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**On anabelian geometry in positive characteristic
via Artin–Schreier Galois evaluation**

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ON ANABELIAN GEOMETRY IN POSITIVE CHARACTERISTIC VIA ARTIN–SCHREIER GALOIS EVALUATION

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ABSTRACT. In the present paper, we investigate anabelian phenomena in positive characteristic from the point of view of Artin–Schreier theory. First, we prove that, for an arbitrary affine variety over a field of positive characteristic, the injectivity portion of the Grothendieck section conjecture holds, under the assumption that the field admits a nontrivial Artin–Schreier extension. While the injectivity portion holds in this greater generality, we observe that the surjectivity portion of the Grothendieck section conjecture, or even its birational version, for smooth curves over positive-characteristic local fields does not hold in general. This observation reflects the substantial difference between the mixed-characteristic situation and the positive-characteristic situation in anabelian geometry. Despite such a difference, by considering Artin–Schreier-theoretic Galois evaluations appropriately, together with various techniques in positive-characteristic anabelian geometry due to A. Tamagawa and S. Mochizuki, we prove that the isomorphism classes of punctured projective lines over positive-characteristic local fields are completely determined, up to pulling back by a power of the Frobenius endomorphism of the base field, by the associated étale fundamental groups in the sense of relative anabelian geometry. This result may be regarded as the first affirmative anabelian result for curves over positive-characteristic local fields.

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INTRODUCTION

A central motivation in anabelian geometry is to understand to what extent a geometric object may be “reconstructed” from its highly non-abelian homotopical data, e.g., the étale fundamental group of a hyperbolic curve over a suitable field. This research area was founded by A. Grothendieck in his famous letter to G. Faltings [cf. [2]], in which several fundamental conjectures, such as the so-called Grothendieck conjecture and section conjecture, were proposed. Roughly speaking,

- the Grothendieck conjecture is a question concerning the rigidity of [suitable morphisms between] the fundamental groups of “anabelian varieties” such as hyperbolic curves, and
- the section conjecture asks whether or not every splitting [i.e., Galois section] of the homotopy exact sequence associated to an “anabelian variety” arises from a unique rational point.

Recall that the base fields of the algebraic varieties considered in the letter are finitely generated fields. With regard to the Grothendieck conjecture, after the contributions due to H. Nakamura and A. Tamagawa [cf. [14], [15], [21]], S. Mochizuki solved the conjecture affirmatively for arbitrary hyperbolic curves over sub- p -adic fields [cf. [9]]. Here, a sub- p -adic field is a field isomorphic to a subfield of a finitely generated extension field of the field of p -adic numbers for some prime number p . This solution is based on the p -adic Hodge theory due to G. Faltings, and its key point was the change of viewpoint from global to local. Moreover, the birational version, i.e., the function field version, of the section conjecture for curves over p -adic local fields was proved by J. Koenigsmann [cf. [7]]. On the other hand, the Grothendieck conjecture for hyperbolic curves over finitely generated fields of positive characteristic was proved by A. Tamagawa, S. Mochizuki, and J. Stix [cf. [21], [11], [19], [20]]. Thus, taking the above situation into account, it is natural to explore anabelian phenomena for smooth curves over positive-characteristic local fields. However, despite its naturalness, no analogous result has been obtained so far.

In the present paper, to overcome this unsatisfactory situation, we investigate the Grothendieck conjecture and the section conjecture in

positive characteristic from the point of view of Artin–Schreier theory. More precisely, we consider these problems by means of Artin–Schreier-theoretic Galois evaluations, i.e., the procedure of obtaining Artin–Schreier classes of elements of the base field by pulling back, by means of the Galois sections under consideration, the Artin–Schreier classes of regular functions. Here, we note that Kummer-theoretic Galois evaluations, i.e., the analogous procedure in which Artin–Schreier classes are replaced by Kummer classes, frequently appear in the study of anabelian geometry [cf., e.g., [11], [13]]. On the other hand, to the knowledge of the authors of the present paper, Artin–Schreier-theoretic Galois evaluations have not been applied substantially in anabelian geometry so far. In this sense, one of the new contributions of the present paper is to show that Artin–Schreier-theoretic Galois evaluations are also effective in anabelian geometry.

First, with regard to the section conjecture, we prove the following result [cf. Theorem 1.8, Remark 1.8.1]:

Theorem A. *Let p be a prime number, A a field of characteristic p , and $\bar{A}$ a separable closure of A . Write $G \stackrel{\text{def}}{=} \text{Gal}(\bar{A}/A)$ for the absolute Galois group of A determined by the separable closure $\bar{A}$. Then the following assertions hold:*

- (i) *Suppose that there exists a finite Galois extension field of A of degree p . Let V be an affine algebraic variety over A [cf. Definition 1.3]. Write Π, Δ for the respective étale fundamental groups of $V, V \times_A \bar{A}$, relative to suitable choices of basepoints. Then the map from the set of A -rational points of V to the set of Δ -conjugacy classes of continuous sections of the natural continuous surjective homomorphism $\Pi \twoheadrightarrow G$ that maps each A -rational point of V to the Galois section-class determined by the A -rational point [cf. Definition 1.6] is injective.*
- (ii) *Let $q > 1$ be a power of p . Suppose that A is the field $\mathbb{F}_q((t))$ — where t is an indeterminate — of formal Laurent series with coefficients in $\mathbb{F}_q$. Then there exists a proper smooth curve C over A [cf. Definition 1.3] of genus ≥ 2 [cf. Definition 1.4] that satisfies the following two conditions:*
 - (1) *The smooth curve C over A has no rational point over the perfect closure of A . In particular, the set $C(A)$ is empty.*
 - (2) *The natural continuous surjective homomorphisms onto G from the étale fundamental group of C and from the absolute Galois group of the function field of C admit continuous sections.*

Theorem A, (i), may be regarded as an application of Artin–Schreier-theoretic Galois evaluations. On the other hand, Theorem A, (ii), is a consequence of the fact that the p -cohomological dimension of the absolute Galois group of a field of characteristic p is ≤ 1 , together with an elementary observation concerning the complete discrete valuation field associated to the generic point of the special fiber of a smooth model of a curve. Thus, while the injectivity portion of the section conjecture holds in the greater generality of Theorem A, (i), the surjectivity portion of the section conjecture, or even its birational version, for smooth curves over positive-characteristic local fields does not hold in general [cf. Theorem A, (ii)]. This observation reflects the substantial difference between the mixed-characteristic situation and the positive-characteristic situation in anabelian geometry.

On the other hand, despite such a difference, by applying Artin–Schreier-theoretic Galois evaluations appropriately, together with various techniques in positive-characteristic anabelian geometry due to A. Tamagawa and S. Mochizuki, we prove a Grothendieck conjecture-type result for punctured projective lines over positive-characteristic local fields. In order to state this result, we introduce some notation. In the remainder of the present Introduction, for each $\square \in \{\circ, \bullet\}$, let $p_\square$ be a prime number, $F_\square$ a finite field of characteristic $p_\square$, and $r_\square \geq 3$ an integer. Write $K_\square \stackrel{\text{def}}{=} F_\square((t))$ — where t is an indeterminate — for the field of formal Laurent series with coefficients in $F_\square$. For each $\square \in \{\circ, \bullet\}$, moreover, let $S_\square \subseteq \mathbb{P}_{K_\square}^1(K_\square)$ be a subset of $\mathbb{P}_{K_\square}^1(K_\square)$ of cardinality $r_\square$. Write

- $X_\square \stackrel{\text{def}}{=} \mathbb{P}_{K_\square}^1 \setminus S_\square$ for the open subscheme of $\mathbb{P}_{K_\square}^1$ obtained by forming the complement in $\mathbb{P}_{K_\square}^1$ of the closed subset [of $\mathbb{P}_{K_\square}^1$ that corresponds to] $S_\square \subseteq \mathbb{P}_{K_\square}^1(K_\square)$,
- $\Pi_\square$ for the étale fundamental group of $X_\square$, relative to a suitable choice of basepoint, and
- $X_\square(n)$, for each nonnegative integer n , for the scheme over $K_\square$ obtained by pulling back the scheme $X_\square$ over $K_\square$ by the absolute $p_\square^n$ -th power Frobenius endomorphism of $\text{Spec}(K_\square)$.

Then our main result concerning the Grothendieck conjecture is as follows [cf. Theorem 6.1]:

Theorem B. *The following two conditions are equivalent:*

- (1) *There exists a nonnegative integer n such that the scheme $X_\circ$ is isomorphic to the scheme $X_\bullet(n)$, or the scheme $X_\bullet$ is isomorphic to the scheme $X_\circ(n)$.*

- (2) *There exists a continuous isomorphism $\Pi_{\circ} \xrightarrow{\sim} \Pi_{\bullet}$ that lies over a continuous isomorphism between the absolute Galois groups of $K_{\circ}, K_{\bullet}$ which arises from an isomorphism between separable closures of $K_{\bullet}, K_{\circ}$ that restricts to an isomorphism $K_{\bullet} \xrightarrow{\sim} K_{\circ}$ of subfields.*

Theorem B may be regarded as the first affirmative anabelian result for smooth curves over positive-characteristic local fields. The main ingredients of the proof of Theorem B are as follows:

- The Grothendieck conjecture for affine hyperbolic curves over finite fields due to A. Tamagawa [cf. [21]].
- The reconstruction of the cuspidal inertia subgroups and various numerical invariants from the étale fundamental groups due to A. Tamagawa [cf. [22]].
- The techniques for reconstructing the log fundamental groups of the log special fibers due to S. Mochizuki [cf. [10]].
- A delicate control, by means of Artin–Schreier-theoretic Galois evaluations, of the relationship between various Artin–Schreier coverings of the affine line over a positive-characteristic local field and their coordinates.

The authors of the present paper expect that this result, together with its proof, will stimulate further investigation of positive-characteristic anabelian geometry.

The present paper is organized as follows: In §1, we introduce several notions concerning algebraic varieties and Galois sections and prove Theorem A. In §2, we discuss generalities concerning models of smooth curves and isomorphisms between the étale fundamental groups of smooth curves. In particular, we prove that, under certain assumptions, an isomorphism between the étale fundamental groups of smooth curves that admit stable reduction induces an isomorphism between the log fundamental groups of the log special fibers of the stable models that arises from an isomorphism between these log schemes. Moreover, we introduce the notions of base-geometric and cyclotomically compatible isomorphisms between the étale fundamental groups of smooth curves. The latter notion is closely related to the control of Frobenius twists, which may be regarded as one of the main issues in positive-characteristic anabelian geometry. In §3, we consider strictly isotrivial smooth curves over positive-characteristic local fields. In particular, we prove that an isomorphism between the étale fundamental groups of strictly isotrivial smooth curves that admit stable reduction induces an isomorphism between the étale fundamental groups of the smooth curves over finite fields from which the given smooth curves arise by

base-change. As a consequence, together with the Grothendieck conjecture for hyperbolic curves over finite fields, one obtains an anabelian result for strictly isotrivial smooth curves over positive-characteristic local fields. In §4, we consider projective lines minus four rational points. In particular, by applying Artin–Schreier-theoretic Galois evaluations, together with the results obtained in §2, we prove that, under certain assumptions, the isomorphism between the étale fundamental groups of the projective lines minus the three points $0, 1, \infty$ induced by a given isomorphism between the étale fundamental groups of projective lines minus four rational points preserves, after a suitable purely inseparable extension of the base field, the Artin–Schreier coverings determined by the nonzero scalar multiples of the standard coordinate. In §5, by applying the result obtained in §4, together with the results obtained in §2 and §3, and by further examining the coordinates of the cusps by means of Artin–Schreier-theoretic Galois evaluations, we prove that the isomorphism between the base fields that arises from the given isomorphism maps the coordinate of the fourth cusp of one of the projective lines minus four rational points to the coordinate of the fourth cusp of the other. Finally, in §6, by applying the results obtained in §2, §3, and §5, we complete the proof of Theorem B.

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1. ANABELIAN INJECTIVITY RESULT

In the present §1, we discuss Artin–Schreier theory for an affine algebraic variety over a field of positive characteristic from the point of view of the étale fundamental group of the algebraic variety [cf. Lemma 1.7 below]. This discussion plays an essential role in §4 and §5. As an application, we prove that the Galois section-class determined by a rational point of such an affine algebraic variety completely determines the rational point, under a certain assumption on the field [cf. Theorem 1.8 below].

In the present §1, let

- p be a prime number,
- A a field of characteristic p , and
- A^{alg} an algebraic closure of A ;

write

- $\overline{A}$ for the separable closure in A^{alg} of A ,
- A_0 for the algebraic closure in A of the prime subfield of A ,
and
- $\overline{A}_0$ for the algebraic closure in $\overline{A}$ of the prime subfield of A .

Definition 1.1. Let B be an algebra over A , and let $q > 1$ be a power of p . Then we shall write

$$\varphi_q: B \longrightarrow B$$

for the *Artin–Schreier endomorphism* of B of degree q — i.e., the endomorphism of the underlying additive group of the algebra B that maps $b \in B$ to $b^q - b \in B$.

Lemma 1.2. *Suppose that there exists a finite Galois extension field of A of degree p . Then the following assertions hold:*

- (i) *The subset $\varphi_p(A) \subseteq A$ of A does not coincide with A .*
- (ii) *Let b be a nonzero element of A . Then there exists an element δ of A such that $\delta \cdot b$ is not contained in the subset $\varphi_p(A) \subseteq A$ of A .*

PROOF. Assertion (i) follows immediately from Artin–Schreier theory, together with our assumption that there exists a finite Galois extension field of A of degree p . Assertion (ii) is an immediate consequence of assertion (i). This completes the proof of Lemma 1.2. $\square$

Definition 1.3. We shall define

- an *algebraic variety* over A to be a scheme that is separated, of finite type, and geometrically integral over A and
- a *smooth curve* over A to be an algebraic variety over A that is of relative dimension one and smooth over A .

Definition 1.4. We shall define the *genus* of a smooth curve C over A to be the dimension over A^{alg} of the first cohomology group of the structure sheaf of the smooth compactification of the smooth curve $C \times_A A^{\text{alg}}$ over A^{alg} . Observe that one verifies easily that this dimension is independent of the choice of the algebraic closure A^{alg} of A .

Definition 1.5. Let V be an algebraic variety over A . Then we shall say that

- (i) the algebraic variety V over A is *isotrivial* if there exist an algebraic variety $\overline{V}_0$ over $\overline{A}_0$ and an isomorphism $V \times_A \overline{A} \xrightarrow{\sim} \overline{V}_0 \times_{\overline{A}_0} \overline{A}$ over $\overline{A}$, and that
- (ii) the algebraic variety V over A is *strictly isotrivial* if there exist an algebraic variety V_0 over A_0 and an isomorphism $V \xrightarrow{\sim} V_0 \times_{A_0} A$ over A .

In the remainder of the present §1, let

- V be an affine algebraic variety over A ;

write

- Π for the étale fundamental group of V , relative to a suitable choice of basepoint,
- Δ for the étale fundamental group of $V \times_A \overline{A}$, relative to a suitable choice of basepoint, and
- $G \stackrel{\text{def}}{=} \text{Gal}(\overline{A}/A)$ for the absolute Galois group of A determined by the separable closure $\overline{A}$.

Thus, the natural morphisms $V \times_A \overline{A} \rightarrow V \rightarrow \text{Spec}(A)$ of schemes induce an exact sequence of topological groups

$$1 \longrightarrow \Delta \longrightarrow \Pi \longrightarrow G \longrightarrow 1.$$

Definition 1.6. Let $v \in V(A)$ be an A -rational point of V . Then we shall refer to the Δ -conjugacy class of continuous sections $G \rightarrow \Pi$ induced by v of the continuous homomorphism $\Pi \rightarrow G$ as the *Galois section-class* determined by $v \in V(A)$.

Lemma 1.7. Let $q > 1$ be a power of p such that A contains a field of cardinality q . Then the following assertions hold:

- (i) The Artin–Schreier sequence of degree q on V determines an isomorphism of groups

$$\mathcal{O}_V(V)/\varphi_q(\mathcal{O}_V(V)) \xrightarrow{\sim} H^1(\Pi, \mathbb{F}_q).$$

- (ii) Let $v \in V(A)$ be an A -rational point of V . Then the diagram of groups

$$\begin{array}{ccccc} A/\varphi_q(A) & \longrightarrow & \mathcal{O}_V(V)/\varphi_q(\mathcal{O}_V(V)) & \longrightarrow & A/\varphi_q(A) \\ \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\ H^1(G, \mathbb{F}_q) & \longrightarrow & H^1(\Pi, \mathbb{F}_q) & \longrightarrow & H^1(G, \mathbb{F}_q) \end{array}$$

— where

- the left-hand upper horizontal arrow is the homomorphism induced by the structure morphism of V over A ,

- the right-hand upper horizontal arrow is the homomorphism induced by $v \in V(A)$,
- the left-hand lower horizontal arrow is the homomorphism induced by the continuous homomorphism $\Pi \rightarrow G$,
- the right-hand lower horizontal arrow is the homomorphism induced by [an arbitrary element $G \rightarrow \Pi$ of] the Galois section-class determined by $v \in V(A)$, and
- the vertical arrows are the isomorphisms discussed in (i) [in the case where we take the “ V ” of (i) to be V and $\text{Spec}(A)$] —
commutes.

PROOF. Assertion (i) follows immediately from our assumption that the algebraic variety V is affine, together with our assumption that A contains a field of cardinality q . Assertion (ii) is a formal consequence of the [easily verified] functoriality of Artin–Schreier classes. This completes the proof of Lemma 1.7. $\square$

Theorem 1.8. *Suppose that there exists a finite Galois extension field of A of degree p . Let $v, w \in V(A)$ be A -rational points of V . Then the following two conditions are equivalent:*

- (1) *The equality $v = w$ holds.*
- (2) *The Galois section-class determined by v coincides with the Galois section-class determined by w .*

PROOF. First, observe that the implication (1) $\Rightarrow$ (2) is immediate. Next, we verify the implication (2) $\Rightarrow$ (1). Suppose that condition (2) is satisfied.

Next, observe that since the algebraic variety V over A is affine, there exists a closed immersion over A from V into the affine space $\mathbb{A}_A^r$ for some positive integer r . Then one verifies easily that, to verify condition (1), we may assume without loss of generality, by replacing V by this affine space, that V is the affine space $\mathbb{A}_A^r$. Next, observe that it is immediate that, to verify condition (1), we may assume without loss of generality, by considering the various projection morphisms $V = \mathbb{A}_A^r \rightarrow \mathbb{A}_A^1$, that V is the affine line $\mathbb{A}_A^1$.

Let $z \in \mathcal{O}_V(V)$ be a global section of $\mathcal{O}_V$ that determines an isomorphism $V \xrightarrow{\sim} \text{Spec}(A[z])$. Write $b \stackrel{\text{def}}{=} z(v) - z(w) \in A$. Then it follows from the commutativity of the right-hand square of the diagram of Lemma 1.7, (ii) [in the case where we take the “ q ” of Lemma 1.7 to be p], together with condition (2), that the homomorphisms $\mathcal{O}_V(V)/\varphi_p(\mathcal{O}_V(V)) \rightarrow A/\varphi_p(A)$ induced by $v, w \in V(A)$ coincide. Let c be an element of A . By considering the images of the

element $c \cdot z \in \mathcal{O}_V(V)$ by these homomorphisms, one concludes that the element $c \cdot b = c \cdot z(v) - c \cdot z(w)$ of A is contained in the subset $\varphi_p(A) \subseteq A$ of A . Thus, since c is an arbitrary element of A , it follows from Lemma 1.2, (ii), that the equality $b = 0$, i.e., the equality $v = w$, holds. This completes the proof of the implication $(2) \Rightarrow (1)$, hence also of Theorem 1.8. $\square$

Remark 1.8.1. Let $q > 1$ be a power of p . Suppose that A is the field $\mathbb{F}_q((t))$ — where t is an indeterminate — of formal Laurent series with coefficients in $\mathbb{F}_q$. Write $R \stackrel{\text{def}}{=} \mathbb{F}_q[[t]] \subseteq A$ for the valuation ring of the discrete valuation field A and $A^{\text{pf}} \subseteq A^{\text{alg}}$ for the perfect closure of A in A^{alg} . Let C be a proper smooth curve over A of genus ≥ 2 . Suppose, moreover, that there exists a semistable curve $\mathcal{C}$ over R that is smooth over R such that the generic fiber $\mathcal{C} \times_R A$ of $\mathcal{C}$ coincides with C , and the special fiber $\mathcal{C}_{\mathbb{F}_q} \stackrel{\text{def}}{=} \mathcal{C} \times_R \mathbb{F}_q$ of $\mathcal{C}$ has no $\mathbb{F}_q$ -rational point. [For instance, by considering suitable deformations over R of a proper smooth curve over $\mathbb{F}_q$ that has no $\mathbb{F}_q$ -rational point, one obtains such a proper smooth curve C over A that is not isotrivial.] Write Π_C , Δ_C for the respective étale fundamental groups of C , $C \times_A \bar{A}$, relative to suitable choices of basepoints. Thus, we have a natural continuous surjective homomorphism $\Pi_C \twoheadrightarrow G$ whose kernel is given by $\Delta_C \subseteq \Pi_C$.

