RIMS-2005

Existence of Nilpotent Ordinary Indigenous Bundles on Hyperbolic Curves of Genus Zero in Characteristic Three

By

Yuichiro HOSHI

September 2026

京都大学　数理解析研究所

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES

KYOTO UNIVERSITY, Kyoto, Japan

Existence of Nilpotent Ordinary Indigenous Bundles on Hyperbolic Curves of Genus Zero in Characteristic Three

Yuichiro Hoshi

September 2026

Abstract. — One of the basic questions in p-adic Teichmüller theory is as follows: Is every hyperbolic curve in odd characteristic hyperbolically ordinary? In the present paper, we prove that every hyperbolic curve of genus zero over an algebraically closed field of characteristic three is hyperbolically ordinary, i.e., has a nilpotent ordinary indigenous bundle. Unlike the previously known results in the case of genus zero in characteristic three, this result is independent of the type of the hyperbolic curve under consideration and imposes no condition whatsoever on the configuration of cusps. The proof proceeds by reducing the problem of the existence of a nilpotent ordinary indigenous bundle to the problem of whether or not an explicit family of polynomials associated to tuples of signs, i.e., the sectors, contains a separable element and then, by means of a phenomenon special to characteristic three, bounding the number of sectors that fail to be separable.

Contents

Introduction

In the present paper, we discuss one of the basic problems in the p-adic Teichmüller theory established by S. Mochizuki [cf. [7], [8]], i.e., the hyperbolic ordinariness of hyperbolic curves, in the case of genus zero in characteristic three.

In the remainder of the present Introduction, let k be an algebraically closed field of characteristic $p > 2$, and let (X, D) be a *hyperbolic curve* of type (g, r) over k — i.e., a pair that consists of a projective smooth geometrically connected curve X over k and a [possibly empty] reduced divisor D on X such that the inequality $2g-2+r > 0$ holds. We shall apply the notions of an *indigenous bundle* on $(X, D)/k$ [cf. [7, Chapter I, Definition 2.2]], of the *nilpotency* [cf. [7, Chapter II, Definition 2.4]], and of the *ordinariness* [cf. [7, Chapter II, Definition 3.1]] of an indigenous bundle. Moreover, we shall say that

2020 Mathematics Subject Classification. — 14G17.
Key words and phrases. — hyperbolic ordinariness, nilpotent ordinary indigenous bundle, hyperbolic curve, p-adic Teichmüller theory, genus zero, characteristic three.

the hyperbolic curve (X, D) is *hyperbolically ordinary* if there exists a nilpotent ordinary indigenous bundle on $(X, D)/k$ [cf. [7, Chapter II, Definition 3.3]].

Here, let us recall the following basic question in p-adic Teichmüller theory discussed in [8, Introduction, §2.1] [cf. [8, Introduction, §2.1, (1)]]:

Is every hyperbolic curve in odd characteristic hyperbolically ordinary?

For a general hyperbolic curve, this problem has an affirmative answer. Indeed, the locus of hyperbolically ordinary curves in the moduli stack of hyperbolic curves is open [cf. [7, Chapter II, Proposition 3.4]] and dense [cf. [7, Chapter II, Corollary 3.8]]. On the other hand, for a given hyperbolic curve, it is not known in general whether or not the hyperbolic curve is hyperbolically ordinary.

Some of the previous results concerning this problem are as follows:

- Every hyperbolic curve of type $(0, 3)$ in odd characteristic is hyperbolically ordinary [cf. [7, Chapter II, Corollary 3.8], [2, Proposition 4.2]].

- Every hyperbolic curve of type $(0, 4)$ in characteristic three, as well as every hyperbolic curve of type $(1, 1)$ in characteristic three or seven, is hyperbolically ordinary [cf. [2, Theorem C]].

- Every hyperbolic curve of type $(1, 1)$ in characteristic five is hyperbolically ordinary [cf. [8, Chapter IV, §1.3], [2, Corollary 5.6]].

- Every hyperbolic curve of type $(2, 0)$ in characteristic three is hyperbolically ordinary [cf. [1, Theorem D]].

- Every hyperbolic curve of type $(2, 0)$, $(3, 0)$, $(4, 0)$, or $(5, 0)$ in characteristic three whose underlying projective smooth curve is hyperelliptic is hyperbolically ordinary [cf. [3, Theorem A]].

- Every hyperbolic curve of type $(0, 4)$ in odd characteristic that has an ordinary Legendre parameter is hyperbolically ordinary [cf. [5, Theorem A]].

- Every hyperbolic curve of type $(0, 5)$ in characteristic three is hyperbolically ordinary [cf. [6, Theorem D]].

Thus, in the case of genus zero in characteristic three, the problem has been settled for the types $(0, 3)$, $(0, 4)$, and $(0, 5)$. Moreover, [4, Corollary 2.5, (1)] settles the problem for an arbitrary type, under the assumption of the existence of a nilpotent admissible [cf. [7, Chapter II, Definition 2.4]] indigenous bundle. On the other hand, it was precisely this existence that was at issue.

The main result of the present paper is as follows [cf. Theorem 4.1]:

Theorem A. — *Let k be an algebraically closed field of characteristic three. Then every hyperbolic curve of genus zero over k is hyperbolically ordinary* [*cf.* [7, Chapter II, Definition 3.3]]*, i.e., there exists a nilpotent ordinary indigenous bundle on the hyperbolic curve.*

Theorem A gives a new partial affirmative answer to the above problem. In particular, Theorem A settles, independently of the type, the case of genus zero in characteristic three. That is to say, Theorem A removes the assumption — in [4, Corollary 2.5, (1)] discussed above — of the existence of a nilpotent admissible indigenous bundle. Moreover, it follows formally from Theorem A that every hyperbolic curve of genus zero over an arbitrary noetherian scheme of characteristic three is hyperbolically ordinary [cf. Corollary 4.2].

We should like to emphasize that Theorem A imposes no condition whatsoever on the configuration of cusps. Indeed, [7, Chapter II, Proposition 3.4] and [7, Chapter II, Corollary 3.8] are assertions concerning a general curve. Moreover, [8, Chapter V, Corollary 1.3, (1)] — whose closed form in characteristic three is given in [6, Theorem 5.3] — gives the number of isomorphism classes of nilpotent ordinary indigenous bundles for a sufficiently general configuration of cusps, hence says nothing concerning a special configuration. By contrast, Theorem A asserts the nonemptiness for an arbitrary configuration.

The proof of Theorem A is explicit. Let us outline this proof. Let us first observe that we may assume without loss of generality that X is the projective line over k with coordinate t and that the support of D is $\{z_1, \ldots, z_{r-1}, \infty\}$ — where $z_1, \ldots, z_{r-1}$ are pairwise distinct elements of k, and we write ∞ for the point at infinity of X. Write $n \stackrel{\text{def}}{=} r-1$ and

$$F_0 \stackrel{\text{def}}{=} \prod_{i=1}^{n}(t-z_i), \quad F_{\neq i} \stackrel{\text{def}}{=} F_0/(t-z_i) \in k[t].$$

Then, for each element $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ of $\{1, -1\}^{\times n}$, we shall refer to the polynomial

$$\mathcal{S}_\varepsilon \stackrel{\text{def}}{=} \sum_{i=1}^{n} \varepsilon_i \cdot F_{\neq i} \in k[t]$$

as the *sector* associated to ε [cf. Definition 1.1, (i)]; moreover, we shall say that ε is *viable* if $\sum_{i=1}^{n} \varepsilon_i$ is nonzero in k [cf. Definition 1.1, (ii)].

