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A Note on Ramification of Finite Flat Commutative Group Schemes

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Abstract. — J.-M. Fontaine has established a ramification upper bound result for finite flat commutative group schemes. The main purpose of the present paper is to establish a ramification lower bound result for finite flat commutative group schemes.

Contents

Introduction

In the present paper,

- let p be a prime number, and
- let R be a *complete discrete valuation ring.*

Write

- K for the field of fractions of R and
- k for the residue field of R.

Suppose that

- the field K is *of characteristic zero*, and that
- the field k is *of characteristic* p and *perfect.*

Let $\overline{K}$ be an algebraic closure of K. Write

- v_0 for the valuation on $\overline{K}$ normalized so that the equality $v_0(p) = 1$ holds,
- $W(k)$ for the ring of Witt vectors with coefficients in k,
- $K_0 \overset{\text{def}}{=} W(k)[1/p]$ for the field of fractions of $W(k)$, and
- $\mathfrak{D}_K \subseteq R$ for the different ideal associated to the finite field extension K/K_0.

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Next, let G be a *finite flat commutative group scheme* over R annihilated by a power of p. Write

- $\mathrm{lv}(G) \stackrel{\mathrm{def}}{=} v_0(\mathrm{Ann}_R(G(\overline{K}) \otimes_{\mathbb{Z}_p} R))$ for the *level* of G [cf. Definition 1.2, (iii)], i.e., the minimal nonnegative integer n such that G is annihilated by p^n,
- G^D for the *Cartier dual* of G,
- $K(G)$, $K(G^D)$ for the [necessarily finite Galois] extension fields of K that correspond to the kernels of the natural Galois actions $\mathrm{Gal}(\overline{K}/K) \to \mathrm{Aut}(G(\overline{K}))$, $\mathrm{Gal}(\overline{K}/K) \to \mathrm{Aut}(G^D(\overline{K}))$, respectively, and
- $\mathfrak{D}_{K(G)/K}$, $\mathfrak{D}_{K(G^D)/K}$ for the different ideals associated to the finite field extensions $K(G)/K$, $K(G^D)/K$, respectively.

In this situation, *J.-M. Fontaine* has established the following *ramification upper bound result* for finite flat commutative group schemes [cf. [2, Corollaire to Théorème A]]. Moreover, *S. Hattori* studied this result in the imperfect residue field case [cf. [3, Theorem 7]].

THEOREM A. — *The inequality*

$$v_0(\mathfrak{D}_{K(G)/K}) < \mathrm{lv}(G) + \frac{1}{p-1}$$

holds.

The main purpose of the present paper is to establish a *ramification lower bound result* for finite flat commutative group schemes. The main result of the present paper is as follows [cf. Theorem 3.2, (ii), (iii)].

THEOREM B. — *The following assertions hold:*

(i) *Write $t^*_G(R)$ for the R-valued cotangent space of G over R [cf. Definition 1.2, (iv)] and $\mathrm{lv}^{\Omega}(G) \stackrel{\mathrm{def}}{=} v_0(\mathrm{Ann}_R(t^*_G(R)))$. Then the inequality*

$$\mathrm{lv}^{\Omega}(G) - \frac{1}{p-1} - v_0(\mathfrak{D}_K) \le v_0(\mathfrak{D}_{K(G)/K})$$

holds.

(ii) *The inequality*

$$\mathrm{lv}(G) - \frac{2}{p-1} - 2v_0(\mathfrak{D}_K) \le \max\{v_0(\mathfrak{D}_{K(G)/K}), v_0(\mathfrak{D}_{K(G^D)/K})\}$$

holds.

The proof of Theorem B can be regarded as an argument obtained simply by carefully generalizing the argument of [4, §1] applied to prove [4, Proposition 1.8], i.e., the *inequalities*

$$\mathrm{lv}(G) - \frac{1}{p-1} \le v_0(\mathfrak{D}_{K(G)/K}) < \mathrm{lv}(G) + \frac{1}{p-1}$$

in the case where the *equality* $K = K_0$ holds, and G is *connected* [cf. also [4, Remark 1.9.1]].