- (i) Since the field A^{pf} is a purely inseparable extension of A , one verifies easily that each A^{pf} -rational point of C determines, in a similar way to Definition 1.6, a Δ_C -conjugacy class of continuous sections of the continuous homomorphism $\Pi_C \twoheadrightarrow G$. On the other hand, since the semistable curve $\mathcal{C}$ over R is proper, and the residue field of the integral closure of R in A^{pf} is $\mathbb{F}_q$, it follows from the valuative criterion of properness, together with our assumption that $\mathcal{C}_{\mathbb{F}_q}$ has no $\mathbb{F}_q$ -rational point, that C has no A^{pf} -rational point.
- (ii) Write $\eta \in \mathcal{C}$ for the generic point of $\mathcal{C}_{\mathbb{F}_q}$, $A(C)$ for the function field of C , L for the completion of $A(C)$ with respect to the discrete valuation determined by η , and k_L for the residue field of L , i.e., the function field of $\mathcal{C}_{\mathbb{F}_q}$. Here, observe that since $\mathcal{C}_{\mathbb{F}_q}$ is reduced, the element t is a uniformizer of L . Moreover, one verifies easily that $\mathbb{F}_q$ is algebraically closed in k_L . Let $\bar{L}$ be a separable closure of L , and let $\bar{A} \hookrightarrow \bar{L}$ be an embedding over A . Then since the absolute Galois group of $\mathbb{F}_q$ is a free profinite group of rank one, hence projective [cf. [16, Proposition 3.5.15]], the natural continuous surjective homomorphism from the absolute Galois group of k_L to the absolute Galois group

of $\mathbb{F}_q$ — where we consider the absolute Galois groups determined by the residue field of $\bar{L}$ — admits a continuous section. Write $M \subseteq \bar{L}$ for the unramified extension field of L that corresponds to the image of such a continuous section and k_M for the residue field of M . Then it is immediate that the natural continuous homomorphism from the absolute Galois group of k_M to the absolute Galois group of $\mathbb{F}_q$ is an isomorphism. Thus, since t is a uniformizer of M , one verifies easily that the natural continuous homomorphism $G_M \stackrel{\text{def}}{=} \text{Gal}(\bar{L}/M) \rightarrow G$ is surjective and induces an isomorphism between the maximal tamely ramified quotients. In particular, the kernel of the continuous homomorphism $G_M \rightarrow G$ is a pro- p group.

- (iii) Since the inequality $\text{cd}_p(G) \leq 1$ holds [cf. [16, Corollary 6.1.3]], it follows from [16, Theorem 3.5.6] that the continuous homomorphism $G_M \rightarrow G$ discussed in (ii) admits a continuous section. Write $G_{A(C)} \stackrel{\text{def}}{=} \text{Gal}(\overline{A(C)}/A(C))$ for the absolute Galois group of $A(C)$ determined by the separable closure $\overline{A(C)}$ of $A(C)$ in $\bar{L}$. Then since the continuous homomorphism $G_M \rightarrow G$ factors as the composite of the natural continuous homomorphisms $G_M \rightarrow G_{A(C)} \twoheadrightarrow \Pi_C \twoheadrightarrow G$ [relative to a suitable choice of basepoint], one concludes that the natural continuous surjective homomorphisms $G_{A(C)} \twoheadrightarrow G$ and $\Pi_C \twoheadrightarrow G$ admit continuous sections. In particular, it follows from (i) that an analogue over A of the Grothendieck section conjecture — even its birational version — does not hold.
- (iv) On the other hand, in contrast to the situation over positive-characteristic local fields discussed in (iii), the birational version of the Grothendieck section conjecture for proper smooth curves over mixed-characteristic local fields holds [cf. [7, Proposition 2.4, (2)]].

2. ÉTALE FUNDAMENTAL GROUPS OF SMOOTH CURVES

In the present §2, we discuss some basic properties of continuous isomorphisms between the étale fundamental groups of smooth curves over fields of formal Laurent series with coefficients in finite fields [cf. Lemma 2.2 below]. In particular, we prove that, under the assumption of stable reduction, such an isomorphism induces an isomorphism between the log fundamental groups of the special fibers of the stable models [cf. Lemma 2.10 below]. This discussion plays an essential role in §3. Moreover, we introduce the notions of base-geometric and cyclotomically compatible continuous isomorphisms [cf. Definition 2.3,

Definition 2.4 below] and prove that, in the case where the smooth curves under consideration are not isotrivial, a base-geometric continuous isomorphism becomes cyclotomically compatible after composing with the isomorphism induced by a suitable relative Frobenius morphism [cf. Lemma 2.12 below].

In the present §2, for each $\square \in \{\circ, \bullet\}$, let

- $p_\square$ be a prime number,
- $F_\square$ a finite field of characteristic $p_\square$,
- $K_\square^{\text{alg}}$ an algebraic closure of the field $K_\square \stackrel{\text{def}}{=} F_\square((t))$ — where t is an indeterminate — of formal Laurent series with coefficients in $F_\square$, and
- $X_\square$ a smooth curve over $K_\square$ [cf. Definition 1.3] that is either not of genus zero [cf. Definition 1.4] or not proper over $K_\square$;

write

- $\overline{K}_\square$ for the separable closure in $K_\square^{\text{alg}}$ of $K_\square$,
- $R_\square \stackrel{\text{def}}{=} F_\square[[t]] \subseteq K_\square$ for the valuation ring of the discrete valuation field $K_\square$,
- $\overline{F}_\square$ for the algebraic closure of $F_\square$ in $\overline{K}_\square$,
- $q_\square \stackrel{\text{def}}{=} \#F_\square$ for the cardinality of $F_\square$,
- $G_\square \stackrel{\text{def}}{=} \text{Gal}(\overline{K}_\square/K_\square)$ for the absolute Galois group of $K_\square$ determined by the separable closure $\overline{K}_\square$,
- $P_\square \subseteq I_\square \subseteq G_\square$ for the wild inertia, inertia subgroups of $G_\square$, respectively,
- $G_\square^{\text{ur}} \stackrel{\text{def}}{=} G_\square/I_\square \leftarrow G_\square^{\text{tm}} \stackrel{\text{def}}{=} G_\square/P_\square (\leftarrow G_\square)$ for the maximal unramified, tamely ramified quotients of $G_\square$, respectively,
- $X_\square^+$ for the regular compactification of $X_\square$ over $K_\square$,
- $D_\square \subseteq X_\square^+$ for the reduced closed subscheme of $X_\square^+$ whose underlying closed subset is given by the complement in $X_\square^+$ of $X_\square$,
- $g_\square$ for the genus of the smooth curve $X_\square$ over $K_\square$ [cf. Definition 1.4],
- $r_\square$ for the cardinality of the [necessarily finite] set $D_\square(K_\square^{\text{alg}})$,
- $\overline{X}_\square \stackrel{\text{def}}{=} X_\square \times_{K_\square} \overline{K}_\square \subseteq \overline{X}_\square^+ \stackrel{\text{def}}{=} X_\square^+ \times_{K_\square} \overline{K}_\square$,
- $\Pi_\square, \Pi_\square^+$ for the respective étale fundamental groups of $X_\square, X_\square^+$, relative to suitable choices of basepoints, and
- $\Delta_\square, \Delta_\square^+$ for the respective étale fundamental groups of $\overline{X}_\square, \overline{X}_\square^+$, relative to suitable choices of basepoints.

Thus, for each $\square \in \{\circ, \bullet\}$, the natural diagram of schemes

$$\begin{array}{ccc} \overline{X}_{\square} & \longrightarrow & X_{\square} \\ \downarrow & & \downarrow \\ \overline{X}_{\square}^+ & \longrightarrow & X_{\square}^+ \end{array}$$

induces a commutative diagram of topological groups

$$\begin{array}{ccccccc} 1 & \longrightarrow & \Delta_{\square} & \longrightarrow & \Pi_{\square} & \longrightarrow & G_{\square} \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \parallel \\ 1 & \longrightarrow & \Delta_{\square}^+ & \longrightarrow & \Pi_{\square}^+ & \longrightarrow & G_{\square} \longrightarrow 1 \end{array}$$

— where the horizontal sequences are exact, and the vertical arrows are surjective. In the present §2, let

$$\alpha: \Pi_{\circ} \xrightarrow{\sim} \Pi_{\bullet}$$

be a continuous isomorphism that lies over some uniquely determined continuous isomorphism $\alpha_G: G_{\circ} \xrightarrow{\sim} G_{\bullet}$, i.e., that fits into a commutative diagram of topological groups

$$\begin{array}{ccc} \Pi_{\circ} & \xrightarrow[\sim]{\alpha} & \Pi_{\bullet} \\ \downarrow & & \downarrow \\ G_{\circ} & \xrightarrow[\alpha_G]{\sim} & G_{\bullet} \end{array}$$

— where the vertical arrows are the natural surjective homomorphisms
— for some uniquely determined continuous isomorphism $\alpha_G: G_{\circ} \xrightarrow{\sim} G_{\bullet}$.

Definition 2.1. Let $\square$ be an element of $\{\circ, \bullet\}$.

- (i) We shall write $\widehat{\mathbb{Z}}_{(\neq p_{\square})}$ for the pro-prime-to- $p_{\square}$ completion of the module $\mathbb{Z}$.
- (ii) We shall write $\Lambda_{\overline{K}_{\square}}$ for the Tate module of the multiplicative group scheme $\mathbb{G}_{m, \overline{K}_{\square}}$ over $\overline{K}_{\square}$ and refer to $\Lambda_{\overline{K}_{\square}}$ as the *cyclotome* associated to $\overline{K}_{\square}$.
- (iii) Suppose that the smooth curve $X_{\square}$ over $K_{\square}$ is of genus ≥ 1 . Then we shall write

$$\Lambda_{X_{\square}} \stackrel{\text{def}}{=} \text{Hom}_{\widehat{\mathbb{Z}}_{(\neq p_{\square})}}(H^2(\Delta_{\square}^+, \widehat{\mathbb{Z}}_{(\neq p_{\square})}), \widehat{\mathbb{Z}}_{(\neq p_{\square})})$$

and refer to $\Lambda_{X_{\square}}$ as the *cyclotome* associated to $X_{\square}$.

Remark 2.1.1. Let $\square$ be an element of $\{\circ, \bullet\}$.

- (i) One verifies easily that the module $\Lambda_{\overline{K}_\square}$ has a natural structure of free $\widehat{\mathbb{Z}}_{(\neq p_\square)}$ -module of rank one.
- (ii) Suppose that the smooth curve $X_\square$ over $K_\square$ is of genus ≥ 1 . Then it follows from [8, Chapter VI, Theorem 11.1, (a)], together with the [well-known — cf. [1, Exposé IX, Théorème 4.10] and footnote 5 of [1, Exposé IX, §4]] fact that the natural morphism to $\overline{X}_\square^+$ from the smooth compactification of the smooth curve $X_\square \times_{K_\square} K_\square^{\text{alg}}$ over $K_\square^{\text{alg}}$ induces a continuous outer isomorphism between the étale fundamental groups, that
 - (a) the module $\Lambda_{X_\square}$ has a natural structure of free $\widehat{\mathbb{Z}}_{(\neq p_\square)}$ -module of rank one, and that
 - (b) there exists a natural $G_\square$ -equivariant isomorphism $\Lambda_{X_\square} \xrightarrow{\sim} \Lambda_{\overline{K}_\square}$.

Lemma 2.2. *The following assertions hold:*

- (i) *The equality $(p_\circ, q_\circ, g_\circ, r_\circ) = (p_\bullet, q_\bullet, g_\bullet, r_\bullet)$ holds.*
- (ii) *The continuous isomorphism α determines a bijective map between the set of cuspidal inertia subgroups [i.e., inertia subgroups associated to cusps] of $\Pi_\circ$ and the set of cuspidal inertia subgroups of $\Pi_\bullet$.*
- (iii) *There exists a uniquely determined continuous outer isomorphism $\alpha^+ : \Pi_\circ^+ \xrightarrow{\sim} \Pi_\bullet^+$ such that the continuous isomorphisms α, α^+ fit into a commutative diagram of topological groups*

$$\begin{array}{ccc} \Pi_\circ & \xrightarrow[\sim]{\alpha} & \Pi_\bullet \\ \downarrow & & \downarrow \\ \Pi_\circ^+ & \xrightarrow[\alpha^+]{\sim} & \Pi_\bullet^+ \end{array}$$

— where the vertical arrows are the $\Delta_\circ^+$ -, $\Delta_\bullet^+$ -conjugacy classes of continuous surjective homomorphisms induced by the natural open immersions $X_\circ \hookrightarrow X_\circ^+$, $X_\bullet \hookrightarrow X_\bullet^+$, respectively.

- (iv) *The equalities*

$$\alpha_G(I_\circ) = I_\bullet, \quad \alpha_G(P_\circ) = P_\bullet$$

hold.

PROOF. First, we verify assertions (i), (ii). First, recall from [1, Exposé IX, Théorème 4.10] and footnote 5 of [1, Exposé IX, §4] that the natural morphism $X_\square \times_{K_\square} K_\square^{\text{alg}} \rightarrow \overline{X}_\square$ induces a continuous outer isomorphism between the étale fundamental groups. Thus, it follows from [22, Theorem 1.9] that the equalities $g_\circ = g_\bullet$, $r_\circ = r_\bullet$ hold. In particular, assertion (ii) is immediate in the case where $r_\circ = 0$ and follows

from [22, Theorem 2.5] in the case where $r_o \neq 0$. Next, the equality $(p_o, q_o) = (p_\bullet, q_\bullet)$ follows from [17, Theorem 2.6, (i)]. This completes the proof of assertions (i), (ii).

Assertion (iii) follows immediately from assertion (ii). Assertion (iv) is a formal consequence of [17, Theorem 2.6, (ii), (iii)]. This completes the proof of Lemma 2.2. $\square$

Definition 2.3. We shall say that the continuous isomorphism α is *base-geometric* if the continuous isomorphism α_G arises from an isomorphism $\bar{K}_\bullet \xrightarrow{\sim} \bar{K}_o$ of fields — which is uniquely determined [cf. the injectivity of the natural homomorphism from the automorphism group of $K_\bullet$ to the outer automorphism group of $G_\bullet$, together with the slimness of $G_\bullet$ — cf. [4, Theorem 2.4], [4, Corollary 4.5], respectively] — that restricts to an isomorphism $K_\bullet \xrightarrow{\sim} K_o$ of subfields.

Definition 2.4. Suppose that the continuous isomorphism α is base-geometric. Then we shall say that the continuous isomorphism α is *cyclotomically compatible* if the following condition is satisfied: Let $J_o \subseteq \Pi_o$ be an open subgroup of Π_o . Write $J_\bullet \stackrel{\text{def}}{=} \alpha(J_o) \subseteq \Pi_\bullet$. For each $\square \in \{o, \bullet\}$, write $Y_\square \rightarrow X_\square$ for the connected finite étale covering of $X_\square$ that arises from the open subgroup $J_\square \subseteq \Pi_\square$ of $\Pi_\square$, which thus implies that $J_\square$ may be identified with the étale fundamental group of $Y_\square$. For each $\square \in \{o, \bullet\}$, we regard $Y_\square$ as a smooth curve over the algebraic closure of $K_\square$ in the function field of $Y_\square$. Suppose that the smooth curve Y_o , hence [cf. Lemma 2.2, (i)] also the smooth curve $Y_\bullet$, is of genus ≥ 1 . Then the diagram of topological modules

$$\begin{array}{ccc} \Lambda_{Y_o} & \xrightarrow{\sim} & \Lambda_{Y_\bullet} \\ \wr \downarrow & & \downarrow \wr \\ \Lambda_{\bar{K}_o} & \xrightarrow{\sim} & \Lambda_{\bar{K}_\bullet} \end{array}$$

— where the upper horizontal arrow is the isomorphism induced by the continuous isomorphism α [cf. Lemma 2.2, (iii)], the vertical arrows are the respective natural isomorphisms of Remark 2.1.1, (ii), (b), and the lower horizontal arrow is an isomorphism determined by the isomorphism $\bar{K}_\bullet \xrightarrow{\sim} \bar{K}_o$ of fields that induces the continuous isomorphism α_G — commutes.

Remark 2.4.1. Suppose that the continuous isomorphism α is base-geometric.

- (i) Let $J'_o \subseteq J_o$ be open subgroups of Π_o . Write $J'_\bullet \stackrel{\text{def}}{=} \alpha(J'_o) \subseteq J_\bullet \stackrel{\text{def}}{=} \alpha(J_o)$. For each $\square \in \{o, \bullet\}$, write $Y'_\square \rightarrow Y_\square \rightarrow X_\square$ for

the connected finite étale coverings of $X_\square$ that arise from the open subgroups $J'_\square \subseteq J_\square \subseteq \Pi_\square$ of $\Pi_\square$, respectively. Suppose that the smooth curve $Y_\circ$, hence [cf. Lemma 2.2, (i)] also the smooth curve $Y_\bullet$, is of genus ≥ 1 , which thus implies that, for each $\square \in \{\circ, \bullet\}$, the smooth curve $Y'_\square$ is of genus ≥ 1 . Then it follows from [8, Chapter VI, Theorem 11.1, (a)] that, for each $\square \in \{\circ, \bullet\}$, the homomorphism $\Lambda_{Y'_\square} \rightarrow \Lambda_{Y_\square}$ induced by the natural finite morphism between the regular compactifications of $Y'_\square$, $Y_\square$ coincides, relative to the natural isomorphisms of Remark 2.1.1, (ii), (b), with the endomorphism of $\Lambda_{\bar{K}_\square}$ given by multiplication by the index of $\Delta_\square \cap J'_\square$ in $\Delta_\square \cap J_\square$. Moreover, since [it is immediate that] the equality $\alpha(\Delta_\circ) = \Delta_\bullet$ holds, these two indices coincide. Thus, since the module $\Lambda_{\bar{K}_\circ}$ is torsion-free [cf. Remark 2.1.1, (i)], the diagram that appears in Definition 2.4 associated to $J_\circ$ commutes if and only if the diagram that appears in Definition 2.4 associated to $J'_\circ$ commutes.

- (ii) Let $J_\circ \subseteq \Pi_\circ$ be an open subgroup of $\Pi_\circ$ such that the corresponding smooth curve is of genus ≥ 1 , and that the diagram that appears in Definition 2.4 associated to $J_\circ$ commutes. Then, for each open subgroup $J'_\circ \subseteq \Pi_\circ$ of $\Pi_\circ$ such that the corresponding smooth curve is of genus ≥ 1 , by applying (i) to the open subgroups $J_\circ \cap J'_\circ \subseteq J_\circ$ and $J_\circ \cap J'_\circ \subseteq J'_\circ$, one concludes that the diagram that appears in Definition 2.4 associated to $J'_\circ$ commutes. In particular, the continuous isomorphism α is cyclotomically compatible.

Definition 2.5. Let $\square$ be an element of $\{\circ, \bullet\}$, and let n be a nonnegative integer.

- (i) We shall write $\text{Fr}_{K_\square}(n)$ for the absolute $p_\square^n$ -th power Frobenius endomorphism of $\text{Spec}(K_\square)$.
- (ii) We shall write $X_\square(n)$ for the smooth curve over $K_\square$ obtained by pulling back the smooth curve $X_\square$ over $K_\square$ by $\text{Fr}_{K_\square}(n)$.
- (iii) We shall write $\text{Fr}_{X_\square/K_\square}(n): X_\square \rightarrow X_\square(n)$ for the relative $p_\square^n$ -th power Frobenius morphism over $K_\square$ [i.e., the morphism determined by the absolute $p_\square^n$ -th power Frobenius endomorphism of $X_\square$].

Remark 2.5.1. Let $\square$ be an element of $\{\circ, \bullet\}$, and let n be a nonnegative integer. Then one verifies easily that the relative $p_\square^n$ -th power Frobenius morphism $\text{Fr}_{X_\square/K_\square}(n): X_\square \rightarrow X_\square(n)$ induces a continuous isomorphism over $G_\square$ from the étale fundamental group $\Pi_\square$ of $X_\square$ to the étale fundamental group of $X_\square(n)$ [which is uniquely determined

up to composition with inner automorphisms determined by elements of the kernel of the natural surjective homomorphism from the étale fundamental group of $X_\square(n)$ onto $G_\square$.