Now let us observe that the differential form $(\mathcal{S}_\varepsilon/F_0) \cdot dt$ coincides with $\sum_{i=1}^{n} \varepsilon_i \cdot \mathrm{dlog}(t - z_i)$, hence is a global section of the sheaf of logarithmic differentials whose residue at z_i is ε_i, and whose residue at ∞ is $-\sum_{i=1}^{n} \varepsilon_i$ [cf. Lemma 1.2]. Put another way, the tuple ε is viable if and only if the residue at each cusp of this differential form belongs to $\{1, -1\}$.

Next, one obtains from [6, Theorem 4.6] the following criterion [cf. Lemma 1.4]:

> If (Y, E) is a hyperbolic curve of type $(0, r)$ over k, and χ is a global section of $\omega_{Y/k}(E)$ such that the residue of χ at each cusp belongs to $\{1, -1\}$, and, moreover, the zero divisor of χ is reduced, then the zero divisor of χ is the supersingular divisor of a nilpotent ordinary indigenous bundle on $(Y, E)/k$.

By applying this criterion to the differential form discussed above, one obtains the following consequence [cf. Corollary 1.5]:

> If there exists a viable tuple ε such that $\mathcal{S}_\varepsilon$ is separable, then (X, D) is hyperbolically ordinary.

Thus, the problem of the existence of a nilpotent ordinary indigenous bundle is reduced to the problem of whether or not an explicit family of polynomials in $k[t]$ contains a separable element. This latter problem is the topic of §2, §3. The case where $r \not\equiv 0 \pmod 3$ (respectively, $r \equiv 0 \pmod 3$) is discussed in §2 (respectively, §3).

The key to this latter problem is the following fact, which is special to characteristic three:

> If one writes
>
> $$\Delta \stackrel{\text{def}}{=} \frac{d}{dt}\left(F_0 \cdot \frac{dF_0}{dt}\right) \in k[t],$$

then the derivative $d\Delta/dt$ is zero, hence Δ is the cube of an element ξ of $k[t]$ [cf. Lemma 1.3, (ii)]. Moreover, the polynomial Δ is nonzero and of degree $\leq 2n-2$ [cf. Lemma 1.3, (iii), (iv)].

In §2, we consider the sector $\mathcal{S}_i \stackrel{\text{def}}{=} dF_0/dt + F_{\neq i}$ obtained by taking precisely one of the signs to be -1 [cf. Definition 2.3, (i)] — where we note that the tuple that corresponds to $\mathcal{S}_i$ is viable precisely when $r \not\equiv 0 \pmod 3$ [cf. Remark 2.3.1]. If $\mathcal{S}_i$ has a multiple root a, then the equality $\xi(a) = 0$ holds, and, moreover, the assignment $i \mapsto a$ is injective [cf. Proposition 2.4]. Thus, the number of elements $i \in \{1, \ldots, n\}$ such that $\mathcal{S}_i$ fails to be separable is bounded by $\deg(\xi) \leq (2n-2)/3$, which implies that $\mathcal{S}_i$ is separable for at least two elements i [cf. Lemma 2.5].

In §3, since the tuple that corresponds to $\mathcal{S}_i$ fails to be viable when $r \equiv 0 \pmod 3$, we consider instead the sector $\mathcal{S}_{ij} \stackrel{\text{def}}{=} dF_0/dt + F_{\neq i} + F_{\neq j}$ obtained by taking precisely two of the signs to be -1 [cf. Definition 2.3, (ii); Remark 2.3.1]. At a multiple root of $\mathcal{S}_{ij}$, if one writes $R_l \stackrel{\text{def}}{=} (t - z_l)^{-1}$, then the values of R_i, R_j are the roots of the quadratic polynomial $Q(y) = y^2 + T_1 \cdot y + \Delta/F_0^2$, where $T_1 \stackrel{\text{def}}{=} (dF_0/dt)/F_0$ [cf. Lemma 3.1]. In the field extension of $k(t)$ generated by a root v of Q, the equality $dv/dt = -v^2$ holds. Thus, $1/v - t$ belongs to the kernel of the derivation d/dt, hence is a cube [cf. Lemma 3.2]. This fact implies that the order at a multiple root of the polynomial

$$G_i \stackrel{\text{def}}{=} F_0^2 + F_0 \cdot \frac{dF_0}{dt} \cdot (t - z_i) + \Delta \cdot (t - z_i)^2$$

is divisible by three [cf. Lemma 3.3, (iv)]. Thus, together with the inequality $\deg(G_i) \leq 2n-1$ — where we apply the fact that the relation $n \equiv 2 \pmod 3$ holds, which follows from the relation $r \equiv 0 \pmod 3$ [cf. Lemma 3.3, (iii)] — one concludes that the number of unordered pairs $\{i, j\}$ of distinct elements of $\{1, \ldots, n\}$ such that $\mathcal{S}_{ij}$ fails to be separable is $\leq n(n-2)/3$, i.e., strictly less than the total number $n(n-1)/2$ of such pairs [cf. Lemma 3.4].

The present paper is organized as follows: In §1, we introduce the sectors associated to a hyperbolic curve of genus zero in characteristic three and give a sufficient condition, in terms of the sectors, for the existence of a nilpotent ordinary indigenous bundle [cf. Corollary 1.5]. In §2, we prove that, in the case where the number of cusps is not divisible by three, there exists a viable sector that is separable [cf. Lemma 2.5]. In §3, we prove the corresponding assertion in the case where the number of cusps is divisible by three [cf. Lemma 3.4]. In §4, we prove the main result of the present paper, i.e., the assertion that every hyperbolic curve of genus zero in characteristic three is hyperbolically ordinary [cf. Theorem 4.1].

Acknowledgments

This research was supported by JSPS KAKENHI Grant Number 24K06668 and by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University. This research was also supported by the CNRS France-Japan "Arithmetic and Homotopic Galois Theory" International Research Network.

1. Sectors and Nilpotent Ordinary Indigenous Bundles

In the present §1, we introduce the sectors associated to a hyperbolic curve of genus zero in characteristic three and give a sufficient condition, in terms of the sectors, for the existence of a nilpotent ordinary indigenous bundle on the hyperbolic curve under consideration [cf. Corollary 1.5 below]. In the present §1, let

- $r \geq 4$ be an integer,
- k an algebraically closed field of characteristic three, and
- $z_1, \ldots, z_{r-1}$ pairwise distinct elements of k.

Moreover, write

- $n \stackrel{\text{def}}{=} r - 1$,
- $m \stackrel{\text{def}}{=} r - 2 = n - 1$,
- $S \stackrel{\text{def}}{=} \mathrm{Spec}(k)$,
- $V \stackrel{\text{def}}{=} \mathrm{Spec}(k[t])$ — where t is an indeterminate — for the affine line over k with coordinate t,
- X ($\supseteq V$) for the [projective line over k obtained by forming the] smooth compactification of the smooth curve V over k,
- ∞ for the unique point of the complement of V in X,
- $D \subseteq X$ for the reduced closed subscheme of X whose support is $\{z_1, \ldots, z_n, \infty\}$,
- $\omega_{X/S}$ for the sheaf of differentials of X over S, and
- $\omega_{X/S}^{\log} \stackrel{\text{def}}{=} \omega_{X/S}(D)$ for the sheaf of logarithmic differentials of the pair (X, D) over S.