Let us also observe that the constants that appear in Theorem B are, to a certain extent, *optimal* [cf. Remark 3.2.1]. Indeed, let $n \geq m$ be positive integers, and suppose that G is multiplicative and cyclic of order p^n, and that the equality $K = K_0(\zeta_{p^m})$ — where $\zeta_{p^m} \in \overline{K}$ is a primitive p^m-th root of unity — holds. Then one verifies easily that the equalities $\mathrm{lv}^{\Omega}(G) = n$, $1/(p-1) + v_0(\mathfrak{D}_K) = m$, and $v_0(\mathfrak{D}_{K(G)/K}) = n - m$ hold, so that the inequality of Theorem B, (i), is, in fact, an *equality*. In particular, the constant "$1/(p-1) + v_0(\mathfrak{D}_K)$" in Theorem B, (i), cannot be replaced by any smaller constant. Moreover, in this situation, the right-hand side of the inequality of Theorem B, (ii), is equal to $\mathrm{lv}(G) - 1/(p-1) - v_0(\mathfrak{D}_K)$, which implies that the constant "$2/(p-1) + 2v_0(\mathfrak{D}_K)$" in Theorem B, (ii), cannot be replaced by any constant $< 1/(p-1) + v_0(\mathfrak{D}_K)$. It is not clear to the authors at the time of writing whether or not this constant "$2/(p-1) + 2v_0(\mathfrak{D}_K)$" may be replaced by "$1/(p-1) + v_0(\mathfrak{D}_K)$".

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1. Level and Ramification

In the present §1, we prepare some notational conventions related to the main results of the present paper. Moreover, we verify some basic facts concerning ramification of finite flat commutative group schemes [cf. Lemma 1.3 below]. Let $\zeta_p \in \overline{K}$ be a primitive p-th root of unity. Write

$$\epsilon_K \stackrel{\text{def}}{=} v_0\big((1 - \zeta_p) \cdot \mathfrak{D}_K\big) = \frac{1}{p-1} + v_0(\mathfrak{D}_K).$$

Moreover, let $d_0 \in R$ be such that the equality $\mathfrak{D}_K = d_0 \cdot R$ holds; thus, the equality $\epsilon_K = v_0((1 - \zeta_p) \cdot d_0)$ holds.

Definition 1.1. — Let $L \subseteq \overline{K}$ be an extension field of K [which is not necessarily finite] contained in $\overline{K}$.

(i) We shall write

$$R_L \subseteq L$$

for the normalization of R in L.

(ii) Let M be an R_L-module. Then we shall write

$$\mathrm{lv}_{R_L}(M) \stackrel{\text{def}}{=} v_0\big(\mathrm{Ann}_{R_L}(M)\big) \ \big(= \inf\{\, v_0(x) \mid x \in \mathrm{Ann}_{R_L}(M)\,\}\big) \ \in \ \mathbb{R} \cup \{\infty\}.$$

(iii) Suppose that the field extension L/K is finite. Then we shall write

$$\mathrm{lv}^{\Omega}(L/K) \stackrel{\text{def}}{=} \mathrm{lv}_{R_L}(\Omega^1_{R_L/R}).$$

In particular, the equality

$$\mathrm{lv}^{\Omega}(L/K) = v_0(\mathfrak{D}_{L/K})$$

holds [cf., e.g., [5, Chapter III, Proposition 14]] — where we write $\mathfrak{D}_{L/K} \subseteq R_L$ for the different ideal associated to the finite field extension L/K.

Definition 1.2.

(i) We shall write

$$K(G) \subseteq \overline{K}$$

for the [necessarily finite Galois] extension field of K that corresponds to the kernel of the natural homomorphism $\mathrm{Gal}(\overline{K}/K) \to \mathrm{Aut}(G(\overline{K}))$.

(ii) We shall write

$$G^D$$

for the *Cartier dual* of G.

(iii) We shall write

$$\mathrm{lv}(G) \stackrel{\mathrm{def}}{=} \mathrm{lv}_R\big(G(\overline{K}) \otimes_{\mathbb{Z}_p} R\big)$$

and refer to $\mathrm{lv}(G)$ as the *level* of G.

(iv) Let M be an R-module. Then we shall write

$$t^*_G(M) \stackrel{\mathrm{def}}{=} (e^*_G \Omega^1_{G/R}) \otimes_R M$$

— where we write e_G for the identity section of G over R — and

$$t_G(M) \stackrel{\mathrm{def}}{=} \mathrm{Hom}_R\big(t^*_G(R), M\big).$$

We shall refer to $t^*_G(M)$ (respectively, $t_G(M)$) as the *M-valued cotangent* (respectively, *tangent*) *space* of G over R.

(v) We shall write

$$\mathrm{lv}^{\Omega}(G) \stackrel{\mathrm{def}}{=} \mathrm{lv}_R\big(t^*_G(R)\big).$$

Lemma 1.3. — *The following assertions hold:*

(i) *The inequality*

$$\max\{\mathrm{lv}^{\Omega}(G), \mathrm{lv}^{\Omega}(G^D)\} \leq \mathrm{lv}(G)$$

holds.

