RIMS-2004

Nilpotent Admissible Indigenous Bundles and Cartier Eigenforms in Characteristic Three

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September 2026

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Abstract. — In the present paper, we study indigenous bundles in characteristic three. In particular, we discuss nilpotent admissible/ordinary indigenous bundles on a hyperbolic curve in characteristic three. The main result of the present paper is a characterization of the supersingular divisors of nilpotent admissible/ordinary indigenous bundles on an arbitrary hyperbolic curve in characteristic three by means of Cartier operators. This result generalizes a characterization obtained in a previous work by the author of the present paper, in which the case of a projective smooth curve — i.e., the case where the divisor of cusps is empty — was discussed. The essential new ingredient is a local theory of the nilpotency of an indigenous bundle. We prove that the square Hasse invariant of a nilpotent active indigenous bundle takes the value -1 at each cusp and, moreover, that this condition describes precisely the image of the Schwarz torsor by the square Hasse invariant morphism. By means of this characterization, we also give, in the case of genus zero, a complete description of the supersingular divisors of nilpotent ordinary indigenous bundles and prove that every hyperbolic curve of type $(0,5)$ in characteristic three is hyperbolically ordinary. This application yields a partial positive answer to a basic question in the p-adic Teichmüller theory. Moreover, in the case of characteristic three, we evaluate in closed form the number of nilpotent ordinary indigenous bundles on a sufficiently general hyperbolic curve of genus zero.

Contents

Introduction

The theory of ordinary p-adic curves initiated by S. Mochizuki [cf. [9], [10]] provides a p-adic and arithmetic analogue of the theory of uniformization of hyperbolic Riemann surfaces. One central object in this theory is the notion of an indigenous bundle. In the present paper, we study indigenous bundles in characteristic three. In particular,

2020 Mathematics Subject Classification. — 14G17.
Key words and phrases. — p-adic Teichmüller theory, hyperbolically ordinary, indigenous bundle, Cartier operator, Cartier eigenform, hyperbolic curve, characteristic three.

we discuss nilpotent admissible/ordinary indigenous bundles on a hyperbolic curve in characteristic three.

In the remainder of the present Introduction, let S be a noetherian scheme over $\mathbb{F}_3$, (X, D) a *hyperbolic curve* of type (g, r) over S [cf. Definition 1.1], and P an indigenous bundle on $(X, D)/S$ [cf. [9, Chapter I, Definition 2.2]]. We shall apply the notions of the *nilpotency* and the *admissibility* of an indigenous bundle [cf. [9, Chapter II, Definition 2.4]], the *ordinariness* of a nilpotent indigenous bundle [cf. [9, Chapter II, Definition 3.1]], and the *activeness* of an indigenous bundle [cf. [10, Chapter II, Definition 1.1]]. Write

$$\varphi_P \in \Gamma\big(X, \omega_{X/S}(D)^{\otimes 2}\big)$$

for the *square Hasse invariant* of P [cf. [9, Chapter II, Proposition 2.6, (1)]] — where we note that the relative cotangent bundle that appears in the case of a projective smooth curve is replaced, in the present situation, by the sheaf $\omega_{X/S}(D)$ of *logarithmic* differentials. Suppose that the indigenous bundle P is nilpotent and admissible. Then it follows from [9, Chapter II, Proposition 2.6, (3)] that there exist an invertible sheaf $\mathcal{H}$ on X, an isomorphism $\mathcal{H}^{\otimes 2} \xrightarrow{\sim} \omega_{X/S}(D)^{\otimes 2}$ of $\mathcal{O}_X$-modules, and a global section χ_P of $\mathcal{H}$ — i.e., the *Hasse invariant* of P — whose square corresponds, by means of this isomorphism, to φ_P. We shall refer to the zero locus of χ_P as the *supersingular divisor* of P [cf. [9, Chapter II, Proposition 2.6, (3)]] and to the invertible sheaf $\mathcal{H}om_{\mathcal{O}_X}(\omega_{X/S}(D), \mathcal{H})$ — whose square is trivial — as the *Hasse defect* of P [cf. Definition 1.8].

Next, let us recall the notions, related to Cartier operators, that play a central role in the present paper. Let $\mathcal{L}$ be an invertible sheaf on X, Θ a trivialization of the square of $\mathcal{L}$, and u a global section of $\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}(D)$. Then, by replacing "$\omega_{X/S}$" by the sheaf $\omega_{X/S}(D)$ of logarithmic differentials in the construction of [2, Definition A.4], one obtains the *Cartier operator* $C_{(\mathcal{L},\Theta)}$ associated to $(\mathcal{L}, \Theta)$ [cf. Definition 1.4, (i)]. We shall say that

- u is a *normalized Cartier eigenform* associated to $(\mathcal{L}, \Theta)$ if the zero locus of u is a relative effective Cartier divisor on X over S, and, moreover, the equality $C_{(\mathcal{L},\Theta)}(u) = -u^F$ — where we write u^F for the base-change of u by the absolute Frobenius endomorphism of S — holds [cf. Definition 1.4, (ii)], and that

- u is a *Cartier eigenform* associated to $\mathcal{L}$ if there exists a trivialization of the square of $\mathcal{L}$ with respect to which u is a normalized Cartier eigenform [cf. Definition 1.4, (iii)].

Moreover, in the case where the square of $\mathcal{L}$ is trivial, we shall say that

- the pair $(\mathcal{L}, u)$ is *of CE-type* if u is a Cartier eigenform associated to $\mathcal{L}$, and, moreover, the zero locus of u is finite étale over S and does not meet D [cf. Definition 1.7, (i)], and that

- the pair $(\mathcal{L}, u)$ is *of CEO-type* if $(\mathcal{L}, u)$ is of CE-type, and, moreover, the invertible sheaf $\mathcal{L}$ is parabolically ordinary [cf. Definition 1.6; Definition 1.7, (ii)].

In the case where the divisor D is empty — i.e., the case where the equality $r = 0$ holds — these notions are precisely the notions introduced in [2, Definition A.8], [2, Definition 5.1]. Moreover, in this situation, the main result of [2] [cf. [2, Theorem 5.2]] asserts that, for a projective smooth curve of genus ≥ 2, a relative effective Cartier divisor is the supersingular divisor of a nilpotent admissible (respectively, nilpotent ordinary) indigenous bundle if and only if the divisor is of CE-type (respectively, of CEO-type). That is to say, the main result of [2] gives a complete characterization, by means of Cartier operators, of the supersingular divisors of nilpotent admissible/ordinary indigenous bundles — but only in the case where the equality $r = 0$ holds.

On the other hand, various portions of the theory of [2] have already been generalized, in [6], to the case of a hyperbolic curve of arbitrary type. For instance,

- the assertion that an indigenous bundle in characteristic three is completely determined by the associated square Hasse invariant is an immediate consequence of [6, Theorem A], from which one may in fact derive [2, Theorem A] [cf. [6, Remark 1.7.2]];

- the criterion [2, Proposition 4.4] for the ordinariness of a nilpotent admissible indigenous bundle is generalized by [6, Theorem 2.4] [cf. [6, Remark 2.4.1]];

- the assertion [2, Theorem C] that the ordinariness of a nilpotent admissible indigenous bundle is not preserved by pull-back by a connected finite log étale covering is generalized by [6, Corollary 2.6] [cf. [6, Remark 2.6.1, (i)]].

The main result of [2] itself, however, has remained restricted to the case where the equality $r = 0$ holds. The goal of the present paper is to remove this restriction.

The first main result of the present paper is as follows [cf. Theorem 3.2, (ii), (iii)]:

Theorem A. — *Let φ be a global section of $\omega_{X/S}(D)^{\otimes 2}$. Then there exists an indigenous bundle on $(X, D)/S$ whose square Hasse invariant is given by φ if and only if the value [cf. Definition 1.3, (iii)] of φ at each cusp is equal to -1. Moreover, in this situation, such an indigenous bundle is uniquely determined.*

Theorem A is the counterpart, for an arbitrary hyperbolic curve, of [2, Theorem A]. Indeed, in the case where the equality $r = 0$ holds, the condition on the values at the cusps is vacuous, which implies that Theorem A is none other than the bijection that appears in [2, Theorem A]. In the language of the Schwarz torsor, Theorem A asserts that the image of the square Hasse invariant morphism is the substack obtained by fixing the values at the cusps [cf. Theorem 3.2, (ii)].

We should like to emphasize, however, that the proofs of the two assertions are of entirely different natures. The proof of [2, Theorem A] proceeds by constructing a dormant indigenous bundle — which, in the case where the equalities $r = 0$, $p = 3$ hold, is unique up to isomorphism, hence trivializes the Schwarz torsor [cf. [2, Theorem 2.1], [2, Corollary 2.4]]. If the divisor D is not empty, then this route is not available. Indeed, since the monodromy of an indigenous bundle at each cusp is a nonzero nilpotent element, every indigenous bundle on $(X, D)/S$ is active [cf. Lemma 1.9], i.e., there exists no dormant indigenous bundle on $(X, D)/S$. Thus, the Schwarz torsor admits no such canonical origin. In the present paper, we replace this canonical origin by an equation. Indeed, the closed immersion of [6, Theorem A] yields the injectivity of the square Hasse invariant morphism, and the local theory that we establish in §2 yields its image. The heart of this local theory is the observation that the *logarithmic* nilpotency equation forces the square Hasse invariant of a nilpotent active indigenous bundle to take the value -1 at each cusp [cf. Lemma 2.3; Lemma 2.6, (iii); Lemma 2.12, (ii)].

The second, and main, result of the present paper is as follows [cf. Theorem 3.6, Corollary 3.7]:

Theorem B. — *The following assertions hold:*

(i) *Suppose that the indigenous bundle P is nilpotent and admissible. Write $\mathcal{L}_P$, χ_P for the Hasse defect, the Hasse invariant of P, respectively. Then the pair $(\mathcal{L}_P, \chi_P)$ is of*

CE-type. Moreover, the indigenous bundle P is nilpotent ordinary if and only if the pair $(\mathcal{L}_P, \chi_P)$ is of CEO-type.

(ii) *Let E be a relative effective Cartier divisor on X over S. Then E is the supersingular divisor of a nilpotent admissible* (*respectively, nilpotent ordinary*) *indigenous bundle on $(X, D)/S$ if and only if E is of CE-type* (*respectively, of CEO-type*).

(iii) *The assignment that maps a nilpotent admissible indigenous bundle to the associated supersingular divisor determines a bijective map between the set of isomorphism classes of nilpotent admissible* (*respectively, nilpotent ordinary*) *indigenous bundles on $(X, D)/S$ and the set of relative effective Cartier divisors on X over S that are of CE-type* (*respectively, of CEO-type*).

Let us make three remarks concerning Theorem B. First, let us observe that Theorem B may be regarded as a linearization of the notion of nilpotency. Indeed, the nilpotency of an indigenous bundle amounts, at the level of the square Hasse invariant, to a nonlinear equation [cf. Definition 2.1, (i); Proposition 2.10]. By way of contrast, since the Cartier operator is additive, the equation that appears in the definition of a normalized Cartier eigenform is $\mathbb{F}_3$-linear [cf. Definition 1.4, (ii)]. Second, we note that the passage from [2, Theorem 5.2] to Theorem B requires, at the level of definitions, the imposition of only one new condition — namely, the condition that the zero locus of the section under consideration does not meet D. Third, unlike [2, Theorem 5.2, (iii)], Theorem B, (iii), does not require the base scheme S to be reduced. This is a consequence of the following assertion [cf. Proposition 3.5], which sharpens [2, Corollary 3.4] — hence also [9, Chapter II, Proposition 2.6, (4)] in characteristic three — by removing the reducedness assumption: The isomorphism class of a nilpotent active indigenous bundle on $(X, D)/S$ is completely determined by the zero locus of the associated square Hasse invariant.

Here, again the nonemptiness of D is of assistance. Indeed, since every indigenous bundle on $(X, D)/S$ is active [cf. Lemma 1.9], this assertion may, in the case where the inequality $r > 0$ holds, be applied to every nilpotent indigenous bundle on $(X, D)/S$.

By applying Theorem B, one obtains the following upper bound [cf. Corollary 3.8]:

Theorem C. — *Suppose that S coincides with the spectrum of an algebraically closed field, and that the inequality $r > 0$ holds. Then, for every invertible sheaf $\mathcal{L}$ on X whose square is trivial, the number of isomorphism classes of nilpotent admissible indigenous bundles on $(X, D)/S$ whose Hasse defects are isomorphic to $\mathcal{L}$ is $\leq \#\mathbb{P}^{g+r-2}(\mathbb{F}_3)$. In particular, the number of isomorphism classes of nilpotent admissible indigenous bundles on $(X, D)/S$ is*

$$\leq\ 2^{2g} \cdot \#\mathbb{P}^{g+r-2}(\mathbb{F}_3) = 2^{2g} \cdot \frac{3^{g+r-1} - 1}{3 - 1}.$$

In the case where the inequality $r > 0$ holds, the dimension of the space of global sections of $\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}(D)$ is equal to $g + r - 1$, independently of the isomorphism class of the invertible sheaf $\mathcal{L}$. Thus, the bound of Theorem C is uniform in $\mathcal{L}$. By way of contrast, the corresponding assertion in the case where the equality $r = 0$ holds — i.e., [2, Corollary 5.3] — takes the form of two distinct bounds, according to whether or not the invertible sheaf $\mathcal{L}$ is trivial. In this connection, let us also recall that, by [8, Theorem B], for an integer $g \geq 2$, the number of isomorphism classes of nilpotent ordinary indigenous bundles on a sufficiently general hyperbolic curve of type $(g, 0)$ coincides with the right-hand side of the inequality that appears in [2, Corollary 5.3, (ii)] [cf. Remark 3.8.1].

Next, in §4, we discuss the case of a hyperbolic curve of genus zero, which plays, in the present paper, the role played in [2, §6] by the case of genus two. In this case, the theory becomes completely explicit. Suppose, moreover, that S coincides with the spectrum of an algebraically closed field k of characteristic three, that the equality $g = 0$ holds, and that X coincides with the projective line $\mathbb{P}^1_k$ over k. Write t for the standard coordinate of $\mathbb{P}^1_k$ and $z_1, \ldots, z_r$ for the elements of k that correspond to the cusps of (X, D) [cf. the discussion at the beginning of §4]. Since the Picard group of $\mathbb{P}^1_k$ has no element of order two, the Hasse defect of a nilpotent admissible indigenous bundle on $(X, D)/k$ is trivial. Observe, moreover, that the notions of admissibility and ordinariness coincide [cf. Lemma 4.1, (ii)]. Moreover, the pairs of CE-type may be completely enumerated: a pair $(\mathcal{O}_X, \chi)$ is of CE-type if and only if the global section χ is a $k^\times$-multiple of the differential form

$$\omega_\varepsilon = \sum_{i=1}^{r} \varepsilon_i \cdot \mathrm{dlog}(t - z_i)$$

— where $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_r)$ denotes an element of $\{\pm 1\}^{\times r}$ such that the congruence $\sum_i \varepsilon_i \equiv 0 \pmod 3$ holds — and, moreover, the zero divisor of χ is reduced [cf. Definition 4.2, (ii), (iii); Lemma 4.5, (ii)]. Thus, Theorem B yields a complete description of the supersingular divisors of the nilpotent ordinary indigenous bundles on a hyperbolic curve of genus zero. They are precisely the reduced zero divisors of the differential forms ω_ε [cf. Theorem 4.6]. By applying this description in the case where the equality $r = 5$ holds, one obtains the following [cf. Theorem 4.12]:

Theorem D. — *Let k be an algebraically closed field of characteristic three. Then every hyperbolic curve of type $(0, 5)$ over k is hyperbolically ordinary* [*cf.* [9, Chapter II, Definition 3.3]]. *Moreover, every hyperbolic curve of type $(0, 5)$ over k has a nilpotent indigenous bundle that is active but not ordinary.*

Let us recall the following two basic questions in p-adic Teichmüller theory [cf. [10, Introduction, §2.1]]:

(1) Is every hyperbolic curve in odd characteristic hyperbolically ordinary?

(2) Is the pull-back of a nilpotent ordinary indigenous bundle by a connected finite log étale covering again ordinary?