Thus, the pair

$$(X, D)$$

forms a hyperbolic curve of type $(0, r)$ over S such that the inclusion $\infty \in D$ holds. Moreover, one verifies easily that every hyperbolic curve of type $(0, r)$ over S is isomorphic to a hyperbolic curve of this form. Also, write

- $F_0 \stackrel{\text{def}}{=} \prod_{i=1}^{n}(t - z_i) \in k[t]$ [so F_0 is monic, separable, and of degree n],
- $R_i \stackrel{\text{def}}{=} (t - z_i)^{-1} \in k(t)$ for each $i \in \{1, \ldots, n\}$,
- $F_{\neq i} \stackrel{\text{def}}{=} F_0 \cdot R_i = F_0/(t - z_i) \in k[t]$ for each $i \in \{1, \ldots, n\}$,
- $\Delta \stackrel{\text{def}}{=} d(F_0 \cdot (dF_0/dt))/dt \in k[t]$,
- $T_1 \stackrel{\text{def}}{=} (dF_0/dt)/F_0 = \sum_{i=1}^{n} R_i \in k(t)$, and
- $T_2 \stackrel{\text{def}}{=} \sum_{i=1}^{n} R_i^2 \in k(t)$.

Definition 1.1.

(i) Let $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ be an element of $\{1, -1\}^{\times n}$. Then we shall write

$$\mathcal{S}_\varepsilon \stackrel{\text{def}}{=} \sum_{i=1}^{n} \varepsilon_i \cdot F_{\neq i} \in k[t]$$

and refer to $\mathcal{S}_\varepsilon$ as the *sector* associated to ε.

(ii) We shall say that the tuple $\varepsilon \in \{1, -1\}^{\times n}$ is *viable* if $\sum_{i=1}^{n} \varepsilon_i$ is nonzero in k, i.e., if $\sum_{i=1}^{n} \varepsilon_i \not\equiv 0 \pmod 3$.

Lemma 1.2. — *Let ε be an element of $\{1, -1\}^{\times n}$. Then the following assertions hold:*

(i) *The polynomials $\mathcal{S}_\varepsilon$, F_0 have no common root.*

(ii) *Suppose that ε is viable. Then $\mathcal{S}_\varepsilon$ is of degree m, and the leading coefficient of $\mathcal{S}_\varepsilon$ is $\sum_{i=1}^{n} \varepsilon_i$.*

(iii) *The differential form $(\mathcal{S}_\varepsilon/F_0) \cdot dt$ has at most simple poles, each of which lies in D.*

(iv) *The residue of the differential form $(\mathcal{S}_\varepsilon/F_0) \cdot dt$ at z_i [where i is an element of $\{1, \dots, n\}$] is ε_i.*

(v) *The residue of the differential form $(\mathcal{S}_\varepsilon/F_0) \cdot dt$ at ∞ is $-\sum_{i=1}^{n} \varepsilon_i$.*

(vi) *The differential form $(\mathcal{S}_\varepsilon/F_0) \cdot dt$ is a global section of $\omega_{X/S}^{\log}$.*

(vii) *The tuple ε is viable if and only if the residue at each cusp of (X, D) of the differential form $(\mathcal{S}_\varepsilon/F_0) \cdot dt$ belongs to $\{1, -1\}$.*

Proof. — Assertions (i), (ii) follow immediately from the [easily verified] observation that, for i, $j \in \{1, \dots, n\}$, the value $F_{\neq i}(z_j)$ is zero (respectively, coincides with $(dF_0/dt)(z_j)$) if $i \neq j$ (respectively, if $i = j$), together with the observation that each $F_{\neq i}$ is monic of degree $n - 1$. Next, we verify assertions (iii), (iv), (v), (vi). First, observe that it is immediate that the equality

$$\frac{\mathcal{S}_\varepsilon}{F_0} \cdot dt = \sum_{i=1}^{n} \varepsilon_i \cdot \mathrm{dlog}(t - z_i)$$

holds. Thus, assertions (iii), (iv), (v), (vi) follow immediately from the [easily verified] observation that, for each $i \in \{1, \dots, n\}$, the differential form $\mathrm{dlog}(t - z_i)$ has simple poles precisely at z_i and ∞, whose respective residues are 1 and -1, together with our assumption that the support of D is $\{z_1, \dots, z_n, \infty\}$. Finally, assertion (vii) follows from assertions (iv), (v), together with the fact that the prime field of k is $\{0, 1, -1\}$ [cf. our assumption that k is of characteristic three]. This completes the proof of Lemma 1.2. $\square$

Remark 1.2.1. — We remark that the logarithmic differential form $(\mathcal{S}_\varepsilon/F_0) \cdot dt$ that appears in Lemma 1.2 should be compared with the differential forms that appear in [6, Definition 4.2]; note, however, that in the present §1, unlike in [6], the point ∞ is a cusp of (X, D).

Lemma 1.3. — *The following assertions hold:*

(i) *The equality*

$$T_1^2 + T_2 = -\frac{\Delta}{F_0^2}$$

holds.

(ii) *The derivative $d\Delta/dt$ is zero. Moreover, there exists a [necessarily uniquely determined] polynomial $\xi \in k[t]$ such that the equality $\Delta = \xi^3$ holds.*

(iii) *The polynomial Δ is nonzero.*

(iv) *The inequality* $\deg(\Delta) \leq 2n - 2$ *holds. If, moreover,* $n \equiv 2 \pmod 3$*, then the inequality* $\deg(\Delta) \leq 2n - 4$ *holds.*

(v) *For each $i \in \{1, \ldots, n\}$, the equality $\Delta(z_i) = (dF_0/dt)(z_i)^2$ holds. In particular, $\Delta(z_i)$ is nonzero.*

Proof. — First, we verify assertion (i). Observe that it is immediate that the equalities $T_1 = (dF_0/dt)/F_0$ and $T_2 = -dT_1/dt$ hold. Thus, since [it is immediate that] the equality $\Delta = (dF_0/dt)^2 + F_0 \cdot (d^2F_0/dt^2)$ holds, one verifies easily that the equalities

$$T_1^2 + T_2 = T_1^2 - \frac{dT_1}{dt} = \frac{2\left(\frac{dF_0}{dt}\right)^2 - F_0 \cdot \frac{d^2F_0}{dt^2}}{F_0^2} = -\frac{\left(\frac{dF_0}{dt}\right)^2 + F_0 \cdot \frac{d^2F_0}{dt^2}}{F_0^2} = -\frac{\Delta}{F_0^2}$$

hold. This completes the proof of assertion (i).

Next, we verify assertion (ii). Since [it is immediate that] the third derivative of every element of $k[t]$ is zero, the equality $d\Delta/dt = 3(dF_0/dt) \cdot (d^2F_0/dt^2) + F_0 \cdot (d^3F_0/dt^3) = 0$ holds. Thus, Δ belongs to $k[t^3]$. In particular, since k is perfect, there exists a polynomial $\xi \in k[t]$ such that $\Delta = \xi^3$. Finally, the uniqueness of such a polynomial follows from the injectivity of the Frobenius endomorphism of $k[t]$. This completes the proof of assertion (ii).

Next, we verify assertion (iii). Observe that it follows from the [easily verified] equality $\Delta = (dF_0/dt)^2 + F_0 \cdot (d^2F_0/dt^2)$, together with the equality $F_0(z_1) = 0$, that the equality $\Delta(z_1) = (dF_0/dt)(z_1)^2$ holds. Thus, since F_0 is separable, the value $(dF_0/dt)(z_1)$, hence also the value $\Delta(z_1)$, is nonzero. In particular, the polynomial Δ is nonzero, as desired. This completes the proof of assertion (iii).

