem 2.3 follows from [17], p. 206, Theorem 4 [or more generally [3]]. $\square$

To construct central regular elements, we combine the above highly nontrivial results with Lie algebra theoretic techniques. For this purpose, we recall the following well-known properties on Lie groups and Lie algebras:

Lemma 2.4. *Let l be a prime number; d a positive integer; $G \subseteq \mathrm{GL}_d(\mathbb{Z}_l)$ a closed subgroup. Then the following hold:*

(i) G admits a structure of a compact l-adic Lie group.

(ii) There exists a normal open uniform pro-l subgroup H of G such that the matrix logarithm gives an isomorphism $\log : \mathcal{L}_H \xrightarrow{\sim} \log(H)$ of Lie algebras over $\mathbb{Z}_l$, where $\mathcal{L}_H$ denotes the canonical Lie algebra over $\mathbb{Z}_l$ associated to H [cf. [4], §4.5; [4], Theorem 9.10].

(iii) Fix H as in (ii). Write $\mathrm{Lie}(H) \stackrel{\mathrm{def}}{=} \log(H) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ ($\subseteq \mathrm{M}_d(\mathbb{Q}_l)$) for the Lie algebra over $\mathbb{Q}_l$. Then it holds that:

(a) Let $W \subseteq \mathbb{Q}_l^{\oplus d}$ be a $\mathbb{Q}_l$-subspace. Then W is H-stable if and only if W is $\mathrm{Lie}(H)$-stable.

(b) $\mathrm{Tr}([\mathrm{Lie}(H), \mathrm{Lie}(H)]) = \{0\}$.

(c) Let $D \in Z(\mathrm{Lie}(H))$. Then, for sufficiently large n, it holds that $l^n D \in Z(\mathrm{Lie}(H)) \cap \log(H)$, and $\exp(l^n D) \in Z(H)$, where we recall that the underlying set of $\mathcal{L}_H$ coincides with the underlying set of H.

Proof. Assertion (i) follows from [4], Theorem 9.6, (i). Assertion (ii) follows immediately from the well-known structure of $\mathrm{GL}_d(\mathbb{Z}_l)$, together with [4], Lemma 7.12; [4], Theorem 7.13; [4], Corollary 7.14. Assertion (iii-a) follows immediately from the explicit presentations of exp and log. Assertion (iii-b) follows immediately from the well-known property of matrix trace, i.e., $\mathrm{Tr}(AB - BA) = 0$ for any pair $A, B \in \mathrm{M}_d(\mathbb{Q}_l)$. Assertion (iii-c) follows immediately from the definition of $\mathrm{Lie}(H)$, together with the easily verified equality $\exp(A)\exp(B) = \exp(B)\exp(A)$ if $AB = BA$. This completes the proof of Lemma 2.4. $\square$

Lemma 2.5. *Let k be a field of characteristic 0; V a finite dimensional vector space over k; $L \subseteq \mathrm{End}_k(V)$ a Lie subalgebra over k. Suppose that V is completely reducible/semisimple as an L-module. Then there exists a natural direct decomposition $L = [L, L] \oplus Z(L)$ of Lie algebras over k.*

Proof. This is a well-known lemma. In light of our assumptions on k and V, after the base extension to $\overline{k}$, one may apply the invariance lemma [cf., e.g., [5], Lemma 5.5], together with Lie's theorem [cf., e.g., [5], Theorem 6.5; [5], Remark 6.7], to deduce that L is reductive. Thus, Lemma 2.5 follows from a well-known structure theorem for reductive Lie algebras over fields of characteristic 0 [cf., e.g., [14], Proposition 6.2; [14], 6.3]. □

Next, we prepare a basic setup.