Definition 2.6. Let $\square$ be an element of $\{\circ, \bullet\}$. Then we shall say that the smooth curve $X_\square$ over $K_\square$ admits *stable reduction* over $R_\square$ if there exist a semistable curve $\mathcal{X}_\square^+$ over $R_\square$ and a [possibly empty] closed subscheme $\mathcal{D}_\square \subseteq \mathcal{X}_\square^+$ of $\mathcal{X}_\square^+$ such that

- the closed subscheme $\mathcal{D}_\square \subseteq \mathcal{X}_\square^+$ of $\mathcal{X}_\square^+$ is contained in the smooth locus of $\mathcal{X}_\square^+$ over $R_\square$,
- the composite $\mathcal{D}_\square \hookrightarrow \mathcal{X}_\square^+ \rightarrow \text{Spec}(R_\square)$ is finite and étale,
- the open subscheme $(\mathcal{X}_\square^+ \setminus \mathcal{D}_\square) \times_{R_\square} K_\square$ of $\mathcal{X}_\square^+$ coincides with $X_\square$, and
- the invertible sheaf $\omega_{\mathcal{X}_\square^+/R_\square}(\mathcal{D}_\square)$ on $\mathcal{X}_\square^+$ — where we write $\omega_{\mathcal{X}_\square^+/R_\square}$ for the dualizing sheaf of $\mathcal{X}_\square^+$ over $R_\square$ — is ample relative to $R_\square$.

Observe that one verifies immediately from [6, Theorem 2.7] that, in this situation, the pair $(\mathcal{X}_\square^+, \mathcal{D}_\square)$ is uniquely determined by $X_\square$.

Remark 2.6.1. Let S be a scheme, $\mathcal{X}_1$ and $\mathcal{X}_2$ semistable curves over S , and $\mathcal{D}_1 \subseteq \mathcal{X}_1$, $\mathcal{D}_2 \subseteq \mathcal{X}_2$ [possibly empty] closed subschemes of $\mathcal{X}_1$, $\mathcal{X}_2$, respectively. Suppose that the closed subschemes $\mathcal{D}_1 \subseteq \mathcal{X}_1$, $\mathcal{D}_2 \subseteq \mathcal{X}_2$ are contained in the smooth loci of $\mathcal{X}_1$, $\mathcal{X}_2$ over S , respectively, that the composites $\mathcal{D}_1 \hookrightarrow \mathcal{X}_1 \rightarrow S$, $\mathcal{D}_2 \hookrightarrow \mathcal{X}_2 \rightarrow S$ are finite and étale, and that the invertible sheaves $\omega_{\mathcal{X}_1/S}(\mathcal{D}_1)$, $\omega_{\mathcal{X}_2/S}(\mathcal{D}_2)$ on $\mathcal{X}_1$, $\mathcal{X}_2$ — where we write $\omega_{\mathcal{X}_1/S}$, $\omega_{\mathcal{X}_2/S}$ for the dualizing sheaves of $\mathcal{X}_1$, $\mathcal{X}_2$ over S , respectively — are ample relative to S . Then one verifies immediately from [6, Theorem 2.7] that the functor that assigns, to a scheme T over S , the set of isomorphisms over T of the pair $(\mathcal{X}_1 \times_S T, \mathcal{D}_1 \times_S T)$ with the pair $(\mathcal{X}_2 \times_S T, \mathcal{D}_2 \times_S T)$ is representable by a scheme that is finite and unramified over S .

Remark 2.6.2. Let $\square$ be an element of $\{\circ, \bullet\}$. Suppose that the inequality $2 - 2g_\square - r_\square < 0$ holds. Then it follows immediately from standard limit arguments that there exists a nonnegative integer n_0 such that, for each integer $n \geq n_0$, the regular compactification of $X_\square(n)$ over $K_\square$ is smooth over $K_\square$, and, moreover, the complement of $X_\square(n)$ in this regular compactification, equipped with the reduced scheme structure, is finite and étale over $K_\square$. In particular, it follows from [6, Theorem 2.7] that, for each integer $n \geq n_0$, there exists a finite extension field $L_\square \subseteq \overline{K}_\square$ of $K_\square$ such that the smooth curve $X_\square(n) \times_{K_\square} L_\square$ over $L_\square$ admits stable reduction over the ring of integers of $L_\square$.

Definition 2.7. Let $\square$ be an element of $\{\circ, \bullet\}$. Then we shall say that the smooth curve $X_\square$ over $K_\square$ admits *good reduction* over $R_\square$ if the smooth curve $X_\square$ over $K_\square$ admits stable reduction over $R_\square$ — which implies that there exists a uniquely determined pair $(\mathcal{X}_\square^+, \mathcal{D}_\square)$ as in Definition 2.6 — and, moreover, the semistable curve $\mathcal{X}_\square^+$ over $R_\square$ is smooth.

Remark 2.7.1. Let $\square$ be an element of $\{\circ, \bullet\}$. Suppose that the inequality $2 - 2g_\square - r_\square < 0$ holds, and that the smooth curve $X_\square$ over $K_\square$ is strictly isotrivial [cf. Definition 1.5, (ii)]. Then one verifies easily that the smooth curve $X_\square$ over $K_\square$ admits good reduction over $R_\square$.

Remark 2.7.2. Let $\square$ be an element of $\{\circ, \bullet\}$. Suppose that the smooth curve $X_\square$ over $K_\square$ is isotrivial [cf. Definition 1.5, (i)] and admits good reduction over $R_\square$. Then one verifies easily — by considering the scheme over $R_\square$ of isomorphisms between the pair $(\mathcal{X}_\square^+, \mathcal{D}_\square)$ of Definition 2.7 and the base-change, via the natural inclusion $F_\square \hookrightarrow R_\square$, of the special fiber of this pair, which is finite and unramified over $R_\square$ [cf. Remark 2.6.1], together with Galois descent — that the smooth curve $X_\square$ over $K_\square$ is strictly isotrivial [cf. Definition 1.5, (ii)].

In the remainder of the present §2, suppose that, for each $\square \in \{\circ, \bullet\}$, the smooth curve $X_\square$ over $K_\square$ admits stable reduction over $R_\square$. In particular, for each $\square \in \{\circ, \bullet\}$, we have a pair $(\mathcal{X}_\square^+, \mathcal{D}_\square)$ as in Definition 2.6.

Definition 2.8. Let $\square$ be an element of $\{\circ, \bullet\}$. Then we shall write

- $\Pi_\square^{\text{tm}}$ for the tame fundamental group of the pair $(X_\square^+, \mathcal{D}_\square)$, relative to a suitable choice of basepoint,
- $\Delta_\square^{\text{tm}}$ for the tame fundamental group of the pair $(\overline{X}_\square^+, \mathcal{D}_\square \times_{K_\square} \overline{K}_\square)$, relative to a suitable choice of basepoint,
- $\underline{\mathcal{X}}_\square^+ \stackrel{\text{def}}{=} \mathcal{X}_\square^+ \times_{R_\square} F_\square$ for the special fiber of the semistable curve $\mathcal{X}_\square^+$ over $R_\square$,
- $\mathcal{X}_\square^{\log}$ for the [necessarily fs] log scheme obtained by equipping $\mathcal{X}_\square^+$ with the log structure determined by the divisor $\underline{\mathcal{X}}_\square^+ \cup \mathcal{D}_\square$ on $\mathcal{X}_\square^+$,
- $\underline{\mathcal{X}}_\square^{\log}$ for the [necessarily fs] log scheme obtained by equipping $\underline{\mathcal{X}}_\square^+$ with the log structure obtained by pulling back the log structure of $\mathcal{X}_\square^{\log}$ by the natural closed immersion $\underline{\mathcal{X}}_\square^+ \hookrightarrow \mathcal{X}_\square^+$,
- $\Pi_\square^{\log}$ for the log fundamental group of $\underline{\mathcal{X}}_\square^{\log}$, relative to a suitable choice of basepoint, and
- $\Delta_\square^{\log}$ for the kernel of the natural continuous outer homomorphism $\Pi_\square^{\log} \twoheadrightarrow G_\square^{\text{tm}}$.

Remark 2.8.1. Let $\square$ be an element of $\{\circ, \bullet\}$. The various results of [5, §6, §7] are stated for stable curves over complete discrete valuation rings whose generic fibers are proper and smooth, i.e., for stable curves without marked points. On the other hand, one verifies easily that the arguments applied in the proofs of [5, Lemma 6.5], [5, Lemma 6.9], [5, Lemma 6.11], [5, Lemma 7.1], and [5, Lemma 7.3] remain valid, without essential modification, in the case of the pair $(\mathcal{X}_{\square}^+, \mathcal{D}_{\square})$, i.e., in the case where marked points are allowed, if we replace the étale fundamental group of the generic fiber and the log schemes of [5, Definition 6.1] by $\Pi_{\square}^{\text{tm}}$ and $\mathcal{X}_{\square}^{\log}, \underline{\mathcal{X}}_{\square}^{\log}$ [cf. Definition 2.8], respectively. Indeed, the properties of log fundamental groups applied in these proofs are consequences of [3, Corollary 1] and [3, Proposition B.7], which are valid for arbitrary log regular log schemes, hence, in particular, for the log scheme $\mathcal{X}_{\square}^{\log}$, whose log structure is determined by the divisor $\mathcal{X}_{\square}^+ \cup \mathcal{D}_{\square}$ [cf. also the proof of [10, Theorem 2.7]].

Lemma 2.9. *Let $\square$ be an element of $\{\circ, \bullet\}$. Then the following assertions hold:*

- (i) *The natural strict closed immersion $\underline{\mathcal{X}}_{\square}^{\log} \hookrightarrow \mathcal{X}_{\square}^{\log}$ induces a continuous outer isomorphism of the log fundamental group $\Pi_{\square}^{\log}$ of $\underline{\mathcal{X}}_{\square}^{\log}$ with the log fundamental group of $\mathcal{X}_{\square}^{\log}$. In particular, the natural inclusions $X_{\square} \hookrightarrow \mathcal{X}_{\square}^+ \hookleftarrow \underline{\mathcal{X}}_{\square}^+$ induce a commutative diagram of topological groups*

$$\begin{array}{ccccccc}
 1 & \longrightarrow & \Delta_{\square} & \longrightarrow & \Pi_{\square} & \longrightarrow & G_{\square} \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \parallel \\
 1 & \longrightarrow & \Delta_{\square}^{\text{tm}} & \longrightarrow & \Pi_{\square}^{\text{tm}} & \longrightarrow & G_{\square} \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 1 & \longrightarrow & \Delta_{\square}^{\log} & \longrightarrow & \Pi_{\square}^{\log} & \longrightarrow & G_{\square}^{\text{tm}} \longrightarrow 1
 \end{array}$$

— where the horizontal sequences are exact, and the vertical arrows are surjective.

- (ii) *The left-hand lower vertical arrow $\Delta_{\square}^{\text{tm}} \twoheadrightarrow \Delta_{\square}^{\log}$ of the diagram of (i) induces a continuous isomorphism between the respective maximal pro-prime-to- $p_{\square}$ quotients.*

PROOF. These assertions follow immediately from [3, Corollary 1], [3, Proposition B.7], [3, Theorem 2], together with the various definitions involved. $\square$

Lemma 2.10. *The following assertions hold:*

- (i) *There exists a uniquely determined continuous isomorphism $\alpha^{\text{tm}}: \Pi_{\circ}^{\text{tm}} \xrightarrow{\sim} \Pi_{\bullet}^{\text{tm}}$ such that the continuous isomorphisms $\alpha, \alpha^{\text{tm}}$ fit into a commutative diagram of topological groups*

$$\begin{array}{ccc} \Pi_{\circ} & \xrightarrow[\sim]{\alpha} & \Pi_{\bullet} \\ \downarrow & & \downarrow \\ \Pi_{\circ}^{\text{tm}} & \xrightarrow[\alpha^{\text{tm}}]{\sim} & \Pi_{\bullet}^{\text{tm}} \end{array}$$

— where the vertical arrows are the middle upper vertical arrows of the diagrams of Lemma 2.9, (i), in the cases where we take the “ $\square$ ” of Lemma 2.9 to be $\circ, \bullet$, respectively.

- (ii) *There exists a uniquely determined continuous isomorphism $\alpha^{\log}: \Pi_{\circ}^{\log} \xrightarrow{\sim} \Pi_{\bullet}^{\log}$ such that the continuous isomorphisms α^{tm} [cf. (i)], $\alpha^{\log}$ fit into a commutative diagram of topological groups*

$$\begin{array}{ccc} \Pi_{\circ}^{\text{tm}} & \xrightarrow[\sim]{\alpha^{\text{tm}}} & \Pi_{\bullet}^{\text{tm}} \\ \downarrow & & \downarrow \\ \Pi_{\circ}^{\log} & \xrightarrow[\alpha^{\log}]{\sim} & \Pi_{\bullet}^{\log} \end{array}$$

— where the vertical arrows are the middle lower vertical arrows of the diagrams of Lemma 2.9, (i), in the cases where we take the “ $\square$ ” of Lemma 2.9 to be $\circ, \bullet$, respectively.

- (iii) *Suppose that the isomorphism α is cyclotomically compatible. Then the isomorphism $\alpha^{\log}: \Pi_{\circ}^{\log} \xrightarrow{\sim} \Pi_{\bullet}^{\log}$ of (ii) arises from an isomorphism $\underline{\mathcal{X}}_{\circ}^{\log} \xrightarrow{\sim} \underline{\mathcal{X}}_{\bullet}^{\log}$ over the isomorphism $F_{\bullet} \xrightarrow{\sim} F_{\circ}$ induced by the isomorphism $K_{\bullet} \xrightarrow{\sim} K_{\circ}$ of Definition 2.3.*

PROOF. Assertion (i) follows immediately from Lemma 2.2, (ii). Next, we verify assertion (ii). The desired uniqueness follows from the surjectivity of the natural continuous homomorphism $\Pi_{\circ}^{\text{tm}} \rightarrow \Pi_{\circ}^{\log}$. To verify the desired existence, for each $\square \in \{\circ, \bullet\}$, write

- $N_{\square} \subseteq \Pi_{\square}^{\text{tm}}$ for the kernel of the natural continuous surjective homomorphism $\Pi_{\square}^{\text{tm}} \rightarrow \Pi_{\square}^{\log}$ that appears in the diagram of Lemma 2.9, (i),
- $\Delta_{\square}^{\text{tm}} \rightarrow \Delta_{\square}^{\text{tm}, (\neq p_{\square})}$ for the maximal pro-prime-to- $p_{\square}$ quotient of $\Delta_{\square}^{\text{tm}}$,
- $Q_{\square}^{\text{tm}}$ for the quotient of $\Pi_{\square}^{\text{tm}}$ by the kernel of $\Delta_{\square}^{\text{tm}} \rightarrow \Delta_{\square}^{\text{tm}, (\neq p_{\square})}$, which thus implies that we have a natural exact sequence of

topological groups

$$1 \longrightarrow \Delta_{\square}^{\text{tm}, (\neq p_{\square})} \longrightarrow Q_{\square}^{\text{tm}} \longrightarrow G_{\square} \longrightarrow 1,$$

- $\tilde{P}_{\square} \subseteq Q_{\square}^{\text{tm}}$ for the inverse image of $P_{\square} \subseteq G_{\square}$ by the natural continuous surjective homomorphism $Q_{\square}^{\text{tm}} \twoheadrightarrow G_{\square}$,
- $\overline{N}_{\square} \subseteq Q_{\square}^{\text{tm}}$ for the image of $N_{\square} \subseteq \Pi_{\square}^{\text{tm}}$ in $Q_{\square}^{\text{tm}}$, and
- $M_{\square} \subseteq \Pi_{\square}^{\text{tm}}$ for the inverse image of $\overline{N}_{\square} \subseteq Q_{\square}^{\text{tm}}$ by the natural surjective homomorphism $\Pi_{\square}^{\text{tm}} \twoheadrightarrow Q_{\square}^{\text{tm}}$.

Then it follows immediately from the commutativity of the diagram of Lemma 2.9, (i), that

- the intersection $N_{\square} \cap \Delta_{\square}^{\text{tm}}$ is the kernel of the natural continuous surjective homomorphism $\Delta_{\square}^{\text{tm}} \twoheadrightarrow \Delta_{\square}^{\log}$, and that
- the image of $N_{\square}$ in $G_{\square}$ coincides with $P_{\square}$.

In particular, since the natural continuous surjective homomorphism $\Delta_{\square}^{\text{tm}} \twoheadrightarrow \Delta_{\square}^{\log}$ induces an isomorphism between the respective maximal pro-prime-to- $p_{\square}$ quotients [cf. Lemma 2.9, (ii)], one verifies easily that

- the normal closed subgroup $\overline{N}_{\square} \subseteq Q_{\square}^{\text{tm}}$ of $Q_{\square}^{\text{tm}}$ is contained in $\tilde{P}_{\square}$, and that
- the intersection $\overline{N}_{\square} \cap \Delta_{\square}^{\text{tm}, (\neq p_{\square})}$ is trivial.

Thus, one concludes that

- (a) for each $\square \in \{\circ, \bullet\}$, the natural surjective homomorphism $Q_{\square}^{\text{tm}} \twoheadrightarrow G_{\square}$ restricts to an isomorphism

$$\overline{N}_{\square} \xrightarrow{\sim} P_{\square}.$$

In particular, since [it is immediate that] both $\overline{N}_{\square}$ and $\Delta_{\square}^{\text{tm}, (\neq p_{\square})}$ are normal subgroups of $Q_{\square}^{\text{tm}}$, every element of $\overline{N}_{\square}$ commutes with every element of $\Delta_{\square}^{\text{tm}, (\neq p_{\square})}$. On the other hand, since [we have assumed that] the invertible sheaf $\omega_{\mathcal{X}_{\square}^+/R_{\square}}(\mathcal{D}_{\square})$ on $\mathcal{X}_{\square}^+$ is ample relative to $R_{\square}$, it follows from [21, Proposition 1.11] that the profinite group $\Delta_{\square}^{\text{tm}, (\neq p_{\square})}$ is center-free. In particular, the restriction to the centralizer of $\Delta_{\square}^{\text{tm}, (\neq p_{\square})}$ in $\tilde{P}_{\square}$ of the natural continuous surjective homomorphism $\tilde{P}_{\square} \twoheadrightarrow P_{\square}$ is injective. Thus, one concludes that

- (b) for each $\square \in \{\circ, \bullet\}$, the closed subgroup $\overline{N}_{\square} \subseteq \tilde{P}_{\square}$ coincides with the centralizer of $\Delta_{\square}^{\text{tm}, (\neq p_{\square})}$ in $\tilde{P}_{\square}$.

Next, observe that it follows from Lemma 2.2, (i), (iv), that the equalities $p_{\circ} = p_{\bullet}$ and $\alpha_G(P_{\circ}) = P_{\bullet}$ hold. Thus, since [it is immediate that] the continuous isomorphism α^{tm} of assertion (i) lies over α_G , it follows from (b) that

- (c) the continuous isomorphism α^{tm} maps $M_{\circ} \subseteq \Pi_{\circ}^{\text{tm}}$ onto $M_{\bullet} \subseteq \Pi_{\bullet}^{\text{tm}}$.

Next, observe that since, for each $\square \in \{\circ, \bullet\}$, the closed subgroup $\overline{N}_{\square} \subseteq Q_{\square}^{\text{tm}}$ is pro- $p_{\square}$ [cf. (a)], and the profinite group $\Delta_{\square}^{\text{tm}, (\neq p_{\square})}$ is pro-prime-to- $p_{\square}$, it follows immediately from [12, Remark 1.1.5] that there exists an open subgroup $J_{\circ} \subseteq \Pi_{\circ}^{\text{tm}}$ of $\Pi_{\circ}^{\text{tm}}$ that contains $M_{\circ}$ such that the stable curve over the residue field that arises from the stable model of the finite étale covering of $X_{\circ}$ that corresponds to $J_{\circ}$ is sturdy [cf. [12, Definition 1.1]]. Write $J_{\bullet} \subseteq \Pi_{\bullet}^{\text{tm}}$ for the image of $J_{\circ}$ by α^{tm} . Then it follows from (c) that $J_{\bullet}$ contains $M_{\bullet}$, hence also $N_{\bullet}$. In particular, for each $\square \in \{\circ, \bullet\}$, it follows from [5, Lemma 6.5, (i)] [cf. also Remark 2.8.1] that the finite étale covering of $X_{\square}$ that corresponds to $J_{\square}$ admits stable reduction, and that the kernel of the natural continuous surjective homomorphism from $J_{\square}$ to the log fundamental group of the log special fiber of the resulting stable model is given by $N_{\square}$. Thus, it follows from a similar argument to the argument applied in the proof of [5, Lemma 7.3, (ii)] [cf. Remark 2.8.1] that α^{tm} maps $N_{\circ}$ onto $N_{\bullet}$. In particular, the continuous isomorphism α^{tm} induces a continuous isomorphism $\Pi_{\circ}^{\log} \xrightarrow{\sim} \Pi_{\bullet}^{\log}$ as in assertion (ii). This completes the proof of assertion (ii).