(ii) *The inequality*

$$\mathrm{lv}^{\Omega}\big(K(G)/K\big) < \mathrm{lv}(G) + \frac{1}{p-1}$$

holds.

PROOF. — Assertion (i) follows from the [easily verified] fact that, for each integer n, the endomorphism of $t^*_G(R)$ induced by the endomorphism of G given by multiplication by n coincides with the endomorphism of $t^*_G(R)$ given by multiplication by n, applied to both G and G^D, together with the [easily verified] equality $\mathrm{lv}(G) = \mathrm{lv}(G^D)$. Assertion (ii) is none other than [2, Corollaire to Théorème A]. $\square$

2. Review of Truncated Hodge–Tate Theory

In the present §2, we give a brief review of the truncated Hodge–Tate theory for finite flat commutative group schemes established in [1, §4].

DEFINITION 2.1. — Let $L \subseteq \overline{K}$ be a Galois extension field of K [which is not necessarily finite] contained in $\overline{K}$ such that the inclusion $K(G) \subseteq L$ holds.

(i) Let g be an element of $G(R_L)$. Then it follows from the definition of the Cartier dual G^D that the element g defines an invertible element of the R-algebra $\Gamma(G^D, \mathcal{O}_{G^D}) \otimes_R R_L$. In particular, we have an element

$$dg/g \in \Omega^1_{G^D \times_R R_L/R} = (\Omega^1_{G^D/R} \otimes_R R_L) \oplus \big(\Omega^1_{R_L/R} \otimes_R \Gamma(G^D, \mathcal{O}_{G^D})\big).$$

Then it follows immediately from a similar argument to the argument applied in [1, §4.5] [cf. also [1, §4.3]] that the image of dg/g in the first component $\Omega^1_{G^D/R} \otimes_R R_L$ is contained in the submodule $t^*_{G^D}(R_L) \subseteq \Omega^1_{G^D/R} \otimes_R R_L$ of $\Omega^1_{G^D/R} \otimes_R R_L$ [cf., e.g., [1, §4.3]]. In particular, we obtain a homomorphism of R_L-modules

$$\mathrm{ev}^0_{G,L} \colon G(R_{\overline{K}}) \otimes_{\mathbb{Z}_p} R_L = G(R_L) \otimes_{\mathbb{Z}_p} R_L \longrightarrow t^*_{G^D}(R_L).$$

(ii) Let g be an element of $G(R_L)$. Then the element g determines a homomorphism $\Omega^1_{G/R} \to \Omega^1_{R_L/R}$ of R-modules, hence also an element $t^*_G(R) \hookrightarrow \Omega^1_{G/R} \to \Omega^1_{R_L/R}$ [cf., e.g., [1, §4.7]] of $t_G(\Omega^1_{R_L/R})$. In particular, we obtain a homomorphism of R_L-modules

$$\mathrm{ev}^1_{G,L} \colon G(R_{\overline{K}}) \otimes_{\mathbb{Z}_p} R_L = G(R_L) \otimes_{\mathbb{Z}_p} R_L \longrightarrow t_G(\Omega^1_{R_L/R}).$$

(iii) We shall write

$$\mathrm{ev}_{G,L} \colon G(R_{\overline{K}}) \otimes_{\mathbb{Z}_p} R_L = G(R_L) \otimes_{\mathbb{Z}_p} R_L \longrightarrow t^*_{G^D}(R_L) \oplus t_G(\Omega^1_{R_L/R})$$

for the homomorphism of R_L-modules determined by $\mathrm{ev}^0_{G,L}$ and $\mathrm{ev}^1_{G,L}$.

LEMMA 2.2. — *Let $L \subseteq \overline{K}$ be a Galois extension field of K [which is not necessarily finite] contained in $\overline{K}$ such that the inclusion $K(G) \subseteq L$ holds. Thus, we have a commutative diagram of modules*

$$\begin{tikzcd} G(R_L) \otimes_{\mathbb{Z}_p} R_L \arrow[r, hook] \arrow[d, "\mathrm{ev}_{G,L}"'] & G(R_L) \otimes_{\mathbb{Z}_p} R_{\overline{K}} \arrow[r, equal] \arrow[d] & G(R_{\overline{K}}) \otimes_{\mathbb{Z}_p} R_{\overline{K}} \arrow[d, "\mathrm{ev}_{G,\overline{K}}"] \\ t^*_{G^D}(R_L) \oplus t_G(\Omega^1_{R_L/R}) \arrow[r] & \big(t^*_{G^D}(R_L) \oplus t_G(\Omega^1_{R_L/R})\big) \otimes_{R_L} R_{\overline{K}} \arrow[r] & t^*_{G^D}(R_{\overline{K}}) \oplus t_G(\Omega^1_{R_{\overline{K}}/R}). \end{tikzcd}$$

Then the following assertions hold:

(i) *The kernel and cokernel of the right-hand vertical arrow* $\mathrm{ev}_{G,\overline{K}}$ *are annihilated by* $(1-\zeta_p)\cdot d_0$.