Definition 2.6. Let K be a generalized global field; l a prime number invertible in K; I a finite set. For each $i \in I$, let G_i be an abelian variety or a torus over K, and write

$$M_i \stackrel{\mathrm{def}}{=} T_l G_i; \quad V_i \stackrel{\mathrm{def}}{=} V_l G_i;$$

$$\rho_i : G_K \longrightarrow \mathrm{Aut}(M_i)$$

for the l-adic Galois representation associated to G_i. Write

$$M \stackrel{\mathrm{def}}{=} \bigoplus_{i \in I} M_i; \quad V \stackrel{\mathrm{def}}{=} \bigoplus_{i \in I} V_i.$$

Let

$$K \subseteq E_l$$

be a finite separable field extension satisfying the following conditions:

- All the tori are split over E_l.
- The image
$$\widetilde{L}_l \stackrel{\mathrm{def}}{=} \mathrm{Im}\Big(G_{E_l} \stackrel{\prod \rho_i}{\longrightarrow} \prod_{i \in I} \mathrm{Aut}(M_i)\Big) \subseteq \mathrm{Aut}(M)$$
is a uniform pro-l subgroup such that the matrix logarithm gives an isomorphism of Lie algebras over $\mathbb{Z}_l$ as in Lemma 2.4, (ii).

[Note that the existence of such an E_l follows immediately from Lemma 2.4, (ii), applied to the image of $G_{E'}$, where E' is a finite Galois extension of K over which all the tori are split.] Write

$$\mathrm{Lie}(\widetilde{L}_l) \stackrel{\mathrm{def}}{=} \log(\widetilde{L}_l) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$$

[cf. Lemma 2.4, (iii)].

In the above situation, it follows from Theorem 2.2, together with our choice of E_l, that V is a semisimple $\widetilde{L}_l$-module. Then, by applying Lemma 2.4, (iii-a), we observe that V is also a semisimple $\mathrm{Lie}(\widetilde{L}_l)$-module. In particular, it follows from Lemma 2.5 that

$$\mathrm{Lie}(\widetilde{L}_l) = [\mathrm{Lie}(\widetilde{L}_l), \mathrm{Lie}(\widetilde{L}_l)] \bigoplus Z(\mathrm{Lie}(\widetilde{L}_l)).$$

Moreover, for each $i \in I$, we consider the decomposition

$$V_i \otimes_{\mathbb{Q}_l} \overline{\mathbb{Q}}_l \;=\; \bigoplus_{\alpha \in I_i} W_\alpha$$

into the irreducible $\mathrm{Lie}(\widetilde{L}_l)$-modules. Again, it follows from Lemma 2.4, (iii-a), that this decomposition is also stable as an $\widetilde{L}_l$-module. Here, we note that Schur's lemma implies that the action of $Z(\mathrm{Lie}(\widetilde{L}_l))$ on W_α is scalar multiplication induced by a homomorphism

$$\lambda_\alpha : Z(\mathrm{Lie}(\widetilde{L}_l)) \;\longrightarrow\; \overline{\mathbb{Q}}_l.$$

Lemma 2.7. *For each $i \in I$ and each $\alpha \in I_i$, it holds that $\lambda_\alpha \neq 0$.*

Proof. Indeed, suppose that $\lambda_\alpha = 0$. Then it follows from Lemma 2.4, (iii-b), that $\mathrm{Tr}(\mathrm{Lie}(\widetilde{L}_l)|_{W_\alpha}) = \{0\}$. Therefore, in light of the well-known equality $\det(\exp A) = \exp(\mathrm{Tr}(A))$, it holds that $\det(\widetilde{L}_l|_{W_\alpha}) = \{1\}$. In the case where G_i is a torus [which is split over E_l], this contradicts the easily verified fact that $\widetilde{L}_l|_{W_\alpha}$ is scalar multiplication via the l-adic cyclotomic character. In the case where G_i is an abelian variety, by considering a suitable reduction, one may observe that this contradicts Theorem 2.3. Thus, we conclude that $\lambda_\alpha \neq 0$. This completes the proof of Lemma 2.7. $\square$

Lemma 2.8. *There exists an element $D \in Z(\mathrm{Lie}(\widetilde{L}_l))$ such that $\lambda_\alpha(D) \neq 0$ for each $i \in I$ and each $\alpha \in I_i$.*

