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# Extension of M-convexity and L-convexity to Polyhedral Convex Functions[1]

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**Abstract**

The concepts of M-convex and L-convex functions were proposed by Murota in 1996 as two mutually conjugate classes of discrete functions over integer lattice points. M/L-convex functions are deeply connected with the well-solvability in nonlinear combinatorial optimization with integer variables. In this paper, we extend the concept of M-convexity and L-convexity to polyhedral convex functions, aiming at clarifying the well-behaved structure in well-solved nonlinear combinatorial optimization problems in real variables. The extended M/L-convexity often appear in nonlinear combinatorial optimization problems with piecewise-linear convex cost. We investigate the structure of polyhedral M-convex and L-convex functions from the dual viewpoint of analysis and combinatorics, and provide some properties and characterizations. It is also shown that polyhedral M/L-convex functions have nice conjugacy relationship.

**Keywords:** combinatorial optimization, matroid, base polyhedron, convex analysis, polyhedral convex function.

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# Contents

# 1 Introduction

In the area of combinatorial optimization, there exist many "well-solved" problems, i.e., the problems which have nice combinatorial structure and which can be solved efficiently (see [5, 24, 38]). Many researchers have been trying to identify the well-behaved structure in combinatorial optimization problems.

The concept of matroid, introduced by Whitney [45], plays an important role in the field of combinatorial optimization (see [44, 46, 47, 48]). Matroidal structure is closely related to the well-solvability of combinatorial optimization problems such as those on graphs and matroids, and can be found in fairly large number of efficiently solvable problems. Matroidal structure yields the tractability of problems in the following way:

- Global optimality is equivalent to local optimality, which implies the success of the so-called greedy algorithm for the problem of optimizing a linear function over a single matroid.
- A nice duality theorem, Edmonds' intersection theorem [9], guarantees the existence of a certificate for the optimality in the matroid intersection problem in terms of dual variables.

In 1970, Edmonds introduced the concept of polymatroid by extending that of matroid to sets of real vectors ([9], see also [44]). A polymatroid $P \subseteq \mathbf{R}_+^V$ is a polyhedron given as

$$P = \{x \in \mathbf{R}_+^V \mid \sum_{w \in V} x(w) \leq \rho(X) \ (\forall X \subseteq V)\}$$

by a submodular set function $\rho : 2^V \to \mathbf{R}$ with certain additional conditions, where $\mathbf{R}_+$ denotes the set of nonnegative reals, and $\rho$ is called submodular if

$$\rho(X) + \rho(Y) \geq \rho(X \cap Y) + \rho(X \cup Y) \qquad (\forall X, Y \subseteq V).$$

Polymatroids share nice combinatorial properties of matroids: for example, the greedy algorithm for matroids still works for polymatroids, and a duality holds for the polymatroid intersection problem. Fujishige, emphasizing the essential role of submodularity of $\rho$, generalized the concept of polymatroid to that of submodular system [16, 17].

In recent years, nonlinear combinatorial optimization problems are investigated more often due to theoretical interest and necessity in practical application. The nonlinear resource allocation problem (see [20, 23]) and the convex cost submodular flow problem (see [17, 22]) are examples of nonlinear combinatorial optimization problems. Both of the problems have nice combinatorial structures, which lead to efficient combinatorial algorithms. These results, however, do not completely fit in the framework of matroid, polymatroid, and submodular system.

The concepts of M-convex and L-convex functions, introduced by Murota [30, 31, 33], afford a nice framework for well-solved nonlinear combinatorial optimization problems. M-convex function is a natural extension of the concept of valuated matroid introduced by Dress–Wenzel [7, 8] (see also [27, 28, 29]) as well as a quantitative generalization of the set of integral points in an integral base polyhedron [17]. L-convex function is an extension of submodular set function.

Let $V$ be a finite set. A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is called M-convex if it satisfies (M-EXC[$\mathbf{Z}$]):

**(M-EXC[Z])** $\forall x, y \in \mathrm{dom}_{\mathbf{Z}} f$, $\forall u \in \mathrm{supp}^+(x - y)$, $\exists v \in \mathrm{supp}^-(x - y)$ such that

$$f(x) + f(y) \geq f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v),$$

where $\mathrm{dom}_{\mathbf{Z}} f = \{x \in \mathbf{Z}^V \mid -\infty < f(x) < +\infty\}$, $\mathrm{supp}^+(x-y) = \{w \in V \mid x(w) > y(w)\}$, $\mathrm{supp}^-(x-y) = \{w \in V \mid x(w) < y(w)\}$, and $\chi_w \in \{0,1\}^V$ is the characteristic vector of $w \in V$. A function $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is called L-convex[4] if it satisfies (LF1[**Z**]) and (LF2[**Z**]):