Next, we verify assertion (iv). The first inequality follows from the definition of Δ, together with the observation that $F_0 \cdot (dF_0/dt)$ is of degree $\leq 2n-1$. Next, observe that it follows from assertion (ii) that $\deg(\Delta)$ is divisible by 3. Thus, since [as one verifies easily, in the case where $n \equiv 2 \pmod 3$] neither $2n-2$ nor $2n-3$ is divisible by 3, the second inequality holds. This completes the proof of assertion (iv). Assertion (v) follows immediately from the [easily verified] equality $\Delta = (dF_0/dt)^2 + F_0 \cdot (d^2F_0/dt^2)$, together with the equality $F_0(z_i) = 0$. This completes the proof of Lemma 1.3. □

Lemma 1.4. — *Let (Y, E) be a hyperbolic curve of type $(0, r)$ over S, and let χ be a global section of $\omega_{Y/S}(E)$ such that the residue of χ at each cusp of (Y, E) belongs to $\{1, -1\}$ and such that the zero divisor of χ is reduced. Then the zero divisor of χ is the supersingular divisor* [*cf.* [7, Chapter II, Proposition 2.6, (3)]] *of a nilpotent* [*cf.* [7, Chapter II, Definition 2.4]] *ordinary* [*cf.* [7, Chapter II, Definition 3.1]] *indigenous bundle on $(Y, E)/S$.*

Proof. — One verifies immediately that, to verify Lemma 1.4, we may assume without loss of generality that Y coincides with X, and that there exist pairwise distinct elements $w_1, \ldots, w_r$ of k such that the support of E is $\{w_1, \ldots, w_r\}$, i.e., that we are in the situation of [6, §4], in the case where we take the "$(r, k, z_1, \ldots, z_r)$" of [6, §4] to be $(r, k, w_1, \ldots, w_r)$ — where we note that the "X", "D" of [6, §4] are then Y, E, respectively. Write $G_0 \stackrel{\text{def}}{=} \prod_{j=1}^{r}(t - w_j) \in k[t]$. Moreover, for each $i \in \{1, \ldots, r\}$, write $G_{\neq i} \stackrel{\text{def}}{=} G_0/(t - w_i) \in k[t]$ [cf. [6, Definition 4.2, (i)]] and $\varepsilon_i \in \{1, -1\}$ for the residue of χ at the cusp of (Y, E) that corresponds to w_i.

Next, observe that it follows from the fact that χ is a global section of $\omega_{Y/S}(E)$ that there exists a polynomial $\mathcal{S} \in k[t]$ of degree $\leq r - 2$ such that the equality $\chi = (\mathcal{S}/G_0) \cdot dt$ holds. Observe, moreover, that it is immediate that, for i, $j \in \{1, \ldots, r\}$, the value $G_{\neq i}(w_j)$ is zero (respectively, coincides with $(dG_0/dt)(w_j)$) if $i \neq j$ (respectively, if $i = j$). Thus, since the residue of χ at the cusp that corresponds to w_j is $\mathcal{S}(w_j)/(dG_0/dt)(w_j)$, the polynomials $\mathcal{S}$, $\sum_{i=1}^{r} \varepsilon_i \cdot G_{\neq i}$ — both of which are of degree $\leq r - 1$ — coincide at the r distinct elements $w_1, \ldots, w_r$ of k, hence coincide. In particular, since the inequality $\deg(\mathcal{S}) \leq r - 2$ holds [cf. our choice of $\mathcal{S}$], the leading coefficient $\sum_{i=1}^{r} \varepsilon_i$ of $\sum_{i=1}^{r} \varepsilon_i \cdot G_{\neq i}$ is zero in k. Thus, one concludes that the tuple $\varepsilon \stackrel{\text{def}}{=} (\varepsilon_1, \ldots, \varepsilon_r)$ is an element of the set "$\mathcal{E}_r$" of [6, Definition 4.2, (ii)]. Moreover, it follows from [6, Lemma 4.3, (ii)] that the differential form χ coincides with the differential form "ω_ε" of [6, Definition 4.2, (iii)]. Thus, it follows from the implication (3) $\Rightarrow$ (2) of [6, Theorem 4.6] that the zero divisor of χ is the supersingular divisor of a nilpotent ordinary indigenous bundle on $(Y, E)/S$, as desired. This completes the proof of Lemma 1.4. □

Corollary 1.5. — *Suppose that there exists a viable tuple $\varepsilon \in \{1, -1\}^{\times n}$ such that $\mathcal{S}_\varepsilon$ is separable. Then there exists a nilpotent ordinary indigenous bundle on $(X, D)/S$.*

Proof. — Write $\chi \stackrel{\text{def}}{=} (\mathcal{S}_\varepsilon/F_0) \cdot dt$. Then it follows from Lemma 1.2, (vi), (vii), that χ is a global section of $\omega_{X/S}^{\log}$ whose residue at each cusp of (X, D) belongs to $\{1, -1\}$. In particular, the differential form χ has a pole at each cusp of (X, D). Thus, since the polynomials $\mathcal{S}_\varepsilon$, F_0 have no common root [cf. Lemma 1.2, (i)], the zero divisor of χ coincides with the zero divisor of $\mathcal{S}_\varepsilon$ — which is of degree m [cf. Lemma 1.2, (ii)] — hence [cf. our assumption that $\mathcal{S}_\varepsilon$ is separable] is reduced. Thus, it follows from Lemma 1.4 that there exists a nilpotent ordinary indigenous bundle on $(X, D)/S$, as desired. This completes the proof of Corollary 1.5. □

2. The Case Where the Number of Cusps Is Not Divisible by Three

In the present §2, we prove that, in the case where r is not divisible by three, there exists a viable sector that is separable, hence also a nilpotent ordinary indigenous bundle on $(X, D)/S$ [cf. Lemma 2.5 below]. In the present §2, we maintain the notational conventions introduced at the beginning of §1.

Definition 2.1. — Let ε be an element of $\{1,-1\}^{\times n}$, and let a be an element of $k \setminus \{z_1,\ldots,z_n\}$. Then, for each $e \in \{1,2,3\}$, we shall write

$$\sigma_e \stackrel{\text{def}}{=} \sum_{i=1}^{n} \varepsilon_i \cdot R_i(a)^e \in k.$$

Lemma 2.2. — *Let ε be an element of $\{1,-1\}^{\times n}$, and let a be an element of $k \setminus \{z_1,\ldots,z_n\}$. Then the following assertions hold:*

(i) *The equalities*

$$\mathcal{S}_\varepsilon(a) = F_0(a)\cdot\sigma_1, \quad \frac{d\mathcal{S}_\varepsilon}{dt}(a) = \frac{dF_0}{dt}(a)\cdot\sigma_1 - F_0(a)\cdot\sigma_2,$$

$$\frac{d^2\mathcal{S}_\varepsilon}{dt^2}(a) = \frac{d^2F_0}{dt^2}(a)\cdot\sigma_1 - 2\frac{dF_0}{dt}(a)\cdot\sigma_2 + 2F_0(a)\cdot\sigma_3$$

hold.

(ii) *The equality $\sigma_3 = \sigma_1^3$ holds.*

(iii) *The following three conditions are equivalent:*

(1) *The element a is a multiple root of $\mathcal{S}_\varepsilon$.*

(2) *The equalities $\sigma_1 = \sigma_2 = 0$ hold.*

(3) *The element $(t-a)^3$ divides $\mathcal{S}_\varepsilon$.*

In particular, every multiple root of $\mathcal{S}_\varepsilon$ is of multiplicity ≥ 3.