Proof. It follows from Lemma 2.7 that $\mathrm{Ker}(\lambda_\alpha)$ is a $\mathbb{Q}_l$-subspace of $Z(\mathrm{Lie}(\widetilde{L}_l))$ with strictly smaller dimension. On the other hand, since $\mathbb{Q}_l$ is infinite, it is immediate that $Z(\mathrm{Lie}(\widetilde{L}_l))$ may not be written as the finite union of proper subspaces. Thus, one may take a desired D. This completes the proof of Lemma 2.8. $\square$

Proposition 2.9. *There exists a sequence $(z_n)_{n \geq 1}$ of elements of $Z(\widetilde{L}_l)$ satisfying the following conditions:*

- *This sequence converges to 1.*
- *Let m be a positive integer. Then, for any sufficiently large positive integer n, the homomorphism $z_n - 1 : M_i \to M_i$ is injective, and $\mathrm{Im}(z_n - 1) \subseteq l^m M_i$ for each $i \in I$.*

Proof. Let $D \in Z(\mathrm{Lie}(\widetilde{L}_l))$ be such that $\lambda_\alpha(D) \neq 0$ for each $i \in I$ and each $\alpha \in I_i$ [cf. Lemma 2.8]. By replacing D by $l^a D$ for a suitable positive integer a, we may assume without loss of generality that $D \in Z(\mathrm{Lie}(\widetilde{L}_l)) \cap \log(\widetilde{L}_l)$. Thus, one may easily verify that the sequence $\{\exp(l^n D)\}_{n \geq 1}$ of elements in $Z(\widetilde{L}_l)$ [cf. Lemma 2.4, (iii-c)] satisfies the desired conditions. This completes the proof of Proposition 2.9. $\square$

Remark 2.9.1. In the number field case, by applying Bogomolov's homothety theorem [cf. [1]], one may give an alternative proof of Proposition 2.9.

3 Grunewald-Segal property for semi-abelian varieties over generalized global fields

In the present section, by applying the results obtained in the previous sections, we prove our main theorems.

First, we begin by discussing some generalities on semi-abelian schemes, i.e., smooth, separated, commutative group schemes over base schemes whose geometric fibers are semi-abelian varieties.

Definition 3.1. Let K be a generalized global field; G a semi-abelian variety over K; U an irreducible normal scheme of finite type over $\mathbb{Z}$ whose function field is K. Then we shall say that U is *G-admissible* if G extends to a semi-abelian scheme $\mathcal{G}$ over U that fits into the following exact sequence of semi-abelian schemes

$$0 \longrightarrow \mathcal{T} \longrightarrow \mathcal{G} \longrightarrow \mathcal{A} \longrightarrow 0,$$

where $\mathcal{T}$ denotes a torus over U [with constant rank]; $\mathcal{A}$ denotes an abelian scheme over U.

Remark 3.1.1. In the notation of Definition 3.1, suppose that U is G-admissible. Then a semi-abelian scheme over U that lifts G is unique up to unique isomorphism over U [cf. [8], Chapter I, Proposition 2.7].

Lemma 3.2. *Let K be a generalized global field; G a semi-abelian variety over K; U an irreducible normal scheme of finite type over $\mathbb{Z}$ whose function field is K. Then there exists a G-admissible open subscheme $V \subseteq U$. Moreover, if l is a prime number invertible in V, then there exists a natural monodromy representation $\Pi_V \to \mathrm{Aut}(T_l G)$.*

Proof. The first assertion follows from a routine discussion in scheme theory. Next, it follows immediately that, for each positive integer n, the respective l^n-torsion subgroup schemes $\mathcal{T}[l^n] \subseteq \mathcal{T}$, $\mathcal{A}[l^n] \subseteq \mathcal{A}$ are finite étale over V. Therefore, the l^n-torsion subgroup scheme $\mathcal{G}[l^n] \subseteq \mathcal{G}$ is also finite étale over V, hence determines a locally constant sheaf on V. Thus, by varying n, we complete the proof of the second assertion, as desired. This completes the proof of Lemma 3.2. □