Assertion (iii) follows immediately — by passing to suitable open subgroups of $\Pi_{\circ}^{\text{tm}}$ that contain $N_{\circ}$ [cf. [5, Lemma 5.9, (iii)], [5, Lemma 6.9], [5, Lemma 6.11]], together with Galois descent — from a similar argument to the argument applied in the proof of [10, Theorem 2.7], where we apply [5, Lemma 7.3, (iv)] [cf. Remark 2.8.1] (respectively, [21, Theorem 0.5]; [11, Theorem 3.12]) in the case where $\mathcal{X}_{\circ}^{+}$ or $\mathcal{X}_{\bullet}^{+}$ is not smooth (respectively, $\mathcal{X}_{\circ}^{+}$, $\mathcal{X}_{\bullet}^{+}$ are smooth, and $\mathcal{D}_{\circ}$ is nonempty; $\mathcal{X}_{\circ}^{+}$, $\mathcal{X}_{\bullet}^{+}$ are smooth, and $\mathcal{D}_{\circ}$ is empty) — where we note that it follows from Lemma 2.2, (ii), that $\mathcal{D}_{\circ}$ is empty if and only if $\mathcal{D}_{\bullet}$ is empty. [Note that our assumption that α is cyclotomically compatible plays the role of [10, Lemma 2.6].] This completes the proof of Lemma 2.10. $\square$

Lemma 2.11. *Let $\square$ be an element of $\{\circ, \bullet\}$. Suppose that the smooth curve $X_{\square}$ over $K_{\square}$ admits good reduction over $R_{\square}$. Then the following two conditions are equivalent:*

- (1) *The smooth curve $X_{\square}$ over $K_{\square}$ is isotrivial.*
- (2) *The natural continuous surjective homomorphism $\Delta_{\square}^{\text{tm}} \twoheadrightarrow \Delta_{\square}^{\log}$ is an isomorphism.*

PROOF. This assertion follows immediately from [23, Theorem 8.1] [cf. also Remark 2.7.2]. $\square$

Lemma 2.12. *Suppose that the following two conditions are satisfied:*

- (1) *The continuous isomorphism α is base-geometric.*
- (2) *The smooth curve $X_\circ$ over $K_\circ$ is not isotrivial.*

Then there exists a uniquely determined integer n that satisfies the following two conditions:

- (3) *If n is nonnegative, then the composite of the continuous isomorphism $\alpha: \Pi_\circ \xrightarrow{\sim} \Pi_\bullet$ with some continuous isomorphism [cf. Remark 2.5.1] over $G_\bullet$ from the étale fundamental group $\Pi_\bullet$ of $X_\bullet$ to the étale fundamental group of $X_\bullet(n)$ induced by the relative $p_\bullet^n$ -th power Frobenius morphism $\mathrm{Fr}_{X_\bullet/K_\bullet}(n): X_\bullet \rightarrow X_\bullet(n)$ is cyclotomically compatible.*
- (4) *If n is negative, then the composite of the inverse of some continuous isomorphism [cf. Remark 2.5.1] over $G_\circ$ from the étale fundamental group $\Pi_\circ$ of $X_\circ$ to the étale fundamental group of $X_\circ(-n)$ induced by the relative $p_\circ^{-n}$ -th power Frobenius morphism $\mathrm{Fr}_{X_\circ/K_\circ}(-n): X_\circ \rightarrow X_\circ(-n)$ with the continuous isomorphism $\alpha: \Pi_\circ \xrightarrow{\sim} \Pi_\bullet$ is cyclotomically compatible.*

PROOF. This assertion follows immediately — in light of Remark 2.4.1, (ii), and Lemma 2.10, (i) — from a similar argument to the argument applied in the proof of [5, Lemma 7.7, (ii)]. $\square$

3. ANABELIAN RESULT FOR STRICTLY ISOTRIVIAL SMOOTH CURVES

In the present §3, we discuss continuous isomorphisms between the étale fundamental groups of smooth curves that admit stable reduction in the case where one of the smooth curves is strictly isotrivial. In particular, we give a characterization of the kernel of the natural continuous surjective homomorphism from the étale fundamental group of a strictly isotrivial smooth curve onto the étale fundamental group of a smooth curve over a finite field from which the given smooth curve arises by base-change in terms of the intersection of this kernel with the geometric fundamental group [cf. Lemma 3.1, (ii), below], together with a characterization of this intersection in terms of finite étale coverings [cf. Lemma 3.2 below]. Next, we prove that the other smooth curve is also strictly isotrivial [cf. Theorem 3.3 below]. Then, by applying the above characterizations, we prove that the given continuous isomorphism induces an isomorphism between the étale fundamental groups of the smooth curves over finite fields [cf. Theorem 3.4 below]. Thus, by applying the anabelian results for smooth curves over finite fields, we conclude that the given smooth curves are isomorphic as schemes [cf. Theorem 3.5 below].

In the present §3, suppose that we are in the situation at the beginning of the preceding §2; suppose, moreover, that, for each $\square \in \{\circ, \bullet\}$, the smooth curve $X_\square$ over $K_\square$ [cf. Definition 1.3] admits stable reduction over $R_\square$ [cf. Definition 2.6]. In the present §3, let $J_\square \subseteq \Pi_\square$ be an open normal subgroup of $\Pi_\square$. Write

- $Y_\square \rightarrow X_\square$ for the connected finite étale Galois covering of $X_\square$ that corresponds to the open normal subgroup $J_\square \subseteq \Pi_\square$,
- $H_\square \subseteq G_\square$ for the image of $J_\square$ in $G_\square$, and
- $L_\square \subseteq \overline{K}_\square$ for the [necessarily finite Galois] extension field of $K_\square$ that corresponds to the open normal subgroup $H_\square \subseteq G_\square$ of $G_\square$, which thus implies that we may regard $Y_\square$ as a smooth curve over $L_\square$.

In the present §3, moreover, for each $\square \in \{\circ, \bullet\}$, if the smooth curve $X_\square$ over $K_\square$ is strictly isotrivial, then fix a smooth curve $X_{\square,0}$ over $F_\square$ and an isomorphism $X_\square \xrightarrow{\sim} X_{\square,0} \times_{F_\square} K_\square$ over $K_\square$ [cf. Definition 1.5, (ii)] and write

- $\Pi_{\square,0}$ for the étale fundamental group of $X_{\square,0}$, relative to a suitable choice of basepoint,
- $\Delta_{\square,0}$ for the étale fundamental group of $X_{\square,0} \times_{F_\square} \overline{F}_\square$, relative to a suitable choice of basepoint,
- $N_\square^{\text{cst}} \subseteq \Pi_\square$ for the kernel of the natural continuous outer homomorphism $\Pi_\square \rightarrow \Pi_{\square,0}$ induced by the composite of the fixed isomorphism $X_\square \xrightarrow{\sim} X_{\square,0} \times_{F_\square} K_\square$ and the projection $X_{\square,0} \times_{F_\square} K_\square \rightarrow X_{\square,0}$ onto the first factor, and
- $\Delta_\square^{\text{cst}} \stackrel{\text{def}}{=} N_\square^{\text{cst}} \cap \Delta_\square$.

Lemma 3.1. *Let $\square$ be an element of $\{\circ, \bullet\}$. Suppose that the smooth curve $X_\square$ over $K_\square$ is strictly isotrivial. Then the following assertions hold:*

- (i) *The natural continuous homomorphisms $\Pi_\square \rightarrow \Pi_{\square,0}$, $\Delta_\square \rightarrow \Delta_{\square,0}$ are surjective. Moreover, the natural continuous homomorphism*

$$\Pi_\square / \Delta_\square^{\text{cst}} \longrightarrow \Pi_{\square,0} \times_{G_\square^{\text{ur}}} G_\square$$

— *where we identify $G_\square^{\text{ur}}$ with $\text{Gal}(\overline{F}_\square / F_\square)$ in the natural way*
 — *is an isomorphism that restricts to an isomorphism of $N_\square^{\text{cst}} / \Delta_\square^{\text{cst}}$ with $\{1\} \times I_\square$ of subgroups.*

- (ii) *The closed subgroup $N_\square^{\text{cst}} \subseteq \Pi_\square$ of $\Pi_\square$ coincides with the set of elements $\gamma \in \Pi_\square$ such that $\gamma \delta \gamma^{-1} \delta^{-1} \in \Delta_\square^{\text{cst}}$ for every $\delta \in \Delta_\square$.*

PROOF. First, we verify assertion (i). Since $\overline{F}_\square$ is algebraically closed, one verifies immediately that every connected finite étale covering of $X_{\square,0} \times_{F_\square} \overline{F}_\square$ remains connected after base-changing via $\overline{F}_\square \hookrightarrow \overline{K}_\square$. In particular, one concludes that the continuous homomorphism $\Delta_\square \rightarrow \Delta_{\square,0}$ is surjective. Moreover, since $K_\square = F_\square((t))$, the natural continuous homomorphism $G_\square \rightarrow \text{Gal}(\overline{F}_\square/F_\square)$ is a surjective homomorphism whose kernel is given by $I_\square$. Thus, the remaining portion of assertion (i) follows immediately from the various definitions involved. This completes the proof of assertion (i).

Next, we verify assertion (ii). It follows immediately from assertion (i) that the action of every element of $\Pi_\square$ on $\Delta_\square/\Delta_\square^{\text{cst}} \xrightarrow{\sim} \Delta_{\square,0}$ by conjugation coincides with the action by conjugation of the image of this element in $\Pi_{\square,0}$. Thus, it follows immediately from the various definitions involved that, to verify assertion (ii), it suffices to verify that the conjugation action of $\Pi_{\square,0}$ on $\Delta_{\square,0}$ is faithful. Let γ be an element of $\Pi_{\square,0}$ that commutes with every element of $\Delta_{\square,0}$. Write $\sigma \in \text{Gal}(\overline{F}_\square/F_\square)$ for the image of γ in $\text{Gal}(\overline{F}_\square/F_\square)$. Now since [we have assumed that] $X_\square$ admits stable reduction over $R_\square$, one verifies easily that there exists an open subgroup $J_{\square,0} \subseteq \Pi_{\square,0}$ of $\Pi_{\square,0}$ such that the smooth curve over $\overline{F}_\square$ that corresponds to the open subgroup $J_{\square,0} \cap \Delta_{\square,0} \subseteq \Delta_{\square,0}$ of $\Delta_{\square,0}$ is of genus ≥ 1 [cf. Definition 1.4, [12, Remark 1.1.5]]. Moreover, since $J_{\square,0}$ is of finite index in $\Pi_{\square,0}$, there exists a positive integer n such that $\gamma^n \in J_{\square,0}$. Since γ^n commutes with every element of $J_{\square,0} \cap \Delta_{\square,0}$ [cf. our choice of γ], the action of γ^n on the second cohomology group, with coefficients in $\widehat{\mathbb{Z}}_{(\neq p_\square)}$, of the maximal pro-prime-to- $p_\square$ quotient of the étale fundamental group of the smooth compactification of the smooth curve over $\overline{F}_\square$ that corresponds to the open subgroup $J_{\square,0} \cap \Delta_{\square,0} \subseteq \Delta_{\square,0}$ of $\Delta_{\square,0}$ is trivial. On the other hand, it follows from [8, Chapter VI, Theorem 11.1, (a)] [cf. also Remark 2.1.1, (ii)] that this action is given by the inverse of the cyclotomic character of σ^n . Thus, since $\overline{F}_\square$ is generated over $F_\square$ by the roots of unity of order prime to $p_\square$, we conclude that σ^n is trivial, which thus implies [cf. the well-known torsion-freeness of $\text{Gal}(\overline{F}_\square/F_\square)$] that σ is trivial, i.e., that $\gamma \in \Delta_{\square,0}$. In particular, one concludes that γ is contained in the center of $\Delta_{\square,0}$. Thus, it follows from [21, Proposition 1.11] that $\gamma = 1$. This completes the proof of assertion (ii), hence also of Lemma 3.1. $\square$

Lemma 3.2. *Let $\square$ be an element of $\{\circ, \bullet\}$. Suppose that the smooth curve $X_\square$ over $K_\square$ is strictly isotrivial. Then the following two conditions are equivalent:*

- (1) *The inclusion $\Delta_\square^{\text{cst}} \subseteq \Delta_\square \cap J_\square$ holds.*

- (2) *The smooth curve $Y_\square$ over $L_\square$ is isotrivial [cf. Definition 1.5, (i)].*

PROOF. First, observe that the implication (1) $\Rightarrow$ (2) follows immediately from Lemma 3.1, (i). Next, we verify the implication (2) $\Rightarrow$ (1). Suppose that condition (2) is satisfied, i.e., that there exist a smooth curve $\bar{V}_0$ over $\bar{F}_\square$ and an isomorphism $Y_\square \times_{L_\square} \bar{K}_\square \xrightarrow{\sim} \bar{V}_0 \times_{\bar{F}_\square} \bar{K}_\square$ over $\bar{K}_\square$. Write $\Gamma \stackrel{\text{def}}{=} \Delta_\square / (\Delta_\square \cap J_\square)$. Then since [we have assumed that] $J_\square$ is normal in $\Pi_\square$, the finite étale covering $Y_\square \times_{L_\square} \bar{K}_\square \rightarrow X_\square \times_{K_\square} \bar{K}_\square$ is Galois with Galois group Γ . Now since [we have assumed that] $X_\square$ admits stable reduction over $R_\square$, it follows from Remark 2.6.1 that the group scheme over $\bar{F}_\square$ of automorphisms of $\bar{V}_0$ is unramified over $\bar{F}_\square$. Thus, since $\bar{F}_\square$ is algebraically closed, the action of Γ on $\bar{V}_0 \times_{\bar{F}_\square} \bar{K}_\square$ [obtained by transporting the action of Γ on $Y_\square \times_{L_\square} \bar{K}_\square$ via the above isomorphism] arises, by base-changing via $\bar{F}_\square \hookrightarrow \bar{K}_\square$, from a [necessarily free] action of Γ on $\bar{V}_0$. Write $W \stackrel{\text{def}}{=} \bar{V}_0 / \Gamma$. Then we have isomorphisms over $\bar{K}_\square$

$$W \times_{\bar{F}_\square} \bar{K}_\square \xrightarrow{\sim} X_\square \times_{K_\square} \bar{K}_\square \xrightarrow{\sim} (X_{\square,0} \times_{F_\square} \bar{F}_\square) \times_{\bar{F}_\square} \bar{K}_\square$$

— where the first arrow is the isomorphism induced by the isomorphisms $W \times_{\bar{F}_\square} \bar{K}_\square \xrightarrow{\sim} (\bar{V}_0 \times_{\bar{F}_\square} \bar{K}_\square) / \Gamma \xrightarrow{\sim} (Y_\square \times_{L_\square} \bar{K}_\square) / \Gamma$. Moreover, since $\bar{F}_\square$ is algebraically closed, it follows from Remark 2.6.1 that the composite isomorphism $W \times_{\bar{F}_\square} \bar{K}_\square \xrightarrow{\sim} (X_{\square,0} \times_{F_\square} \bar{F}_\square) \times_{\bar{F}_\square} \bar{K}_\square$ arises, by base-changing via $\bar{F}_\square \hookrightarrow \bar{K}_\square$, from an isomorphism $W \xrightarrow{\sim} X_{\square,0} \times_{F_\square} \bar{F}_\square$ over $\bar{F}_\square$. Thus, the finite étale covering $Y_\square \times_{L_\square} \bar{K}_\square \rightarrow X_\square \times_{K_\square} \bar{K}_\square$ arises, by base-changing via $\bar{F}_\square \hookrightarrow \bar{K}_\square$, from the connected finite étale covering $\bar{V}_0 \rightarrow \bar{V}_0 / \Gamma = W \xrightarrow{\sim} X_{\square,0} \times_{F_\square} \bar{F}_\square$. In particular, it follows from Lemma 3.1, (i), that the inclusion $\Delta_\square^{\text{cst}} \subseteq \Delta_\square \cap J_\square$ holds. This completes the proof of the implication (2) $\Rightarrow$ (1), hence also of Lemma 3.2. $\square$

Remark 3.2.1. Let $\square$ be an element of $\{\circ, \bullet\}$. Suppose that the smooth curve $X_\square$ over $K_\square$ is strictly isotrivial. Then it follows immediately from Lemma 3.2 that the closed subgroup $\Delta_\square^{\text{cst}} \subseteq \Delta_\square$ of $\Delta_\square$ does not depend on the choice of the smooth curve $X_{\square,0}$ over $F_\square$ and the isomorphism $X_\square \xrightarrow{\sim} X_{\square,0} \times_{F_\square} K_\square$ fixed in the discussion at the beginning of §3. Thus, it follows from Lemma 3.1, (ii), that the closed subgroup $N_\square^{\text{cst}} \subseteq \Pi_\square$ of $\Pi_\square$ also does not depend on these choices.

Theorem 3.3. *Suppose that the smooth curve $X_\circ$ over $K_\circ$ is strictly isotrivial. Then the smooth curve $X_\bullet$ over $K_\bullet$ is strictly isotrivial.*

PROOF. First, we verify that the smooth curve $X_\bullet$ over $K_\bullet$ admits good reduction over $R_\bullet$ [cf. Definition 2.7]. Since [we have assumed that] the smooth curve $X_\circ$ over $K_\circ$ admits stable reduction over $R_\circ$, the inequality $2 - 2g_\circ - r_\circ < 0$ holds. Thus, since [we have assumed that] the smooth curve $X_\circ$ over $K_\circ$ is strictly isotrivial, it follows from Remark 2.7.1 that the smooth curve $X_\circ$ over $K_\circ$ admits good reduction over $R_\circ$. In particular, it follows from [21, Theorem 5.3] [cf. our assumption that, for each $\square \in \{\circ, \bullet\}$, the smooth curve $X_\square$ over $K_\square$ admits stable reduction over $R_\square$], together with Lemma 2.2, (i), (iv), that the smooth curve $X_\bullet$ over $K_\bullet$ admits good reduction over $R_\bullet$, as desired.

Next, we verify that the smooth curve $X_\bullet$ over $K_\bullet$ is isotrivial. Since the smooth curve $X_\circ$ over $K_\circ$ is isotrivial [cf. our assumption that the smooth curve $X_\circ$ over $K_\circ$ is strictly isotrivial] and admits good reduction over $R_\circ$ [cf. the argument of the first paragraph of the present proof], it follows from the implication (1) $\Rightarrow$ (2) of Lemma 2.11 that the natural continuous surjective homomorphism $\Delta_\circ^{\text{tm}} \twoheadrightarrow \Delta_\circ^{\text{log}}$ is an isomorphism. Thus, since the continuous isomorphisms $\alpha^{\text{tm}}, \alpha^{\text{log}}$ of Lemma 2.10, (i), (ii), lie over the continuous isomorphisms $G_\circ \xrightarrow{\sim} G_\bullet$, $G_\circ^{\text{tm}} \xrightarrow{\sim} G_\bullet^{\text{tm}}$ determined by α_G , respectively [cf. Lemma 2.2, (iv)], one verifies easily that the natural continuous surjective homomorphism $\Delta_\bullet^{\text{tm}} \twoheadrightarrow \Delta_\bullet^{\text{log}}$ is an isomorphism. In particular, since the smooth curve $X_\bullet$ over $K_\bullet$ admits good reduction over $R_\bullet$ [cf. the argument of the first paragraph of the present proof], it follows from the implication (2) $\Rightarrow$ (1) of Lemma 2.11 that the smooth curve $X_\bullet$ over $K_\bullet$ is isotrivial, as desired.

Finally, since the smooth curve $X_\bullet$ over $K_\bullet$ is isotrivial and admits good reduction over $R_\bullet$ [cf. the arguments of the first and second paragraphs of the present proof], it follows from Remark 2.7.2 that the smooth curve $X_\bullet$ over $K_\bullet$ is strictly isotrivial, as desired. This completes the proof of Theorem 3.3. $\square$

Theorem 3.4. *Suppose that the smooth curve $X_\circ$ over $K_\circ$ is strictly isotrivial [which thus implies that the smooth curve $X_\bullet$ over $K_\bullet$ is strictly isotrivial — cf. Theorem 3.3]. Then there exists a continuous isomorphism $\alpha_0: \Pi_{\circ,0} \xrightarrow{\sim} \Pi_{\bullet,0}$ [which is uniquely determined up to composition with inner automorphisms of $\Pi_{\bullet,0}$ determined by elements*

of $\Delta_{\bullet,0}]$ such that the continuous isomorphisms α, α_0 fit into a commutative diagram of topological groups

$$\begin{array}{ccc} \Pi_{\circ} & \xrightarrow[\sim]{\alpha} & \Pi_{\bullet} \\ \downarrow & & \downarrow \\ \Pi_{\circ,0} & \xrightarrow[\alpha_0]{\sim} & \Pi_{\bullet,0} \end{array}$$

— where the vertical arrows are the $\Delta_{\circ,0}$ -, $\Delta_{\bullet,0}$ -conjugacy classes of continuous surjective homomorphisms induced by the natural morphisms $X_{\circ} \rightarrow X_{\circ,0}$, $X_{\bullet} \rightarrow X_{\bullet,0}$, respectively.