(iv) *The equalities*

$$\sigma_1 = T_1(a) + \sum_{i;\,\varepsilon_i=-1} R_i(a), \quad \sigma_2 = T_2(a) + \sum_{i;\,\varepsilon_i=-1} R_i(a)^2$$

— *where the sums are taken over the elements i of $\{1,\ldots,n\}$ such that $\varepsilon_i = -1$ — hold.*

Proof. — First, we verify assertion (i). Observe that, for each $i \in \{1,\ldots,n\}$, since [one verifies easily that] the equalities $F_{\neq i} = F_0 \cdot R_i$ and $dR_i/dt = -R_i^2$ hold, the equalities

$$F_{\neq i}(a) = F_0(a)\cdot R_i(a), \quad \frac{dF_{\neq i}}{dt}(a) = \frac{dF_0}{dt}(a)\cdot R_i(a) - F_0(a)\cdot R_i(a)^2,$$

$$\frac{d^2F_{\neq i}}{dt^2}(a) = \frac{d^2F_0}{dt^2}(a)\cdot R_i(a) - 2\frac{dF_0}{dt}(a)\cdot R_i(a)^2 + 2F_0(a)\cdot R_i(a)^3$$

hold. Thus, assertion (i) follows from the definition of $\mathcal{S}_\varepsilon$. This completes the proof of assertion (i). Assertion (ii) follows immediately from the additivity of the Frobenius endomorphism of k, together with the equality $\varepsilon_i^3 = \varepsilon_i$.

Next, we verify assertion (iii). First, observe that since $F_0(a)$ is nonzero, the implication (1) $\Rightarrow$ (2) follows from the first two equalities of assertion (i). Next, observe that if condition (2) is satisfied, then it follows from assertions (i), (ii) that the values at a of $\mathcal{S}_\varepsilon$, $d\mathcal{S}_\varepsilon/dt$, and $d^2\mathcal{S}_\varepsilon/dt^2$ are zero, which thus implies that condition (3) is satisfied. The implication (3) $\Rightarrow$ (1) is immediate. This completes the proof of assertion (iii).

Finally, we verify assertion (iv). Let e be an element of $\{1, 2\}$. Then one verifies easily that the equalities

$$\sum_{i=1}^{n} \varepsilon_i \cdot R_i(a)^e = \sum_{i=1}^{n} R_i(a)^e - 2 \sum_{i;\, \varepsilon_i = -1} R_i(a)^e = \sum_{i=1}^{n} R_i(a)^e + \sum_{i;\, \varepsilon_i = -1} R_i(a)^e$$

hold. Thus, assertion (iv) follows from the definitions of T_1, T_2. This completes the proof of assertion (iv), hence also of Lemma 2.2. □

DEFINITION 2.3.

(i) For each $i \in \{1, \ldots, n\}$, we shall write

$$\mathcal{S}_i \stackrel{\text{def}}{=} \frac{dF_0}{dt} + F_{\neq i} \in k[t].$$

(ii) For each unordered pair $\{i, j\}$ of distinct elements of $\{1, \ldots, n\}$, we shall write

$$\mathcal{S}_{ij} \stackrel{\text{def}}{=} \frac{dF_0}{dt} + F_{\neq i} + F_{\neq j} \in k[t].$$

REMARK 2.3.1. — One verifies easily that the equality $\sum_{i=1}^{n} F_{\neq i} = dF_0/dt$ holds. In particular, since $-2 = 1$ in k, the polynomial $\mathcal{S}_i$ (respectively, $\mathcal{S}_{ij}$) coincides with the sector $\mathcal{S}_\varepsilon$ associated to the $\varepsilon \in \{1, -1\}^{\times n}$ whose i-th entry (respectively, whose i-th and j-th entries) is -1 (respectively, are -1) and whose other entries are 1. Moreover, for this ε, the sum $\sum_{l=1}^{n} \varepsilon_l = n - 2$ (respectively, $= n - 4$) is $\equiv r \pmod 3$ (respectively, $\equiv r + 1 \pmod 3$). Thus, this ε is viable if and only if $r \not\equiv 0 \pmod 3$ (respectively, $r \not\equiv 2 \pmod 3$).

PROPOSITION 2.4. — *Let i be an element of $\{1, \ldots, n\}$, and let $a \in k$ be a multiple root of $\mathcal{S}_i$. Then the following assertions hold:*

(i) *The equalities $R_i(a) = -T_1(a)$ and $R_i(a)^2 = -T_2(a)$ hold.*

(ii) *The equality $\xi(a) = 0$ holds.*

(iii) *The value $(dF_0/dt)(a)$ is nonzero, and, moreover, the equality*

$$z_i = a + \frac{F_0(a)}{\frac{dF_0}{dt}(a)}$$

holds.

(iv) *For each $b \in k$, there exists at most one element $i' \in \{1, \ldots, n\}$ such that b is a multiple root of $\mathcal{S}_{i'}$.*

PROOF. — First, we verify assertion (i). Observe that it follows from Lemma 1.2, (i), that $a \notin \{z_1, \ldots, z_n\}$. Next, observe that, for the ε that appears in Remark 2.3.1, i.e., such that $\mathcal{S}_i = \mathcal{S}_\varepsilon$, it follows from Lemma 2.2, (iv), that the equalities

$$\sigma_1 = T_1(a) + R_i(a), \quad \sigma_2 = T_2(a) + R_i(a)^2$$

hold. Thus, it follows from Lemma 2.2, (iii), that the equalities $R_i(a) = -T_1(a)$ and $R_i(a)^2 = -T_2(a)$ hold, as desired. This completes the proof of assertion (i).

Next, we verify assertion (ii). It follows from assertion (i) that the equality $T_1(a)^2 + T_2(a) = 0$ holds. In particular, it follows from Lemma 1.3, (i), that the equality $\Delta(a) = 0$, hence [cf. Lemma 1.3, (ii)] also the equality $\xi(a) = 0$, holds. This completes the proof of assertion (ii).

Next, we verify assertion (iii). First, observe that since [it is immediate that] $R_i(a)$ is nonzero, it follows from assertion (i) that $T_1(a)$ $(= (dF_0/dt)(a)/F_0(a))$ is nonzero, i.e., that $(dF_0/dt)(a)$ is nonzero. Moreover, again by assertion (i), the equalities $z_i = a - R_i(a)^{-1} = a + T_1(a)^{-1} = a + F_0(a)/(dF_0/dt)(a)$ hold, as desired. This completes the proof of assertion (iii). Assertion (iv) is a formal consequence of assertion (iii). This completes the proof of Proposition 2.4. $\square$

LEMMA 2.5. — *Suppose that* $r \not\equiv 0 \pmod 3$. *Then the number of elements* $i \in \{1, \ldots, n\}$ *such that* $\mathcal{S}_i$ *is separable is* $\geq n - \lfloor (2n-2)/3 \rfloor \geq 2$. *In particular* [*cf. Remark* 2.3.1*; Corollary* 1.5], *there exists a nilpotent ordinary indigenous bundle on* $(X, D)/S$.