As already observed above, a negative answer to question (2) — for an arbitrary hyperbolic curve in characteristic three — is given in [6, Corollary 2.6] [cf. [6, Remark 2.6.1, (ii)]; also [2, Theorem C]]. The first assertion of Theorem D gives a new partial positive answer to question (1), i.e., the answer in the case of a hyperbolic curve of type $(0, 5)$ in characteristic three; some of the previous results concerning question (1) are collected in Remark 4.12.1. The second assertion of Theorem D — whose proof consists of an explicit construction [cf. Lemma 4.9, (vi)] — shows that, for a hyperbolic curve of type $(0, 5)$ in characteristic three, the nilpotent indigenous bundles are not exhausted by the nilpotent ordinary ones.

Finally, in §5, we observe that, in the case of characteristic three, the combinatorial sum that appears in [10, Chapter V, Corollary 1.3, (1)] — which computes the number of isomorphism classes of nilpotent ordinary indigenous bundles on a sufficiently general hyperbolic curve of type $(0, r)$ — admits the closed form $(2^{r-1} - (-1)^{r-1})/3$ [cf. Lemma 5.2, (iii); Theorem 5.3]. By comparing this number with one half of the number of elements ε of $\{\pm 1\}^{\times r}$ that satisfy the congruence $\sum_i \varepsilon_i \equiv 0 \pmod 3$ [cf. Lemma 5.4, (iii)], one concludes that, in the case of a sufficiently general hyperbolic curve of genus zero, the zero divisor of the differential form ω_ε is the supersingular divisor of a nilpotent ordinary indigenous bundle for every such element ε [cf. Corollary 5.5].

The present paper is organized as follows: In §1, we recall generalities on indigenous bundles in characteristic three and introduce the logarithmic versions of Cartier operators and Cartier eigenforms. In §2, we establish the local theory of the nilpotency of an indigenous bundle [cf. Proposition 2.10]. In §3, we give a description of the image of the Schwarz torsor by the square Hasse invariant morphism [cf. Theorem 3.2, (ii)] and characterize the square Hasse invariants of nilpotent active indigenous bundles by means of Cartier eigenforms [cf. Theorem 3.4]. Moreover, we prove the main theorem of the present paper [cf. Theorem 3.6, Corollary 3.7]. In §4, we discuss the case of a hyperbolic curve of genus zero. In particular, we characterize the supersingular divisors of nilpotent ordinary indigenous bundles on hyperbolic curves of genus zero [cf. Theorem 4.6] and prove that every hyperbolic curve of type $(0, 5)$ in characteristic three is hyperbolically ordinary [cf. Theorem 4.12, (i)]. In §5, we evaluate, in the case of characteristic three, the combinatorial sum of [10, Chapter V, Corollary 1.3, (1)] and apply this evaluation to the case of a sufficiently general hyperbolic curve of genus zero [cf. Lemma 5.2, Theorem 5.3, Corollary 5.5].

Acknowledgments

This research was supported by JSPS KAKENHI Grant Number 24K06668 and by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University. This research was also supported by the CNRS France-Japan "Arithmetic and Homotopic Galois Theory" International Research Network.

1. Indigenous Bundles and Cartier Operators in Characteristic Three

In the present §1, we recall generalities on indigenous bundles in characteristic three and introduce the logarithmic versions of Cartier operators and Cartier eigenforms. In the present §1, let S be a noetherian scheme over $\mathbb{F}_3$.

Definition 1.1. — Let (X, D) be a pair that consists of a projective smooth curve X over S and a [possibly empty] relative effective Cartier divisor $D \subseteq X$ on X over S.

(i) Suppose that S is the spectrum of an algebraically closed field. Then we shall say that the pair (X, D) is a *hyperbolic curve* if the following two conditions are satisfied:

- The Cartier divisor $D \subseteq X$ is reduced.

- If one writes g for the genus of the projective smooth curve X over S and r for the degree of the Cartier divisor $D \subseteq X$ over S, then the inequality $2 - 2g - r < 0$ holds.

In this situation, the *type* of the hyperbolic curve (X, D) over S is defined to be the uniquely determined pair (g, r).

(ii) We shall say that the pair (X, D) is a *hyperbolic curve* (respectively, *hyperbolic curve of type* (g, r)) over S if every geometric fiber of (X, D) over S is a hyperbolic curve (respectively, hyperbolic curve of type (g, r)) in the sense of (i).

In the remainder of the present §1, let

- (X, D) be a hyperbolic curve over S, and let
- P be an indigenous bundle on (X, D) over S [cf. [9, Chapter I, Definition 2.2]].

DEFINITION 1.2. — Let x be a point of X, and let t be a local section of $\mathcal{O}_X$ at x.

(i) We shall say that an open neighborhood $U \subseteq X$ of $x \in X$ is *t-line-like* if the restriction to U of $\omega_{X/S}$ is a free $\mathcal{O}_U$-module of rank one generated by $dt|_U$.

(ii) We shall say that the local section t is a *local coordinate* at x [over S] if there exists a t-line-like open neighborhood of $x \in X$.

(iii) Suppose that t is a local coordinate at x. Then since, for each t-line-like open neighborhood $U \subseteq X$ of $x \in X$, the restriction to U of $\omega_{X/S}$ is a free $\mathcal{O}_U$-module of rank one generated by $dt|_U$, there exists a unique local derivation ∂ at x of $\mathcal{O}_X$ over $\mathcal{O}_S$ such that the local equality $d\phi = \partial\phi \cdot dt$ holds for every local section ϕ of $\mathcal{O}_X$ at x. We shall write d/dt for this unique local derivation at x of $\mathcal{O}_X$ over $\mathcal{O}_S$ and, for each positive integer n, d^n/dt^n for the n-fold composite of the local derivation d/dt. Moreover, for each positive integer n and each local section ϕ of $\mathcal{O}_X$ at x, we shall write $d^n\phi/dt^n$ for the local section of $\mathcal{O}_X$ at x obtained by applying d^n/dt^n to ϕ.

(iv) We shall say that the local section t is a *local log-coordinate* at x [over S] if there exists a t-line-like open neighborhood $U \subseteq X$ of $x \in X$ [which implies that the local section t is a local coordinate at x] such that the Cartier divisor on U determined by the intersection of U and D is zero (respectively, coincides with the Cartier divisor determined by t) whenever $x \notin D$ (respectively, $x \in D$).

DEFINITION 1.3.

(i) We shall refer to a point of X that lies on D as a *cusp* of the hyperbolic curve (X, D) over S.

(ii) Let x be a cusp of (X, D), and let t be a local log-coordinate at x. Then one verifies easily that the local section dt/t of $\omega_{X/S}(D)$ determines a local trivialization of $\omega_{X/S}(D)$ at x. We shall refer to the isomorphism

$$\mathrm{Res}_x \colon \omega_{X/S}(D) \otimes_{\mathcal{O}_X} \kappa(x) \xrightarrow{\sim} \kappa(x)$$

— where we write $\kappa(x)$ for the residue field of X at x — as the *residue isomorphism* at x. Observe that one verifies easily that the residue isomorphism at x does not depend on the choice of the local log-coordinate t.

(iii) Let x be a cusp of (X, D), and let ϕ be a local section of $\omega_{X/S}(D)^{\otimes 2}$ at x. Then we shall refer to the image by $\mathrm{Res}_x^{\otimes 2}$ of ϕ as the *value* of ϕ at x.

Definition 1.4. — Let $\mathcal{L}$ be an invertible sheaf on X, Θ a trivialization of the square of $\mathcal{L}$, and $u \in \Gamma(X, \mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}(D))$ a global section of $\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}(D)$. Write $f\colon X \to S$ for the structure morphism of X. Write, moreover, $f^F\colon X^F \to S$, D^F, $\mathcal{L}^F$, and u^F for the respective base-changes of $f\colon X \to S$, D, $\mathcal{L}$, and u by the absolute Frobenius endomorphism of S.

(i) We shall write

$$C_{(\mathcal{L},\Theta)}\colon f_*\big(\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}(D)\big) \to f^F_*\big(\mathcal{L}^F \otimes_{\mathcal{O}_{X^F}} \omega_{X^F/S}(D^F)\big)$$

for the *Cartier operator* associated to $(\mathcal{L}, \Theta)$, i.e., the homomorphism obtained by replacing "$\omega_{X/S}$" by $\omega_{X/S}(D)$ in the construction of the Cartier operator defined in [2, Definition A.4]. [Note that it is well-known that the Cartier operator for logarithmic differential forms is compatible with logarithmic poles and residues.]

(ii) We shall say that the global section $u \in \Gamma(X, \mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}(D))$ is a *normalized Cartier eigenform* associated to $(\mathcal{L}, \Theta)$ if the zero locus of u is a relative effective Cartier divisor on X over S, and, moreover, the equality $C_{(\mathcal{L},\Theta)}(u) = -u^F$ holds.

(iii) We shall say that the global section $u \in \Gamma(X, \mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}(D))$ is a *Cartier eigenform* associated to $\mathcal{L}$ if there exists a trivialization Θ' of the square of $\mathcal{L}$ such that the global section u is a normalized Cartier eigenform associated to the pair $(\mathcal{L}, \Theta')$.

Lemma 1.5. — *Let $\mathcal{L}$ be an invertible sheaf on X, Θ a trivialization of the square of $\mathcal{L}$, $u \in \Gamma(X, \mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}(D))$ a global section of $\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}(D)$, x a point of $X \setminus D$, t a local coordinate at x, and l a local trivialization of $\mathcal{L}$ at x. Write $\theta \stackrel{\mathrm{def}}{=} \Theta(l \otimes l)$ for the local trivialization of $\mathcal{O}_X$ at x determined by $l \otimes l$ and Θ, b for the local section of $\mathcal{O}_X$ at x uniquely determined by the equality $u = b \cdot (l \otimes dt)$, and u^F for the base-change of u by the absolute Frobenius endomorphism of S. Then the following two conditions are equivalent:*

(1) *The local equality $C_{(\mathcal{L},\Theta)}(u) = -u^F$ holds at x.*

(2) *The local equality*

$$b^3 = \frac{d^2}{dt^2}\left(\frac{b}{\theta}\right)$$

holds at x.

Proof. — This assertion follows immediately from [2, Lemma A.9, (ii)]. □

Definition 1.6. — We shall say that an invertible sheaf $\mathcal{L}$ on X whose square is trivial is *parabolically ordinary* [cf. [2, Definition A.7]] if there exists a trivialization Θ of the square of $\mathcal{L}$ such that the Cartier operator $C_{(\mathcal{L},\Theta)}$ is injective at each fiber of (X, D) over S.

Definition 1.7. — Let $\mathcal{L}$ be an invertible sheaf on X whose square is trivial, and let $\chi \in \Gamma(X, \mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}(D))$ be a global section of $\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}(D)$.

(i) We shall say that the pair $(\mathcal{L}, \chi)$ is *of CE-type* if χ is a Cartier eigenform associated to $\mathcal{L}$, and, moreover, the zero locus of χ is finite étale over S and does not meet D.

(ii) We shall say that the pair $(\mathcal{L}, \chi)$ is *of CEO-type* if $(\mathcal{L}, \chi)$ is of CE-type, and, moreover, the invertible sheaf $\mathcal{L}$ is parabolically ordinary.

(iii) We shall say that a relative effective Cartier divisor E on X over S is *of CE-type* (respectively, *of CEO-type*) if there exists a pair $(\mathcal{F}, \chi_{\mathcal{F}})$ of CE-type (respectively, of CEO-type) such that E coincides with the zero divisor of $\chi_{\mathcal{F}}$.

Definition 1.8. — Suppose that the indigenous bundle P is nilpotent [cf. [9, Chapter II, Definition 2.4]] and admissible [cf. [9, Chapter II, Definition 2.4]]. Thus, it follows from [9, Chapter II, Proposition 2.6, (3)] that there exist an invertible sheaf $\mathcal{H}$ on X — which is uniquely determined up to tensoring with the pullback of an invertible sheaf on S — and an isomorphism $\mathcal{H}^{\otimes 2} \xrightarrow{\sim} \omega_{X/S}(D)^{\otimes 2}$ of $\mathcal{O}_X$-modules such that the Hasse invariant of P [cf. [9, Chapter II, Proposition 2.6, (3)]] is a global section of $\mathcal{H}$. Then we shall refer to the invertible sheaf $\mathcal{H}om_{\mathcal{O}_X}(\omega_{X/S}(D), \mathcal{H})$ on X — whose square is trivial, and which is uniquely determined up to tensoring with the pullback of an invertible sheaf on S — as the *Hasse defect* of P.

Lemma 1.9. — *Suppose that the composite of the natural inclusion $D \hookrightarrow X$ and the structure morphism $X \to S$ of X is surjective. Then the indigenous bundle P is active* [*cf.* [10, Chapter II, Definition 1.1]].

Proof. — This assertion follows immediately from [10, Chapter II, Proposition 1.4], together with the fact that the monodromy at each cusp of an indigenous bundle on $(X, D)/S$ is nonzero and nilpotent [cf. [9, Chapter I, Definition 2.2]], i.e., that the "ρ" of [10, Chapter II, §1.2] associated to each cusp is zero. $\square$

2. Local Theory of a Nilpotent Indigenous Bundle

In the present §2, we establish the local theory of the nilpotency of an indigenous bundle [cf. Proposition 2.10 below].

Definition 2.1. — Let R be a ring, A an R-algebra, ∂ a derivation of A over R, and φ an element of A.

(i) We shall say that the element $\varphi \in A$ satisfies the *interior nilpotency equation* with respect to ∂ if the equality

$$(\partial\varphi)^2 + \varphi \cdot \partial^2\varphi + \varphi^3 = 0$$

holds.

(ii) We shall say that the element $\varphi \in A$ satisfies the *logarithmic nilpotency equation* with respect to ∂ if the equality

$$(\partial\varphi + \varphi)^2 + \varphi \cdot (\partial^2\varphi + \partial\varphi) + \varphi^3 = 0$$

holds.

LEMMA 2.2. — *Let R be an $\mathbb{F}_3$-algebra, A an R-algebra, ∂ a derivation of A over R, π an element of A such that the equality $\partial\pi = 1$ holds, and φ an element of A. Write $\psi \overset{\text{def}}{=} \pi^2 \cdot \varphi$. Then the following assertions hold:*

(i) *The equality*

$$\big((\pi \cdot \partial)\psi + \psi\big)^2 + \psi \cdot \big((\pi \cdot \partial)^2\psi + (\pi \cdot \partial)\psi\big) + \psi^3 = \pi^6 \cdot \big((\partial\varphi)^2 + \varphi \cdot \partial^2\varphi + \varphi^3\big)$$

holds.

(ii) *Suppose that the element $\pi \in A$ is not a zero-divisor. Then φ satisfies the interior nilpotency equation with respect to ∂ if and only if ψ satisfies the logarithmic nilpotency equation with respect to $\pi \cdot \partial$.*

PROOF. — Assertion (i) follows immediately from a straightforward computation in characteristic three. [Note that the equalities $(\pi \cdot \partial) \cdot \psi + \psi = \pi^3 \cdot \partial\varphi$ and $(\pi \cdot \partial)^2 \cdot \psi + (\pi \cdot \partial) \cdot \psi = \pi^4 \cdot \partial^2\varphi$ hold.] Assertion (ii) is a formal consequence of assertion (i). This completes the proof of Lemma 2.2. □

REMARK 2.2.1. — In the situation of Lemma 2.2, the factor π^2 that appears in the definition of ψ arises from the easily verified equality "$f \cdot (dt)^{\otimes 2} = (t^2 \cdot f) \cdot (\mathrm{dlog}t)^{\otimes 2}$".

LEMMA 2.3. — *Let R be a ring, A an R-algebra, ∂ a derivation of A over R, π an element of A, and ψ an element of A that satisfies the logarithmic nilpotency equation with respect to $\pi \cdot \partial$. Write $\overline{\psi}$ for the image of ψ in the quotient $A/\pi A$ of A. Then the following assertions hold:*

(i) *The equality $\overline{\psi}^2 \cdot (1 + \overline{\psi}) = 0$ holds.*

(ii) *If the element $\overline{\psi}$ of $A/\pi A$ is a unit, then the equality $\overline{\psi} = -1$ holds.*

PROOF. — Let us first observe that since the derivation $\pi \cdot \partial$ of A maps A into the ideal $\pi A \subseteq A$, both $(\pi \cdot \partial)\psi$ and $(\pi \cdot \partial)^2\psi$ are contained in πA. Thus, by considering the image in $A/\pi A$ of the left-hand side of the logarithmic nilpotency equation with respect to $\pi \cdot \partial$, one obtains the equality $\overline{\psi}^2 + \overline{\psi}^3 = 0$, i.e., the equality $\overline{\psi}^2 \cdot (1 + \overline{\psi}) = 0$. This completes the proof of assertion (i). Assertion (ii) follows immediately from assertion (i), together with the fact that if the element $\overline{\psi}$ is a unit, then the element $\overline{\psi}^2$ is a unit. This completes the proof of Lemma 2.3. □

In the remainder of the present §2, let

- k be an algebraically closed field of characteristic three,
- O a discrete valuation ring over k such that the structure morphism $k \to O$ determines an isomorphism of k with the residue field of O, by means of which we identify k with the residue field of O,
- $\pi \in O$ a uniformizer of O,
- ∂ a derivation of O over k such that the equality $\partial\pi = 1$ holds — which uniquely determines a derivation of the field $O[1/\pi]$ of fractions of O — and
- φ an element of the field $O[1/\pi]$ of fractions of O.