Next, we record the following group-theoretic lemma, which leads to the verification of the ECR-property [cf. Proposition 3.4]:

Lemma 3.3. *Let n be a positive integer; $\{l_i\}_{1\leq i\leq n}$ n distinct prime numbers. For each positive integer $i \leq n$, let P_i be a pro-l_i group. Let $P \subseteq \prod_{1\leq i\leq n} P_i$ be a closed subgroup whose projection homomorphisms $\{P \to P_i\}_{1\leq i\leq n}$ are surjective. Then $P = \prod_{1\leq i\leq n} P_i$.*

Proof. For each positive integer $i \leq n$, by considering a Sylow-l_i subgroup of P, together with our assumptions, we observe that $\{1\} \times \cdots \times \{1\} \times P_i \times \{1\} \times \cdots \times \{1\} \subseteq P$. Thus, by varying i, we conclude that $P = \prod_{1\leq i\leq n} P_i$, as desired. This completes the proof of Lemma 3.3. □

Proposition 3.4. *Let K be a generalized global field; U an irreducible normal scheme of finite type over $\mathbb{Z}$ whose function field is K. In the notation of Definition 1.4, suppose that:*

- $\Gamma = \Pi_U$, *and any element of* S *is invertible in* U.
- *For each* $l \in S$, M_l *is a finite product of the* l*-adic Tate modules associated to abelian varieties or tori over* K. *Moreover, the scheme* U *is admissible with respect to these semi-abelian varieties, and* $\rho_l : \Pi_U \to \mathrm{Aut}(M_l)$ *is the natural continuous homomorphism induced by the associated* l*-adic Galois representations.*

Then there exists a finite Galois extension $K \subseteq E$ *such that* $\{M_l, \rho_l|_{G_E}\}_{l\in S}$ *satisfies the* ECR*-property.*

Proof. For each $l \in S$, fix E_l as in Definition 2.6 [applied to the abelian varieties and tori whose Tate modules constitute M_l]. Write E for the Galois closure over K of the composite field of E_l's. Then, for each $l \in S$, it follows immediately from the second condition on E_l that the image L'_l of G_E in $\mathrm{Aut}(M_l)$ is a pro-l group. Therefore, by applying Lemma 3.3, we observe that the image of G_E in $\prod_{l\in S} \mathrm{Aut}(M_l)$ coincides with $\prod_{l\in S} L'_l$. Thus, since $L'_l \subseteq \widetilde{L}_l$ is an open subgroup, Proposition 3.4 follows immediately from Proposition 2.9, together with the various definitions involved. □

Next, we discuss some elementary properties on cohomology groups associated to semi-abelian varieties to obtain our main results. Before proceeding, we recall the following result:

Lemma 3.5. *Let* G *be a profinite group;* M *a compact* G*-module that may be written as a countable inverse limit* $M = \varprojlim_n M_n$ *of finite, discrete* G*-modules. Then the natural homomorphism*

$$H^1(G, M) \longrightarrow \varprojlim_n H^1(G, M_n)$$

is an isomorphism.

Proof. Lemma 3.5 follows immediately from [18], Corollary 2.7.6. □

Lemma 3.6. *Let* K *be a generalized global field;* G *a semi-abelian variety over* K*;* U *a* G*-admissible, irreducible normal scheme of finite type over* $\mathbb{Z}$ *whose function field is* K*;* l *a prime number invertible in* U*;* $K \subseteq E$ *a finite Galois extension. Then the natural homomorphism*

$$H^1(\Pi_U, T_l G) \longrightarrow H^1(G_E, T_l G)$$

is injective.