PROOF. — For each $i \in \{1, \ldots, n\}$ such that $\mathcal{S}_i$ is not separable, let $a_i \in k$ be a multiple root of $\mathcal{S}_i$. Then it follows from Proposition 2.4, (ii), that each a_i is a root of the nonzero [cf. Lemma 1.3, (iii)] polynomial ξ [cf. Lemma 1.3, (ii)]. Moreover, it follows from Proposition 2.4, (iv), that the assignment $i \mapsto a_i$ is injective. Thus, since the polynomial ξ is of degree $\leq (2n-2)/3$ [cf. Lemma 1.3, (ii), (iv)], the number of elements $i \in \{1, \ldots, n\}$ such that $\mathcal{S}_i$ is not separable is $\leq \lfloor (2n-2)/3 \rfloor$. Moreover, one verifies easily that the inequalities $n - \lfloor (2n-2)/3 \rfloor \geq (n+2)/3 \geq 5/3$ [cf. our assumption that $r \geq 4$], hence also the inequality $n - \lfloor (2n-2)/3 \rfloor \geq 2$, hold. Finally, since $r \not\equiv 0 \pmod 3$, it follows from Remark 2.3.1 that, for each $i \in \{1, \ldots, n\}$ such that $\mathcal{S}_i$ is separable, the "ε" associated to $\mathcal{S}_i$ is viable. Thus, the final portion of Lemma 2.5 follows from Corollary 1.5. This completes the proof of Lemma 2.5. $\square$

3. The Case Where the Number of Cusps Is Divisible by Three

In the present §3, we prove the assertion corresponding to Lemma 2.5 in the case where r is divisible by three [cf. Lemma 3.4 below]. In the present §3, we maintain the notational conventions introduced at the beginning of §1. Moreover, write

- $Q(y) \stackrel{\text{def}}{=} y^2 + T_1 \cdot y + \Delta/F_0^2 \in k(t)[y]$ — where y is an indeterminate — and
- $G_i \stackrel{\text{def}}{=} F_0^2 + F_0 \cdot (dF_0/dt) \cdot (t - z_i) + \Delta \cdot (t - z_i)^2 \in k[t]$ for each $i \in \{1, \ldots, n\}$.

LEMMA 3.1. — *Let* $i, j \in \{1, \ldots, n\}$ *be distinct elements, and let* $a \in k$ *be a multiple root of* $\mathcal{S}_{ij}$. *Then the following assertions hold:*

(i) *The value* $F_0(a)$ *is nonzero* [*i.e.,* $a \notin \{z_1, \ldots, z_n\}$], *and, moreover, the equality*

$$\big(y - R_i(a)\big) \cdot \big(y - R_j(a)\big) = y^2 + T_1(a) \cdot y + \frac{\Delta(a)}{F_0(a)^2}$$

holds.

(ii) *The value* $\Delta(a)$ *is nonzero.*

(iii) *For each* $b \in k$, *there exists at most one unordered pair* $\{i', j'\}$ *of distinct elements of* $\{1, \ldots, n\}$ *such that* b *is a multiple root of* $\mathcal{S}_{i'j'}$.

Proof. — First, we verify assertion (i). Observe that it follows from Lemma 1.2, (i), that $a \notin \{z_1, \ldots, z_n\}$. Next, observe that, for the ε that appears in Remark 2.3.1, i.e., such that $\mathcal{S}_{ij} = \mathcal{S}_\varepsilon$, it follows from Lemma 2.2, (iv), that the equalities

$$\sigma_1 = T_1(a) + R_i(a) + R_j(a), \quad \sigma_2 = T_2(a) + R_i(a)^2 + R_j(a)^2$$

hold. Thus, it follows from Lemma 2.2, (iii), that the equalities

$$R_i(a) + R_j(a) = -T_1(a), \quad R_i(a)^2 + R_j(a)^2 = -T_2(a)$$

hold, which thus [cf. Lemma 1.3, (i)] imply that the equalities

$$R_i(a) \cdot R_j(a) = -\big(T_1(a)^2 + T_2(a)\big) = \frac{\Delta(a)}{F_0(a)^2}$$

hold. Assertion (i) follows immediately from these equalities. This completes the proof of assertion (i).

Assertion (ii) follows from assertion (i), together with the observation that $R_i(a)$ and $R_j(a)$ are nonzero. Finally, we verify assertion (iii). Observe that it follows from assertion (i) that the set $\{R_{i'}(b), R_{j'}(b)\}$ of the pair under consideration is determined by b. Moreover, since [it is immediate that], for each $l \in \{1, \ldots, n\}$, the equality $z_l = b - R_l(b)^{-1}$ holds, the set $\{z_{i'}, z_{j'}\}$, hence also the unordered pair $\{i', j'\}$, is determined by b. This completes the proof of assertion (iii), hence also of Lemma 3.1. $\square$

Lemma 3.2. — *Let* v *be a root of* Q *in an algebraic closure of* $k(t)$. *Then the following assertions hold:*

(i) *The field extension* $k(t)(v)/k(t)$ *is separable of degree* ≤ 2. *In particular, the derivation* d/dt *of* $k(t)$ *extends uniquely to a derivation of* $k(t)(v)$, *which we shall denote again by* d/dt.

(ii) *The root* v *is nonzero, and, moreover, the equality* $dv/dt = -v^2$ *holds.*

(iii) *The subfield* $k(t)(v)^3 \subseteq k(t)(v)$ *that consists of the cubes of the elements of* $k(t)(v)$ *is of index three and, moreover, coincides with the kernel of* d/dt.

(iv) *There exists an element* $Y \in k(t)(v)$ *such that the equality* $1/v - t = Y^3$ *holds.*

(v) *For each* $c \in k$, *the element* $t - 1/v - c \in k(t)(v)$ *is the cube of an element of* $k(t)(v)$.

Proof. — First, we verify assertion (i). If v is contained in $k(t)$, then assertion (i) is immediate. If v is not contained in $k(t)$, then the extension $k(t)(v)/k(t)$ is of degree two, hence separable. This completes the proof of assertion (i).

Next, we verify assertion (ii). Since the value $Q(0) = \Delta/F_0^2$ is nonzero [cf. Lemma 1.3, (iii)], the root v is nonzero. Next, observe that since the equalities $dT_1/dt = -T_2$ and $d(\Delta/F_0^2)/dt = (\Delta/F_0^2) \cdot T_1$ hold [cf. Lemma 1.3, (ii)], one concludes, by differentiating the equality $v^2 + T_1 \cdot v + \Delta/F_0^2 = 0$, the equalities

$$(2v + T_1) \cdot \frac{dv}{dt} = -\frac{dT_1}{dt} \cdot v - \frac{d}{dt}\left(\frac{\Delta}{F_0^2}\right) = T_2 \cdot v - \frac{\Delta}{F_0^2} \cdot T_1.$$

Now suppose that $2v + T_1$ is nonzero. Then, to verify the equality $dv/dt = -v^2$, it suffices to verify the equality $(2v+T_1)\cdot(-v^2) = T_2 \cdot v - (\Delta/F_0^2)\cdot T_1$. However, this follows from a straightforward computation by means of the equalities $v^2 = -T_1 \cdot v - \Delta/F_0^2$ and $T_1^2 + T_2 = -\Delta/F_0^2$ [cf. Lemma 1.3, (i)].

Next, suppose that $2v + T_1$ is zero, i.e., that $v = T_1$. Then, substituting $v = T_1$ into the equality $v^2 + T_1 \cdot v + \Delta/F_0^2 = 0$, one obtains the equality $2T_1^2 = -\Delta/F_0^2$, hence [cf. Lemma 1.3, (i)] also the equality $T_2 = T_1^2$. Thus, the equality $dv/dt = dT_1/dt = -T_2 = -T_1^2 = -v^2$ holds. This completes the proof of assertion (ii).