Definition 2.4.

(i) We shall write $\mathrm{ord}_\pi\colon O[1/\pi]^\times \to \mathbb{Z}$ for the valuation of $O[1/\pi]$ normalized so that the equality $\mathrm{ord}_\pi(\pi) = 1$ holds.

(ii) Let m be an integer, and let ψ be an element of $O[1/\pi]$ such that the element ψ/π^m of $O[1/\pi]$ is contained in $O \subseteq O[1/\pi]$. Then we shall refer to the image of the element $\psi/\pi^m \in O$ in the residue field k as the *coefficient of* π^m in ψ.

Lemma 2.5. — *Suppose that the element $\varphi \in O[1/\pi]$ is nonzero and satisfies the interior nilpotency equation with respect to ∂. Write $n \stackrel{\mathrm{def}}{=} \mathrm{ord}_\pi(\varphi)$ for the order of φ and $a_n \in k$ for the coefficient of π^n in φ. Then the following assertions hold:*

(i) *The inequality $n \geq -2$ holds.*

(ii) *Suppose that the inequality $n \geq -1$ holds. Then $n-1$ is not divisible by* 3.

(iii) *Suppose that the equality $n = -2$ holds. Then the equality $a_n = -1$ holds.*

Proof. — Let us first observe that it follows immediately from a straightforward computation that

(a) each of the first two terms of the left-hand side of the interior nilpotency equation is contained in $\pi^{2n-2}O$, and the coefficient of π^{2n-2} in the sum of these two terms is given by $n(2n-1)a_n^2$, and, moreover,

(b) the equality $\mathrm{ord}_\pi(\varphi^3) = 3n$ holds, and the coefficient of π^{3n} in φ^3 is given by a_n^3.

First, we verify assertion (i). Assume that the inequality $n < -2$, i.e., that the inequality $3n < 2n-2$, holds. Then one verifies easily from (a), (b) that the coefficient of π^{3n} in the left-hand side of the interior nilpotency equation is given by a_n^3, which thus implies that $a_n = 0$, in contradiction to the definition of n. This completes the proof of assertion (i).

Next, we verify assertion (ii). Assume that $n-1$ is divisible by 3. Then since [we have assumed that] the inequality $n \geq -1$ holds, the inequality $2n-2 < 3n$ holds. Thus, one verifies easily from (a), (b) that the coefficient of π^{2n-2} in the left-hand side of the interior nilpotency equation is given by $n(2n-1)a_n^2$. Then since $n(2n-1) \neq 0$ in k [cf. our assumption that $n-1$ is divisible by 3], one concludes that $a_n = 0$, in contradiction to the definition of n. This completes the proof of assertion (ii).

Finally, we verify assertion (iii). Since [we have assumed that] the equality $n = -2$ holds, the equality $2n-2 = 3n$ holds. Then one verifies easily from (a), (b) that the coefficient of π^{2n-2} in the left-hand side of the interior nilpotency equation is given by $n(2n-1)a_n^2 + a_n^3 = a_n^2 \cdot (1 + a_n)$. Thus, since $a_n \neq 0$, one concludes that the equality $a_n = -1$ holds, as desired. This completes the proof of assertion (iii), hence also of Lemma 2.5. $\square$

Lemma 2.6. — *Suppose that the element $\varphi \in O[1/\pi]$ satisfies the logarithmic nilpotency equation with respect to $\pi \cdot \partial$. Then the following assertions hold:*

(i) *Suppose that φ is nonzero, and that the inequality $\mathrm{ord}_\pi(\varphi) \geq 1$ holds. Then $\mathrm{ord}_\pi(\varphi)$ is not divisible by* 3.

(ii) *For every positive integer* m *not divisible by* 3*, there exists a formal power series* $\psi \in k[[s]]$ *— where* s *is an indeterminate — of order* m *that satisfies the logarithmic nilpotency equation with respect to* $s \cdot d/ds$*.*

(iii) *Suppose that the element* φ *is contained in* O*. Then the image in* k *of* φ *is contained in* $\{0, -1\}$*.*

Proof. — Assertion (i) is a formal consequence of Lemma 2.2, (ii), and Lemma 2.5, (ii). Assertion (ii) follows immediately from the successive determination of the coefficients of a formal solution. Indeed, if one writes $\psi = \sum_j c_j \cdot s^j$, then one verifies easily that the logarithmic nilpotency equation with respect to $s \cdot d/ds$ amounts to the equalities

$$\sum_{a+b=3i} c_a \cdot c_b = -c_i^3$$

— where i ranges over the nonnegative integers — so that the only obstruction to such a determination arises in degree $2m$, which does not occur [cf. our assumption that m is not divisible by three].

Finally, we verify assertion (iii). If the image in k of φ is zero, then assertion (iii) is immediate. Suppose that the image in k of φ is nonzero, i.e., that the equality $\mathrm{ord}_\pi(\varphi) = 0$ holds. Then assertion (iii) is a formal consequence of Lemma 2.2, (ii), and Lemma 2.5, (iii). This completes the proof of assertion (iii), hence also of Lemma 2.6. ☐

In the remainder of the present §2, let

- S be a noetherian scheme over $\mathbb{F}_3$,
- (X, D) a hyperbolic curve over S, and
- P an indigenous bundle on (X, D) over S.

Write $\varphi_P \in \Gamma(X, \omega_{X/S}(D)^{\otimes 2})$ for the square Hasse invariant of P [cf. [9, Chapter II, Proposition 2.6, (1)]].

Definition 2.7. — Suppose that the indigenous bundle P is active. Then it follows from [9, Chapter II, Proposition 2.6, (1), (2)] — whose proofs only require the activeness of the indigenous bundle under consideration — that the zero locus of the square Hasse invariant φ_P forms a relative effective Cartier divisor on X over S. We shall refer to this relative effective Cartier divisor as the *double supersingular divisor* of P.

Remark 2.7.1. — In the case where the indigenous bundle P is admissible [hence also active], the divisor defined in Definition 2.7 is none other than the divisor discussed in [9, Chapter II, Proposition 2.6, (2)], where it is referred to as the double supersingular divisor. In the case where S coincides with the spectrum of an algebraically closed field, and the indigenous bundle P is nilpotent and active, the divisor defined in Definition 2.7 is none other than the divisor discussed in [3, Definition A.2, (i)], where it is denoted by "E_{sH}".

Definition 2.8. — Let x be a point of X, t a local coordinate at x [cf. Definition 1.2, (ii)], and ϕ a local section of $\mathcal{O}_X$ at x. Then we shall say that the local section ϕ satisfies the *interior nilpotency equation* with respect to t if the element ϕ of the local ring $\mathcal{O}_{X,x}$

satisfies the interior nilpotency equation with respect to d/dt [cf. Definition 1.2, (iii)] in the sense of Definition 2.1, (i).

Remark 2.8.1. — Let x be a point of X, ψ a local section of $\omega_{X/S}^{\otimes 2}$ at x, and t, t' local coordinates at x. Write ϕ, ϕ' for the local sections of $\mathcal{O}_X$ at x uniquely determined by the local equalities $\psi = \phi \cdot (dt)^{\otimes 2} = \phi' \cdot (dt')^{\otimes 2}$ at x and $u \stackrel{\text{def}}{=} dt'/dt$. Thus, one verifies easily that u is a unit of $\mathcal{O}_{X,x}$, and, moreover, the local equality $\phi' = u^{-2} \cdot \phi$ at x holds. Then since the cube of a derivation in characteristic three is again a derivation, the operator d^3/dt^3 is an $\mathcal{O}_S$-derivation of $\mathcal{O}_X$ at x that annihilates t, hence is zero, which thus implies that the equality $d^2u/dt^2 = d^3t'/dt^3 = 0$ holds. Thus, one verifies easily that the equality

$$\left(\frac{d\phi'}{dt'}\right)^2 + \phi' \cdot \frac{d^2\phi'}{dt'^2} + \phi'^3 = u^{-6} \cdot \left(\left(\frac{d\phi}{dt}\right)^2 + \phi \cdot \frac{d^2\phi}{dt^2} + \phi^3\right)$$

holds. In particular, the local section ϕ satisfies the interior nilpotency equation with respect to t if and only if the local section ϕ' satisfies the interior nilpotency equation with respect to t', i.e., the condition that appears in Definition 2.8 does not depend on the choice of the local coordinate under consideration.

Lemma 2.9. — *Suppose that the indigenous bundle P arises from an indigenous vector bundle $(\mathcal{E}, \nabla_{\mathcal{E}})$ on (X, D) over S* [*cf.* [9, Chapter I, Definition 2.2]]. *Let x be a point of X, and let t be a local log-coordinate at x* [*cf. Definition* 1.2, (iv)]. *If $x \notin D$ (respectively, $x \in D$), then write $\nu_t \stackrel{\text{def}}{=} dt$ (respectively, $\nu_t \stackrel{\text{def}}{=} dt/t$) for the local trivialization of $\omega_{X/S}(D)$ at x determined by t, $\partial_t \stackrel{\text{def}}{=} d/dt$ (respectively, $\partial_t \stackrel{\text{def}}{=} t \cdot d/dt$) for the local trivialization of $\tau_{X/S}(-D)$ at x determined by t, ϕ_P for the local section of $\mathcal{O}_X$ at x uniquely determined by the local equality $\varphi_P = \phi_P \cdot \nu_t^{\otimes 2}$ at x, and $\Psi_{\mathcal{E}} \colon \tau_{X/S}(-D)^{\otimes 3} \to \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})$ for the p-curvature homomorphism of $(\mathcal{E}, \nabla_{\mathcal{E}})$. Then the following assertions hold:*

(i) *Suppose that x does not lie on D. Then the local equality*

$$\det\Big(\Psi_{\mathcal{E}}(\partial_t^{\otimes 3}) + \mathrm{tr}\big(\Psi_{\mathcal{E}}(\partial_t^{\otimes 3})\big)\Big) = -\left(\left(\frac{d\phi_P}{dt}\right)^2 + \phi_P \cdot \frac{d^2\phi_P}{dt^2} + \phi_P^3\right)$$

at x holds.

(ii) *Suppose that x does not lie on D. Write $\Psi_P \colon \tau_{X/S}(-D)^{\otimes 3} \to \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})$ for the p-curvature homomorphism of P, i.e., the homomorphism obtained by composing $\Psi_{\mathcal{E}}$ with the projection homomorphism of $\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})$ onto the subsheaf of $\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})$ that consists of endomorphisms whose traces are zero* [*cf. the discussion preceding* [9, Chapter II, Proposition 2.1]]. *Then the local equality*

$$\det\big(\Psi_P(\partial_t^{\otimes 3})\big) = -\left(\left(\frac{d\phi_P}{dt}\right)^2 + \phi_P \cdot \frac{d^2\phi_P}{dt^2} + \phi_P^3\right)$$

at x holds.

(iii) *Suppose that S coincides with the spectrum of the field k, and that the indigenous bundle P is nilpotent and active. Write $E_{\mathrm{sH}} \subseteq X$ for the double supersingular divisor of P, E_{gss} for the generalized supersingular divisor of P* [*cf.* [3, Definition A.2, (iii)]], *and* E_{spk} *for the spiked locus of* P [*cf.* [3, Definition A.2, (iv)]]. *Then the inclusion* $\mathrm{Supp}(E_{\mathrm{sH}}) \subseteq \mathrm{Supp}(E_{\mathrm{gss}}) \cup \mathrm{Supp}(E_{\mathrm{spk}})$ *holds.*

(iv) *Suppose that* S *coincides with the spectrum of the field* k*, that* x *lies on* D*, and that the indigenous bundle* P *is nilpotent and active. Write* $E_{\mathrm{sH}} \subseteq X$ *for the double supersingular divisor of* P*. Suppose, moreover, that* x *does not lie on* E_{sH}*. Then the equality* $\phi_P(x) = -1$ *holds.*

PROOF. — Write τ, γ for the local endomorphisms of $\mathcal{E}$ at x defined by $\nabla_{\mathcal{E}}(\partial_t)$, $\Psi_{\mathcal{E}}(\partial_t^{\otimes 3})$, respectively, which implies that

(a) if $x \notin D$, then the local equality $\tau^3 = \gamma$ at x holds.

Let e_H, e_C be local trivializations at x of the Hodge, conjugate filtrations of $\mathcal{E}$, respectively [cf. the discussions preceding [3, Lemma A.6], [3, Lemma A.8], respectively].

First, we verify assertion (i). Write $e_E \stackrel{\mathrm{def}}{=} \tau(e_H)$. Thus, it follows from the discussion preceding [3, Lemma A.6] [i.e., from the fact that the Kodaira–Spencer homomorphism associated to $(\mathcal{E}, \nabla_{\mathcal{E}})$ and the Hodge filtration of $\mathcal{E}$ is an isomorphism] that the pair (e_H, e_E) is a local trivialization of $\mathcal{E}$ at x. Write c_H, c_E for the local sections of $\mathcal{O}_X$ at x uniquely determined by the local equality

$$\tau(e_E) = c_H \cdot e_H + c_E \cdot e_E$$

at x. Write, moreover,

$$A \stackrel{\mathrm{def}}{=} \frac{dc_H}{dt} + c_H \cdot c_E, \quad B \stackrel{\mathrm{def}}{=} c_H + \frac{dc_E}{dt} + c_E^2.$$

Then it follows immediately by a straightforward computation [cf. also the proof of [6, Lemma 1.5]] that

(b) the local equality

$$\tau^3 \begin{pmatrix} e_H \\ e_E \end{pmatrix} = \begin{pmatrix} A & B \\ \frac{dA}{dt} + B \cdot c_H & A + \frac{dB}{dt} + B \cdot c_E \end{pmatrix} \begin{pmatrix} e_H \\ e_E \end{pmatrix}$$

at x holds.

Moreover, it follows from [6, Lemma 1.5] that

(c) the local equality

$$\phi_P = B$$

at x holds.

Next, write $\Psi_{\det(\mathcal{E})} \colon \tau_{X/S}(-D)^{\otimes 3} \to \mathcal{E}nd_{\mathcal{O}_X}(\det(\mathcal{E})) = \mathcal{O}_X$ for the p-curvature homomorphism of $(\det(\mathcal{E}), \nabla_{\det(\mathcal{E})})$ and δ for the local endomorphism of $\det(\mathcal{E})$ at x defined by $\Psi_{\det(\mathcal{E})}(\partial_t^{\otimes 3})$. Then since [it is immediate that] the local equality $\tau(e_H \wedge e_E) = c_E \cdot (e_H \wedge e_E)$ at x holds, it follows immediately by a straightforward computation, together with (c), that the local equality $\delta = d^2 c_E/dt^2 + c_E^3$ at x holds, which implies

(d) the local equality

$$A + \delta = \frac{d\phi_P}{dt} + \phi_P \cdot c_E$$

at x and

(e) the local equalities

$$\frac{d\delta}{dt} = \frac{d^3 c_E}{dt^3} + 3c_E^2 \cdot \frac{dc_E}{dt} = 0$$

at x.

In particular, it follows from (d), (e) that

(f) the local equality

$$\frac{dA}{dt} = \frac{d^2\phi_P}{dt^2} + c_E \cdot \frac{d\phi_P}{dt} + \phi_P \cdot \frac{dc_E}{dt}$$

at x holds.

Moreover, it follows from (a), (b), (d) that

(g) the trace of γ is given by $2A + d\phi_P/dt + \phi_P \cdot c_E = 3A + \delta = \delta$.