Proof. First, since the natural homomorphism $G_K \twoheadrightarrow \Pi_U$ is surjective, the induced homomorphism $H^1(\Pi_U, T_l G) \to H^1(G_K, T_l G)$ is injective. Then, in light of the inflation-restriction exact sequence [induced by the inflation-restriction exact sequences of finite quotients, together with Lemma 3.5]

$$0 \longrightarrow H^1(\mathrm{Gal}(E/K), (T_l G)^{G_E}) \longrightarrow H^1(G_K, T_l G) \longrightarrow H^1(G_E, T_l G)$$

associated to the natural exact sequence

$$1 \longrightarrow G_E \longrightarrow G_K \longrightarrow \mathrm{Gal}(E/K) \longrightarrow 1,$$

it suffices to verify that $(T_l G)^{G_E} = \{0\}$. On the other hand, it follows immediately from the Mordell-Weil-Lang-Néron theorem [cf. [12], [21]] that the torsion subgroup of $G(E)$ is finite. Thus, we conclude that $(T_l G)^{G_E} = \{0\}$, as desired. This completes the proof of Lemma 3.6. □

Lemma 3.7. *Let K be a generalized global field; G a semi-abelian variety over K; U a G-admissible, irreducible normal scheme of finite type over $\mathbb{Z}$ whose function field is K; l a prime number invertible in U; $v \in |U|$. Write T for the torus over K determined by G; A for the quotient abelian variety of G by T. Then there exist natural exact sequences*

$$0 \longrightarrow H^1(\Pi_U, T_l T) \longrightarrow H^1(\Pi_U, T_l G) \longrightarrow H^1(\Pi_U, T_l A),$$

$$0 \longrightarrow H^1(G_{k(v)}, T_l T) \longrightarrow H^1(G_{k(v)}, T_l G) \longrightarrow H^1(G_{k(v)}, T_l A).$$

Proof. In light of Lemma 3.5, Lemma 3.7 follows immediately from the cohomology long exact sequences associated to the exact sequence of Π_U-/$G_{k(v)}$-modules

$$0 \longrightarrow T_l T \longrightarrow T_l G \longrightarrow T_l A \longrightarrow 0,$$

together with the easily verified fact that $H^0(\Pi_U, T_l A) \subseteq H^0(G_{k(v)}, T_l A) = \{0\}$. □

Proposition 3.8. *In the notation of Lemma 3.7, let $\xi \in H^1(\Pi_U, T_l G)$ be a nonzero element. Write $\xi_A \in H^1(\Pi_U, T_l A)$ for the image of ξ via the natural homomorphism $H^1(\Pi_U, T_l G) \to H^1(\Pi_U, T_l A)$. In a slight abuse of notation, we shall use sp_v for the natural homomorphisms obtained by restricting to $G_{k(v)}$. Then the following hold:*

(i) Suppose that $\xi_A \neq 0$. Then, for each $v \in |U|$, it holds that

$$\mathrm{sp}_v(\xi_A) \neq 0 \ \Rightarrow \ \mathrm{sp}_v(\xi) \neq 0.$$

(ii) Suppose that $\xi_A = 0$. Then ξ arises from a unique nonzero element $\xi_T \in H^1(\Pi_U, T_l T)$. Moreover, for each $v \in |U|$, it holds that

$$\mathrm{sp}_v(\xi_T) \neq 0 \ \Rightarrow \ \mathrm{sp}_v(\xi) \neq 0.$$

Proof. Proposition 3.8 follows immediately from Lemma 3.7, together with the various definitions involved. □

Next, by combining the above discussions, we obtain our first main result:

Theorem 3.9. *Let K be a generalized global field; n a positive integer. For each positive integer $i \leq n$, let G_i be a semi-abelian variety over K; l_i a prime number. Let U be an irreducible normal scheme of finite type over $\mathbb{Z}$ whose function field is K such that l_i is invertible in U, and U is G_i-admissible for each $i \leq n$ [cf. Lemma 3.2]. For each positive integer $i \leq n$, let*

$$\xi_i \ \in \ H^1(\Pi_U, T_{l_i} G_i)$$

be a nonzero element. Write

$$T \ \subseteq \ |U|$$

for the subset of closed points $v \in U$ such that the image of ξ_i via the natural homomorphism

$$H^1(\Pi_U, T_{l_i} G_i) \ \longrightarrow \ H^1(G_{k(v)}, T_{l_i} G_i)$$

is still nonzero for any positive integer $i \leq n$. Then T contains a set with positive Dirichlet density.