PROOF. Since the smooth curve $X_{\bullet}$ over $K_{\bullet}$ is strictly isotrivial [cf. Theorem 3.3], it follows immediately from Lemma 3.1, (i), that, to verify Theorem 3.4, it suffices to verify the equality $\alpha(N_{\circ}^{\text{cst}}) = N_{\bullet}^{\text{cst}}$. Moreover, since $\alpha(\Delta_{\circ}) = \Delta_{\bullet}$, it follows from Lemma 3.1, (ii), that, to verify the equality $\alpha(N_{\circ}^{\text{cst}}) = N_{\bullet}^{\text{cst}}$, it suffices to verify the equality $\alpha(\Delta_{\circ}^{\text{cst}}) = \Delta_{\bullet}^{\text{cst}}$. Thus, it follows immediately from Lemma 3.2 [cf. also Remark 3.2.1], together with a similar argument applied to the inverse of α , that, to verify Theorem 3.4, it suffices to verify the following assertion:

- (a) If $J_{\bullet} = \alpha(J_{\circ})$, and the smooth curve $Y_{\circ}$ over $L_{\circ}$ is isotrivial, then the smooth curve $Y_{\bullet}$ over $L_{\bullet}$ is isotrivial.

Suppose that $J_{\bullet} = \alpha(J_{\circ})$, and that the smooth curve $Y_{\circ}$ over $L_{\circ}$ is isotrivial. Then one verifies easily that there exists a finite extension field $L'_{\circ} \subseteq \overline{K}_{\circ}$ of $L_{\circ}$ such that the smooth curve $Y_{\circ} \times_{L_{\circ}} L'_{\circ}$ over $L'_{\circ}$ is strictly isotrivial. Write $L'_{\bullet} \subseteq \overline{K}_{\bullet}$ for the finite extension field of $L_{\bullet}$ that corresponds, relative to α_G , to $L'_{\circ}$. Now since [we have assumed that] the smooth curve $X_{\bullet}$ over $K_{\bullet}$ admits stable reduction over $R_{\bullet}$, we may assume without loss of generality, by replacing $L'_{\circ}$ by a suitable finite extension field of $L'_{\circ}$ [cf. Remark 2.6.2], that there exists a nonnegative integer n such that the smooth curve $Y_{\bullet}(n) \times_{L_{\bullet}} L'_{\bullet}$ over $L'_{\bullet}$ admits stable reduction over the ring of integers of $L'_{\bullet}$. On the other hand, since the smooth curve $Y_{\circ} \times_{L_{\circ}} L'_{\circ}$ over $L'_{\circ}$ is strictly isotrivial, and [we have assumed that] the smooth curve $X_{\circ}$ over $K_{\circ}$ admits stable reduction over $R_{\circ}$, it follows from Remark 2.7.1 that the smooth curve $Y_{\circ} \times_{L_{\circ}} L'_{\circ}$ over $L'_{\circ}$ admits good reduction over the ring of integers of $L'_{\circ}$. Thus, by applying Theorem 3.3 — where we take the “ $(K_{\circ}, K_{\bullet}, X_{\circ}, X_{\bullet})$ ” of Theorem 3.3 to be $(L'_{\circ}, L'_{\bullet}, Y_{\circ} \times_{L_{\circ}} L'_{\circ}, Y_{\bullet}(n) \times_{L_{\bullet}} L'_{\bullet})$ and the “ α ” of Theorem 3.3 to be the continuous isomorphism from the étale fundamental group of $Y_{\circ} \times_{L_{\circ}} L'_{\circ}$ to the étale fundamental group of $Y_{\bullet}(n) \times_{L_{\bullet}} L'_{\bullet}$ obtained by composing the restriction of α to the open subgroup of $J_{\circ}$

that corresponds to $Y_{\circ} \times_{L_{\circ}} L'_{\circ}$ with a continuous isomorphism as in Remark 2.5.1 [in the case where we take the “ $(K_{\bullet}, X_{\bullet})$ ” of Remark 2.5.1 to be $(L'_{\bullet}, Y_{\bullet} \times_{L_{\bullet}} L'_{\bullet})$] — one concludes that the smooth curve $Y_{\bullet}(n) \times_{L_{\bullet}} L'_{\bullet}$ over $L'_{\bullet}$ is strictly isotrivial, hence that the smooth curve $Y_{\bullet}(n)$ over $L_{\bullet}$ is isotrivial. Next, write

- $X_{\bullet,0}(n)$ for the smooth curve over $F_{\bullet}$ obtained by pulling back the smooth curve $X_{\bullet,0}$ over $F_{\bullet}$ by the absolute $p_{\bullet}^n$ -th power Frobenius endomorphism of $\mathrm{Spec}(F_{\bullet})$ and
- $J_{\bullet}(n)$ for the image of $J_{\bullet}$ by a continuous isomorphism from $\Pi_{\bullet}$ to the étale fundamental group of $X_{\bullet}(n)$ as in Remark 2.5.1.

Then one verifies easily that the isomorphism $X_{\bullet} \xrightarrow{\sim} X_{\bullet,0} \times_{F_{\bullet}} K_{\bullet}$ fixed in the discussion at the beginning of the present §3 determines an isomorphism $X_{\bullet}(n) \xrightarrow{\sim} X_{\bullet,0}(n) \times_{F_{\bullet}} K_{\bullet}$ over $K_{\bullet}$ that is compatible with the relative $p_{\bullet}^n$ -th power Frobenius morphisms of $X_{\bullet}$ over $K_{\bullet}$ and $X_{\bullet,0}$ over $F_{\bullet}$, which thus implies that the smooth curve $X_{\bullet}(n)$ over $K_{\bullet}$ is strictly isotrivial and admits good reduction over $R_{\bullet}$, and that the above continuous isomorphism maps $\Delta_{\bullet}^{\mathrm{cst}}$ onto the closed subgroup of the étale fundamental group of $X_{\bullet}(n)$ obtained by applying the construction of $\Delta_{\bullet}^{\mathrm{cst}}$ in the discussion at the beginning of the present §3 to $X_{\bullet}(n)$ and this isomorphism [cf. Remark 3.2.1]. Moreover, one verifies easily that the connected finite étale covering of $X_{\bullet}(n)$ that corresponds to $J_{\bullet}(n)$ is naturally isomorphic to $Y_{\bullet}(n)$. Thus, since the smooth curve $Y_{\bullet}(n)$ over $L_{\bullet}$ is isotrivial, it follows from the implication (2) $\Rightarrow$ (1) of Lemma 3.2 — where we take the “ $(X_{\bullet}, X_{\bullet,0}, J_{\bullet})$ ” of Lemma 3.2 to be $(X_{\bullet}(n), X_{\bullet,0}(n), J_{\bullet}(n))$ — that this closed subgroup is contained in $J_{\bullet}(n)$, which thus implies that the inclusion $\Delta_{\bullet}^{\mathrm{cst}} \subseteq \Delta_{\bullet} \cap J_{\bullet}$ holds. In particular, it follows from the implication (1) $\Rightarrow$ (2) of Lemma 3.2 that the smooth curve $Y_{\bullet}$ over $L_{\bullet}$ is isotrivial, as desired. This completes the proof of (a), hence also of Theorem 3.4. $\square$

Theorem 3.5. *Suppose that the smooth curve $X_{\circ}$ over $K_{\circ}$ is strictly isotrivial. Then the following assertions hold:*

- (i) *There exists a uniquely determined isomorphism of schemes $\theta_0: X_{\circ,0} \xrightarrow{\sim} X_{\bullet,0}$ such that the continuous isomorphism α_0 of Theorem 3.4 belongs to the continuous outer isomorphism $\Pi_{\circ,0} \xrightarrow{\sim} \Pi_{\bullet,0}$ induced by θ_0 . In particular, the scheme $X_{\circ}$ is isomorphic to the scheme $X_{\bullet}$.*
- (ii) *Suppose that the continuous isomorphism α is cyclotomically compatible [cf. Definition 2.4]. Then the isomorphism θ_0 of (i) lies over the isomorphism $\mathrm{Spec}(F_{\circ}) \xrightarrow{\sim} \mathrm{Spec}(F_{\bullet})$ of schemes determined by the isomorphism $F_{\bullet} \xrightarrow{\sim} F_{\circ}$ of fields obtained*

by restricting the isomorphism $\overline{K}_\bullet \xrightarrow{\sim} \overline{K}_\circ$ of fields of Definition 2.3. Moreover, the continuous isomorphism α_0 of Theorem 3.4 is contained in the $\Delta_{\bullet,0}$ -conjugacy class of continuous isomorphisms $\Pi_{\circ,0} \xrightarrow{\sim} \Pi_{\bullet,0}$ induced by the isomorphism $X_{\circ,0} \times_{F_\circ} \overline{F}_\circ \xrightarrow{\sim} X_{\bullet,0} \times_{F_\bullet} \overline{F}_\bullet$ of schemes determined by θ_0 and the isomorphism $\overline{F}_\bullet \xrightarrow{\sim} \overline{F}_\circ$ of fields obtained by restricting the isomorphism $\overline{K}_\bullet \xrightarrow{\sim} \overline{K}_\circ$ of fields of Definition 2.3.

- (iii) Suppose that the continuous isomorphism α is cyclotomically compatible. Then there exists an isomorphism $X_\circ \xrightarrow{\sim} X_\bullet$ of schemes that lies over the isomorphism $\mathrm{Spec}(K_\circ) \xrightarrow{\sim} \mathrm{Spec}(K_\bullet)$ of schemes determined by the isomorphism $K_\bullet \xrightarrow{\sim} K_\circ$ of fields of Definition 2.3.

PROOF. First, we verify assertion (i). Since the smooth curve $X_\bullet$ over $K_\bullet$ is strictly isotrivial [cf. Theorem 3.3], for each $\square \in \{\circ, \bullet\}$, the smooth curve $X_{\square,0}$ over $F_\square$ is defined [cf. the discussion at the beginning of the present §3]. Moreover, since [we have assumed that], for each $\square \in \{\circ, \bullet\}$, the smooth curve $X_\square$ over $K_\square$ admits stable reduction over $R_\square$, the existence and uniqueness of the isomorphism θ_0 of assertion (i) follow from Theorem 3.4, together with [21, Theorem 0.6] (respectively, [11, Theorem 3.12]), in the case where the smooth curve $X_\circ$ over $K_\circ$ is not proper (respectively, is proper) [cf. also Lemma 2.2, (i)]. In particular, since [one verifies easily that], for each $\square \in \{\circ, \bullet\}$, the field $F_\square$ is the algebraic closure of the prime field in the function field of $X_{\square,0}$, and the equality $K_\square = F_\square((t))$ holds, the scheme $X_\circ$ is isomorphic to the scheme $X_\bullet$, as desired. This completes the proof of assertion (i).

Assertion (ii) follows immediately — in light of Lemma 3.1, (i), and Definition 2.4 — from a similar argument to the argument applied in the proof of [24, Theorem 2.6]. Assertion (iii) follows immediately from assertion (ii), together with the fact that, for each $\square \in \{\circ, \bullet\}$, the smooth curve $X_\square$ over $K_\square$ is isomorphic over $K_\square$ to $X_{\square,0} \times_{F_\square} K_\square$ [cf. the discussion at the beginning of the present §3]. This completes the proof of Theorem 3.5. $\square$

4. PRESERVATION OF SUITABLE ARTIN–SCHREIER COVERINGS

In the present §4, we consider the case where the smooth curves under consideration are obtained by removing four rational points from the projective line, and the given continuous isomorphism is cyclotomically compatible and satisfies a certain compatibility condition concerning the cusps. In this situation, we prove that the continuous isomorphism between the étale fundamental groups of the projective lines minus the

three points $0, 1, \infty$ induced by the given continuous isomorphism preserves, after a suitable purely inseparable extension of the base field, the Artin–Schreier coverings determined by the nonzero scalar multiples of the standard coordinate [cf. Lemma 4.4 below]. This result plays an essential role in §5.

In the present §4, suppose that we are in the situation at the beginning of §2. In particular, we have a continuous isomorphism

$$\alpha: \Pi_{\circ} \xrightarrow{\sim} \Pi_{\bullet}.$$

In the present §4, suppose, moreover, that this continuous isomorphism α is cyclotomically compatible [which thus implies that α is base-geometric — cf. Definition 2.3, Definition 2.4]. In the present §4, for each $\square \in \{\circ, \bullet\}$, let $a_{\square}$ be an element of $K_{\square} \setminus \{0, 1\}$; write

- $T_{\square}$ for the spectrum of the $K_{\square}$ -algebra

$$K_{\square} \left[x, \frac{1}{x(1-x)} \right]$$

— where x is an indeterminate — and

- $\Pi_{T_{\square}}, \Delta_{T_{\square}}$ for the respective étale fundamental groups of $T_{\square}$, $T_{\square} \times_{K_{\square}} \overline{K}_{\square}$, relative to suitable choices of basepoints.

In the present §4, suppose, moreover, that the following two conditions are satisfied:

- (1) For each $\square \in \{\circ, \bullet\}$, the equality $X_{\square} = T_{\square} \setminus \{a_{\square}\}$ holds, i.e., the smooth curve $X_{\square}$ over $K_{\square}$ [cf. Definition 1.3] is the open subscheme of $T_{\square}$ obtained by removing the $K_{\square}$ -rational point [that corresponds to] $a_{\square} \in K_{\square}$.
- (2) The continuous isomorphism α maps each decomposition subgroup of $\Pi_{\circ}$ associated to the cusp 0 (respectively, $1; \infty; a_{\circ}$) of $X_{\circ}$ onto a decomposition subgroup of $\Pi_{\bullet}$ associated to the cusp 0 (respectively, $1; \infty; a_{\bullet}$) of $X_{\bullet}$.

Thus, it follows immediately from conditions (1), (2), together with the various definitions involved, that there exists a uniquely determined continuous isomorphism

$$\alpha_T: \Pi_{T_{\circ}} \xrightarrow{\sim} \Pi_{T_{\bullet}}$$

that satisfies the following three conditions:

- (3) The continuous isomorphisms α , α_T fit into a commutative diagram of topological groups

$$\begin{array}{ccc} \Pi_{\circ} & \xrightarrow[\sim]{\alpha} & \Pi_{\bullet} \\ \downarrow & & \downarrow \\ \Pi_{T_{\circ}} & \xrightarrow[\alpha_T]{\sim} & \Pi_{T_{\bullet}} \end{array}$$

— where the vertical arrows are the natural continuous surjective homomorphisms induced by the natural open immersions $X_{\circ} \hookrightarrow T_{\circ}$, $X_{\bullet} \hookrightarrow T_{\bullet}$, respectively.

- (4) The continuous isomorphism α_T lies over α_G .
 (5) Let ζ be an element of $\{0, 1, \infty\}$. Then the continuous isomorphism α_T maps each decomposition subgroup of $\Pi_{T_{\circ}}$ associated to the cusp ζ of $T_{\circ}$ onto a decomposition subgroup of $\Pi_{T_{\bullet}}$ associated to the cusp ζ of $T_{\bullet}$.

In the remainder of the present §4, we identify $K_{\bullet}^{\text{alg}}$ with $K_{\circ}^{\text{alg}}$ by means of the [necessarily uniquely determined] isomorphism $K_{\bullet}^{\text{alg}} \xrightarrow{\sim} K_{\circ}^{\text{alg}}$ of fields that restricts to the isomorphism $\overline{K}_{\bullet} \xrightarrow{\sim} \overline{K}_{\circ}$ of fields that induces α_G [cf. Definition 2.3]. Thus, we identify $\overline{K}_{\bullet}$, $K_{\bullet}$, $F_{\bullet}$, $\overline{F}_{\bullet}$, $R_{\bullet}$, $G_{\bullet}$, $T_{\bullet}$, $\Pi_{T_{\bullet}}$, $\Delta_{T_{\bullet}}$ with $\overline{K}_{\circ}$, $K_{\circ}$, $F_{\circ}$, $\overline{F}_{\circ}$, $R_{\circ}$, $G_{\circ}$, $T_{\circ}$, $\Pi_{T_{\circ}}$, $\Delta_{T_{\circ}}$, respectively, in the natural way. In particular, we regard $a_{\bullet}$ as an element of $K_{\circ} \setminus \{0, 1\}$, the continuous isomorphism α_G is the identity automorphism of $G_{\circ} = G_{\bullet}$, and the continuous isomorphism α_T is an automorphism of $\Pi_{T_{\circ}} = \Pi_{T_{\bullet}}$. In the remainder of the present §4, write

- $p \stackrel{\text{def}}{=} p_{\circ} = p_{\bullet}$ [cf. Lemma 2.2, (i)],
- $F \stackrel{\text{def}}{=} F_{\circ} = F_{\bullet}$,
- $\overline{F} \stackrel{\text{def}}{=} \overline{F}_{\circ} = \overline{F}_{\bullet}$,
- $K \stackrel{\text{def}}{=} K_{\circ} = K_{\bullet}$ [which thus implies that $K = F((t))$],
- $R \stackrel{\text{def}}{=} R_{\circ} = R_{\bullet}$,
- $K^{\text{alg}} \stackrel{\text{def}}{=} K_{\circ}^{\text{alg}} = K_{\bullet}^{\text{alg}}$,
- $\overline{K} \stackrel{\text{def}}{=} \overline{K}_{\circ} = \overline{K}_{\bullet}$,
- $G \stackrel{\text{def}}{=} G_{\circ} = G_{\bullet}$,
- $T \stackrel{\text{def}}{=} T_{\circ} = T_{\bullet}$,
- $\Pi_T \stackrel{\text{def}}{=} \Pi_{T_{\circ}} = \Pi_{T_{\bullet}}$, and
- $\Delta_T \stackrel{\text{def}}{=} \Delta_{T_{\circ}} = \Delta_{T_{\bullet}}$.

Definition 4.1. Let n be a nonnegative integer.

- (i) We shall write $K_n \subseteq K^{\text{alg}}$ for the subfield of K^{alg} generated over K by the [necessarily uniquely determined] p^n -th root of t

in K^{alg} and $\overline{K}_n \stackrel{\text{def}}{=} \overline{K} \cdot K_n \subseteq K^{\text{alg}}$. Observe that K_n is a purely inseparable extension field of K of degree p^n , and that $\overline{K}_n$ is a separable closure of K_n . Moreover, since the natural morphism $T \times_K K_n \rightarrow T$ is a universal homeomorphism, we identify Π_T , Δ_T , G with the étale fundamental groups of $T \times_K K_n$, $T \times_K \overline{K}_n$, relative to suitable choices of basepoints, and $\text{Gal}(\overline{K}_n/K_n)$, respectively, in the natural way.

- (ii) Let h be an element of $\mathcal{O}_T(T) \otimes_K K_n$. Then we shall write

$$\eta_h \in H^1(\Pi_T, \mathbb{F}_p)$$

for the class of h relative to the isomorphism of Lemma 1.7, (i), in the case where we take the “ (A, V, q) ” of Lemma 1.7 to be $(K_n, T \times_K K_n, p)$ [cf. (i)]. Observe that one verifies easily that, for each integer $n' \geq n$, the class η_h coincides with the class obtained by applying the above construction to the image of h in $\mathcal{O}_T(T) \otimes_K K_{n'}$ [in the case where we take the “ n ” to be n'].

- (iii) Let c be an element of $K_n^\times$. Then we shall write

$$J_c \subseteq \Pi_T$$

for the kernel of the continuous homomorphism $\Pi_T \rightarrow \mathbb{F}_p$ determined by $\eta_{c \cdot x}$.

Remark 4.1.1. Let n be a nonnegative integer.

- (i) Let h be an element of $\mathcal{O}_T(T) \otimes_K K_n$ such that the class η_h is nontrivial. Then one verifies immediately from the various definitions involved that the connected finite étale covering of $T \times_K K_n$ that corresponds to the kernel of the continuous homomorphism $\Pi_T \rightarrow \mathbb{F}_p$ determined by η_h is naturally isomorphic over $T \times_K K_n$ to the affine scheme

$$\text{Spec}((\mathcal{O}_T(T) \otimes_K K_n)[z]/(z^p - z - h))$$

— where z is an indeterminate.

- (ii) Let c be an element of $K_n^\times$. Then it follows immediately from [18, Proposition 3.7.8, (c)] — applied to the Artin–Schreier extension of the function field of $T \times_K K^{\text{alg}}$ determined by $c \cdot x$ — that the restriction of the class $\eta_{c \cdot x}$ to each cuspidal inertia subgroup of Π_T associated to the cusp ∞ of T is nontrivial. In particular, the restriction of the class $\eta_{c \cdot x}$ to Δ_T is nontrivial, which thus implies that the image of J_c in G coincides with G .

Lemma 4.2. *Let n be a nonnegative integer, ζ an element of $\{0, 1\}$, $\mathfrak{D} \subseteq \Pi_T$ a decomposition subgroup of Π_T associated to the cusp ζ*

of T , and h an element of $\mathcal{O}_T(T) \otimes_K K_n$ whose image in the field $K_n((x - \zeta))$ of formal Laurent series in $x - \zeta$ with coefficients in K_n is contained in $K_n[[x - \zeta]]$. Write $h(\zeta) \in K_n$ for the image of h in $K_n[[x - \zeta]]/(x - \zeta) = K_n$. Then the restriction of the class η_h to $\mathfrak{D}$ coincides with the image, by the homomorphism $H^1(G, \mathbb{F}_p) \rightarrow H^1(\mathfrak{D}, \mathbb{F}_p)$ induced by the composite $\mathfrak{D} \hookrightarrow \Pi_T \twoheadrightarrow G$, of the class of $h(\zeta)$ relative to the isomorphism of Lemma 1.7, (i), in the case where we take the “ (A, V, q) ” of Lemma 1.7 to be $(K_n, \text{Spec}(K_n), p)$.