(ii) *The cokernel of the natural homomorphism* $t_G(\Omega^1_{R_L/R})\otimes_{R_L} R_{\overline{K}} \to t_G(\Omega^1_{R_{\overline{K}}/R})$ *is annihilated by* $(1-\zeta_p)\cdot d_0$.

Proof. — Assertion (i) follows from [1, Corollaire to Théorème 3] [cf. also [1, Proposition 6]]. Assertion (ii) is a formal consequence of assertion (i). □

3. Proof of Main Results

In the present §3, we give the proof of the main result of the present paper [cf. Theorem 3.2 below].

Lemma 3.1. — *Let* $L \subseteq \overline{K}$ *be a Galois extension field of* K [*which is not necessarily finite*] *contained in* $\overline{K}$ *such that the inclusion* $K(G) \subseteq L$ *holds. Then the following assertions hold:*

(i) *The inequality*

$$\mathrm{lv}(G) - \epsilon_K \leq \max\{\mathrm{lv}^{\Omega}(G), \mathrm{lv}^{\Omega}(G^D)\}$$

holds.

(ii) *Suppose that the field extension* L/K *is finite. Then the inequality*

$$\mathrm{lv}^{\Omega}(G) - \epsilon_K \leq \mathrm{lv}^{\Omega}(L/K)$$

holds.

Proof. — First, we verify assertion (i). Let $a \in R_{\overline{K}}$ be such that the equality

$$v_0(a) = \max\{\mathrm{lv}^{\Omega}(G), \mathrm{lv}^{\Omega}(G^D)\}$$

holds. [Note that it is immediate that such an element always exists.] Then observe that it is immediate that the submodule $a\cdot(G(R_{\overline{K}})\otimes_{\mathbb{Z}_p} R_{\overline{K}}) \subseteq G(R_{\overline{K}})\otimes_{\mathbb{Z}_p} R_{\overline{K}}$ is contained in the kernel of the homomorphism $\mathrm{ev}_{G,\overline{K}}$. Thus, it follows from Lemma 2.2, (i), that $a\cdot(G(R_{\overline{K}})\otimes_{\mathbb{Z}_p} R_{\overline{K}})$ is annihilated by $(1-\zeta_p)\cdot d_0$, i.e., that $G(R_{\overline{K}})\otimes_{\mathbb{Z}_p} R_{\overline{K}}$ is annihilated by $(1-\zeta_p)\cdot d_0\cdot a$. Thus, we conclude that

$$\epsilon_K + \max\{\mathrm{lv}^{\Omega}(G), \mathrm{lv}^{\Omega}(G^D)\} = v_0\big((1-\zeta_p)\cdot d_0\cdot a\big) \geq \mathrm{lv}(G),$$

as desired. This completes the proof of assertion (i).

Next, we verify assertion (ii). Let $b \in R_{\overline{K}}$ be such that the equality

$$v_0(b) = \mathrm{lv}^{\Omega}(L/K)$$

holds. [Note that it is immediate that such an element always exists.] Then observe that it follows from Lemma 2.2, (ii), that the submodule $(1-\zeta_p)\cdot d_0\cdot t_G(\Omega^1_{R_{\overline{K}}/R}) \subseteq t_G(\Omega^1_{R_{\overline{K}}/R})$ is contained in the image of the natural homomorphism $t_G(\Omega^1_{R_L/R})\otimes_{R_L} R_{\overline{K}} \to t_G(\Omega^1_{R_{\overline{K}}/R})$. In particular, it follows from our choice of $b \in R_{\overline{K}}$ [cf. also our assumption that the field extension L/K is finite] that $(1-\zeta_p)\cdot d_0\cdot t_G(\Omega^1_{R_{\overline{K}}/R})$ is annihilated by b, i.e., that $t_G(\Omega^1_{R_{\overline{K}}/R})$ is annihilated by $(1-\zeta_p)\cdot d_0\cdot b$. Thus, we conclude that

$$\epsilon_K + \mathrm{lv}^{\Omega}(L/K) = v_0\big((1-\zeta_p)\cdot d_0\cdot b\big) \geq \mathrm{lv}_{R_{\overline{K}}}\big(t_G(\Omega^1_{R_{\overline{K}}/R})\big).$$