Proof. In light of Proposition 3.8, we may assume without loss of generality that, for each positive integer $i \leq n$, the semi-abelian variety G_i is an abelian variety or a torus over K. Then it follows immediately from Lemma 3.6 and Proposition 3.4 that the assumptions of Proposition 1.9 hold in our setting, where we take "S", "M_l", "C_l", to be $\{l_i\}_{1\leq i\leq n}$, $\bigoplus_{l_i=l} T_l G_i$, $\{\xi_i\}_{l_i=l}$, respectively. Thus, by applying Proposition 1.9, we obtain the desired conclusion. This completes the proof of Theorem 3.9. $\square$

As a consequence, we obtain our second main result:

Theorem 3.10. *Let K be a generalized global field; X an irreducible normal scheme of finite type over $\mathbb{Z}$ whose function field is K; $Z \subseteq |X|$ a subset of Dirichlet density 0;*

$$|X| \ = \ \coprod_{1\leq i\leq r} S_i$$

a finite partition. Then there exists a positive integer $j \leq r$ such that, for each semi-abelian variety G over K, each G-admissible open subscheme $U \subseteq X$, and each prime number l invertible in U, the natural homomorphism

$$H^1(\Pi_U, T_l G) \ \longrightarrow \ \prod_{v\in|U|\cap(S_j\setminus Z)} H^1(G_{k(v)}, T_l G)$$

is injective.

Proof. Suppose that there exists no such positive integer $j \leq r$. Then, for each positive integer $i \leq r$, there exist a semi-abelian variety G_i, a G_i-admissible open subscheme $U_i \subseteq X$, a prime number l_i invertible in U_i, and a nonzero element $\xi_i \in H^1(\Pi_{U_i}, T_{l_i} G_i)$ that vanishes in $\prod_{v\in|U_i|\cap(S_i\setminus Z)} H^1(G_{k(v)}, T_{l_i} G_i)$. Write $U \overset{\mathrm{def}}{=} \cap_{1\leq i\leq r} U_i$. Then it is immediate that l_i is invertible in U, U is G_i-admissible, and ξ_i vanishes in $\prod_{v\in|U|\cap(S_i\setminus Z)} H^1(G_{k(v)}, T_{l_i} G_i)$ for each $i \leq r$. On the other hand, for each positive integer $i \leq r$, since the natural homomorphism $\Pi_U \to \Pi_{U_i}$ [induced by the natural open immersion $U \subseteq U_i$] is surjective, the image of ξ_i in $H^1(\Pi_U, T_{l_i} G_i)$ via the inflation map $H^1(\Pi_{U_i}, T_{l_i} G_i) \hookrightarrow H^1(\Pi_U, T_{l_i} G_i)$ is still nonzero. Thus, by applying Theorem 3.9, we conclude that Z contains a set with positive Dirichlet density [in U]. However, this contradicts the fact that Z has Dirichlet density 0 [cf. Remark 1.8.1]. This completes the proof of Theorem 3.10. $\square$

Next, in order to generalize the result of Grunewald and Segal, we prove the following lemma.

Lemma 3.11. *Let K be a generalized global field; G a semi-abelian variety over K; U a G-admissible, irreducible normal scheme of finite type over $\mathbb{Z}$ whose function field is K; l a prime number invertible in U; $\Gamma \subseteq \mathcal{G}(U)$ a finitely generated subgroup, where $\mathcal{G}$ denotes the semi-abelian scheme that appears in the definition of G-admissibility. Then the natural Kummer map*

$$\varprojlim_{n\geq 1} \ \Gamma/l^n\Gamma \ \longrightarrow \ H^1(\Pi_U, T_l G)$$

is injective.