PROOF. First, observe that one verifies easily that the assertion under consideration is invariant under replacing $\mathfrak{D}$ by a Π_T -conjugate of $\mathfrak{D}$. Thus, we may assume without loss of generality, by replacing $\mathfrak{D}$ by a suitable Π_T -conjugate of $\mathfrak{D}$, that $\mathfrak{D}$ is the image of the continuous homomorphism to Π_T from the absolute Galois group of the field $K_n((x - \zeta))$, relative to a suitable choice of separable closure, induced by the natural morphism $\text{Spec}(K_n((x - \zeta))) \rightarrow T \times_K K_n$ [cf. Definition 4.1, (i)]. In particular, it follows from the [easily verified] functoriality of Artin–Schreier classes that the image of the restriction of η_h to $\mathfrak{D}$ by the [necessarily injective] homomorphism from $H^1(\mathfrak{D}, \mathbb{F}_p)$ to the first cohomology group of the absolute Galois group of $K_n((x - \zeta))$ induced by the continuous surjective homomorphism from this absolute Galois group onto $\mathfrak{D}$ determined by the above continuous homomorphism coincides with the class of the image of h in $K_n((x - \zeta))$ relative to the isomorphism of Lemma 1.7, (i), in the case where we take the “ (A, V, q) ” of Lemma 1.7 to be $(K_n((x - \zeta)), \text{Spec}(K_n((x - \zeta))), p)$.

Next, observe that since [one verifies easily that] $h - h(\zeta) \in (x - \zeta) \cdot K_n[[x - \zeta]]$, it follows from Hensel’s lemma [applied to the polynomial $z^p - z - (h - h(\zeta))$ in the indeterminate z with coefficients in the complete local ring $K_n[[x - \zeta]]$, whose derivative with respect to z is the unit -1] that $h - h(\zeta) \in \varphi_p(K_n[[x - \zeta]])$. Thus, the image of h in $K_n((x - \zeta))/\varphi_p(K_n((x - \zeta)))$ coincides with the image of $h(\zeta) \in K_n$. In particular, the desired equality follows immediately from the [easily verified] functoriality of Artin–Schreier classes with respect to the natural morphism $\text{Spec}(K_n((x - \zeta))) \rightarrow \text{Spec}(K_n)$. This completes the proof of Lemma 4.2. $\square$

Lemma 4.3. *Let c be an element of $K^\times$. Write $\eta' \in H^1(\Pi_T, \mathbb{F}_p)$ for the image of $\eta_{c \cdot x}$ by the automorphism of $H^1(\Pi_T, \mathbb{F}_p)$ induced by α_T^{-1} . Then the following assertions hold:*

- (i) *The restriction of η' to Δ_T is nontrivial.*
- (ii) *For each $\zeta \in \{0, 1\}$, the restriction of η' to each cuspidal inertia subgroup of Π_T associated to the cusp ζ of T is trivial.*

- (iii) *There exist a nonnegative integer n and an element $h \in K_n[x]$ such that the equality $\eta' = \eta_h$ holds, and, moreover, the degree N of the polynomial h is positive and prime to p .*
- (iv) *Let $n, h \in K_n[x]$, and N be as in (iii). Then the equality $N = 1$ holds.*

PROOF. First, we verify assertion (i). Observe that since the continuous automorphism α_T lies over the identity automorphism of G [cf. (4)], the equality $\alpha_T(\Delta_T) = \Delta_T$ holds. Thus, assertion (i) follows from Remark 4.1.1, (ii). This completes the proof of assertion (i).

Next, we verify assertion (ii). Let ζ be an element of $\{0, 1\}$, and let $\mathfrak{D} \subseteq \Pi_T$ be a decomposition subgroup of Π_T associated to the cusp ζ of T . Then it follows from (5) that $\alpha_T^{-1}(\mathfrak{D})$ is a decomposition subgroup of Π_T associated to the cusp ζ of T . Thus, it follows from Lemma 4.2 — where we take the “ $(n, \zeta, \mathfrak{D}, h)$ ” of Lemma 4.2 to be $(0, \zeta, \alpha_T^{-1}(\mathfrak{D}), c \cdot x)$ — that the restriction of $\eta_{c \cdot x}$ to $\alpha_T^{-1}(\mathfrak{D}) \cap \Delta_T = \alpha_T^{-1}(\mathfrak{D} \cap \Delta_T)$ is trivial, which thus implies that the restriction of η' to $\mathfrak{D} \cap \Delta_T$ is trivial. In particular, since the cuspidal inertia subgroups of Π_T associated to the cusp ζ of T are the subgroups of the form “ $\mathfrak{D} \cap \Delta_T$ ”, assertion (ii) holds. This completes the proof of assertion (ii).

Next, we verify assertion (iii). First, observe that it follows from Lemma 1.7, (i) — where we take the “ (A, V, q) ” of Lemma 1.7 to be (K, T, p) — that there exists an element h of $\mathcal{O}_T(T) = K[x, x^{-1}, (1-x)^{-1}]$ such that the equality $\eta' = \eta_h$ holds. Write $h = h_\infty + h_0 + h_1$ — where $h_\infty \in K[x]$, $h_0 \in x^{-1} \cdot K[x^{-1}]$, and $h_1 \in (1-x)^{-1} \cdot K[(1-x)^{-1}]$. Now it follows immediately from assertion (ii) that the Artin–Schreier extension of the function field $K^{\text{alg}}(x)$ of $T \times_K K^{\text{alg}}$ determined by h is unramified over the place $x = 0$. Thus, it follows from [18, Proposition 3.7.8, (b)] that there exists an element $u \in K^{\text{alg}}(x)$ such that the image of $h - \varphi_p(u)$ in the field $K^{\text{alg}}((x))$ of formal Laurent series in x with coefficients in K^{alg} is contained in $K^{\text{alg}}[[x]]$. Write $u_0 \in x^{-1} \cdot K^{\text{alg}}[x^{-1}]$ for the principal part at $x = 0$ of u . Then since the images of h_∞ , h_1 , and $u - u_0$ in $K^{\text{alg}}((x))$ are contained in $K^{\text{alg}}[[x]]$, one verifies easily that the equality $h_0 = \varphi_p(u_0)$ holds. Moreover, by comparing the coefficients of the various powers of x^{-1} , one verifies easily that $u_0 \in x^{-1} \cdot K[x^{-1}]$. Observe that it follows immediately from a similar argument to this argument that there exists an element $u_1 \in (1-x)^{-1} \cdot K[(1-x)^{-1}]$ such that the equality $h_1 = \varphi_p(u_1)$ holds. Thus, since $u_0, u_1 \in \mathcal{O}_T(T)$, the equalities $\eta' = \eta_h = \eta_{h_\infty}$ hold. In particular, we may assume without loss of generality, by replacing h by h_∞ , that $h \in K[x]$. Next, observe that, for each nonnegative integer n , each element $\beta \in K_n$, and each positive integer k , the equality $\beta \cdot x^{p^k} - \beta^{1/p} \cdot x^k = \varphi_p(\beta^{1/p} \cdot x^k)$ holds,

and $\beta^{1/p} \in K_{n+1}$. Thus, by repeatedly replacing, whenever the degree of the polynomial under consideration is positive and divisible by p , its leading monomial $\beta \cdot x^{p \cdot k}$ by $\beta^{1/p} \cdot x^k$, one verifies easily — in light of Lemma 1.7, (i), and Definition 4.1, (ii) — that we may assume without loss of generality, by replacing h by a suitable element of $K_n[x]$ for a suitable nonnegative integer n , that the polynomial h is either constant or of degree prime to p . On the other hand, if h is constant, i.e., $h \in K_n$, then it follows from Lemma 1.7, (ii) — where we take the “ (A, V, q) ” of Lemma 1.7 to be $(K_n, T \times_K K_n, p)$ — that the class $\eta' = \eta_h$ is contained in the image of the homomorphism $H^1(G, \mathbb{F}_p) \rightarrow H^1(\Pi_T, \mathbb{F}_p)$ induced by the natural continuous surjective homomorphism $\Pi_T \twoheadrightarrow G$, which thus implies that the restriction of η' to Δ_T is trivial — in contradiction to assertion (i). This completes the proof of assertion (iii).

Next, we verify assertion (iv). Write $Y \rightarrow T$, $Y' \rightarrow T$ for the connected finite étale coverings of T that correspond to the open subgroups J_c , $\alpha_T(J_c) \subseteq \Pi_T$, respectively. Then it follows from Remark 4.1.1, (i) — where we take the “ n ” of Remark 4.1.1 to be 0, n , respectively — that Y is naturally isomorphic over T to the affine scheme

$$\mathrm{Spec}(\mathcal{O}_T(T)[z]/(z^p - z - c \cdot x))$$

and that $Y' \times_K K_n$ is naturally isomorphic over $T \times_K K_n$ to the affine scheme

$$\mathrm{Spec}((\mathcal{O}_T(T) \otimes_K K_n)[z]/(z^p - z - h))$$

— where z is an indeterminate. Moreover, it follows from assertion (i) and Remark 4.1.1, (ii), that the images of J_c , $\alpha_T(J_c)$ in G coincide with G , which thus implies that we may regard each of Y , Y' as a smooth curve over K . Now it follows from [18, Proposition 3.7.8, (d)] — applied to the Artin–Schreier extensions of the function field $K^{\mathrm{alg}}(x)$ of $T \times_K K^{\mathrm{alg}}$ determined by $c \cdot x$, h — that the genus [cf. Definition 1.4] of the smooth curve Y (respectively, Y') over K is 0 (respectively, $(p-1)(N-1)/2$). On the other hand, since the continuous automorphism α_T lies over the identity automorphism of G [cf. (4)], it follows from Lemma 2.2, (i) — where we take the “ $(K_\circ, K_\bullet, X_\circ, X_\bullet, \alpha)$ ” of Lemma 2.2 to be $(K, K, Y, Y', \alpha_T|_{J_c})$ — that the genus of Y coincides with the genus of Y' . Thus, the equality $(p-1)(N-1)/2 = 0$, i.e., the equality $N = 1$, holds. This completes the proof of assertion (iv), hence also of Lemma 4.3. $\square$

Lemma 4.4. *Let c be an element of $K^\times$. Then there exist a non-negative integer n and an element $c' \in K_n^\times$ such that the equality $\alpha_T(J_c) = J_{c'}$ holds.*

PROOF. First, observe that one verifies easily that, to verify Lemma 4.4, it suffices to verify that there exist a nonnegative integer n and an element $c' \in K_n^\times$ such that, in the case where we take the “ c ” of Lemma 4.3 to be c , the element η' of Lemma 4.3 coincides with $\eta_{c' \cdot x}$. Next, to this end, let $n, h \in K_n[x]$ be as in Lemma 4.3, (iii). Then it follows from Lemma 4.3, (iv), that there exist elements $c' \in K_n^\times, e \in K_n$ such that the equality $h = c' \cdot x + e$ holds. Let $\mathfrak{D} \subseteq \Pi_T$ be a decomposition subgroup of Π_T associated to the cusp 0 of T . Then it follows from (5) that $\alpha_T^{-1}(\mathfrak{D})$ is a decomposition subgroup of Π_T associated to the cusp 0 of T . Thus, it follows from Lemma 4.2 — where we take the “ $(n, \zeta, \mathfrak{D}, h)$ ” of Lemma 4.2 to be $(0, 0, \alpha_T^{-1}(\mathfrak{D}), c \cdot x)$ — that the restriction of $\eta_{c \cdot x}$ to $\alpha_T^{-1}(\mathfrak{D})$ is trivial, which thus implies that the restriction of η' to $\mathfrak{D}$ is trivial. On the other hand, since $\eta' = \eta_h$ [cf. Lemma 4.3, (iii)], it follows from Lemma 4.2 — where we take the “ $(n, \zeta, \mathfrak{D}, h)$ ” of Lemma 4.2 to be $(n, 0, \mathfrak{D}, h)$ — that the restriction of η' to $\mathfrak{D}$ coincides with the image of the class of e by the homomorphism $H^1(G, \mathbb{F}_p) \rightarrow H^1(\mathfrak{D}, \mathbb{F}_p)$ induced by the composite $\mathfrak{D} \hookrightarrow \Pi_T \twoheadrightarrow G$. Thus, since [one verifies easily that] this composite is surjective, the class of e is trivial, i.e., $e \in \varphi_p(K_n)$. In particular, the equalities $\eta' = \eta_h = \eta_{c' \cdot x}$ hold, as desired. This completes the proof of Lemma 4.4. $\square$

5. PRESERVATION OF COORDINATES OF CUSPS

In the present §5, we continue our discussion of the situation of the preceding §4. In particular, we prove a variant of Theorem 1.8 for a suitable open subscheme of the projective line [cf. Lemma 5.2 below]. Next, by observing that the cusp 1 and the fourth cusp of each of the smooth curves, i.e., the cusp other than 0, 1, ∞ , determine rational points of the Artin–Schreier coverings discussed in §4, we prove that the given continuous isomorphism induces an automorphism of the étale fundamental group of this open subscheme that maps the Galois section-classes determined by these rational points compatibly [cf. Lemma 5.4 below]. Thus, by applying Lemma 5.2, we conclude that the isomorphism between the base fields that arises from the given continuous isomorphism [cf. Definition 2.3] maps the coordinate of the fourth cusp of one of the smooth curves to the coordinate of the fourth cusp of the other [cf. Lemma 5.5 below]. This result plays an essential role in §6.

In the present §5, suppose that we are in the situation at the beginning of the preceding §4. In the present §5, write

- $U_0 \stackrel{\text{def}}{=} \text{Spec}(F[z, (z^p - z)^{-1}])$ — where z is an indeterminate —
- $U \stackrel{\text{def}}{=} U_0 \times_F K,$

- U_0^+, U^+ for the smooth compactifications of U_0, U over F, K , respectively [which thus may be identified with the projective lines over F, K with standard coordinate z , respectively],
- Π_U, Π_{U_0} for the respective étale fundamental groups of U, U_0 , relative to suitable choices of basepoints, and
- Δ_U, Δ_{U_0} for the respective étale fundamental groups of $U \times_K \overline{K}, U_0 \times_F \overline{F}$, relative to suitable choices of basepoints.

Write, moreover, for each positive integer d ,

- $F_d \subseteq \overline{F}$ for the [necessarily uniquely determined] extension field of F of degree d [so we have a natural identification $K \cdot F_d = F_d((t))$],
- $G_d \subseteq G$ for the open subgroup $\text{Gal}(\overline{K}/K \cdot F_d)$ of G , and
- $\Pi_{U,d} \subseteq \Pi_U, \Pi_{U_0,d} \subseteq \Pi_{U_0}$ for the respective inverse images of the open subgroups $G_d \subseteq G, \text{Gal}(\overline{F}/F_d) \subseteq \text{Gal}(\overline{F}/F)$ in Π_U, Π_{U_0} [which thus may be identified with the étale fundamental groups of $U \times_K F_d((t)) = U_0 \times_F F_d((t)), U_0 \times_F F_d$, respectively].

Lemma 5.1. *The following assertions hold:*

- (i) *Let d be a positive integer, ω an element of F_d , and l, j integers such that $l > 0$ and $j \geq 0$. Then the element $\omega \cdot t^{-p^j \cdot l} - \omega^{1/p^j} \cdot t^{-l}$ of $F_d((t))$ is contained in $\varphi_p(F_d((t)))$.*
- (ii) *Let d be a positive integer, and let y be an element of $F_d((t))$ of the form $\sum_{l>0, p \nmid l} y_l \cdot t^{-l}$ — where $y_l \in F_d$, and $y_l = 0$ for all but finitely many l . Suppose that y is contained in $\varphi_p(F_d((t))) + F_d[[t]]$. Then the equality $y = 0$ holds.*
- (iii) *Let y be an element of K . Suppose that, for each positive integer d and each element $\omega \in F_d$, the element $\omega \cdot y$ of $F_d((t))$ is contained in the subset $\varphi_p(F_d((t))) + F_d[[t]] \subseteq F_d((t))$ of $F_d((t))$. Then the element y is contained in R .*

PROOF. Assertion (i) follows immediately, by induction on j , from the equality $\omega \cdot t^{-p \cdot l} - \omega^{1/p} \cdot t^{-l} = \varphi_p(\omega^{1/p} \cdot t^{-l})$, together with the fact that the finite field F_d is perfect. Next, we verify assertion (ii). Write ord_t for the discrete valuation of $F_d((t))$ normalized by the equality $\text{ord}_t(t) = 1$. Assume that $y \neq 0$. Write l_0 for the maximal positive integer l such that $y_l \neq 0$, which thus implies that $\text{ord}_t(y) = -l_0$. Let $u \in F_d((t)), \delta \in F_d[[t]]$ be elements such that the equality $y = \varphi_p(u) + \delta$ holds. If $\text{ord}_t(u) \geq 0$, then $\varphi_p(u) \in F_d[[t]]$, which thus implies that $y \in F_d[[t]]$ — in contradiction to the inequality $\text{ord}_t(y) = -l_0 < 0$. If $\text{ord}_t(u) < 0$, then $\text{ord}_t(\varphi_p(u)) = p \cdot \text{ord}_t(u)$, which thus implies that $-l_0 = \text{ord}_t(y - \delta) = p \cdot \text{ord}_t(u)$ — in contradiction to the fact that l_0 is prime to p . This completes the proof of assertion (ii).

Next, we verify assertion (iii). Assume that $y \notin R$. Let N be a positive integer, and let $y_1, \dots, y_N$ be elements of F such that $y - \sum_{i=1}^N y_i \cdot t^{-i} \in R$. For each integer $i > N$, write $y_i \stackrel{\text{def}}{=} 0$. Let E be a nonnegative integer such that $p^{E+1} > N$. Then, for each positive integer d and each element $\omega \in F_d$, it follows from assertion (i) that

$$\omega \cdot y - \sum_{l>0, p \nmid l} \left(\sum_{j=0}^E y_{p^j \cdot l}^{1/p^j} \cdot \omega^{1/p^j} \right) \cdot t^{-l} \in \varphi_p(F_d((t))) + F_d[[t]].$$

Thus, it follows from the assumption on y that appears in the statement of assertion (iii), together with assertion (ii) — where we take the “ y ” of assertion (ii) to be $\sum_{l>0, p \nmid l} (\sum_{j=0}^E y_{p^j \cdot l}^{1/p^j} \cdot \omega^{1/p^j}) \cdot t^{-l}$ — that, for each positive integer l prime to p , each positive integer d , and each element $\omega \in F_d$, the equality

$$\sum_{j=0}^E y_{p^j \cdot l}^{1/p^j} \cdot \omega^{1/p^j} = 0$$

holds. Now let d be a positive integer such that $d > E$. Then since the extension degree of F_d over the prime field is $\geq d > E$, the automorphisms of the field F_d given by $\omega \mapsto \omega^{1/p^j}$ — where j ranges over the integers such that $0 \leq j \leq E$ — are distinct. Thus, it follows from the well-known linear independence of distinct characters that, for each positive integer l prime to p and each $0 \leq j \leq E$, the equality $y_{p^j \cdot l} = 0$ holds. In particular, since each positive integer $i \leq N$ may be written as $p^j \cdot l$ for some $0 \leq j \leq E$ and some positive integer l prime to p , it follows that $y_i = 0$ for each $1 \leq i \leq N$, i.e., $y \in R$ — in contradiction to our assumption that $y \notin R$. This completes the proof of assertion (iii), hence also of Lemma 5.1. $\square$

Lemma 5.2. *Let μ be an element of $\mathbb{F}_p^\times$, and let λ be an element of $\mathbb{F}_p$. Write θ_0 for the automorphism of U_0 over F given by $z \mapsto \mu \cdot z + \lambda$. Let ψ be a continuous automorphism of Π_U that lies over the identity automorphism of G , and let $v, w \in U(K)$ be K -rational points of U . Suppose that the following three conditions are satisfied:*

- (1) *The continuous automorphism ψ fits into a commutative diagram of topological groups*

$$\begin{array}{ccc} \Pi_U & \xrightarrow[\sim]{\psi} & \Pi_U \\ \downarrow & & \downarrow \\ \Pi_{U_0} & \xrightarrow[\sim]{} & \Pi_{U_0} \end{array}$$

— where the vertical arrows are the Δ_{U_0} -conjugacy class of continuous surjective homomorphisms induced by the projection morphism $U = U_0 \times_F K \rightarrow U_0$ onto the first factor, and the lower horizontal arrow is the Δ_{U_0} -conjugacy class of continuous automorphisms of Π_{U_0} induced by the automorphism of $U_0 \times_F \bar{F}$ determined by θ_0 and the identity automorphism of $\bar{F}$.