Thus, it follows from (a), (b), (c), (d), (f), (g) that the local equalities

$$\det\big(\gamma + \mathrm{tr}(\gamma)\big) = \det\begin{pmatrix} A+\delta & \phi_P \\ \frac{dA}{dt} + \phi_P \cdot c_H & -A-\delta \end{pmatrix} = -(A+\delta)^2 - \phi_P \cdot \frac{dA}{dt} - \phi_P^2 \cdot c_H$$

$$= -\left(\frac{d\phi_P}{dt} + \phi_P \cdot c_E\right)^2 - \phi_P \cdot \left(\frac{d^2\phi_P}{dt^2} + c_E \cdot \frac{d\phi_P}{dt} + \phi_P \cdot \frac{dc_E}{dt}\right) - \phi_P^2 \cdot c_H$$

$$= -\left(\frac{d\phi_P}{dt}\right)^2 - \phi_P \cdot \frac{d^2\phi_P}{dt^2} - \phi_P^2 \cdot \left(c_H + \frac{dc_E}{dt} + c_E^2\right).$$

at x hold. In particular, since the local equalities $\phi_P = B = c_H + dc_E/dt + c_E^2$ at x hold [cf. (c) and the definition of B], one concludes that the local equality

$$\det\big(\gamma + \mathrm{tr}(\gamma)\big) = -\left(\left(\frac{d\phi_P}{dt}\right)^2 + \phi_P \cdot \frac{d^2\phi_P}{dt^2} + \phi_P^3\right)$$

at x holds, as desired. This completes the proof of assertion (i). Assertion (ii) is a formal consequence of assertion (i).

Next, we verify assertion (iii). To this end, suppose that $x \notin \mathrm{Supp}(E_{\mathrm{gss}}) \cup \mathrm{Supp}(E_{\mathrm{spk}})$. Write γ' for the local endomorphism of $\mathcal{E}$ at x defined by $\Psi_P(\partial_t^{\otimes 3})$ [cf. assertion (ii)], i.e., the trace-free part of γ. Thus, the endomorphisms γ and γ' differ by a scalar, which implies that, for a local section e of $\mathcal{E}$ at x, the images of $\gamma(e)$ and $\gamma'(e)$ in the quotient of $\mathcal{E}$ by the Hodge filtration coincide modulo the image of e. Let us first observe that since [we have assumed that] the indigenous bundle P is nilpotent, and S is the spectrum of a field, the endomorphism $\gamma' \otimes_{\mathcal{O}_X} \kappa(x)$ — where we write $\kappa(x)$ for the residue field of X at x — is nilpotent [cf. [9, Chapter II, Proposition 2.1]]; moreover, since $x \notin \mathrm{Supp}(E_{\mathrm{spk}})$, this endomorphism is nonzero. Thus, since [it is immediate that] the kernel of a nonzero nilpotent endomorphism of a vector space of dimension two is of dimension one and coincides with the image of the endomorphism, one concludes that $\mathrm{Ker}(\gamma' \otimes_{\mathcal{O}_X} \kappa(x)) = \mathrm{Im}(\gamma' \otimes_{\mathcal{O}_X} \kappa(x))$, and that this subspace is of dimension one.

Next, observe that since the conjugate filtration of $\mathcal{E}$ is horizontal, it is preserved by γ, hence also by γ'. Thus, since $\gamma' \otimes_{\mathcal{O}_X} \kappa(x)$ is nilpotent, the line generated by $e_C \otimes_{\mathcal{O}_X} \kappa(x)$ is contained in $\mathrm{Ker}(\gamma' \otimes_{\mathcal{O}_X} \kappa(x))$. In particular, one concludes that $\mathrm{Ker}(\gamma' \otimes_{\mathcal{O}_X} \kappa(x)) = \mathrm{Im}(\gamma' \otimes_{\mathcal{O}_X} \kappa(x))$ is generated by $e_C \otimes_{\mathcal{O}_X} \kappa(x)$.

Next, observe that since $x \notin \mathrm{Supp}(E_{\mathrm{gss}})$, the elements $e_H \otimes_{\mathcal{O}_X} \kappa(x)$ and $e_C \otimes_{\mathcal{O}_X} \kappa(x)$ are linearly independent, which thus implies that $(\gamma' \otimes_{\mathcal{O}_X} \kappa(x))(e_H \otimes_{\mathcal{O}_X} \kappa(x))$ is a nonzero element of the line generated by $e_C \otimes_{\mathcal{O}_X} \kappa(x)$, hence that $e_H \otimes_{\mathcal{O}_X} \kappa(x)$ and $(\gamma' \otimes_{\mathcal{O}_X} \kappa(x))(e_H \otimes_{\mathcal{O}_X} \kappa(x))$ — hence also [cf. the above observation on γ, γ'] $e_H \otimes_{\mathcal{O}_X} \kappa(x)$ and $(\gamma \otimes_{\mathcal{O}_X} \kappa(x))(e_H \otimes_{\mathcal{O}_X} \kappa(x))$ — are linearly independent. In particular, it follows immediately from the definition of E_{sH} [cf. also the second display of the statement of [3, Lemma A.14]] that $x \notin \mathrm{Supp}(E_{\mathrm{sH}})$, as desired. This completes the proof of assertion (iii).

Next, we verify assertion (iv). Let us first observe that since [we have assumed that] the scheme S coincides with the spectrum of the field k, the local ring $\mathcal{O}_{X,x}$ is a discrete valuation ring over k such that the structure morphism $k \to \mathcal{O}_{X,x}$ determines an isomorphism of k with the residue field of $\mathcal{O}_{X,x}$. Moreover, since the local section t is a local log-coordinate at the cusp x, the local section t is a uniformizer of $\mathcal{O}_{X,x}$, and the local derivation d/dt at x of $\mathcal{O}_X$ over $\mathcal{O}_S$ determines a derivation over k of $\mathcal{O}_{X,x}$ such that the equality $dt/dt = 1$ holds [cf. Definition 1.2, (iii)].

Next, let us observe that it follows from assertion (ii) — applied to the various points of $X \setminus D$ [cf. our assumption that the indigenous bundle P is nilpotent] — together with the [easily verified] fact that $X \setminus D$ is dense in X, that the element ϕ_P/t^2 of the field of fractions of $\mathcal{O}_{X,x}$ satisfies the interior nilpotency equation with respect to d/dt, hence [cf. Lemma 2.2, (ii)] that the local section ϕ_P satisfies the logarithmic nilpotency equation with respect to $t \cdot d/dt$. Next, let us observe that since [we have assumed that] the point x does not lie on E_{sH}, one concludes from the definition of E_{sH} that the inequality $\phi_P(x) \neq 0$ holds. Thus, it follows from Lemma 2.6, (iii) — in the case where we take the "$(O, \pi, \partial, \varphi)$" in the discussion following Definition 2.1 to be $(\mathcal{O}_{X,x}, t, d/dt, \phi_P)$ — that the equality $\phi_P(x) = -1$ holds, as desired. This completes the proof of assertion (iv), hence also of Lemma 2.9. □

PROPOSITION 2.10. — *The following two conditions are equivalent:*

(1) *The indigenous bundle P is nilpotent.*

(2) *Let x be a point of $X \setminus D$, and let t be a local coordinate at x. Write ϕ_P for the local section of $\mathcal{O}_X$ at x uniquely determined by the local equality $\varphi_P = \phi_P \cdot (dt)^{\otimes 2}$ at x. Then the local section ϕ_P satisfies the interior nilpotency equation with respect to t.*

PROOF. — Let us first observe that since [we have assumed that] $D \subseteq X$ is a relative effective Cartier divisor over S, the open subscheme $X \setminus D \subseteq X$ is schematically dense. Thus, the assertion follows immediately — in light of [9, Chapter I, Proposition 2.6] [cf. also the discussion entitled "The Definition of the Verschiebung" of [9, Chapter II, §2] concerning the fact that it suffices to work Zariski locally on X] — from Lemma 2.9, (ii). This completes the proof of Proposition 2.10. □

REMARK 2.10.1. — Suppose that S coincides with the spectrum of the field k. Let x be a point of $X \setminus D$, and let t be a local coordinate at x. Write ϕ_P for the local section of $\mathcal{O}_X$ at x uniquely determined by the local equality $\varphi_P = \phi_P \cdot (dt)^{\otimes 2}$ at x. Then one verifies easily from Proposition 2.10 — by considering the local rings of X at the various points of $X \setminus D$ — that the indigenous bundle P is nilpotent if and only if the local section ϕ_P satisfies the interior nilpotency equation with respect to t.

Remark 2.10.2. — Observe that the nilpotency of an indigenous bundle, i.e., the vanishing of the determinant of the p-curvature homomorphism, is, in general, a condition strictly stronger than the condition that the image of the p-curvature homomorphism consist of nilpotent endomorphisms. Also, observe that these two conditions are equivalent if the scheme S is reduced [cf. [9, Chapter II, Proposition 2.1]]. In particular, Proposition 2.10 is valid over an arbitrary S, i.e., without the assumption that S is reduced.

Lemma 2.11. — *Suppose that S coincides with the spectrum of the field k, and that the indigenous bundle P is nilpotent and active. Let x be a point of $X \setminus D$. Then the order at x of φ_P is not congruent to* 1 *modulo* 3.

Proof. — Let us first observe that since [we have assumed that] the scheme S coincides with the spectrum of the field k, the local ring $\mathcal{O}_{X,x}$ is a discrete valuation ring over k such that the structure morphism $k \to \mathcal{O}_{X,x}$ determines an isomorphism of k with the residue field of $\mathcal{O}_{X,x}$. Let t be a uniformizer of $\mathcal{O}_{X,x}$. Thus, the local section t is a local coordinate at x, and, moreover, the local derivation d/dt at x of $\mathcal{O}_X$ over $\mathcal{O}_S$ determines a derivation over k of $\mathcal{O}_{X,x}$ such that the equality $dt/dt = 1$ holds. In particular, one may apply Lemma 2.5 to $\mathcal{O}_{X,x}$, equipped with the uniformizer t and the derivation d/dt. Thus, the assertion is a formal consequence of Proposition 2.10 and Lemma 2.5, (ii) [cf. our assumption that the indigenous bundle P is active]. This completes the proof of Lemma 2.11. □

Lemma 2.12. — *Suppose that S coincides with the spectrum of the field k, and that the indigenous bundle P is nilpotent and active. Write $E_{\mathrm{sH}} \subseteq X$ for the double supersingular divisor of P. Then the following assertions hold:*

(i) *The intersection* $\mathrm{Supp}(E_{\mathrm{sH}}) \cap \mathrm{Supp}(D)$ *is empty.*

(ii) *The value of φ_P at each cusp* [*cf. Definition* 1.3, (iii)] *is equal to* -1.

Proof. — In light of [9, Chapter I, Proposition 2.6], assertion (i) is a formal consequence of Lemma 2.9, (iii), and [3, Proposition A.3, (i), (ii)]. Moreover, again in light of [9, Chapter I, Proposition 2.6], assertion (ii) is a formal consequence of assertion (i) and Lemma 2.9, (iv). This completes the proof of Lemma 2.12. □

Remark 2.12.1. — Observe that the proofs, hence also the conclusions, of Lemma 2.9, (iii), and Lemma 2.12, (i), are valid even in the case where the characteristic of the base field is an odd prime number that is not necessarily equal to three, i.e., over an arbitrary algebraically closed field of odd characteristic. We leave the routine details to the interested reader.

3. A Classification of Nilpotent Admissible Indigenous Bundles

In the present §3, we give a description of the image of the Schwarz torsor by the square Hasse invariant morphism [cf. Theorem 3.2, (ii), below] and characterize the square Hasse invariants of nilpotent active indigenous bundles by means of Cartier eigenforms [cf. Theorem 3.4 below]. Moreover, we prove the main theorem of the present paper [cf.

Theorem 3.6, Corollary 3.7 below]. In the present §3, let (g, r) be a pair of nonnegative integers such that the inequality $2 - 2g - r < 0$ holds.

DEFINITION 3.1.

(i) We shall write

$$\mathcal{M}$$

for the *moduli stack of hyperbolic curves* of type (g, r) in characteristic three.

(ii) We shall write

$$(\mathcal{C}, \mathcal{D})$$

for the *universal hyperbolic curve* of type (g, r) over $\mathcal{M}$ and

$$\pi \colon \mathcal{C} \longrightarrow \mathcal{M}$$

for the structure morphism of $\mathcal{C}$ over $\mathcal{M}$.

(iii) We shall write

$$\mathcal{S} \longrightarrow \mathcal{M}$$

for the *Schwarz torsor* over $\mathcal{M}$ [cf. [10, Introduction, §0.4]], i.e., the moduli stack of hyperbolic curves of type (g, r) in characteristic three equipped with indigenous bundles.

(iv) We shall write

$$\mathcal{G} \stackrel{\text{def}}{=} \pi_*\big(\omega_{\mathcal{C}/\mathcal{M}}^{\otimes 2}(\mathcal{D})\big)$$

for the [necessarily locally free coherent] $\mathcal{O}_{\mathcal{M}}$-module obtained by forming the direct image sheaf of $\omega_{\mathcal{C}/\mathcal{M}}^{\otimes 2}(\mathcal{D})$ by π.

(v) We shall write

$$\mathcal{V} \longrightarrow \mathcal{M}$$

for the vector bundle over $\mathcal{M}$ associated to the locally free coherent $\mathcal{O}_{\mathcal{M}}$-module obtained by forming the $\mathcal{O}_{\mathcal{M}}$-dual of the [necessarily locally free coherent] $\mathcal{O}_{\mathcal{M}}$-module $\pi_*(\omega_{\mathcal{C}/\mathcal{M}}(\mathcal{D})^{\otimes 2})$, i.e., the vector bundle over $\mathcal{M}$ such that, for every scheme S over $\mathcal{M}$, there exists a natural bijective map between the set of morphisms $S \to \mathcal{V}$ over $\mathcal{M}$ and the [underlying set of the] module $\Gamma(S, \pi_*(\omega_{\mathcal{C}/\mathcal{M}}(\mathcal{D})^{\otimes 2})|_S)$.

(vi) By considering the square Hasse invariants of the indigenous bundles parametrized by the Schwarz torsor, we obtain a morphism $\mathcal{S} \to \mathcal{V}$ of stacks over $\mathcal{M}$. We shall write

$$\text{sq-Hss} \colon \mathcal{S} \longrightarrow \mathcal{V}$$

for this morphism, to which we shall refer as the *square Hasse invariant morphism*.

REMARK 3.1.1.

(i) Let us recall from [10, Introduction, §0.4] [cf. also [9, Chapter I, Corollary 2.9]] that the Schwarz torsor $\mathcal{S}$ admits a natural structure of $\mathcal{G}$-torsor over $\mathcal{M}$.

(ii) The natural inclusion $\mathcal{O}_{\mathcal{C}} \hookrightarrow \mathcal{O}_{\mathcal{C}}(\mathcal{D})$ determines an injective homomorphism

$$\mathcal{G} = \pi_*\big(\omega_{\mathcal{C}/\mathcal{M}}^{\otimes 2}(\mathcal{D})\big) \hookrightarrow \pi_*(\omega_{\mathcal{C}/\mathcal{M}}(\mathcal{D})^{\otimes 2}).$$

In particular, one obtains an action of $\mathcal{G}$ on the vector bundle $\mathcal{V}$ over $\mathcal{M}$.

Theorem 3.2. — *The following assertions hold:*

(i) *The square Hasse invariant morphism* sq-Hss: $\mathcal{S} \to \mathcal{V}$ *is a closed immersion and compatible with the respective natural actions on* $\mathcal{S}$, $\mathcal{V}$ [*cf. Remark* 3.1.1, (i), (ii)].

(ii) *The image of the square Hasse invariant morphism* sq-Hss: $\mathcal{S} \to \mathcal{V}$ *coincides with the substack of* $\mathcal{V}$ *that consists of* φ *whose value at each cusp* [*cf. Definition* 1.3, (iii)] *is given by* -1.

(iii) *Let* S *be a noetherian scheme over* $\mathbb{F}_3$, *and let* (X, D) *be a hyperbolic curve of type* (g, r) *over* S. *Then, for every* $\varphi \in \Gamma(X, \omega_{X/S}(D)^{\otimes 2})$ *whose value at each cusp is* -1, *there exists a uniquely determined indigenous bundle* P *on* $(X, D)/S$ *such that the square Hasse invariant of* P *is given by* φ.