Proof. First, it follows immediately from the Mordell-Weil-Lang-Néron theorem [cf. [12], [21]], together with a higher-dimensional analogue of Dirichlet's unit theorem [cf., e.g., [13], Theorem 7.4], that $\mathcal{G}(U)$ is finitely generated. Then since $\mathbb{Z}_l$ is flat over $\mathbb{Z}$, we observe that the natural homomorphism $\varprojlim_{n\geq 1} \Gamma/l^n\Gamma \to \varprojlim_{n\geq 1} \mathcal{G}(U)/l^n\mathcal{G}(U)$ is injective. On the other hand, since the morphism $\mathcal{G} \to \mathcal{G}$ obtained by multiplication by l^n is finite étale, the natural Kummer map $\mathcal{G}(U)/l^n\mathcal{G}(U) \to H^1(\Pi_U, T_l G/l^n T_l(G))$ is injective. Thus, by applying Lemma 3.5, we obtain the desired conclusion. This completes the proof of Lemma 3.11. $\square$

Finally, we prove our final main result, i.e., a generalization of the result of Grunewald and Segal to the groups of rational points of semi-abelian varieties over generalized global fields.

Definition 3.12. Let K be a generalized global field; G a semi-abelian variety over K; $\Gamma \subseteq G(K)$ a finitely generated subgroup. Then, if K is of characteristic 0 (respectively, positive characteristic p), then we shall write

$$\widehat{\Gamma}$$

for the profinite (respectively, pro-prime-to-p) completion of Γ.

Corollary 3.13. *Let K be a generalized global field; X an irreducible normal scheme of finite type over $\mathbb{Z}$ whose function field is K; $Z \subseteq |X|$ a subset of Dirichlet density 0;*

$$|X| \ = \ \coprod_{1 \leq i \leq r} S_i$$

a finite partition. Fix a positive integer $j \leq r$ as in Theorem 3.10. Let G be a semi-abelian variety over K;

$$\Gamma \ \subseteq \ G(K)$$

a finitely generated subgroup. Note that there exists a G-admissible open subscheme $U \subseteq X$ such that any element of Γ extends to a U-valued point of the semi-abelian scheme $\mathcal{G}$ over U that lifts G. Then, for each such open subscheme U, the natural homomorphism

$$\widehat{\Gamma} \ \longrightarrow \ \prod_{v \in |U| \cap (S_j \setminus Z)} \mathcal{G}_{k(v)}(k(v))'$$

is injective, where $\mathcal{G}_{k(v)} \stackrel{\text{def}}{=} \mathcal{G} \times_U \operatorname{Spec} k(v)$; $\mathcal{G}_{k(v)}(k(v))' \stackrel{\text{def}}{=} \mathcal{G}_{k(v)}(k(v))$ (respectively, $\mathcal{G}_{k(v)}(k(v))'$ denotes the prime-to-p part of $\mathcal{G}_{k(v)}(k(v))$) if K is of characteristic 0 (respectively, positive characteristic p).

Proof. First, since Γ is a finitely generated abelian group, one may reduce to the pro-l setting, where l is a prime number invertible in K. If K is of characteristic 0 (respectively, positive characteristic), then write $U_l \subseteq U$ for the open subscheme obtained by removing closed points of U that lie over l (respectively, $U_l \stackrel{\text{def}}{=} U$). Then, in light of Lemma 3.11, together with a specialization argument, it suffices to verify that the natural homomorphism

$$H^1(\Pi_{U_l}, T_l G) \ \longrightarrow \ \prod_{v \in |U_l| \cap (S_j \setminus Z)} H^1(G_{k(v)}, T_l G)$$

is injective. However, this follows immediately from Theorem 3.10. This completes the proof of Corollary 3.13. $\square$

Remark 3.13.1. Corollary 3.13 may be regarded as a semi-abelian variety generalization of the original result of Grunewald and Segal for the $\mathbb{G}_{\mathrm{m}}$-case. In fact, in the $\mathbb{G}_{\mathrm{m}}$-case, we do not need most of the discussions in §2, §3. In particular, one may regard our discussion as a simpler alternative proof of the original result.