On the other hand, it follows immediately from [1, Corollaire 1, (1)] that the equality $\mathrm{lv}^{\Omega}(G) = \mathrm{lv}_{R_{\overline{K}}}(t_G(\Omega^1_{R_{\overline{K}}/R}))$ holds. This completes the proof of assertion (ii), hence also of Lemma 3.1. □

REMARK 3.1.1. — Lemma 3.1, (ii), may be regarded as a generalization of [4, Proposition 1.4]. Moreover, the proof of the main result of the present paper may be regarded as an argument obtained simply by carefully generalizing the argument of [4, §1] applied to prove [4, Proposition 1.8], i.e., the *inequalities*

$$\mathrm{lv}(G) - \frac{1}{p-1} \leq v_0(\mathfrak{D}_{K(G)/K}) < \mathrm{lv}(G) + \frac{1}{p-1}$$

in the case where the *equality* $K = K_0$ holds, and G is *connected* [cf. also [4, Remark 1.9.1]].

THEOREM 3.2. — *The following assertions hold:*

(i) *The inequalities*

$$\mathrm{lv}(G) - \epsilon_K \leq \max\{\mathrm{lv}^{\Omega}(G), \mathrm{lv}^{\Omega}(G^D)\} \leq \mathrm{lv}(G)$$

hold.

(ii) *The inequalities*

$$\mathrm{lv}^{\Omega}(G) - \epsilon_K \leq \mathrm{lv}^{\Omega}\big(K(G)/K\big) < \mathrm{lv}(G) + \frac{1}{p-1}$$

hold.

(iii) *The inequalities*

$$\mathrm{lv}(G) - 2\epsilon_K \leq \max\{\mathrm{lv}^{\Omega}\big(K(G)/K\big), \mathrm{lv}^{\Omega}\big(K(G^D)/K\big)\} < \mathrm{lv}(G) + \frac{1}{p-1}$$

hold.

PROOF. — Assertion (i) (respectively, (ii)) follows from Lemma 1.3, (i) (respectively, (ii)), and Lemma 3.1, (i) (respectively, (ii)). Assertion (iii) follows from assertion (i), together with assertion (ii) — applied to both G and G^D — and the [easily verified] equality $\mathrm{lv}(G) = \mathrm{lv}(G^D)$. □

REMARK 3.2.1. — Let n, m be positive integers. Suppose that

- the finite flat commutative group scheme G is *étale* and *cyclic of order* p^n, and that
- the *equality* $K = K_0(\zeta_{p^m})$ — where $\zeta_{p^m} \in \overline{K}$ is a primitive p^m-th root of unity — holds.

Then one verifies immediately that

$$\mathrm{lv}(G) = n, \qquad K(G) = K \subseteq K(G^D) = K(\zeta_{p^n}), \qquad \epsilon_K = m,$$

and, moreover, that

$$\mathrm{lv}^{\Omega}(G) = 0, \qquad \mathrm{lv}^{\Omega}(G^D) = n, \qquad \mathrm{lv}^{\Omega}\big(K(G^D)/K\big) = \max\{0, n-m\}.$$

Thus, the *inequalities* of Theorem 3.2, (iii), may be described as follows:

$$n - 2m \leq \max\{0, n-m\} < n + \frac{1}{p-1}.$$

This example shows, moreover, that the main results of the present paper are *sharp* in the following sense:

(i) Suppose that the inequality $n \geq m$ holds. Then, by applying the above computations to the Cartier dual G^D, we obtain the *equality*

$$\mathrm{lv}^{\Omega}(G^D) - \epsilon_K \ = \ n - m \ = \ \mathrm{lv}^{\Omega}\big(K(G^D)/K\big),$$

i.e., the first inequality of Theorem 3.2, (ii), is, in fact, an *equality* for G^D. In particular, the constant ϵ_K in the first inequality of Theorem 3.2, (ii), cannot be improved.

(ii) Since [cf. the above computations] the *equality*

$$\max\{\mathrm{lv}^{\Omega}\big(K(G)/K\big), \mathrm{lv}^{\Omega}\big(K(G^D)/K\big)\} = \mathrm{lv}(G) - \epsilon_K$$

holds whenever the inequality $n \geq m$ holds, the inequality obtained by replacing the quantity "$\mathrm{lv}(G) - 2\epsilon_K$" in Theorem 3.2, (iii), by "$\mathrm{lv}(G) - c$" *fails*, in general, for every $c < \epsilon_K$. In particular, the "optimal constant" in the first inequality of Theorem 3.2, (iii), lies in the closed interval $[\epsilon_K, 2\epsilon_K]$.

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