Proof. — Assertion (i) is none other than [6, Theorem A]. Next, we verify assertion (ii). Let us first observe that if (X, D) is a hyperbolic curve of type (g, r) over a noetherian scheme S over $\mathbb{F}_3$, then every section of the subsheaf $\omega_{X/S}^{\otimes 2}(D) \subseteq \omega_{X/S}(D)^{\otimes 2}$ has value zero at each cusp. Thus, it follows from assertion (i) that the image of sq-Hss is the substack of $\mathcal{V}$ obtained by fixing the values at the cusps. Moreover, these values coincide with the values of the square Hasse invariant of any single indigenous bundle. Next, observe that it is well-known [cf. [9, Chapter II, Theorem 2.3]] that the substack of $\mathcal{S}$ that parametrizes nilpotent indigenous bundles is finite flat of degree 3^{3g-3+r} and surjective onto $\mathcal{M}$. In particular, every hyperbolic curve of type (g, r) in characteristic three admits a nilpotent indigenous bundle. On the other hand, in the case where the inequality $r > 0$ holds, such a bundle is necessarily active [cf. Lemma 1.9], hence [cf. Lemma 2.12, (ii)] the values at the cusps of the associated square Hasse invariant are all -1 [note that since the moduli stack $\mathcal{M}$ is smooth — cf. [1, Theorem 5.2] — hence reduced, it suffices to verify this assertion at the geometric points of $\mathcal{M}$] [in the case where the equality $r = 0$ holds, the condition on the values at the cusps is vacuous]. This completes the proof of assertion (ii). Assertion (iii) is a formal consequence of assertions (i), (ii). This completes the proof of Theorem 3.2. □

Lemma 3.3. — *Let* S *be a noetherian scheme over* $\mathbb{F}_3$, (X, D) *a hyperbolic curve of type* (g, r) *over* S, $\mathcal{L}$ *an invertible sheaf on* X, Θ *a trivialization of the square of* $\mathcal{L}$, $\chi \in \Gamma(X, \mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}(D))$ *a global section of* $\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}(D)$ *whose zero locus is a relative effective Cartier divisor on* X *over* S, x *a point of* $X \setminus D$, t *a local coordinate at* x, *and* l *a local trivialization of* $\mathcal{L}$ *at* x. *Write* $\varphi \stackrel{\mathrm{def}}{=} \Theta(\chi \otimes \chi) \in \Gamma(X, \omega_{X/S}(D)^{\otimes 2})$ *for the image of* $\chi \otimes \chi$ *by* Θ, $\theta \stackrel{\mathrm{def}}{=} \Theta(l \otimes l)$ *for the local trivialization of* $\mathcal{O}_X$ *at* x *determined by* $l \otimes l$ *and* Θ, b *for the local section of* $\mathcal{O}_X$ *at* x *uniquely determined by the local equality* $\chi = b \cdot (l \otimes dt)$ *at* x, *and* ϕ *for the local section of* $\mathcal{O}_X$ *at* x *given by* $b^2 \cdot \theta$. [*In particular, the local equality* $\varphi = \phi \cdot (dt)^{\otimes 2}$ *at* x *holds.*] *Then the following assertions hold:*

(i) *The local equality*

$$\left(\frac{d\phi}{dt}\right)^2 + \phi \cdot \frac{d^2\phi}{dt^2} + \phi^3 = b^3 \cdot \theta^3 \cdot \left(b^3 - \frac{d^2}{dt^2}\left(\frac{b}{\theta}\right)\right)$$

at x *holds.*

(ii) *Write* χ^F *for the base-change of* χ *by the absolute Frobenius endomorphism of* S. *Then the following two conditions are equivalent:*

(1) *The local section ϕ satisfies the interior nilpotency equation with respect to t* [*cf. Definition* 2.8].

(2) *The local equality $C_{(\mathcal{L},\Theta)}(\chi) = -\chi^F$* [*cf. Definition* 1.4, (i)] *at x holds.*

Proof. — Assertion (i) may be verified by a straightforward computation. Assertion (ii) follows immediately from assertion (i), together with Lemma 1.5. This completes the proof of Lemma 3.3. □

Theorem 3.4. — *Let S be a noetherian scheme over $\mathbb{F}_3$, (X, D) a hyperbolic curve of type (g, r) over S, $\mathcal{L}$ an invertible sheaf on X, Θ a trivialization of the square of $\mathcal{L}$, and $\chi \in \Gamma(X, \mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}(D))$ a global section of $\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}(D)$ whose zero locus is a relative effective Cartier divisor on X over S. Write $\varphi \overset{\mathrm{def}}{=} \Theta(\chi \otimes \chi) \in \Gamma(X, \omega_{X/S}(D)^{\otimes 2})$ for the image of $\chi \otimes \chi$ by Θ. Then the following two conditions are equivalent:*

(1) *The global section $\chi \in \Gamma(X, \mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}(D))$ is a normalized Cartier eigenform associated to $(\mathcal{L}, \Theta)$* [*cf. Definition* 1.4, (ii)]. *Moreover, the intersection of D and the zero locus of χ is empty.*

(2) *The global section $\varphi \in \Gamma(X, \omega_{X/S}(D)^{\otimes 2})$ is the square Hasse invariant of a* [*necessarily uniquely determined — cf. Theorem* 3.2, (i)] *nilpotent active indigenous bundle on $(X, D)/S$.*

Proof. — Let x be a point of $X \setminus D$, and let t be a local coordinate at x. Write ϕ for the local section of $\mathcal{O}_X$ at x uniquely determined by the local equality $\varphi = \phi \cdot (dt)^{\otimes 2}$ at x.

First, we verify the implication (1) ⇒ (2). Suppose that condition (1) is satisfied. Then observe that it follows from Lemma 3.3, (ii), that ϕ satisfies the interior nilpotency equation with respect to t. Next, observe that since [we have assumed that] the intersection of D and the zero locus of χ is empty, the value of φ at each cusp is invertible. Thus, it follows from Lemma 2.2, (ii), and Lemma 2.3, (ii) — applied to the local ring of X at each cusp [as well as to the localization of this local ring obtained by inverting the local log-coordinate under consideration], equipped with a local log-coordinate at the cusp under consideration and the corresponding derivation — that the value of φ at each cusp is given by -1. In particular, it follows from Theorem 3.2, (iii), that there exists an indigenous bundle P on $(X, D)/S$ such that the square Hasse invariant of P is given by φ. Moreover, it follows from Proposition 2.10, together with our assumption that the zero locus of χ is a relative effective Cartier divisor on X over S [which implies that the restriction of φ to each geometric fiber of the structure morphism $X \to S$ is nonzero], that P is nilpotent and active. This completes the proof of the implication (1) ⇒ (2).

Next, we verify the implication (2) ⇒ (1). Suppose that condition (2) is satisfied. Then observe that it follows from Proposition 2.10 that ϕ satisfies the interior nilpotency equation with respect to t. Thus, it follows from Lemma 3.3, (ii) — applied to the various points of $X \setminus D$ — together with the [easily verified] fact that $X \setminus D$ is schematically dense in X, that the equality $C_{(\mathcal{L},\Theta)}(\chi) = -\chi^F$ holds. Thus, since [we have assumed that] the zero locus of χ is a relative effective Cartier divisor on X over S, one concludes that χ is a normalized Cartier eigenform associated to $(\mathcal{L}, \Theta)$. Next, observe that the zero divisor of φ coincides with twice the zero divisor of χ. Thus, it follows from Lemma 2.12, (i) — applied to the various geometric fibers of the structure morphism $X \to S$ [note that the assertion under consideration is an assertion concerning the supports of closed subschemes of X] — that the intersection of D and the zero locus of χ is empty, as desired. This completes the proof of the implication (2) $\Rightarrow$ (1), hence also of Theorem 3.4. $\square$

In the remainder of the present §3, let

- S be a noetherian scheme over $\mathbb{F}_3$, and let
- (X, D) be a hyperbolic curve of type (g, r) over S.

PROPOSITION 3.5. — *The isomorphism class of a nilpotent active indigenous bundle on $(X, D)/S$ is completely determined by the zero locus of the associated square Hasse invariant.*

PROOF. — Let P, P' be nilpotent active indigenous bundles on $(X, D)/S$. Write φ_P, $\varphi_{P'}$ for the square Hasse invariants of P, P', respectively. Suppose that the zero loci of φ_P, $\varphi_{P'}$ coincide. Then since φ_P, $\varphi_{P'}$ are global sections of the invertible sheaf $\omega_{X/S}(D)^{\otimes 2}$ whose zero loci coincide, there exists a global section $c \in \Gamma(X, \mathcal{O}_X^\times)$ such that the equality $\varphi_{P'} = c \cdot \varphi_P$ holds. Moreover, since X is proper and geometrically connected over S, hence satisfies the equality that the direct image sheaf of $\mathcal{O}_X$ coincides with $\mathcal{O}_S$, the global section c is the pull-back of a section of $\mathcal{O}_S^\times$. In particular, one concludes that

(a) the equality $dc/dt = 0$ holds for an arbitrary local coordinate t at an arbitrary point of X.

Next, let us observe that if x is a point of X, t is a local coordinate at x, and ϕ is an invertible local section of $\mathcal{O}_X$ at x that satisfies the interior nilpotency equation with respect to t, then the local equality

$$\phi = R(\phi) \stackrel{\text{def}}{=} -\frac{1}{\phi^2} \cdot \left(\left(\frac{d\phi}{dt} \right)^2 + \phi \cdot \frac{d^2\phi}{dt^2} \right)$$

at x holds. Observe, moreover, that since [one verifies easily that] both the numerator and the denominator of $R(c \cdot \phi)$ are obtained from those of $R(\phi)$ by multiplication by c^2 [cf. (a)],

(b) the equality $R(c \cdot \phi) = R(\phi)$ holds.

Thus, since the local expressions of φ_P, $\varphi_{P'}$ with respect to an arbitrary local coordinate t at an arbitrary point of $X \setminus D$ satisfy the interior nilpotency equation with respect to t [cf. Proposition 2.10], one concludes from (b) that, on the complement of D and of the zero locus of φ_P, the equalities

$$\varphi_{P'} = R(\varphi_{P'}) = R(c \cdot \varphi_P) = R(\varphi_P) = \varphi_P$$

hold. On the other hand, since [we have assumed that] P is active, the zero locus of φ_P is a relative effective Cartier divisor on X over S [cf. Definition 2.7]. Thus, the complement of D and of the zero locus of φ_P is schematically dense in X. In particular, one concludes that the equalities $c = 1$ and $\varphi_{P'} = \varphi_P$ hold on X, hence [cf. Theorem 3.2, (i)] that P is isomorphic to P', as desired. This completes the proof of Proposition 3.5. $\square$

THEOREM 3.6. — *The following assertions hold:*

(i) *Let P be a nilpotent admissible indigenous bundle on $(X,D)/S$. Write $\mathcal{L}_P$ for the Hasse defect of P* [*cf. Definition* 1.8] *and χ_P for the Hasse invariant of P. Then the pair $(\mathcal{L}_P,\chi_P)$ is of CE-type* [*cf. Definition* 1.7, (i)]. *Moreover, in this situation, the indigenous bundle P is nilpotent ordinary* [*cf.* [9, Chapter II, Definition 3.1]] *if and only if the pair $(\mathcal{L}_P,\chi_P)$ is of CEO-type* [*cf. Definition* 1.7, (ii)].

(ii) *Let E be a relative effective Cartier divisor on X over S. Then E is the supersingular divisor* [*cf.* [9, Chapter II, Proposition 2.6, (3)]] *of a nilpotent admissible* (*respectively, nilpotent ordinary*) *indigenous bundle on $(X,D)/S$ if and only if E is of CE-type* (*respectively, of CEO-type*) [*cf. Definition* 1.7, (iii)].

PROOF. — First, we verify assertion (i). Let us first recall that the square of $\mathcal{L}_P$ is trivial [cf. Definition 1.8]. Next, observe that it follows from [3, Proposition A.5], together with — for the assertion relative to S — an openness argument similar to the argument applied in the proof of [2, Proposition 4.2] that the zero locus of χ_P is finite étale over S. Moreover, since [one verifies easily that] a nilpotent admissible indigenous bundle is necessarily active, and the nilpotency and the admissibility of an indigenous bundle are preserved by base-change, it follows from Lemma 2.12 — applied to the various geometric fibers of the structure morphism $X \to S$ — that the zero locus of χ_P does not meet D.

Next, we verify that χ_P is a normalized Cartier eigenform. To this end, fix a trivialization Θ of the square of $\mathcal{L}_P$, and write φ_P for the square Hasse invariant of P. Then $\Theta(\chi_P \otimes \chi_P)$ and φ_P differ by a global unit [cf. Definition 1.8, [9, Chapter II, Proposition 2.6, (3)], also [2, Proposition 3.2]]. Thus, one verifies immediately that, to verify that χ_P is a normalized Cartier eigenform, we may assume without loss of generality, by replacing Θ by a suitable unit multiple of Θ, that the equality $\Theta(\chi_P \otimes \chi_P) = \varphi_P$ holds. In particular, since the zero locus of χ_P is a relative effective Cartier divisor on X over S [cf. the finite étaleness observed above], it follows from Theorem 3.4 that χ_P is a normalized Cartier eigenform associated to $(\mathcal{L}_P,\Theta)$. Finally, the asserted equivalence concerning the ordinariness is none other than [6, Theorem 2.4]. This completes the proof of assertion (i).

Next, we verify assertion (ii). The necessity follows from assertion (i). To verify the sufficiency, let $(\mathcal{L},\chi)$ be a pair of CE-type. Write E for the zero divisor of χ. Let Θ be a trivialization of the square of $\mathcal{L}$ such that χ is a normalized Cartier eigenform associated to $(\mathcal{L},\Theta)$. Write $\varphi \stackrel{\text{def}}{=} \Theta(\chi \otimes \chi)$. Then it follows from Theorem 3.4 that there exists a nilpotent active indigenous bundle P on $(X,D)/S$ whose square Hasse invariant is given by φ. Moreover, since [it is immediate that] the zero divisor of φ is $2E$, and the divisor E is finite étale over S [cf. Definition 1.7, (i)], it follows from [3, Proposition A.5], together with — for the assertion relative to S — an openness argument similar to the argument applied in the proof of [2, Proposition 4.2], that P is a nilpotent admissible indigenous bundle whose supersingular divisor is given by E. The ordinariness in the case of a pair of CEO-type follows from [6, Theorem 2.4]. This completes the proof of assertion (ii), hence also of Theorem 3.6. $\square$

COROLLARY 3.7. — *The assignment that maps a nilpotent admissible indigenous bundle to the associated supersingular divisor determines a bijective map between the set of isomorphism classes of nilpotent admissible* (*respectively, nilpotent ordinary*) *indigenous bundles on $(X,D)/S$ and the set of relative effective Cartier divisors on X over S that are of CE-type* (*respectively, of CEO-type*).

PROOF. — The surjectivity follows from Theorem 3.6, (ii). The injectivity follows from Proposition 3.5, together with the [easily verified] fact that a nilpotent admissible indigenous bundle is necessarily active. This completes the proof of Corollary 3.7. $\square$

COROLLARY 3.8. — *Suppose that S coincides with the spectrum of an algebraically closed field k, and that the inequality $r > 0$ holds. Then the following assertions hold:*

(i) *Let $\mathcal{L}$ be an invertible sheaf on X whose square is trivial. Then the number of isomorphism classes of nilpotent admissible indigenous bundles on $(X, D)/S$ whose Hasse defects are isomorphic to $\mathcal{L}$ is*

$$\leq \#\mathbb{P}^{g+r-2}(\mathbb{F}_3) = \frac{3^{g+r-1} - 1}{3 - 1}.$$

(ii) *The number of isomorphism classes of nilpotent admissible indigenous bundles on $(X, D)/S$ is*

$$\leq 2^{2g} \cdot \frac{3^{g+r-1} - 1}{3 - 1}.$$

PROOF. — First, we verify assertion (i). Fix a trivialization Θ of the square of $\mathcal{L}$. Let us first observe that it follows from Corollary 3.7, together with Theorem 3.6, (i), that the assignment that maps a nilpotent admissible indigenous bundle on $(X, D)/S$ whose Hasse defect is isomorphic to $\mathcal{L}$ to the subspace of $\Gamma(X, \mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}(D))$ [necessarily of dimension one] generated by the Hasse invariant of the indigenous bundle under consideration — where we regard this Hasse invariant as a global section of $\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}(D)$ by means of an isomorphism between the Hasse defect of the indigenous bundle under consideration and $\mathcal{L}$ — determines an injective map from the set of isomorphism classes under consideration to the set of subspaces of $\Gamma(X, \mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}(D))$ of dimension one that are preserved and not annihilated by the Cartier operator $C_{(\mathcal{L},\Theta)}$ [cf. Definition 1.4, (i)]. Next, let us observe that since [we have assumed that] the inequality $r > 0$ holds, the degree of the invertible sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}(D), \omega_{X/S})$ is negative, which thus implies — by means of the Riemann–Roch theorem, together with Serre duality — that the equality

$$\dim_k\bigl(\Gamma(X, \mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}(D))\bigr) = \deg(\mathcal{L} \otimes_{\mathcal{O}_X} \omega_{X/S}(D)) + 1 - g = g + r - 1$$

holds. Thus, assertion (i) follows immediately from elementary linear algebra [cf. also [11, Corollary, p. 143]]. This completes the proof of assertion (i).