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References

[1] F. Bogomolov, Points of finite order on an abelian variety, *Math. USSR Izv.* **17** (1981), pp. 55–72.

[2] C. Chevalley, Deux théorèmes d'arithmétique, *J. Math. Soc. Japan* **3** (1951), pp. 36–44.

[3] P. Deligne, La conjecture de Weil II, *Publ. Math. IHÉS* **52** (1980), pp.137–252.

[4] J. D. Dixon, M.P.F. du Sautoy, A. Mann, and D. Segal, *Analytic Pro-p Groups (Second edition)*, *Cambridge Studies in Advanced Mathematics* **61**, Cambridge University Press (1999).

[5] K. Erdmann and M. J. Wildon, *Introduction to Lie algebras*, *Springer Undergraduate Mathematics Series*, Springer-Verlag (2006).

[6] G. Faltings, Endlichkeitssätze für abelsche Varietäten über Zahlkörpern, *Invent. Math.* **73** (1983), pp. 349–366.

[7] G. Faltings, Complements to Mordell: in *Rational Points (Second edition)*, *Aspects of Mathematics* **6**, Vieweg (1986).

[8] G. Faltings and C.-L. Chai, *Degeneration of Abelian Varieties*, *Ergebnisse der Mathematik und ihrer Grenzgebiete* **22**, Springer-Verlag (1990).

[9] A. Grothendieck and M. Raynaud, *Revêtements étales et groupe fondamental* (SGA1), *Lecture Notes in Math.* **224** (1971), Springer-Verlag.

[10] F. Grunewald and D. Segal, On congruence topologies in number fields, *J. Reine Angew. Math.* **311/312** (1979), pp. 389–396.

[11] M. Jarden and A. Razon, Generalized transfer theorem, *New York J. Math.* **29** (2023), pp. 488–506.

[12] S. Lang and A. Néron, Rational points of abelian varieties over function field, *Amer. J. Math.* **81** (1959), pp. 95–118.

[13] S. Lang, *Fundamentals of Diophantine Geometry*, Springer-Verlag (1983).

[14] J. S. Milne, *Lie Algebras, Algebraic Groups, and Lie Groups*, available at `https://www.jmilne.org/math/CourseNotes/ala.html`

[15] J. S. Milne, *Arithmetic Duality Theorems (Second edition)*, BookSurge, LLC, Charleston, SC (2006).

[16] L. Moret-Bailly, Pinceaux de variétés abéliennes, *Astérisque*, **129** (1985).

[17] D. Mumford, *Abelian Varieties*, Oxford Univ. Press (1974).

[18] J. Neukirch, A. Schmidt, and K. Wingberg, *Cohomology of Number Fields*, *Grundlehren der Mathematischen Wissenschaften* **323**, Springer-Verlag (2000).

[19] F. K. Schmidt, Zur Klassenkörpertheorie im Kleinen, *J. Reine Angew. Math.* **162** (1930), pp. 155–168.

[20] J.-P. Serre, Zeta and L-functions. Arithmetical Algebraic Geometry, (Proc. Conf. Purdue Univ., 1963), pp. 82–92, Harper and Row, New York (1965).

[21] J.-P. Serre, *Lectures on the Mordell-Weil Theorem (Third edition)*, Vieweg (1997).

[22] R. Tachihara and S. Tsujimura, *Purely mono-anabelian reconstruction of number fields I: solvably closed case*, *RIMS preprint* **2002** (September 2026).

[23] B. A. F. Wehrfritz, Nilpotence in groups of semilinear maps, *Proc. London. Math. Soc* **36** (1978), pp. 448–479.

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