Next, we verify assertion (iii). Observe that since k is perfect, the equality $k(t)^3 = k(t^3)$ holds. Thus, since 1, t, t^2 form a basis of the finite field extension $k(t)/k(t)^3$, the degree $[k(t) : k(t)^3]$ is three. On the other hand, the Frobenius endomorphism determines an isomorphism of $k(t)(v)$ onto $k(t)(v)^3$ that maps $k(t)$ onto $k(t)^3$, which implies that the equality $[k(t)(v)^3 : k(t)^3] = [k(t)(v) : k(t)]$ holds. Thus, by dividing $[k(t)(v) : k(t)^3] = [k(t)(v) : k(t)]\cdot[k(t) : k(t)^3]$ by $[k(t)(v)^3 : k(t)^3]$, one concludes that the equality $[k(t)(v) : k(t)(v)^3] = 3$ holds, as desired. Next, observe that since the derivative of the cube of an element of $k(t)(v)$ is zero [cf. our assumption that the characteristic of k is three], the subfield $k(t)(v)^3$ is contained in the kernel of d/dt. Moreover, since the kernel of d/dt does not coincide with $k(t)(v)$ [cf. the equality $dt/dt = 1$], we conclude that the kernel of d/dt coincides with $k(t)(v)^3$, as desired. This completes the proof of assertion (iii).

Next, we verify assertion (iv). First, observe that it follows from assertion (ii) that the equalities

$$\frac{d}{dt}\left(\frac{1}{v} - t\right) = -\frac{1}{v^2}\cdot\frac{dv}{dt} - 1 = \frac{v^2}{v^2} - 1 = 0$$

hold, i.e., that $1/v - t$ belongs to the kernel of the derivation d/dt of $k(t)(v)$. Thus, it follows from assertion (iii) that there exists an element $Y \in k(t)(v)$ such that the equality $1/v - t = Y^3$ holds, as desired. This completes the proof of assertion (iv).

Finally, we verify assertion (v). Since the Frobenius endomorphism is additive, it follows from assertion (iv) that there exists an element $Y \in k(t)(v)$ such that the equality

$$t - 1/v - c = -(Y + c^{1/3})^3$$

— where we write $c^{1/3} \in k$ for the uniquely determined cube root of c — holds. This completes the proof of assertion (v), hence also of Lemma 3.2. ☐

Lemma 3.3. — *Let i be an element of $\{1, \ldots, n\}$. Then the following assertions hold:*

(i) *For each $l \in \{1, \ldots, n\}$ such that $l \neq i$, the equality*

$$G_i(z_l) = \frac{dF_0}{dt}(z_l)^2 \cdot (z_l - z_i)^2$$

holds. In particular, G_i is nonzero.

(ii) *The inequality $\mathrm{ord}_{z_i}(G_i) \geq 3$ holds.*

(iii) *Suppose that $n \equiv 2 \pmod 3$. Then the inequality $\deg(G_i) \leq 2n - 1$ holds.*

(iv) *Let j be an element of $\{1, \ldots, n\} \setminus \{i\}$, and let $a \in k$ be a multiple root of $\mathcal{S}_{ij}$. Then the order $\mathrm{ord}_a(G_i)$ is divisible by three, and the inequality $\mathrm{ord}_a(G_i) \geq 3$ holds.*

PROOF. — First, we verify assertion (i). Since [it is immediate that] $F_0(z_l) = 0$, the equality $G_i(z_l) = \Delta(z_l) \cdot (z_l - z_i)^2$ holds. Thus, assertion (i) follows from Lemma 1.3, (v). This completes the proof of assertion (i).

Next, we verify assertion (ii). Write $s \stackrel{\text{def}}{=} t - z_i$ and $c_1 \stackrel{\text{def}}{=} (dF_0/dt)(z_i)$ [which is nonzero]. Then since $F_0 \equiv c_1 \cdot s \pmod{s^2}$ and $dF_0/dt \equiv c_1 \pmod{s}$, one verifies immediately from Lemma 1.3, (v), that each of the three terms of the definition of G_i is $\equiv c_1^2 \cdot s^2 \pmod{s^3}$. Thus, the congruence $G_i \equiv 3c_1^2 \cdot s^2 \equiv 0 \pmod{s^3}$ holds. This completes the proof of assertion (ii).

Next, we verify assertion (iii). First, observe that it follows from Lemma 1.3, (iv), that the coefficient of t^{2n} in F_0^2 (respectively, $F_0 \cdot (dF_0/dt) \cdot (t - z_i)$; $\Delta \cdot (t - z_i)^2$) is 1 (respectively, n; 0). Thus, since [we have assumed that] $n \equiv 2 \pmod 3$, the coefficient of t^{2n} in G_i is $1 + n \equiv 0 \pmod 3$, i.e., zero. This completes the proof of assertion (iii).

Finally, we verify assertion (iv). First, observe that it follows from Lemma 3.1, (i), that $a \notin \{z_1, \ldots, z_n\}$, and that the two roots of the polynomial $y^2 + T_1(a) \cdot y + \Delta(a)/F_0(a)^2$ are the [necessarily distinct — cf. the immediate injectivity of the assignment "$l \mapsto R_l(a)$"] elements $R_i(a)$, $R_j(a)$. Next, observe that since [it is immediate that] the equality $F_0 \cdot (dF_0/dt) = T_1 \cdot F_0^2$ holds, one verifies easily that the equality

$$G_i = (t - z_i)^2 \cdot F_0^2 \cdot \left(R_i^2 + T_1 \cdot R_i + \frac{\Delta}{F_0^2} \right)$$

holds.

Next, let v be a root of Q in an algebraic closure of $k(t)$. Write $\overline{v} \stackrel{\text{def}}{=} -T_1 - v$ for the other root of Q [so $Q(y) = (y - v)(y - \overline{v})$]. Then since [one verifies immediately that] the equalities $R_i - v = -R_i \cdot v \cdot (t - 1/v - z_i)$, $(t - z_i)^2 \cdot R_i^2 = 1$, and $v \cdot \overline{v} = \Delta/F_0^2$ hold, one concludes that the equality

$$G_i = \Delta \cdot \left(t - \frac{1}{v} - z_i \right) \cdot \left(t - \frac{1}{\overline{v}} - z_i \right)$$

holds.

Next, write $L \stackrel{\text{def}}{=} k(t)(v)$, C for the projective smooth curve over k whose function field is L, and $\pi \colon C \to X$ for the morphism determined by the inclusion $k(t) \subseteq L$. First, observe that since $v + \overline{v} = -T_1$ and $v \cdot \overline{v} = \Delta/F_0^2$ are regular at a [cf. Lemma 3.1, (i)], the element v of L is integral over the local ring of X at a, hence regular at each point of $\pi^{-1}(a)$, and the value of v at each point of $\pi^{-1}(a)$ is a root of the polynomial $y^2 + T_1(a) \cdot y + \Delta(a)/F_0(a)^2$, i.e., is $R_i(a)$ or $R_j(a)$ [cf. Lemma 3.1, (i)]. In particular, since these two roots are distinct, the morphism π is étale over a. Thus, we may assume without loss of generality, by interchanging v and $\overline{v}$, that there exists a point P of $\pi^{-1}(a)$ such that the equalities $v(P) = R_i(a)$, $\overline{v}(P) = R_j(a)$ hold [cf. the equality $v + \overline{v} = -T_1$]. Next, one verifies easily that the orders at P of the three factors Δ, $t - 1/v - z_i$, $t - 1/\overline{v} - z_i$ of the final display are, respectively, zero [cf. Lemma 3.1, (ii)], ≥ 1 [cf. the equality $a - 1/R_i(a) = z_i$], and zero [cf. the equality $a - 1/R_j(a) = z_j \neq z_i$]. Thus, since the [nonzero — cf. assertion (i), together with the final display] element $t - 1/v - z_i$ of L is the cube of an element of L [cf. Lemma 3.2, (v)], the order at P of this element is divisible by three, hence ≥ 3, which thus implies that the order at P of G_i is divisible by three and ≥ 3. Thus, since G_i is contained in $k[t]$, and π is étale over a, the order $\mathrm{ord}_a(G_i)$ is divisible by three and ≥ 3, as desired. This completes the proof of assertion (iv), hence also of Lemma 3.3. $\square$