- (2) The element $z(v) \in K$ is not contained in R .
- (3) The composite of ψ with the Galois section-class determined by v [cf. Definition 1.6] coincides with the Galois section-class determined by w .

Then the equality $z(w) = \mu \cdot z(v) + \lambda$ holds.

PROOF. Let d be a positive integer. For each $h \in F_d[z, (z^p - z)^{-1}]$, write

- $\eta_{U,h} \in H^1(\Pi_{U,d}, \mathbb{F}_p)$ for the class of h , relative to the isomorphism of Lemma 1.7, (i) — in the case where we take the “ (A, V, q) ” of Lemma 1.7 to be $(F_d((t)), U_0 \times_F F_d((t)), p)$ —
- $\eta_{U_0,h} \in H^1(\Pi_{U_0,d}, \mathbb{F}_p)$ for the class of h , relative to the isomorphism of Lemma 1.7, (i) — in the case where we take the “ (A, V, q) ” of Lemma 1.7 to be $(F_d, U_0 \times_F F_d, p)$ — and
- $h \circ \theta_0 \in F_d[z, (z^p - z)^{-1}]$ for the image of h by the automorphism of $F_d[z, (z^p - z)^{-1}]$ that maps z to $\mu \cdot z + \lambda$.

Then it follows from the [easily verified] functoriality of Artin–Schreier classes that

- (a) the class $\eta_{U,h}$ coincides with the image of $\eta_{U_0,h}$ by the homomorphism $H^1(\Pi_{U_0,d}, \mathbb{F}_p) \rightarrow H^1(\Pi_{U,d}, \mathbb{F}_p)$ induced by [the restriction to $\Pi_{U,d}$ of] a vertical arrow of the diagram of condition (1), and that
- (b) the homomorphism $H^1(\Pi_{U_0,d}, \mathbb{F}_p) \rightarrow H^1(\Pi_{U_0,d}, \mathbb{F}_p)$ induced by [the restriction to $\Pi_{U_0,d}$ of] the lower horizontal arrow of the diagram of condition (1) maps $\eta_{U_0,h}$ to $\eta_{U_0,h \circ \theta_0}$.

Thus, since an inner automorphism of the profinite group $\Pi_{U_0,d}$ induces the identity automorphism on the cohomology of $\Pi_{U_0,d}$, it follows from the commutativity of the diagram of condition (1), together with (a), (b), that

- (c) the homomorphism $H^1(\Pi_{U,d}, \mathbb{F}_p) \rightarrow H^1(\Pi_{U,d}, \mathbb{F}_p)$ induced by the restriction to $\Pi_{U,d}$ of ψ — which is an automorphism of $\Pi_{U,d}$ since ψ lies over the identity automorphism of G — maps $\eta_{U,h}$ to $\eta_{U,h \circ \theta_0}$.

Next, write $s_v: G \rightarrow \Pi_U$ for the Galois section-class determined by v and $s_w \stackrel{\text{def}}{=} \psi \circ s_v: G \rightarrow \Pi_U$, which thus [cf. condition (3)] is the Galois section-class determined by w . Write, moreover, $\xi \stackrel{\text{def}}{=} \mu \cdot z(v) + \lambda \in K$. Then one verifies easily that the restriction of s_v (respectively, s_w) to G_d determines the Galois section-class determined by the $F_d((t))$ -rational point of $U \times_K F_d((t))$ determined by v (respectively, w). Thus, it follows from the commutativity of the right-hand square of the diagram of Lemma 1.7, (ii), in the case where we take the “ (A, V, q) ” of Lemma 1.7 to be $(F_d((t)), U_0 \times_F F_d((t)), p)$, together with (c), that, for each $h \in F_d[z, (z^p - z)^{-1}]$, the images in $F_d((t))/\varphi_p(F_d((t)))$ of the values $h(z(w))$, $h(\xi) = (h \circ \theta_0)(z(v))$ at $z(w)$, ξ of h [regarded as a rational function in z] coincide. By applying this observation to the elements $h = \omega \cdot z^k$ — where ω is an element of F_d , and k is a positive integer — one concludes that, for each positive integer d , each element $\omega \in F_d$, and each positive integer k , the element $\omega \cdot (z(w)^k - \xi^k)$ is contained in $\varphi_p(F_d((t)))$. Thus, it follows from Lemma 5.1, (iii) — where we take the “ y ” of Lemma 5.1, (iii), to be $z(w)^k - \xi^k$ — that,

- (d) for each positive integer k , the element $z(w)^k - \xi^k$ is contained in R .

Next, observe that since $z(v) \notin R$ [cf. condition (2)], $\mu \in \mathbb{F}_p^\times \subseteq R^\times$, and $\lambda \in \mathbb{F}_p \subseteq R$, the element ξ is not contained in R . Write ord_t for the discrete valuation of K normalized by the equality $\text{ord}_t(t) = 1$ and $\epsilon \stackrel{\text{def}}{=} z(w) - \xi$. Then the inequality $\text{ord}_t(\xi) < 0$ holds, and it follows from (d) in the case where $k = 1$ that $\epsilon \in R$.

Assume that $\epsilon \neq 0$. Let k be a positive integer prime to p . Then since

$$z(w)^k - \xi^k = \sum_{i=1}^k \binom{k}{i} \cdot \xi^{k-i} \cdot \epsilon^i,$$

and $\text{ord}_t(\epsilon) \geq 0 > \text{ord}_t(\xi)$, one verifies easily that $\text{ord}_t(k \cdot \xi^{k-1} \cdot \epsilon) = (k-1) \cdot \text{ord}_t(\xi) + \text{ord}_t(\epsilon)$, and that, for each integer $2 \leq i \leq k$ such that $\binom{k}{i} \cdot \xi^{k-i} \cdot \epsilon^i \neq 0$, the inequalities

$$\text{ord}_t \left(\binom{k}{i} \cdot \xi^{k-i} \cdot \epsilon^i \right) \geq (k-i) \cdot \text{ord}_t(\xi) + i \cdot \text{ord}_t(\epsilon),$$

$$(k-i) \cdot \text{ord}_t(\xi) + i \cdot \text{ord}_t(\epsilon) > (k-1) \cdot \text{ord}_t(\xi) + \text{ord}_t(\epsilon)$$

hold. In particular, the equality $\text{ord}_t(z(w)^k - \xi^k) = (k-1) \cdot \text{ord}_t(\xi) + \text{ord}_t(\epsilon)$ holds. Thus, since $\text{ord}_t(\xi) < 0$, for each sufficiently large positive integer k prime to p , the element $z(w)^k - \xi^k$ is not contained in

R — in contradiction to (d). This completes the proof of the equality $\epsilon = 0$, i.e., the equality $z(w) = \mu \cdot z(v) + \lambda$, hence also of Lemma 5.2. $\square$

In the remainder of the present §5, let

- m be a positive integer such that $a_\circ$ is not contained in $t^m \cdot R$ [cf. the fact that $a_\circ \neq 0$];

write

$$c \stackrel{\text{def}}{=} t^{-m} \in K^\times,$$

which thus implies that neither c nor $c \cdot a_\circ$ is contained in R . In the remainder of the present §5, moreover, let

- n be a nonnegative integer and $c' \in K_n^\times$ an element of $K_n^\times$ such that the equality $\alpha_T(J_c) = J_{c'}$ holds [cf. Lemma 4.4],
- $K' \subseteq \overline{K}_n$ a finite extension field of K_n that contains a root of each of the four polynomials $z^p - z - c$, $z^p - z - c \cdot a_\circ$, $z^p - z - c'$, $z^p - z - c' \cdot a_\bullet$ in the indeterminate z with coefficients in K_n [cf. the fact that these four polynomials are separable], and
- $\sigma_c, \sigma_{ca_\circ}, \sigma_{c'}, \sigma_{c'a_\bullet} \in K'$ roots of these four polynomials, respectively;

write

- $R' \subseteq K'$ for the valuation ring of K' ,
- $F' \subseteq K'$ for the algebraic closure in K' of the prime field of K' ,
- $\overline{F}' \subseteq \overline{K}_n$ for the algebraic closure of F' in $\overline{K}_n$, and
- $G' \stackrel{\text{def}}{=} \text{Gal}(\overline{K}_n/K')$ for the absolute Galois group of K' determined by the separable closure $\overline{K}_n$, which we regard as an open subgroup of the profinite group $G = \text{Gal}(\overline{K}_n/K_n)$ [cf. Definition 4.1, (i)].

Thus, the field K' is a complete discrete valuation field of characteristic p whose residue field is finite, and the field $\overline{K}_n$ is a separable closure of K' . Moreover, one verifies easily that the field F' is finite, and that there exists an isomorphism $K' \xrightarrow{\sim} F'((t'))$ of fields — where t' is an indeterminate — that restricts to the identity automorphism of F' . In the remainder of the present §5, we fix such an isomorphism $K' \xrightarrow{\sim} F'((t'))$, by means of which we regard K' as a field of the form “ $K_\square$ ” of §2. In the remainder of the present §5, write, moreover,

- $U_0, U, U_0^+, U^+, \Pi_U, \Pi_{U_0}, \Delta_U, \Delta_{U_0}$ for the objects of the discussion at the beginning of the present §5 in the case where we take the “ $(F, K, \overline{K})$ ” to be $(F', K', \overline{K}_n)$ [relative to the isomorphism $K' \xrightarrow{\sim} F'((t'))$ fixed above],

- $\Pi'_T \subseteq \Pi_T$, $\Pi'_\circ \subseteq \Pi_\circ$, $\Pi'_\bullet \subseteq \Pi_\bullet$ for the inverse images of $G' \subseteq G$ in Π_T , $\Pi_\circ$, $\Pi_\bullet$, respectively, which thus may be identified with the étale fundamental groups of $T \times_K K'$, $X_\circ \times_K K'$, $X_\bullet \times_K K'$, relative to suitable choices of basepoints, respectively,
- $J_\circ^\dagger \subseteq \Pi'_\circ$ (respectively, $J_\bullet^\dagger \subseteq \Pi'_\bullet$) for the inverse image of $J_c \cap \Pi'_T$ (respectively, $J_{c'} \cap \Pi'_T$) by the natural continuous surjective homomorphism $\Pi_\circ \twoheadrightarrow \Pi_T$ (respectively, $\Pi_\bullet \twoheadrightarrow \Pi_T$) of condition (3) in the discussion at the beginning of §4, and
- $Z_\circ \subseteq U$ (respectively, $Z_\bullet \subseteq U$) for the open subscheme of U obtained by forming the complement in U of the closed subset [of U that corresponds to the subset of $U(K')$] that consists of the $2p$ K' -rational points $z = \sigma_c + \lambda$, $z = \sigma_{ca_\circ} + \lambda$ (respectively, $z = \sigma_{c'} + \lambda$, $z = \sigma_{c'a_\bullet} + \lambda$) — where λ ranges over the elements of $\mathbb{F}_p$.

Lemma 5.3. *The following assertions hold:*

- (i) *Neither σ_c nor $\sigma_{ca_\circ}$ is contained in R' .*
- (ii) *The equality $\alpha(J_\circ^\dagger) = J_\bullet^\dagger$ holds.*
- (iii) *The connected finite étale covering of $X_\circ \times_K K'$ that corresponds to the open subgroup $J_\circ^\dagger \subseteq \Pi'_\circ$ is naturally isomorphic over K' to $Z_\circ$ in such a way that the covering morphism $Z_\circ \rightarrow X_\circ \times_K K'$ extends to the finite morphism $Z_\circ^+ = U^+ \rightarrow X_\circ^+ \times_K K'$ given by $x = c^{-1} \cdot (z^p - z)$ — where we regard z , x as the standard coordinates on the source, target, respectively. Moreover, this finite morphism is étale over $(X_\circ^+ \times_K K') \setminus \{\infty\}$ and totally ramified over ∞ . The assertions obtained by replacing $(X_\circ, X_\circ^+, J_\circ^\dagger, \Pi'_\circ, Z_\circ, Z_\circ^+, c)$ in the above two sentences by $(X_\bullet, X_\bullet^+, J_\bullet^\dagger, \Pi'_\bullet, Z_\bullet, Z_\bullet^+, c')$ also hold. In particular, for each $\square \in \{\circ, \bullet\}$, we may identify $J_\square^\dagger$ with the étale fundamental group of $Z_\square$, relative to a suitable choice of basepoint.*

PROOF. Since [it is immediate that] $R = R' \cap K$, and $\varphi_p(\sigma_c) = c$, $\varphi_p(\sigma_{ca_\circ}) = c \cdot a_\circ$, assertion (i) follows immediately from the fact that neither c nor $c \cdot a_\circ$ is contained in R . Next, we verify assertion (ii). Since the continuous automorphism α_T lies over the identity automorphism of G [cf. condition (4) in the discussion at the beginning of §4], the equality $\alpha_T(\Pi'_T) = \Pi'_T$ holds. Thus, assertion (ii) follows immediately from the commutativity of the diagram of condition (3) in the discussion at the beginning of §4, together with the equality $\alpha_T(J_c) = J_{c'}$.

Next, we verify assertion (iii). It follows from Remark 4.1.1, (i), (ii), that the connected finite étale covering of $T \times_K K'$ that corresponds

to $J_c \cap \Pi'_T$ is naturally isomorphic over $T \times_K K'$ to the affine scheme

$$\mathrm{Spec}(K'[x, x^{-1}, (1-x)^{-1}, z]/(z^p - z - c \cdot x)),$$

which is naturally isomorphic, by means of the equality $x = c^{-1} \cdot (z^p - z)$, to the affine scheme

$$\mathrm{Spec}(K'[z, (z^p - z)^{-1}, (z^p - z - c)^{-1}]).$$

Thus, one verifies easily that the connected finite étale covering of $X_\circ \times_K K' = (T \setminus \{a_\circ\}) \times_K K'$ that corresponds to $J_\circ^\dagger$ is naturally isomorphic to the affine scheme $\mathrm{Spec}(K'[z, (z^p - z)^{-1}, (z^p - z - c)^{-1}, (z^p - z - c \cdot a_\circ)^{-1}]) = Z_\circ$ in such a way that the covering morphism is given by $x = c^{-1} \cdot (z^p - z)$. Moreover, since the derivative with respect to z of $z^p - z - c \cdot x$ is -1 , and the pole of $c^{-1} \cdot (z^p - z)$ at $z = \infty$ is of order p , one verifies easily that the finite morphism $Z_\circ^+ = U^+ \rightarrow X_\circ^+ \times_K K'$ given by $x = c^{-1} \cdot (z^p - z)$ is étale over $(X_\circ^+ \times_K K') \setminus \{\infty\}$ and totally ramified over ∞ . A similar argument implies the assertion of (iii) concerning $J_\bullet^\dagger, Z_\bullet$. This completes the proof of assertion (iii), hence also of Lemma 5.3. $\square$

Lemma 5.4. *The following assertions hold:*

- (i) *There exists a continuous automorphism ψ of Π_U that lies over the identity automorphism of G' and fits into a commutative diagram of topological groups*

$$\begin{array}{ccc} J_\circ^\dagger & \xrightarrow[\sim]{\alpha|_{J_\circ^\dagger}} & J_\bullet^\dagger \\ \downarrow & & \downarrow \\ \Pi_U & \xrightarrow[\psi]{\sim} & \Pi_U \end{array}$$

— where the vertical arrows are the continuous surjective homomorphisms induced by the natural open immersions $Z_\circ \hookrightarrow U, Z_\bullet \hookrightarrow U$, respectively [cf. Lemma 5.3, (ii), (iii)].

- (ii) *Let ψ be as in (i). Then the continuous automorphism ψ maps*
- *each cuspidal inertia subgroup of Π_U associated to a cusp of U contained in $\mathbb{F}_p \subseteq U^+(K')$ onto a cuspidal inertia subgroup of Π_U associated to a cusp of U contained in $\mathbb{F}_p \subseteq U^+(K')$ and*
 - *each cuspidal inertia subgroup of Π_U associated to the cusp ∞ of U onto a cuspidal inertia subgroup of Π_U associated to the cusp ∞ of U .*
- (iii) *Let ψ be as in (i). Then the continuous automorphism ψ is cyclotomically compatible [cf. Definition 2.4].*

- (iv) Let ψ be as in (i). Then there exist elements $\mu \in \mathbb{F}_p^\times$, $\lambda_0 \in \mathbb{F}_p$ that satisfy condition (1) of Lemma 5.2 in the case where we take the “ (μ, λ, ψ) ” of Lemma 5.2 to be (μ, λ_0, ψ) and the “ $(F, K, \overline{K})$ ” of Lemma 5.2 to be $(F', K', \overline{K}_n)$ [relative to the isomorphism $K' \xrightarrow{\sim} F'((t'))$ fixed in the discussion preceding Lemma 5.3].
- (v) Let ψ be as in (i), $(\zeta_\circ, \zeta_\bullet)$ an element of $\{(1, 1), (a_\circ, a_\bullet)\}$, and $v \in U(K')$ a K' -rational point of U such that the equality $z(v)^p - z(v) = c \cdot \zeta_\circ$ holds. Then there exists a K' -rational point $w \in U(K')$ of U such that the equality $z(w)^p - z(w) = c' \cdot \zeta_\bullet$ holds, and, moreover, the composite of ψ with the Galois section-class determined by v coincides with the Galois section-class determined by w .

PROOF. First, we verify assertion (i). Observe that one verifies easily that, for each $\square \in \{\circ, \bullet\}$, the kernel of the natural continuous surjective homomorphism $J_\square^\dagger \twoheadrightarrow \Pi_U$ is topologically normally generated by the cuspidal inertia subgroups of $J_\square^\dagger$ associated to the cusps of $Z_\square$ that lie over the cusps 1, $a_\square$ of $X_\square$. Moreover, since [it follows from Lemma 5.3, (iii), that] the finite morphism $U^+ \rightarrow X_\square^+ \times_K K'$ of Lemma 5.3, (iii), is étale over $(X_\square^+ \times_K K') \setminus \{\infty\}$, one verifies easily that these cuspidal inertia subgroups of $J_\square^\dagger$ are precisely the cuspidal inertia subgroups of $\Pi_\square$ associated to the cusps 1, $a_\square$ of $X_\square$. Thus, since [it is immediate that] the equality $\alpha(\Delta_\circ) = \Delta_\bullet$ holds, it follows from condition (2) in the discussion at the beginning of §4, together with Lemma 5.3, (ii), that α maps the kernel of $J_\circ^\dagger \twoheadrightarrow \Pi_U$ onto the kernel of $J_\bullet^\dagger \twoheadrightarrow \Pi_U$. In particular, since the continuous isomorphism α lies over the identity automorphism of G , assertion (i) holds. This completes the proof of assertion (i).

In the remainder of the present proof, let ψ be as in assertion (i). Then since the continuous automorphism ψ lies over the identity automorphism of G' , it is immediate that ψ is base-geometric [cf. Definition 2.3], and that the isomorphism of fields of Definition 2.3 in the case of ψ is the identity automorphism of $\overline{K}_n$.

Next, we verify assertions (ii), (iii). First, observe that it follows from Lemma 5.3, (iii), that, for each $\square \in \{\circ, \bullet\}$, the cusps of U contained in $\mathbb{F}_p$ lie over the cusp 0 of $X_\square$, and that the cusp ∞ of U lies over the cusp ∞ of $X_\square$. Thus, since [it follows from Lemma 5.3, (iii), that] the finite morphism $U^+ \rightarrow X_\square^+ \times_K K'$ of Lemma 5.3, (iii), is étale over $(X_\square^+ \times_K K') \setminus \{\infty\}$ and totally ramified over ∞ , one verifies easily that, for each $\square \in \{\circ, \bullet\}$, the cuspidal inertia subgroups of Π_U associated to the cusps of U contained in $\mathbb{F}_p$ (respectively, to the cusp ∞ of

U) are precisely the images by the natural continuous surjective homomorphism $J_{\square}^{\dagger} \twoheadrightarrow \Pi_U$ of the cuspidal inertia subgroups of $\Pi_{\square}$ associated to the cusp 0 of $X_{\square}$ (respectively, of the intersections with $J_{\square}^{\dagger}$ of the cuspidal inertia subgroups of $\Pi_{\square}$ associated to the cusp ∞ of $X_{\square}$). In particular, assertion (ii) follows immediately from condition (2) in the discussion at the beginning of §4 and assertion (i), together with the equality $\alpha(\Delta_{\circ}) = \Delta_{\bullet}$.