Next, we verify assertion (ii). Let us first recall that the square of the Hasse defect of a nilpotent admissible indigenous bundle on $(X, D)/S$ is trivial [cf. Definition 1.8]. Next, let us observe that since the characteristic of k is three, the set of isomorphism classes of invertible sheaves on X whose squares are trivial may be identified with the set of elements of order dividing two of the Jacobian variety of X, hence is of cardinality 2^{2g}. Thus, assertion (ii) follows immediately from assertion (i). This completes the proof of assertion (ii), hence also of Corollary 3.8. $\square$

REMARK 3.8.1. — In the case where the equality $r = 0$ holds — i.e., the case where the divisor D is empty — an assertion similar to Corollary 3.8 may be found in [2, Corollary 5.3]. Moreover, it follows from [8, Theorem B] that, for an integer $g \geq 2$, the number of isomorphism classes of nilpotent ordinary [hence also nilpotent admissible —

cf. [9, Chapter II, Proposition 3.2]] indigenous bundles on a sufficiently general hyperbolic curve of type $(g, 0)$ coincides with the right-hand side of the inequality that appears in [2, Corollary 5.3, (ii)].

4. The Case of a Hyperbolic Curve of Genus Zero I

In the present §4, we discuss the case of a hyperbolic curve of genus zero. In particular, we characterize the supersingular divisors of nilpotent ordinary indigenous bundles on hyperbolic curves of genus zero [cf. Theorem 4.6 below]. Moreover, we prove [cf. Theorem 4.12, (i)] that every hyperbolic curve of type $(0, 5)$ over k is hyperbolically ordinary [cf. [9, Chapter II, Definition 3.3]]. In the present §4, let

- $r \geq 3$ be an integer,
- k an algebraically closed field of characteristic three, and
- $z_1, \ldots, z_r$ distinct elements of k.

Write

$$U \stackrel{\text{def}}{=} \operatorname{Spec}\left(k\left[\frac{1}{t - z_1}, \ldots, \frac{1}{t - z_r}\right]\right)$$

— where t is an indeterminate, X for the smooth compactification of U over k, and $D \subseteq X$ for the reduced closed subscheme of X whose support is given by the complement $X \setminus U$ of U in X. Thus, the pair

$$(X, D)$$

forms a hyperbolic curve of type $(0, r)$ over k.

Lemma 4.1. — *The following assertions hold:*

(i) *Let $(\mathcal{L}, \chi)$ be a pair of CE-type [cf. Definition 1.7, (i)]. Then the invertible sheaf $\mathcal{L}$ is trivial.*

(ii) *Let P be a [necessarily active — cf. Lemma 1.9] nilpotent indigenous bundle on $(X, D)/k$ and write E_{sH} for the double supersingular divisor of P [cf. Definition 2.7]. Then the following three conditions are equivalent:*

(1) *The indigenous bundle P is admissible.*

(2) *The indigenous bundle P is ordinary.*

(3) *The divisor E_{sH} is of the form $2E$ for some reduced effective Cartier divisor E on X.*

Proof. — Assertion (i) follows immediately from the fact that the Picard group of $\mathbb{P}^1_k$ has no element of order two [cf. our assumption that the square of $\mathcal{L}$ is trivial]. Next, we verify assertion (ii). The implication (1) $\Rightarrow$ (2) follows from [6, Corollary 2.5, (1)]. The implication (2) $\Rightarrow$ (1) follows from the [well-known — cf. [9, Chapter II, Proposition 3.2]] fact that a nilpotent ordinary indigenous bundle is necessarily admissible. The equivalence (1) $\Leftrightarrow$ (3) follows from the equivalence (1) $\Leftrightarrow$ (2) of [3, Proposition A.5] [cf. also Remark 2.7.1]. This completes the proof of assertion (ii), hence also of Lemma 4.1. $\square$

DEFINITION 4.2.

(i) We shall write

$$F_U \stackrel{\text{def}}{=} \prod_{j=1}^{r}(t - z_j) \in k[t].$$

Moreover, for each element i of $\{1, \ldots, r\}$, we shall write

$$F_{U,\neq i} \stackrel{\text{def}}{=} F_U/(t - z_i) \in k[t].$$

(ii) Let n be a positive integer. Then we shall write

$$\mathcal{E}_n \subseteq \{\pm 1\}^{\times n}$$

for the subset of $\{\pm 1\}^{\times n}$ that consists of $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{\pm 1\}^{\times n}$ such that the congruence $\sum_{i=1}^{n} \varepsilon_i \equiv 0 \pmod 3$ holds.

(iii) Let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_r)$ be an element of $\mathcal{E}_r$. Then we shall write

$$\omega_\varepsilon \stackrel{\text{def}}{=} \sum_{i=1}^{r} \varepsilon_i \cdot \mathrm{dlog}(t - z_i) \in \Gamma\big(X, \omega_{X/k}(D)\big)$$

— where we note that the differential form ω_ε has at most a simple pole at the point at infinity "∞" of X, whose residue [cf. Definition 1.3, (ii)] is given by the image in k of $-\sum_{i=1}^{r} \varepsilon_i$, hence is zero [cf. the definition of $\mathcal{E}_r$], which implies that the differential form ω_ε may be regarded as a global section of the invertible sheaf $\omega_{X/k}(D)$.

(iv) Let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_r)$ be an element of $\mathcal{E}_r$. Then we shall write

$$F_{U,\varepsilon} \stackrel{\text{def}}{=} \sum_{i=1}^{r} \varepsilon_i \cdot F_{U,\neq i} \in k[t].$$

LEMMA 4.3. — *Let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_r)$ be an element of $\mathcal{E}_r$. Then the following assertions hold:*

(i) *The polynomial $F_{U,\varepsilon}$ is of degree $\leq r - 2$.*

(ii) *The equality $\omega_\varepsilon = (F_{U,\varepsilon}/F_U) \cdot dt$ holds.*

(iii) *Let i be an element of $\{1, \ldots, r\}$. Then the residue at the cusp that corresponds to z_i of ω_ε is given by ε_i, hence nonzero.*

(iv) *The zero divisor of ω_ε — which is of degree* $\deg(\omega_{X/k}(D)) = r - 2$ *— coincides with the sum of the zero divisor of the polynomial $F_{U,\varepsilon}$ and the point at infinity "∞" of X with multiplicity* $r - 2 - \deg(F_{U,\varepsilon})$.

PROOF. — Assertion (i) follows immediately from the definition of $F_{U,\varepsilon}$, together with our assumption that the congruence $\sum_i \varepsilon_i \equiv 0 \pmod 3$ holds. Assertion (ii) is immediate from the definitions of F_U, $F_{U,\varepsilon}$, and ω_ε. Assertion (iii) follows immediately from assertion (ii), together with the [easily verified] fact that the equalities $F_{U,\neq j}(z_i) = 0$ hold for $j \in \{1, \ldots, r\} \setminus \{i\}$. Assertion (iv) follows immediately from assertions (ii), (iii), together with the various definitions involved. This completes the proof of Lemma 4.3. $\square$

Lemma 4.4. — *Let ε, ε' be elements of $\mathcal{E}_r$. Then the following two conditions are equivalent:*

(1) *The inclusion $\varepsilon \in \{\varepsilon', -\varepsilon'\}$ holds.*

(2) *The zero divisor of ω_ε coincides with the zero divisor of $\omega_{\varepsilon'}$.*

Proof. — The implication (1) $\Rightarrow$ (2) is immediate. Next, to verify the implication (2) $\Rightarrow$ (1), suppose that the zero divisor of ω_ε coincides with the zero divisor of $\omega_{\varepsilon'}$. Then it is immediate that the global sections ω_ε, $\omega_{\varepsilon'}$ of $\omega_{X/k}(D)$ differ by a $k^\times$-multiple. Thus, by comparing the residues at the cusps [cf. Lemma 4.3, (iii)], one concludes that the inclusion $\varepsilon \in \{\varepsilon', -\varepsilon'\}$ holds, as desired. This completes the proof of the implication (2) $\Rightarrow$ (1), hence also of Lemma 4.4. □

Lemma 4.5. — *Let χ be a global section of $\omega_{X/k}(D)$. Then the following assertions hold:*

(i) *Suppose that the global section χ is a $k^\times$-multiple of ω_ε for some element $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_r)$ of $\mathcal{E}_r$. Then the global section χ is a Cartier eigenform associated to $\mathcal{O}_X$ [cf. Definition* 1.4, (iii)] *whose zero locus does not meet D.*

(ii) *The following two conditions are equivalent:*

(1) *The pair $(\mathcal{O}_X, \chi)$ is of CE-type.*

(2) *The global section χ is a $k^\times$-multiple of ω_ε for some element $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_r)$ of $\mathcal{E}_r$, and, moreover, the zero divisor of χ [necessarily of degree $r-2$] is reduced.*

Proof. — Let us first observe that it is immediate that

(a) each of the conditions that appear in the statement of Lemma 4.5 is invariant with respect to the operation of replacing χ by a $k^\times$-multiple of χ.

Let us also observe that since k is a field,

(b) the zero locus of χ is finite étale over k if and only if the zero divisor of χ is reduced.

In the remainder of the present proof, let $a \in k$ be a square root of -1 [cf. our assumption that k is algebraically closed].

First, we verify assertion (i). First, observe that, in light of (a), to verify assertion (i), we may assume without loss of generality, by replacing χ by a suitable $k^\times$-multiple of χ, that the equality $\chi = a \cdot \omega_\varepsilon$ holds. Next, observe that since the residue of χ at each cusp is nonzero [cf. Lemma 4.3, (iii)], the global section χ is nonzero, and, moreover, the zero locus of χ — which is thus a relative effective Cartier divisor on X over k — does not meet D. Next, let us observe that since [one verifies easily that] the Cartier operator fixes $\mathrm{dlog}(t - z_i)$ for each $i \in \{1, \ldots, r\}$, and the equality $C(b^3 \cdot \omega) = b \cdot C(\omega)$ holds for each $b \in k$, one concludes that the equality $C(\omega_\varepsilon) = \omega_\varepsilon$, hence the equality $C(\chi) = -\chi$, holds. Thus, the global section χ is a normalized Cartier eigenform associated to the pair that consists of $\mathcal{O}_X$ and the identity automorphism of $\mathcal{O}_X$, hence a Cartier eigenform associated to $\mathcal{O}_X$, as desired. This completes the proof of assertion (i).

Next, we verify assertion (ii). To verify the implication (1) $\Rightarrow$ (2), suppose that the pair $(\mathcal{O}_X, \chi)$ is of CE-type. Then, in light of (a), to verify the implication (1) $\Rightarrow$ (2), we may assume without loss of generality, by replacing χ by a suitable $k^\times$-multiple of χ, that χ is a normalized Cartier eigenform associated to the pair that consists of $\mathcal{O}_X$ and the identity automorphism of $\mathcal{O}_X$, i.e., that the equality $C(\chi) = -\chi$ holds. Then it follows from Theorem 3.4 — together with our assumption that the zero locus of χ does not meet D — that the global section $\chi^{\otimes 2}$ of $\omega_{X/k}(D)^{\otimes 2}$ is the square Hasse invariant of a nilpotent active indigenous bundle on $(X, D)/k$, which thus [cf. Lemma 2.12, (ii)] implies that the value of $\chi^{\otimes 2}$ at each cusp [cf. Definition 1.3, (iii)] is equal to -1. In particular, there exists an element $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_r)$ of $\{\pm 1\}^{\times r}$ such that the equality $\mathrm{Res}_{z_i}(\chi) = a \cdot \varepsilon_i$ [cf. Definition 1.3, (ii)] holds for each $i \in \{1, \ldots, r\}$. Next, let us observe that since [one verifies easily that] the point at infinity "∞" of X does not lie on D, the sum of the residues of χ is zero, which thus implies that the congruence $\sum_i \varepsilon_i \equiv 0 \pmod 3$ holds. Thus, since the residue of ω_ε at each cusp is given by the corresponding ε_i [cf. Lemma 4.3, (iii)], and there exists no nonzero regular differential form on $\mathbb{P}^1_k$, one concludes that the equality $\chi = a \cdot \omega_\varepsilon$ holds. In particular, the global section χ is a $k^\times$-multiple of ω_ε as in the statement of Lemma 4.5. Finally, since [we have assumed that] the zero locus of χ is finite étale over k, it follows from (b) that the zero divisor of χ is reduced. This completes the proof of the implication (1) $\Rightarrow$ (2).

Finally, we verify the implication (2) $\Rightarrow$ (1). Suppose that condition (2) is satisfied. Then it follows from assertion (i) that the global section χ is a Cartier eigenform associated to $\mathcal{O}_X$ whose zero locus does not meet D. Thus, it follows from (b), together with the various definitions involved, that the pair $(\mathcal{O}_X, \chi)$ is of CE-type, as desired. This completes the proof of the implication (2) $\Rightarrow$ (1), hence also of Lemma 4.5. □

THEOREM 4.6. — *Let E be a divisor on X. Then the following three conditions are equivalent:*

(1) *The divisor E is the supersingular divisor of a nilpotent admissible indigenous bundle on $(X, D)/k$.*

(2) *The divisor E is the supersingular divisor of a nilpotent ordinary indigenous bundle on $(X, D)/k$.*

(3) *The divisor E coincides with the zero divisor of a global section χ of the invertible sheaf $\omega_{X/k}(D)$ that satisfies the following condition: The global section χ is a $k^\times$-multiple of ω_ε for some element $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_r)$ of $\mathcal{E}_r$, and, moreover, the zero divisor of χ [necessarily of degree $r - 2$] is reduced.*

PROOF. — The equivalence (1) $\Leftrightarrow$ (2) follows from the equivalence (1) $\Leftrightarrow$ (2) of Lemma 4.1, (ii). The equivalence (1) $\Leftrightarrow$ (3) follows from Corollary 3.7 and Lemma 4.5, (ii), together with Lemma 4.1, (i). This completes the proof of Theorem 4.6. □

LEMMA 4.7. — *Let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_r)$ be an element of $\mathcal{E}_r$. Then the zero divisor of ω_ε has no point of multiplicity two.*

PROOF. — Let us first observe that it follows from Lemma 4.3, (iii), that the zero divisor of ω_ε does not meet D. Next, let us observe that it follows from Theorem 3.4 and Lemma 4.5, (i), that the global section $\omega_\varepsilon^{\otimes 2}$ of $\omega_{X/k}(D)^{\otimes 2}$ is a $k^\times$-multiple of the square Hasse invariant of a nilpotent active indigenous bundle on $(X, D)/k$. Thus, if the zero divisor of ω_ε had a point x of multiplicity two, then we would obtain a contradiction to Lemma 2.11. This completes the proof of Lemma 4.7. □

DEFINITION 4.8. — Suppose that the equality $r = 5$ holds.

(i) We shall write $e_1, \ldots, e_5$ for the elements of k determined by the equality

$$F_U = t^5 + e_1 t^4 + e_2 t^3 + e_3 t^2 + e_4 t + e_5$$

and

$$q_6 \stackrel{\text{def}}{=} e_2 - e_1^2, \quad q_3 \stackrel{\text{def}}{=} e_1 e_4 + e_2 e_3 + e_5, \quad q_0 \stackrel{\text{def}}{=} e_3 e_5 - e_4^2 \in k.$$

(ii) We shall write

$$Q \stackrel{\text{def}}{=} -\left(\left(\frac{dF_U}{dt}\right)^2 + F_U \cdot \frac{d^2 F_U}{dt^2}\right) \in k[t], \quad \varphi_{\text{spk}} \stackrel{\text{def}}{=} \frac{Q}{F_U^2} \cdot (dt)^{\otimes 2}.$$

Lemma 4.9. — *In the situation of Definition* 4.8, *the following assertions hold:*

(i) *The equality* $Q = q_6 t^6 + q_3 t^3 + q_0$ *holds.*

(ii) *The congruence* $Q \equiv -(dF_U/dt)^2 \pmod{F_U}$ *holds. In particular, for each* $i \in \{1, \ldots, 5\}$, *the element* $Q(z_i)$ *of* k, *hence also the polynomial* Q, *is nonzero.*

(iii) *The element* φ_{spk} *is a global section of* $\omega_{X/k}(D)^{\otimes 2}$. *Moreover, the value of* φ_{spk} *at each cusp is equal to* -1. *Finally, the zero divisor of* φ_{spk} *is of the form* $3Z$ *for a* [*necessarily uniquely determined*] *effective divisor* Z *on* X *of degree two.*

(iv) *Write* a_6, a_3, a_0 *for the elements of* k *whose third powers are* q_6, q_3, q_0, *respectively, and* $G \stackrel{\text{def}}{=} a_6 t^2 + a_3 t + a_0$. *Then the following three conditions are equivalent:*

(1) *The divisor* Z *of* (iii) *is not reduced.*

(2) *Either the polynomial* G *is of degree two and has a multiple root, or the polynomial* G *is of degree zero.*

(3) *The element* $q_3^2 - q_6 q_0$ *of* k *is zero.*

(v) *The following two conditions are equivalent:*

(a) *There exist an invertible sheaf* $\mathcal{H}$ *on* X *and a global section* χ *of* $\mathcal{H}$ *such that an isomorphism* $\mathcal{H}^{\otimes 2} \xrightarrow{\sim} \omega_{X/k}(D)^{\otimes 2}$ *of* $\mathcal{O}_X$*-modules maps* $\chi^{\otimes 2}$ *to* φ_{spk} [*cf.* (iii)].