Lemma 3.4. — *Suppose that* $r \equiv 0 \pmod 3$ [*so* $n \equiv 2 \pmod 3$]. *Then the number of unordered pairs* $\{i, j\}$ *of distinct elements of* $\{1, \ldots, n\}$ *such that* $\mathcal{S}_{ij}$ *is not separable is* $\leq n(n-2)/3$, *which is* $< n(n-1)/2$, *i.e., the total number of such pairs. In particular* [*cf. Remark* 2.3.1*; Corollary* 1.5], *there exists a nilpotent ordinary indigenous bundle on* $(X, D)/S$.

Proof. — For each unordered pair $\{i, j\}$ of distinct elements of $\{1, \ldots, n\}$ such that $\mathcal{S}_{ij}$ is not separable, let $a_{ij} \in k$ be a multiple root of $\mathcal{S}_{ij}$. Let i be an element of $\{1, \ldots, n\}$. Then it follows from Lemma 3.1, (iii), that the elements a_{ij} — where j ranges over the elements of $\{1, \ldots, n\} \setminus \{i\}$ such that $\mathcal{S}_{ij}$ is not separable — are pairwise distinct; moreover, it follows from Lemma 3.1, (i), that none of these elements is contained in $\{z_1, \ldots, z_n\}$. Thus, it follows from Lemma 3.3, (ii), (iii), (iv), together with the nonvanishing of G_i [cf. Lemma 3.3, (i)], that the inequalities

$$3 + 3 \cdot \#\{\, j \mid \mathcal{S}_{ij} \text{ is not separable}\,\} \;\leq\; \deg(G_i) \;\leq\; 2n - 1$$

hold. In particular, the number of elements j such that $\mathcal{S}_{ij}$ is not separable is $\leq (2n-4)/3$. Thus, by summing over $i \in \{1, \ldots, n\}$ — where we note that, in the resulting sum, each unordered pair under consideration is counted twice — we conclude that the number of unordered pairs $\{i, j\}$ of distinct elements of $\{1, \ldots, n\}$ such that $\mathcal{S}_{ij}$ is not separable is $\leq n(2n-4)/6 = n(n-2)/3$. Moreover, since $2(n-2) < 3(n-1)$, the inequality $n(n-2)/3 < n(n-1)/2$ holds. Finally, since $r \equiv 0 \not\equiv 2 \pmod 3$, it follows from Remark 2.3.1 that, for each unordered pair $\{i, j\}$ of distinct elements of $\{1, \ldots, n\}$ such that $\mathcal{S}_{ij}$ is separable, the "ε" associated to $\mathcal{S}_{ij}$ is viable. Thus, the final portion of Lemma 3.4 follows from Corollary 1.5. This completes the proof of Lemma 3.4. ☐

4. The Existence of a Nilpotent Ordinary Indigenous Bundle

In the present §4, we give a proof of the main result of the present paper, i.e., the assertion that every hyperbolic curve of genus zero in characteristic three is hyperbolically ordinary [cf. Theorem 4.1 below]. Moreover, we observe that the corresponding assertion over an arbitrary noetherian scheme of characteristic three follows formally [cf. Corollary 4.2 below].

Theorem 4.1. — *Let* $r \geq 3$ *be an integer,* k *an algebraically closed field of characteristic three, and* (X, D) *a hyperbolic curve of type* $(0, r)$ *over* $S \overset{\text{def}}{=} \operatorname{Spec}(k)$. *Then the hyperbolic curve* (X, D) *over* k *is hyperbolically ordinary* [*cf.* [7, Chapter II, Definition 3.3]], *i.e., there exists a nilpotent ordinary indigenous bundle on* $(X, D)/S$.

Proof. — If the equality $r = 3$ holds, then Theorem 4.1 follows from [7, Chapter II, Corollary 3.8]. Suppose that the inequality $r \geq 4$ holds. Then one verifies immediately that, to verify Theorem 4.1, we may assume without loss of generality that (X, D) coincides with the "(X, D)" of §1 for suitable choices of the pairwise distinct elements "$z_1, \ldots, z_{r-1}$" of §1. Then Theorem 4.1 follows immediately from Lemma 2.5 and Lemma 3.4. This completes the proof of Theorem 4.1. ☐

Remark 4.1.1. — We remark that Theorem 4.1 should be compared with the fact that a general hyperbolic curve of arbitrary type is hyperbolically ordinary [cf. [7, Chapter II, Proposition 3.4], [7, Chapter II, Corollary 3.8]]. In the case of genus zero in characteristic three, the genericity assumption may be removed. We also remark that, by [8, Chapter V, Corollary 1.3, (1)], for a sufficiently general configuration of cusps, the number of isomorphism classes of nilpotent ordinary indigenous bundles on such a hyperbolic curve of genus zero is given by a combinatorial formula [cf. also [6, Theorem 5.3]]. On the other hand, Theorem 4.1 asserts, by way of contrast, the nonemptiness for an arbitrary configuration of cusps.

Corollary 4.2. — *Every hyperbolic curve of genus zero over a noetherian scheme of characteristic three is hyperbolically ordinary.*

Proof. — This assertion is a formal consequence of Theorem 4.1, together with [7, Chapter II, Proposition 3.4]. □

References

[1] Y. Hoshi, Nilpotent admissible indigenous bundles via Cartier operators in characteristic three, *Kodai Math. J.* **38** (2015), no. 3, 690–731.

[2] Y. Hoshi, On the supersingular divisors of nilpotent admissible indigenous bundles, *Kodai Math. J.* **42** (2019), no. 1, 1–47.

[3] Y. Hoshi, Hyperbolic ordinariness of hyperelliptic curves of lower genus in characteristic three, *Kyushu J. Math.* **73** (2019), no. 2, 317–335.

[4] Y. Hoshi, *On indigenous bundles in characteristic three*, RIMS Preprint **1918** (June 2020).

[5] Y. Hoshi, *Hyperbolic ordinariness of tetrapods with ordinary Legendre parameters*, RIMS Preprint **1996** (July 2026).

[6] Y. Hoshi, *Nilpotent admissible indigenous bundles and Cartier eigenforms in characteristic three*, RIMS Preprint **2004** (September 2026).

[7] S. Mochizuki, A theory of ordinary p-adic curves, *Publ. Res. Inst. Math. Sci.* **32** (1996), no. 6, 957–1152.

[8] S. Mochizuki, *Foundations of p-adic Teichmüller theory*, AMS/IP Stud. Adv. Math. **11**, Amer. Math. Soc./Int. Press, 1999.

(Yuichiro Hoshi) Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, JAPAN

Email address: yuichiro@kurims.kyoto-u.ac.jp