Next, let $J \subseteq \Pi_U$ be an open subgroup of Π_U . Write $J' \subseteq J_{\circ}^{\dagger}$ for the inverse image of J by the natural continuous surjective homomorphism $J_{\circ}^{\dagger} \twoheadrightarrow \Pi_U$. Then it follows from the commutativity of the diagram of assertion (i) that $\alpha(J') \subseteq J_{\bullet}^{\dagger}$ is the inverse image of $\psi(J)$ by the natural continuous surjective homomorphism $J_{\bullet}^{\dagger} \twoheadrightarrow \Pi_U$. Moreover, one verifies easily that the connected finite étale covering of $X_{\circ} \times_K K'$ (respectively, $X_{\bullet} \times_K K'$) that corresponds to J' (respectively, $\alpha(J')$) is an open subscheme of the connected finite étale covering of U that corresponds to J (respectively, $\psi(J)$), which thus implies that these two finite étale coverings have the same regular compactification. In particular, it follows from [1, Exposé IX, Théorème 4.10] and footnote 5 of [1, Exposé IX, §4] that the cyclotome [cf. Definition 2.1, (iii)] associated to the connected finite étale covering of $X_{\circ}$ (respectively, $X_{\bullet}$) that corresponds to J' (respectively, $\alpha(J')$) may be naturally identified with the cyclotome associated to the connected finite étale covering of U that corresponds to J (respectively, $\psi(J)$). Thus, since the continuous isomorphism α is cyclotomically compatible, and the isomorphisms of fields of Definition 2.3 in the case of α, ψ are the identity automorphisms of $\overline{K}, \overline{K}_n$, respectively, assertion (iii) follows immediately. This completes the proof of assertions (ii), (iii).

Next, we verify assertion (iv). Observe that one verifies easily that the smooth curve $U = U_0 \times_{F'} K'$ over K' [cf. Definition 1.3] is strictly isotrivial [cf. Definition 1.5, (ii)] and admits good reduction over R' [cf. Definition 2.7]. Thus, it follows from assertion (iii), together with Theorem 3.4 and Theorem 3.5, (i), (ii) — where we take the “ $(K_{\circ}, K_{\bullet}, X_{\circ}, X_{\bullet}, \alpha)$ ” of §3 to be (K', K', U, U, ψ) [relative to the isomorphism $K' \xrightarrow{\sim} F'((t'))$ fixed in the discussion preceding Lemma 5.3] and, for each $\square \in \{\circ, \bullet\}$, the isomorphism “ $X_{\square} \xrightarrow{\sim} X_{\square,0} \times_{F_{\square}} K_{\square}$ ” fixed in the discussion at the beginning of §3 to be the identity automorphism of $U = U_0 \times_{F'} K'$ — that there exist a continuous automorphism ψ_0 of Π_{U_0} and an automorphism θ_0 of U_0 over F' such that

- the continuous automorphisms ψ, ψ_0 fit into a commutative diagram of topological groups as in condition (1) of Lemma 5.2, and, moreover,

- the continuous automorphism ψ_0 of Π_{U_0} is contained in the Δ_{U_0} -conjugacy class of continuous automorphisms of Π_{U_0} induced by the automorphism of $U_0 \times_{F'} \bar{F}'$ determined by θ_0 and the identity automorphism of $\bar{F}'$.

Moreover, one verifies easily that a cuspidal inertia subgroup of Π_{U_0} associated to a cusp of U_0 is not a cuspidal inertia subgroup of Π_{U_0} associated to any other cusp of U_0 [cf. [22, Corollary 2.2]]. Thus, since [one verifies easily that] the image in Π_{U_0} of a cuspidal inertia subgroup of Π_U associated to a cusp of U is a cuspidal inertia subgroup of Π_{U_0} associated to the corresponding cusp of U_0 , it follows from assertion (ii), together with the commutativity of the diagram of condition (1) of Lemma 5.2, that the automorphism of U_0^+ over F' that extends θ_0 maps ∞ to ∞ and the subset $\mathbb{F}_p \subseteq U_0^+(F')$ onto $\mathbb{F}_p \subseteq U_0^+(F')$. In particular, since the cardinality of $\mathbb{F}_p$ is ≥ 2 , there exist elements $\mu \in \mathbb{F}_p^\times$, $\lambda_0 \in \mathbb{F}_p$ such that θ_0 is given by $z \mapsto \mu \cdot z + \lambda_0$. This completes the proof of assertion (iv).

Finally, we verify assertion (v). First, observe that it follows from Lemma 5.3, (iii), that v is a cusp of $Z_\circ$ that lies over the cusp $\zeta_\circ$ of $X_\circ$. Let $\mathfrak{D}_v \subseteq J_\circ^\dagger$ be a decomposition subgroup of $J_\circ^\dagger$ associated to the cusp v of $Z_\circ$. Then since [it follows from Lemma 5.3, (iii), that] the finite morphism $U^+ \rightarrow X_\circ^+ \times_K K'$ of Lemma 5.3, (iii), is étale over $(X_\circ^+ \times_K K') \setminus \{\infty\}$, and v is K' -rational, one verifies easily that there exists a decomposition subgroup $\mathfrak{D} \subseteq \Pi_\circ$ of $\Pi_\circ$ associated to the cusp $\zeta_\circ$ of $X_\circ$ such that the equality $\mathfrak{D}_v = \mathfrak{D} \cap \Pi_\circ'$ holds. Thus, it follows from condition (2) in the discussion at the beginning of §4, together with Lemma 5.3, (ii), that $\alpha(\mathfrak{D}_v) = \alpha(\mathfrak{D}) \cap \Pi_\bullet'$ is a decomposition subgroup of $\Pi_\bullet'$ associated to the cusp $\zeta_\bullet$ of $X_\bullet \times_K K'$ that is contained in $J_\bullet^\dagger$, which thus implies that $\alpha(\mathfrak{D}_v)$ is a decomposition subgroup of $J_\bullet^\dagger$ associated to a K' -rational cusp of $Z_\bullet$ that lies over the cusp $\zeta_\bullet$ of $X_\bullet$. Write $w \in U(K')$ for this cusp. Then it follows from Lemma 5.3, (iii), that the equality $z(w)^p - z(w) = c' \cdot \zeta_\bullet$ holds. On the other hand, one verifies easily that the image of $\mathfrak{D}_v$ (respectively, $\alpha(\mathfrak{D}_v)$) in Π_U by the natural continuous surjective homomorphism $J_\circ^\dagger \twoheadrightarrow \Pi_U$ (respectively, $J_\bullet^\dagger \twoheadrightarrow \Pi_U$) is the image of a member of the Galois section-class determined by v (respectively, w). Thus, assertion (v) follows immediately from the commutativity of the diagram of assertion (i). This completes the proof of assertion (v), hence also of Lemma 5.4. $\square$

Lemma 5.5. *The equality $a_\circ = a_\bullet$ holds, i.e., the isomorphism $K_\bullet \xrightarrow{\sim} K_\circ$ of fields obtained by restricting the isomorphism $\bar{K}_\bullet \xrightarrow{\sim} \bar{K}_\circ$ of fields of Definition 2.3 maps $a_\bullet \in K_\bullet$ to $a_\circ \in K_\circ$.*

PROOF. Let ψ be as in Lemma 5.4, (i), and let $\mu \in \mathbb{F}_p^\times$, $\lambda_0 \in \mathbb{F}_p$ be as in Lemma 5.4, (iv). Write $v_1, v_2 \in U(K')$ for the K' -rational points $z = \sigma_c$, $z = \sigma_{ca_\circ}$ of U , respectively [cf. the fact that $c \neq 0$, $c \cdot a_\circ \neq 0$]. Then it follows from Lemma 5.4, (v) — where we take the “ $(\zeta_\circ, \zeta_\bullet, v)$ ” of Lemma 5.4, (v), to be $(1, 1, v_1)$ (respectively, $(a_\circ, a_\bullet, v_2)$) — that

- there exists a K' -rational point w_1 (respectively, w_2) of U such that the equality $z(w_1)^p - z(w_1) = c'$ (respectively, $z(w_2)^p - z(w_2) = c' \cdot a_\bullet$) holds, and, moreover,
- the composite of ψ with the Galois section-class determined by v_1 (respectively, v_2) coincides with the Galois section-class determined by w_1 (respectively, w_2).

Thus, it follows from Lemma 5.3, (i), and Lemma 5.4, (iv), together with Lemma 5.2 — where we take the “ $(\mu, \lambda, \psi, v, w)$ ” of Lemma 5.2 to be $(\mu, \lambda_0, \psi, v_1, w_1)$ (respectively, $(\mu, \lambda_0, \psi, v_2, w_2)$) and the “ $(F, K, \overline{K})$ ” of Lemma 5.2 to be $(F', K', \overline{K}_n)$ [relative to the isomorphism $K' \xrightarrow{\sim} F'((t'))$ fixed in the discussion preceding Lemma 5.3] — that the equality $z(w_1) = \mu \cdot \sigma_c + \lambda_0$ (respectively, $z(w_2) = \mu \cdot \sigma_{ca_\circ} + \lambda_0$) holds. In particular, since the Artin–Schreier endomorphism φ_p of K' [cf. Definition 1.1] is $\mathbb{F}_p$ -linear and vanishes on $\mathbb{F}_p$, the equalities

$$c' = \varphi_p(z(w_1)) = \mu \cdot \varphi_p(\sigma_c) = \mu \cdot c,$$

$$c' \cdot a_\bullet = \varphi_p(z(w_2)) = \mu \cdot \varphi_p(\sigma_{ca_\circ}) = \mu \cdot c \cdot a_\circ$$

hold. Thus, since $\mu \cdot c \neq 0$, the equality $a_\circ = a_\bullet$ holds, as desired. This completes the proof of Lemma 5.5. $\square$

6. ANABELIAN RESULT FOR SMOOTH CURVES OF GENUS ZERO

In the present §6, we prove the main theorem of the present paper, i.e., the assertion that a smooth curve obtained by removing at least three rational points from the projective line over a field of formal Laurent series with coefficients in a finite field is determined, up to pulling back by a power of the Frobenius endomorphism of the base field, by its étale fundamental group equipped with the structure of a “base-geometric extension” of the absolute Galois group of the base field [cf. Theorem 6.1 below].

Theorem 6.1. *For each $\square \in \{\circ, \bullet\}$, let*

- $p_\square$ be a prime number,
- $F_\square$ a finite field of characteristic $p_\square$, and
- $r_\square \geq 3$ an integer.

Write $K_\square \stackrel{\text{def}}{=} F_\square((t))$ — where t is an indeterminate — for the field of formal Laurent series with coefficients in $F_\square$. For each $\square \in \{\circ, \bullet\}$, moreover, let $S_\square \subseteq \mathbb{P}_{K_\square}^1(K_\square)$ be a subset of $\mathbb{P}_{K_\square}^1(K_\square)$ of cardinality $r_\square$. Write

- $X_\square \stackrel{\text{def}}{=} \mathbb{P}_{K_\square}^1 \setminus S_\square$ for the open subscheme of $\mathbb{P}_{K_\square}^1$ obtained by forming the complement in $\mathbb{P}_{K_\square}^1$ of the closed subset [of $\mathbb{P}_{K_\square}^1$ that corresponds to] $S_\square \subseteq \mathbb{P}_{K_\square}^1(K_\square)$ and
- $\Pi_\square$ for the étale fundamental group of $X_\square$, relative to a suitable choice of basepoint.

Then the following two conditions are equivalent:

- (1) There exists a nonnegative integer n such that the scheme $X_\circ$ is isomorphic to the scheme $X_\bullet(n)$, or the scheme $X_\bullet$ is isomorphic to the scheme $X_\circ(n)$ — where, for each $\square \in \{\circ, \bullet\}$, we write $X_\square(n)$ for the scheme over $K_\square$ obtained by pulling back the scheme $X_\square$ over $K_\square$ by the absolute $p_\square^n$ -th power Frobenius endomorphism of $\text{Spec}(K_\square)$ [cf. Definition 2.5, (ii)].
- (2) There exists a base-geometric [cf. Definition 2.3] continuous isomorphism $\Pi_\circ \xrightarrow{\sim} \Pi_\bullet$ — i.e., a continuous isomorphism $\Pi_\circ \xrightarrow{\sim} \Pi_\bullet$ that lies over a continuous isomorphism between the absolute Galois groups of $K_\circ, K_\bullet$ which arises from an isomorphism between separable closures of $K_\bullet, K_\circ$ that restricts to an isomorphism $K_\bullet \xrightarrow{\sim} K_\circ$ of subfields.

PROOF. Observe that, for each $\square \in \{\circ, \bullet\}$, since $r_\square \geq 3$, the smooth curve $X_\square$ over $K_\square$ [cf. Definition 1.3] is not proper over $K_\square$. Thus, for each $\square \in \{\circ, \bullet\}$, by fixing an algebraic closure of $K_\square$, we may regard the data $(p_\square, F_\square, X_\square)$ as data as in the discussion at the beginning of §2. In the remainder of the present proof, we apply the notational conventions introduced in §2 to the data $(p_\square, F_\square, X_\square)$. Next, observe that one verifies easily that, for each $\square \in \{\circ, \bullet\}$,

- (a) for each nonnegative integer n , the smooth curve $X_\square(n)$ over $K_\square$ [cf. Definition 2.5, (ii)] is the open subscheme of $\mathbb{P}_{K_\square}^1$ obtained by forming the complement in $\mathbb{P}_{K_\square}^1$ of the closed subset [of $\mathbb{P}_{K_\square}^1$ that corresponds to] the subset of $\mathbb{P}_{K_\square}^1(K_\square)$ obtained by applying the $p_\square^n$ -th power map to the homogeneous coordinates of the elements of $S_\square \subseteq \mathbb{P}_{K_\square}^1(K_\square)$,

and, moreover,

- (b) the smooth curve $X_\square$ over $K_\square$ admits stable reduction over $R_\square$ [cf. Definition 2.6],

- (c) the smooth curve $X_\square$ over $K_\square$ is isotrivial [cf. Definition 1.5, (i)] if and only if the smooth curve $X_\square$ over $K_\square$ is strictly isotrivial [cf. Definition 1.5, (ii)], and
- (d) every subfield of the ring $\mathcal{O}_{X_\square}(X_\square)$ is contained in the subfield $K_\square$.

In particular, it follows from (a) that, for each $\square \in \{\circ, \bullet\}$ and each nonnegative integer n , assertions (b), (c), (d) also hold for the smooth curve $X_\square(n)$ over $K_\square$.

First, we verify the implication $(1) \Rightarrow (2)$. Since the inverse of a base-geometric continuous isomorphism $\Pi_\circ \xrightarrow{\sim} \Pi_\bullet$ is base-geometric, we may assume without loss of generality that there exist a nonnegative integer n and an isomorphism $X_\circ \xrightarrow{\sim} X_\bullet(n)$ of schemes. Then it follows from (d) that this isomorphism lies over an isomorphism $\text{Spec}(K_\circ) \xrightarrow{\sim} \text{Spec}(K_\bullet)$ of schemes, which thus implies that this isomorphism induces a continuous isomorphism from $\Pi_\circ$ to the étale fundamental group of $X_\bullet(n)$ that lies over the continuous isomorphism $G_\circ \xrightarrow{\sim} G_\bullet$ induced by an isomorphism $\overline{K}_\bullet \xrightarrow{\sim} \overline{K}_\circ$ of fields that restricts to an isomorphism $K_\bullet \xrightarrow{\sim} K_\circ$ of subfields. Thus, by composing this continuous isomorphism with the inverse of a continuous isomorphism as in Remark 2.5.1 in the case where we take the “ $(\square, n)$ ” of Remark 2.5.1 to be $(\bullet, n)$ [which thus lies over the identity automorphism of $G_\bullet$], one obtains a base-geometric continuous isomorphism $\Pi_\circ \xrightarrow{\sim} \Pi_\bullet$. This completes the proof of the implication $(1) \Rightarrow (2)$.

Next, we verify the implication $(2) \Rightarrow (1)$. Suppose that condition (2) is satisfied. Let $\alpha: \Pi_\circ \xrightarrow{\sim} \Pi_\bullet$ be a base-geometric continuous isomorphism. In the remainder of the present proof, we regard α as the “ α ” of the discussion at the beginning of §2. Then it follows from (b) that we are in the situation at the beginning of §3.

Suppose that the smooth curve $X_\circ$ over $K_\circ$ is isotrivial. Then it follows from (c) that the smooth curve $X_\circ$ over $K_\circ$ is strictly isotrivial. Thus, it follows from Theorem 3.5, (i), that the scheme $X_\circ$ is isomorphic to the scheme $X_\bullet = X_\bullet(0)$, as desired.

Next, suppose that the smooth curve $X_\circ$ over $K_\circ$ is not isotrivial. Then it follows from Lemma 2.12 that there exists an integer n as in Lemma 2.12. Thus, we may assume without loss of generality — by replacing $X_\bullet$ by $X_\bullet(n)$ and α by the composite of α with a continuous isomorphism as in Lemma 2.12 if n is nonnegative (respectively, $X_\circ$ by $X_\circ(-n)$ and α by the composite of the inverse of a continuous isomorphism as in Lemma 2.12 with α if n is negative) [cf. (a)] — that the continuous isomorphism α is cyclotomically compatible [cf. Definition 2.4].

Next, observe that it follows from Lemma 2.2, (ii), together with [22, Corollary 2.2], that there exists a bijective map $\beta: S_\circ \xrightarrow{\sim} S_\bullet$ such that, for each $x \in S_\circ$, the continuous isomorphism α maps each cuspidal inertia subgroup of $\Pi_\circ$ associated to the cusp x of $X_\circ$ onto a cuspidal inertia subgroup of $\Pi_\bullet$ associated to the cusp $\beta(x)$ of $X_\bullet$. Moreover, one verifies easily from [22, Corollary 2.2] [cf. also [1, Exposé IX, Théorème 4.10] and footnote 5 of [1, Exposé IX, §4]] that, for each $\square \in \{\circ, \bullet\}$, the decomposition subgroups of $\Pi_\square$ associated to a cusp of $X_\square$ are precisely the normalizers in $\Pi_\square$ of the cuspidal inertia subgroups of $\Pi_\square$ associated to the cusp. Thus, one concludes that

- (e) for each $x \in S_\circ$, the continuous isomorphism α maps each decomposition subgroup of $\Pi_\circ$ associated to the cusp x of $X_\circ$ onto a decomposition subgroup of $\Pi_\bullet$ associated to the cusp $\beta(x)$ of $X_\bullet$.

Let $x_1, x_2, x_3 \in S_\circ$ be three distinct elements of $S_\circ$ [cf. our assumption that $r_\circ \geq 3$]. Then since there exists an automorphism of $\mathbb{P}_{K_\circ}^1$ over $K_\circ$ (respectively, $\mathbb{P}_{K_\bullet}^1$ over $K_\bullet$) that maps x_1, x_2, x_3 (respectively, $\beta(x_1), \beta(x_2), \beta(x_3)$) to $0, 1, \infty$, respectively, one verifies easily that we may assume without loss of generality, by replacing $S_\circ, S_\bullet, \alpha$, and β suitably, that, for each $\square \in \{\circ, \bullet\}$, $\{0, 1, \infty\} \subseteq S_\square$, and that $\beta(\zeta) = \zeta$ for each $\zeta \in \{0, 1, \infty\}$. Since β is bijective, to verify that the scheme $X_\circ$ is isomorphic to the scheme $X_\bullet$, it suffices to verify the following assertion:

- (*) For each $x \in S_\circ \setminus \{0, 1, \infty\}$, the isomorphism $K_\bullet \xrightarrow{\sim} K_\circ$ of fields obtained by restricting the isomorphism $\overline{K}_\bullet \xrightarrow{\sim} \overline{K}_\circ$ of fields of Definition 2.3 maps $\beta(x) \in K_\bullet$ to $x \in K_\circ$.

In particular, it is immediate that, to verify assertion (*), we may assume without loss of generality, by replacing $S_\square$ by a suitable subset of $S_\square$ for each $\square \in \{\circ, \bullet\}$, that $S_\square$ is of cardinality four. For each $\square \in \{\circ, \bullet\}$, write $a_\square \in S_\square$ for the unique element of $S_\square \setminus \{0, 1, \infty\}$.

Then, for each $\square \in \{\circ, \bullet\}$, the equality $S_\square = \{0, 1, \infty, a_\square\}$ holds, which thus implies that the smooth curve $X_\square$ over $K_\square$ is the open subscheme $T_\square \setminus \{a_\square\}$ of the smooth curve $T_\square$ over $K_\square$ [cf. the discussion at the beginning of §4]. Moreover, since β is bijective, and $\beta(\zeta) = \zeta$ for each $\zeta \in \{0, 1, \infty\}$, the equality $\beta(a_\circ) = a_\bullet$ holds. Thus, it follows from (e) that the continuous isomorphism α maps each decomposition subgroup of $\Pi_\circ$ associated to the cusp 0 (respectively, $1; \infty; a_\circ$) of $X_\circ$ onto a decomposition subgroup of $\Pi_\bullet$ associated to the cusp 0 (respectively, $1; \infty; a_\bullet$) of $X_\bullet$. In particular, since the continuous isomorphism α is cyclotomically compatible, we are in the situation at the beginning of §4 [cf. conditions (1), (2) in the discussion at the beginning of §4]. Thus, it follows from Lemma 5.5 that the isomorphism $K_\bullet \xrightarrow{\sim} K_\circ$

of fields obtained by restricting the isomorphism $\overline{K}_\bullet \xrightarrow{\sim} \overline{K}_\circ$ of fields of Definition 2.3 maps $a_\bullet = \beta(a_\circ)$ to $a_\circ$. This completes the proof of assertion (*), hence also of Theorem 6.1. $\square$

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