(b) *The divisor* Z *of* (iii) *is not reduced.*

(vi) *There exists a* [*necessarily uniquely determined — cf. Theorem* 3.2, (iii)] *nilpotent active indigenous bundle* P_{spk} *on* $(X, D)/k$ *whose square Hasse invariant coincides with* φ_{spk} [*cf.* (iii)]. *Moreover, the indigenous bundle* P_{spk} *is not admissible, hence* [*cf. Lemma* 4.1, (ii)] *not ordinary.*

Proof. — First, we verify assertion (i). Let us first observe that one verifies easily that the equality $Q = d^2(F_U^2)/dt^2$ holds, which thus implies that, for each nonnegative integer n, the coefficient of t^n in Q is the product of $(n+2)(n+1)$ and the coefficient of t^{n+2} in F_U^2. Thus, since [it is immediate that] the polynomial F_U^2 is of degree 10, and the integer $(n+2)(n+1)$ is divisible by 3 whenever the integer n is not divisible by 3, assertion (i) follows immediately from a straightforward computation of the coefficients of t^2, t^5, t^8 in F_U^2. This completes the proof of assertion (i). Assertion (ii) follows immediately from the definition of the polynomial Q, together with the fact that the elements $z_1, \ldots, z_5$ of k are distinct [which implies that the element $(dF_U/dt)(z_i)$ of k is nonzero for each $i \in \{1, \ldots, 5\}$].

Next, we verify assertions (iii), (iv). First, observe that it follows from assertion (i) that the equality $Q = G^3$ holds [cf. the statement of assertion (iv)]. Next, let us observe that since the polynomial Q is of degree ≤ 6 [cf. assertion (i)], and [it is immediate that] the zero divisor of the polynomial F_U is given by D, the element φ_{spk} is a global section of $\omega_{X/k}(D)^{\otimes 2}$. Moreover, recall from the various definitions involved that the order of φ_{spk} at the point at infinity "∞" of X is equal to $6 - \deg(Q)$. Thus, since [one verifies easily that] the restriction of the zero divisor of φ_{spk} to the complement of the point at infinity "∞" of X coincides with three times the zero divisor of the polynomial G, and the equality $6 - \deg(Q) = 3(2 - \deg(G))$ holds, one concludes that the order of the zero divisor of φ_{spk} at each point of X is divisible by 3. Thus, since the degree of the zero divisor of φ_{spk} is equal to $\deg(\omega_{X/k}(D)^{\otimes 2}) = 2r - 4 = 6$, the assertion concerning the zero divisor of φ_{spk} in assertion (iii) holds. Moreover, the divisor Z of assertion (iii) is obtained by adding, to the zero divisor of the polynomial G, the point at infinity "∞" of X with multiplicity $2 - \deg(G)$. In particular, the equivalence (1) $\Leftrightarrow$ (2) of assertion (iv) holds.

Next, we verify the assertion concerning the value of φ_{spk} at each cusp in assertion (iii). Let i be an element of $\{1, \ldots, 5\}$. Then since [one verifies easily that] the local section $dt/(t - z_i)$ of $\omega_{X/k}(D)$ at the cusp z_i is a local trivialization [cf. Definition 1.3, (ii)], and [one also verifies easily that] the equality

$$\varphi_{\mathrm{spk}} = \frac{Q}{F_{U,\neq i}^2} \cdot \left(\frac{dt}{t - z_i}\right)^{\otimes 2}$$

holds, the value of φ_{spk} at z_i coincides with $Q(z_i)/F_{U,\neq i}(z_i)^2$. Thus, since [it is immediate that] the equality $F_{U,\neq i}(z_i) = (dF_U/dt)(z_i)$ holds, it follows from assertion (ii) that the value of φ_{spk} at z_i is equal to -1, as desired. This completes the proof of assertion (iii).

Next, we verify the equivalence (2) $\Leftrightarrow$ (3) of assertion (iv). Let us first observe that since [one verifies immediately that] the equality $(a_3^2 - a_6 a_0)^3 = q_3^2 - q_6 q_0$ holds, the element $q_3^2 - q_6 q_0$ of k is zero if and only if the element $a_3^2 - a_6 a_0$ of k is zero. Thus, the equivalence (2) $\Leftrightarrow$ (3) of assertion (iv) follows from a straightforward computation. This completes the proof of assertion (iv).

Next, we verify assertion (v). First, we verify the implication (a) $\Rightarrow$ (b). Suppose that condition (a) is satisfied. Then it is immediate that the zero divisor of φ_{spk} coincides with twice the zero divisor of χ, which thus implies that the order of the divisor $3Z$ [cf. assertion (iii)] at each point of X is even, hence — since the divisor Z is of degree two — that the divisor Z is not reduced, as desired. This completes the proof of the implication (a) $\Rightarrow$ (b).

Next, we verify the implication (b) $\Rightarrow$ (a). Suppose that condition (b) is satisfied, i.e., that the equality $Z = 2x$ holds for some point x of X. Then one verifies easily — in light of the fact that the isomorphism class of an invertible sheaf on $\mathbb{P}^1_k$ is completely determined by the degree, together with our assumption that the field k is algebraically closed — that there exist an invertible sheaf $\mathcal{H}$ on X and a global section χ of $\mathcal{H}$ whose zero divisor is $3x$ such that an isomorphism $\mathcal{H}^{\otimes 2} \xrightarrow{\sim} \omega_{X/k}(D)^{\otimes 2}$ of $\mathcal{O}_X$-modules maps $\chi^{\otimes 2}$ to φ_{spk}, as desired. This completes the proof of the implication (b) $\Rightarrow$ (a), hence also of assertion (v).

Finally, we verify assertion (vi). Let us first observe that it follows from assertion (iii), together with Theorem 3.2, (iii), that there exists a uniquely determined indigenous bundle P_{spk} on $(X, D)/k$ whose square Hasse invariant coincides with φ_{spk}.

Next, let us verify that the indigenous bundle $P_{\rm spk}$ is nilpotent. Let us first observe that since the polynomial Q is contained in $k[t^3]$ [cf. assertion (i)], and [it is immediate that] the polynomial F_U^3 is contained in $k[t^3]$, the equalities $dQ/dt = 0$ and $d(F_U^3)/dt = 0$ hold. Thus, since [it is immediate that] the equality $Q/F_U^2 = QF_U/F_U^3$ holds, the equalities

$$\frac{d}{dt}\left(\frac{Q}{F_U^2}\right) = \frac{Q}{F_U^3}\cdot\frac{dF_U}{dt}, \quad \frac{d^2}{dt^2}\left(\frac{Q}{F_U^2}\right) = \frac{Q}{F_U^3}\cdot\frac{d^2F_U}{dt^2},$$

hence also the equalities

$$\left(\frac{d}{dt}\left(\frac{Q}{F_U^2}\right)\right)^2 + \frac{Q}{F_U^2}\cdot\frac{d^2}{dt^2}\left(\frac{Q}{F_U^2}\right) + \left(\frac{Q}{F_U^2}\right)^3 = \frac{Q^2}{F_U^6}\left(\left(\frac{dF_U}{dt}\right)^2 + F_U\cdot\frac{d^2F_U}{dt^2} + Q\right) = 0,$$

hold [cf. the definition of the polynomial Q]. In particular, for each point x of $\mathbb{A}^1_k \setminus D$, the local section Q/F_U^2 of $\mathcal{O}_X$ — i.e., the local section determined by the local equality $\varphi_{\rm spk} = (Q/F_U^2)\cdot(dt)^{\otimes 2}$ — at x satisfies the interior nilpotency equation with respect to the local coordinate t at x [cf. Definition 2.8]. Thus, it follows from Remark 2.10.1 that the indigenous bundle $P_{\rm spk}$ is nilpotent, as desired.

Next, let us observe that since the divisor D is nonempty, it follows from Lemma 1.9 that the indigenous bundle $P_{\rm spk}$ is active. Next, let us observe that it follows from assertion (iii) that the double supersingular divisor of $P_{\rm spk}$ [cf. Definition 2.7] coincides with $3Z$, hence that the order at each point of X of this divisor is divisible by 3. Thus, since [it is immediate that] the divisor Z is nonzero, and the order at each point of X of a divisor of the form $2E$, where E is a reduced effective Cartier divisor on X, is either zero or two, one concludes that the divisor $3Z$ is not of the form $2E$ for a reduced effective Cartier divisor E on X, hence that the indigenous bundle $P_{\rm spk}$ is not admissible, hence [cf. Lemma 4.1, (ii)] also not ordinary, as desired. This completes the proof of assertion (vi), hence also of Lemma 4.9. ☐

Lemma 4.10. — *In the situation of Definition 4.8, the following assertions hold:*

(i) *Let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_5)$ be an element of $\mathcal{E}_5$. Then the following four conditions are equivalent:*

(1) *The zero divisor of ω_ε — which is of degree three — is not reduced.*

(2) *The zero divisor of ω_ε is of the form $3x$ for some point x of X.*

(3) *The equality $-\omega_\varepsilon^{\otimes 2} = \varphi_{\rm spk}$ holds.*

(4) *The equality $-F_{U,\varepsilon}^2 = Q$ holds.*

(ii) *The set $\mathcal{E}_5$ is of cardinality ten.*

(iii) *There exist at most two elements ε of $\mathcal{E}_5$ such that the zero divisor of ω_ε is not reduced.*

(iv) *The following two conditions are equivalent:*

(1) *There exists an element ε of $\mathcal{E}_5$ such that the zero divisor of ω_ε is not reduced.*

(2) *The equality $q_3^2 = q_6q_0$ holds.*

(v) *The number of the elements ε of $\mathcal{E}_5$ whose associated ω_ε has reduced zero divisor is equal to eight (respectively, ten) if the equality $q_3^2 = q_6q_0$ holds (respectively, does not* *hold). In particular, there exists an element ε of $\mathcal{E}_5$ whose associated ω_ε has reduced zero divisor.*

Proof. — First, we verify assertion (i). To verify the equivalence (1) $\Leftrightarrow$ (2), let us first observe that it follows from Lemma 4.3, (iv), that the zero divisor of ω_ε is of degree $r-2=3$. Thus, since this divisor has no point of multiplicity two [cf. Lemma 4.7], this divisor fails to be reduced if and only if this divisor has a point of multiplicity three, i.e., if and only if this divisor is of the form $3x$ for some point x of X. This completes the proof of the equivalence (1) $\Leftrightarrow$ (2).

Next, we verify the equivalence (3) $\Leftrightarrow$ (4). Let us first observe that it follows from Lemma 4.3, (ii), that the equality $\omega_\varepsilon = (F_{U,\varepsilon}/F_U)\cdot dt$, hence the equality $-\omega_\varepsilon^{\otimes 2} = (-F_{U,\varepsilon}^2/F_U^2)\cdot(dt)^{\otimes 2}$, holds. Thus, the equivalence (3) $\Leftrightarrow$ (4) holds. This completes the proof of the equivalence (3) $\Leftrightarrow$ (4).

Next, we verify the implication (4) $\Rightarrow$ (2). Suppose that the equality $-F_{U,\varepsilon}^2 = Q$ holds. Then since the zero divisor of φ_{spk} coincides with twice the zero divisor of ω_ε, and this zero divisor is of the form $3Z$ [cf. Lemma 4.9, (iii)], the order at each point of X of the zero divisor of ω_ε is divisible by 3. Thus, since the zero divisor of ω_ε is of degree three, this divisor is of the form $3x$ for some point x of X. This completes the proof of the implication (4) $\Rightarrow$ (2).

Next, we verify the implication (2) $\Rightarrow$ (4). Suppose that the zero divisor of ω_ε is of the form $3x$ for some point x of X. Then it follows from Lemma 4.3, (iv), that if x is the point at infinity "∞" of X, then the polynomial $F_{U,\varepsilon}$ is a nonzero constant; if x is a point of $\mathbb{A}^1_k$, then the polynomial $F_{U,\varepsilon}$ is a $k^\times$-multiple of $(t-x)^3$. In particular, in either case, the polynomial $F_{U,\varepsilon}^2$ is contained in $k[t^3]$. Next, let us observe that since [it is immediate that] the equality $F_{U,\neq j}(z_i) = 0$ holds whenever $j \neq i$, the equality $F_{U,\varepsilon}(z_i) = \varepsilon_i \cdot F_{U,\neq i}(z_i) = \varepsilon_i \cdot (dF_U/dt)(z_i)$ holds for each $i \in \{1,\ldots,5\}$, which thus implies that the congruence $-F_{U,\varepsilon}^2 \equiv -(dF_U/dt)^2 \pmod{F_U}$ holds [cf. the equality $\varepsilon_i^2 = 1$, together with the fact that the polynomial F_U is of degree five and has five distinct roots]. Thus, it follows from Lemma 4.9, (ii), that the congruence $-F_{U,\varepsilon}^2 \equiv Q \pmod{F_U}$, hence the equality $-F_{U,\varepsilon}^2 - Q = (b+ct)\cdot F_U$ for some elements b, c of k [cf. the fact that the polynomials $F_{U,\varepsilon}^2$, Q are of degree ≤ 6, while the polynomial F_U is of degree five], holds. Next, let us observe that since the polynomials $F_{U,\varepsilon}^2$, Q are contained in $k[t^3]$ [cf. the above discussion, together with Lemma 4.9, (i)], the polynomial $(b+ct)\cdot F_U$ is contained in $k[t^3]$. Thus, since [it is immediate that] the polynomial F_U is of degree five, the equality $c=0$ implies the equality $b=0$. Assume that the inequality $c \neq 0$ holds. Then since the polynomial $(b+ct)\cdot F_U$ is of degree six and is contained in $k[t^3]$, this polynomial is the cube of a polynomial of degree two, which thus implies that the element $-b/c$ of k — i.e., the root of the polynomial $b+ct$ — is a root of $(b+ct)\cdot F_U$ of multiplicity ≥ 3, hence a root of F_U of multiplicity ≥ 2, in contradiction to the fact that the elements $z_1,\ldots,z_5$ of k are distinct. Thus, the equalities $b=c=0$, hence also the equality $-F_{U,\varepsilon}^2 = Q$, hold. This completes the proof of the implication (2) $\Rightarrow$ (4), hence also of assertion (i). Assertion (ii) is immediate.

Next, we verify assertion (iii). Let us first observe that if ε, ε' are elements of $\mathcal{E}_5$ whose associated ω_ε, $\omega_{\varepsilon'}$ both have non-reduced zero divisor, then it follows from the implication (1) $\Rightarrow$ (3) of assertion (i) that the equality $\omega_\varepsilon^{\otimes 2} = \omega_{\varepsilon'}^{\otimes 2}$ holds, hence that the zero divisor of ω_ε coincides with the zero divisor of $\omega_{\varepsilon'}$, which thus implies that the inclusion $\varepsilon \in \{\varepsilon', -\varepsilon'\}$ holds [cf. Lemma 4.4]. In particular, there exist at most two elements of $\mathcal{E}_5$ whose associated ω_ε has non-reduced zero divisor, as desired. This completes the proof of assertion (iii).

Next, we verify assertion (iv). Let $a \in k$ be a square root of -1 [cf. our assumption that the field k is algebraically closed]. To verify the implication (1) $\Rightarrow$ (2), suppose that there exists an element ε of $\mathcal{E}_5$ such that the zero divisor of ω_ε is not reduced. Then it follows from the implication (1) $\Rightarrow$ (3) of assertion (i) that the equality $(a \cdot \omega_\varepsilon)^{\otimes 2} = -\omega_\varepsilon^{\otimes 2} = \varphi_{\mathrm{spk}}$ holds, which thus implies that the equality $q_3^2 = q_6 q_0$ holds [cf. Lemma 4.9, (iv), (v)]. This completes the proof of the implication (1) $\Rightarrow$ (2).

Next, we verify the implication (2) $\Rightarrow$ (1). Suppose that the equality $q_3^2 = q_6 q_0$ holds. Then it follows from Lemma 4.9, (iv), (v), that there exists a pair $(\mathcal{H}, \chi)$ as in condition (a) of Lemma 4.9, (v). Moreover, since [it is immediate that] every invertible sheaf on $\mathbb{P}^1_k$ is completely determined by the degree, and the equality $2\deg(\mathcal{H}) = \deg(\omega_{X/k}(D)^{\otimes 2})$ holds, the invertible sheaf $\mathcal{H}$ is isomorphic to $\omega_{X/k}(D)$, so that we may regard χ as a global section of $\omega_{X/k}(D)$ such that the equality $-(a \cdot \chi)^{\otimes 2} = \varphi_{\mathrm{spk}}$ holds. Next, let us observe that since the value of φ_{spk} at each cusp is equal to -1 [cf. Lemma 4.9, (iii)], the square of the residue of $a \cdot \chi$ at the cusp z_i is equal to one for each $i \in \{1, \ldots, 5\}$. Write $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_5)$ for the element of $\{\pm 1\}^{\times 5}$ determined by these residues. Then since the sum of the residues of $a \cdot \chi$ is zero, the congruence $\sum_i \varepsilon_i \equiv 0 \pmod 3$ holds, i.e., the inclusion $\varepsilon \in \mathcal{E}_5$ holds. Moreover, since there exists no nonzero regular differential form on $\mathbb{P}^1_k$, the equality $a \cdot \chi = \omega_\varepsilon$ holds. Thus, it follows from the implication (3) $\Rightarrow$ (1) of assertion (i) that the zero divisor of the ω_ε thus obtained fails to be reduced, as desired. This completes the proof of the implication (2) $\Rightarrow$ (1), hence also of assertion (iv). Assertion (v) follows from assertions (ii), (iii), (iv), together with the observation that since two elements of $\mathcal{E}_5$ determine the same zero divisor if and only if they coincide up to the operation of replacing ε by $-\varepsilon$ [cf. Lemma 4.4], and [it is immediate that] the assignment $\varepsilon \mapsto -\varepsilon$ determines a fixed-point-free involution of $\mathcal{E}_5$, the number of elements ε of $\mathcal{E}_5$ whose associated ω_ε has non-reduced zero divisor is necessarily even. This completes the proof of Lemma 4.10. □

THEOREM 4.11. — *In the situation of Definition* 4.8, *the number of isomorphism classes of nilpotent ordinary indigenous bundles on* $(X, D)/k$ *is equal to four* (*respectively, five*) *if the equality* $q_3^2 = q_6 q_0$ *holds* (*respectively, does not hold*).

PROOF. — First, observe that it follows from Corollary 3.7 that the assignment that maps a nilpotent ordinary indigenous bundle on $(X, D)/k$ to the associated supersingular divisor determines an injective map from the set of isomorphism classes under consideration to the set of divisors on X. Moreover, it follows from the equivalence (2) $\Leftrightarrow$ (3) of Theorem 4.6 that the image of this injective map coincides with the set of zero divisors of the various ω_ε, where ε ranges over the elements of $\mathcal{E}_5$, that are reduced. Thus, since two such ε determine the same zero divisor if and only if they coincide up to the operation of replacing ε by $-\varepsilon$ [cf. Lemma 4.4], and [it is immediate that] the assignment $\varepsilon \mapsto -\varepsilon$ determines a fixed-point-free involution of $\mathcal{E}_5$, the number of isomorphism classes under consideration is equal to one half of the number of elements ε of $\mathcal{E}_5$ whose associated ω_ε has reduced zero divisor. Thus, Theorem 4.11 follows from Lemma 4.10, (v). □

THEOREM 4.12. — *The following assertions hold:*

(i) *Every hyperbolic curve of type* $(0, 5)$ *over* k *is hyperbolically ordinary.*

(ii) *Every hyperbolic curve of type* $(0, 5)$ *over* k *has a nilpotent indigenous bundle that is active but not ordinary.*

Proof. — Let us first observe that one concludes — by applying a suitable automorphism of $\mathbb{P}^1_k$ — that, to verify assertion (i) (respectively, (ii)), it suffices to verify that the hyperbolic curve (X, D) — which was introduced in the discussion at the beginning of the present §4 — in the case where $r = 5$ has a nilpotent ordinary indigenous bundle (respectively, nilpotent indigenous bundle that is active but not ordinary).

Now we verify assertion (i). It follows from Lemma 4.10, (v), that there exists an element ε of $\mathcal{E}_5$ such that the zero divisor of ω_ε is reduced. Thus, it follows from the implication (3) $\Rightarrow$ (2) of Theorem 4.6 that the zero divisor of this ω_ε is the supersingular divisor of a nilpotent ordinary indigenous bundle on $(X, D)/k$. This completes the proof of assertion (i). Assertion (ii) follows from Lemma 4.9, (vi). This completes the proof of Theorem 4.12. $\square$

Remark 4.12.1. — Let us recall the following basic question in p-adic Teichmüller theory discussed in [10, Introduction, §2.1] [cf. [10, Introduction, §2.1, (1)]]: Is every hyperbolic curve in odd characteristic hyperbolically ordinary? Some of the previous results concerning this problem are as follows:

- Every hyperbolic curve of type $(0, 3)$ in odd characteristic is hyperbolically ordinary [cf. [9, Chapter II, Theorem 2.3], [3, Proposition 4.2]].
- Every hyperbolic curve of type $(0, 4)$ in characteristic three, as well as every hyperbolic curve of type $(1, 1)$ in characteristic three or seven, is hyperbolically ordinary [cf. [3, Theorem C]]. Note that, in the case of the type $(0, 4)$ in characteristic three, concrete descriptions of the canonical liftings of level two of hyperbolic curves, as well as of the canonical Frobenius lifting of level two, were given in [5, Theorem A].
- Every hyperbolic curve of type $(1, 1)$ in characteristic five is hyperbolically ordinary [cf. [10, Chapter IV, §1.3], [3, Corollary 5.6]].
- Every hyperbolic curve of type $(2, 0)$ in characteristic three is hyperbolically ordinary [cf. [2, Theorem D]].
- Every hyperbolic curve of type $(2, 0)$, $(3, 0)$, $(4, 0)$, or $(5, 0)$ in characteristic three whose underlying projective smooth curve is hyperelliptic is hyperbolically ordinary [cf. [4, Theorem A]].
- Every hyperbolic curve of genus zero in characteristic three that has a nilpotent admissible indigenous bundle is hyperbolically ordinary [cf. [6, Corollary 2.5, (1)]].
- Every hyperbolic curve of type $(0, 4)$ in odd characteristic that has an ordinary Legendre parameter is hyperbolically ordinary [cf. [7, Theorem A]].

Observe that Theorem 4.12, (i), gives an affirmative answer to the above question in the case of hyperbolic curves of type $(0, 5)$ in characteristic three.

5. The Case of a Hyperbolic Curve of Genus Zero II

In the present §5, we continue our discussion of the case of a hyperbolic curve of genus zero. In particular, we give an explicit closed-form evaluation, in the case of characteristic three, of the combinatorial sum that appears in [10, Chapter V, Corollary 1.3, (1)] [cf. Lemma 5.2, (iii), below]. Thus, we obtain a closed formula for the number of isomorphism classes of nilpotent ordinary indigenous bundles on a sufficiently general hyperbolic curve of genus zero in characteristic three [cf. Theorem 5.3 below]. Moreover, we apply this formula to show that, in the case of such a curve, the zero divisor of every differential form of the sort discussed in the preceding §4 is indeed the supersingular divisor of a nilpotent ordinary indigenous bundle [cf. Corollary 5.5 below]. In the present §5, let

- $r \geq 3$ be an integer, and
- p be an odd prime number.

DEFINITION 5.1.

(i) We shall write $S_p(r)$ for the sum that appears in [10, Chapter V, Corollary 1.3, (1)], i.e., the sum

$$S_p(r) \stackrel{\text{def}}{=} \sum_{(a_1,\ldots,a_{r-3})} 2^{N(a_1,\ldots,a_{r-3})}$$

— where $(a_1, \ldots, a_{r-3})$ ranges over the tuples of even integers that satisfy the following two conditions (1), (2) and $N(a_1, \ldots, a_{r-3})$ for the number of nonzero entries of the tuple $(a_1, \ldots, a_{r-3})$:

(1) The inequalities $0 \leq a_i \leq p - 1$ hold for each $i \in \{1, \ldots, r-3\}$.

(2) The inequality $a_i + a_{i+1} \leq p - 1$ holds for each $i \in \{1, \ldots, r-4\}$.

[Here, we note that the inequalities $0 \leq a_i \leq p - 1$ are implicit in [10, Chapter V, Corollary 1.3, (1)]. That these inequalities are indeed to be imposed may be verified by observing that, with these inequalities, the equalities $S_p(4) = p$, $S_p(5) = (p^2 + 1)/2$, and $S_p(6) = p(p^2 + 2)/3$ of [10, Chapter V, Corollary 1.3, (3)] hold. Here, we note also that, in the case where the inequality $r \geq 5$ holds, condition (2), together with the inequalities $0 \leq a_i$, implies the inequalities $a_i \leq p - 1$, so that these upper bounds are essential only in the case where the equality $r = 4$ holds.]

(ii) We shall write

$$n^{\mathrm{ord}}_{0,r}$$

for the generic degree, over the moduli stack of hyperbolic curves of type $(0, r)$ in characteristic p, of the moduli stack of nilpotent ordinary indigenous bundles of type $(0, r)$ in characteristic p [cf. [10, Chapter V, Corollary 1.3]]. Thus, since the latter moduli stack is étale over the former moduli stack [cf. the discussion following [9, Chapter II, Proposition 3.4]], hence finite étale over a dense open substack of the former moduli stack [cf. [10, Chapter V, Corollary 1.2]], the integer $n^{\mathrm{ord}}_{0,r}$ coincides with the number of isomorphism classes of nilpotent ordinary indigenous bundles on a sufficiently general hyperbolic curve of type $(0, r)$ over an algebraically closed field of characteristic p.

REMARK 5.1.1. — The content of [10, Chapter V, Corollary 1.3, (1)] is that, for every odd prime number p and every integer $r \geq 3$, the equality

$$n^{\mathrm{ord}}_{0,r} = S_p(r)$$

holds. That is to say, the degree $n^{\mathrm{ord}}_{0,r}$ under consideration is given by a purely combinatorial sum, i.e., by $S_p(r)$. This formula is obtained by degenerating the hyperbolic curve under consideration to a totally degenerate curve; the tuples $(a_1, \ldots, a_{r-3})$ of Definition 5.1, (i), are then the tuples of radii [cf. [10, Chapter II, Proposition 1.5]] attached to the $r-3$ nodes of such a curve, and the inequalities $a_i + a_{i+1} \leq p-1$ are the conditions imposed at the $r-4$ irreducible components that meet two nodes.

Lemma 5.2. — *The following assertions hold:*

(i) *The equalities* $S_3(3) = 1$ *and* $S_3(4) = 3$ *hold.*

(ii) *If the inequality* $r \geq 5$ *holds, then the equality* $S_3(r) = S_3(r-1) + 2S_3(r-2)$ *holds.*

(iii) *The equality*

$$S_3(r) = \frac{2^{r-1} - (-1)^{r-1}}{3}$$

holds.

Proof. — Assertion (i) follows immediately from a straightforward computation. Next, we verify assertion (ii). Let $(a_1, \ldots, a_{r-3})$ be a tuple that satisfies conditions (1), (2) that appear in Definition 5.1, (i), in the case where $p = 3$. Then since [we have assumed that] each a_i is even, it is immediate that condition (1) implies that

- the integer a_i belongs to $\{0, 2\}$ for each $i \in \{1, \ldots, r-3\}$.

Moreover, it is immediate that condition (2) implies that

- the integers a_i, a_{i+1} are not both equal to 2 for each $i \in \{1, \ldots, r-4\}$.

[Here, the fact that each entry of such a tuple is subject to precisely two possibilities is a reflection of the fact that the set of radii of [10, Chapter II, Proposition 1.5] — i.e., the quotient of the set $\mathbb{F}_3$ by the action of ± 1 — consists of two elements.] Now let us consider the final entry a_{r-3} of such a tuple:

- If the equality $a_{r-3} = 0$ holds, then $(a_1, \ldots, a_{r-4})$ is an arbitrary such tuple of length $r-4$, and the integer "$2^{N(-)}$" is unchanged. Thus, the contribution of this case is $S_3(r-1)$.

- If the equality $a_{r-3} = 2$ holds, then the equality $a_{r-4} = 0$ is forced, and $(a_1, \ldots, a_{r-5})$ is an arbitrary such tuple of length $r-5$, while the integer "$2^{N(-)}$" acquires a factor 2. Thus, the contribution of this case is $2S_3(r-2)$.

In particular, the equality $S_3(r) = S_3(r-1) + 2S_3(r-2)$ holds, as desired. This completes the proof of assertion (ii). Assertion (iii) follows immediately from assertions (i), (ii). This completes the proof of Lemma 5.2. $\square$

Theorem 5.3. — *Suppose that the equality* $p = 3$ *holds. Then the number of isomorphism classes of nilpotent ordinary indigenous bundles on a sufficiently general hyperbolic curve of type* $(0, r)$ *in characteristic three is given by*

$$\frac{2^{r-1} - (-1)^{r-1}}{3}.$$

Proof. — This assertion is a formal consequence of Remark 5.1.1 and Lemma 5.2. $\square$

Lemma 5.4. — *Let n be a positive integer. Then the following assertions hold:*

(i) *The equality $\#\mathcal{E}_1 = 0$ [cf. Definition 4.2, (ii)] holds.*

(ii) *If the inequality $n \geq 2$ holds, then the equality $\#\mathcal{E}_n = 2^{n-1} - \#\mathcal{E}_{n-1}$ holds.*

(iii) *The equality*

$$\#\mathcal{E}_n = \frac{2^n + 2 \cdot (-1)^n}{3}$$

holds. If, moreover, the inequality $n \geq 3$ holds, then the equality $\#\mathcal{E}_n = 2S_3(n)$ holds.

Proof. — Assertion (i) follows immediately from a straightforward computation. Next, we verify assertion (ii). First, observe that, for each element $(\varepsilon_1, \ldots, \varepsilon_{n-1})$ of $\{\pm 1\}^{\times(n-1)}$, there exists an element ε_n of $\{\pm 1\}$ such that the congruence $\sum_{i=1}^{n} \varepsilon_i \equiv 0$ modulo 3 holds if and only if the congruence $\sum_{i=1}^{n-1} \varepsilon_i \not\equiv 0$ modulo 3 holds; moreover, if such an element ε_n exists, then it is uniquely determined. Thus, the assignment that maps $(\varepsilon_1, \ldots, \varepsilon_n)$ to $(\varepsilon_1, \ldots, \varepsilon_{n-1})$ determines a bijection of $\mathcal{E}_n$ onto the complement in $\{\pm 1\}^{\times(n-1)}$ of $\mathcal{E}_{n-1}$, which implies assertion (ii). This completes the proof of assertion (ii). Assertion (iii) follows immediately from assertions (i), (ii), and Lemma 5.2, (iii). This completes the proof of Lemma 5.4. $\square$

Corollary 5.5. — *Suppose that we are in the situation of the discussion at the beginning of the preceding §4. Suppose, moreover, that the hyperbolic curve (X, D) of type $(0, r)$ over k is sufficiently general. Then, for every $\varepsilon \in \mathcal{E}_r$, the zero divisor of the differential form ω_ε [cf. Definition 4.2, (iii)] is the supersingular divisor of a nilpotent ordinary indigenous bundle on $(X, D)/k$.*

Proof. — Since [one verifies easily that] the assignment $\varepsilon \mapsto -\varepsilon$ determines a fixed-point-free involution of $\mathcal{E}_r$, this assertion is a formal consequence of Corollary 3.7; Lemma 4.4; Theorem 4.6; Theorem 5.3; Lemma 5.4, (iii). $\square$

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