**(LF1[Z])** $\quad g(p) + g(q) \geq g(p \wedge q) + g(p \vee q) \qquad (\forall p, q \in \mathrm{dom}_{\mathbf{Z}}\, g)$,
**(LF2[Z])** $\quad \exists r \in \mathbf{R}$ such that $g(p + \lambda \mathbf{1}) = g(p) + \lambda r \quad (\forall p \in \mathrm{dom}_{\mathbf{Z}}\, g,\ \lambda \in \mathbf{Z})$,

where $p \wedge q, p \vee q\ (\in \mathbf{R}^V)$ denote the vectors with $(p \wedge q)(v) = \min\{p(v), q(v)\}$, $(p \vee q)(v) = \max\{p(v), q(v)\}$ $(v \in V)$, and $\mathbf{1}\ (\in \mathbf{R}^V)$ is the vector with each component being equal to one. M/L-convex functions have nice properties:

- local optimality is equivalent to global optimality.
- M/L-convex functions can be extended to ordinary convex functions.
- M/L-convex functions are conjugate to each other.
- a (discrete) separation theorem and a Fenchel-type duality theorem hold for a pair of M-convex/M-concave (L-convex/L-concave) functions.

The minimization of M/L-convex functions can be done in polynomial time [11, 42]. Application of M-convex functions can be found in system analysis through polynomial matrices [26, 32, 34], and in mathematical economics [6].

M-convexity and L-convexity appear in various nonlinear combinatorial optimization problems with integer variables. Such nice combinatorial properties, however, are enjoyed not only by combinatorial optimization problems in integer variables but also by those in real variables. We dwell on this point by considering the minimum cost flow/tension problems.

Let $G = (V, A)$ be a directed graph with a specified vertex subset $T \subseteq V$. Suppose we are given a family of piecewise-linear convex functions $f_a : \mathbf{R} \to \mathbf{R} \cup \{+\infty\}$ $(a \in A)$, each of which represents the cost of flow on the arc $a$. A function $\xi : A \to \mathbf{R}$ is called a flow. The boundary $\partial\xi : V \to \mathbf{R}$ of a flow $\xi$ is given by

$$\partial\xi(v) = \sum\{\xi(a) \mid a(\in A) \text{ leaves } v\} - \sum\{\xi(a) \mid a(\in A) \text{ enters } v\} \qquad (v \in V). \tag{1.1}$$

Then, the cost function $f : \mathbf{R}^T \to \mathbf{R} \cup \{\pm\infty\}$ of the minimum cost flow that realizes a supply/demand vector $x \in \mathbf{R}^T$ is defined by

$$f(x) = \inf\{\sum_{a \in A} f_a(\xi(a)) \mid \xi \in \mathbf{R}^A,\ \partial\xi(w) = -x(w)\ (w \in T),\ \partial\xi(w) = 0\ (w \in V \setminus T)\} \quad (x \in \mathbf{R}^T). \tag{1.2}$$

Suppose we are given another family of piecewise-linear convex functions $g_a : \mathbf{R} \to \mathbf{R} \cup \{+\infty\}$ $(a \in A)$, each of which represents the cost of tension on the arc $a$. Any function $p : V \to \mathbf{R}$ is called a potential. Given a potential $p$, its coboundary $\delta p : A \to \mathbf{R}$ is defined by

$$\delta p(a) = p(u) - p(v) \qquad (a = (u, v) \in A). \tag{1.3}$$

Then, the cost function $g : \mathbf{R}^T \to \mathbf{R} \cup \{\pm\infty\}$ of the minimum cost tension that realizes a potential vector $p' \in \mathbf{R}^T$ is written as

$$g(p') = \inf\{\sum_{a \in A} g_a(-\delta p(a)) \mid p \in \mathbf{R}^V,\ p(w) = p'(w)\ (w \in T)\} \qquad (p' \in \mathbf{R}^T). \tag{1.4}$$

[4] In the original definition [33], an L-convex function is assumed to be integer-valued. See Remark 6.1.

It is well-known that the minimum cost flow/tension problems with piecewise-linear convex cost can be solved efficiently by various combinatorial algorithms (see [1, 41]). It can be shown that both $f$ and $g$ are polyhedral convex functions, which is a direct extension of results in Iri [21] and Rockafellar [41] for the case of $|T| = 2$.

We consider here the cost functions $f_{\mathbf{Z}}$ and $g_{\mathbf{Z}}$ for the integer version of the minimum cost flow/tension problems:

$$\begin{aligned} f_{\mathbf{Z}}(x) &= \inf\{\sum_{a \in A} f_a(\xi(a)) \mid \xi \in \mathbf{Z}^A,\ \partial\xi(w) = -x(w)\ (w \in T),\ \partial\xi(w) = 0\ (w \in V \setminus T)\} \quad (x \in \mathbf{Z}^T), \\ g_{\mathbf{Z}}(p') &= \inf\{\sum_{a \in A} g_a(-\delta p(a)) \mid p \in \mathbf{Z}^V,\ p(w) = p'(w)\ (w \in T)\} \qquad (p' \in \mathbf{Z}^T). \end{aligned}$$

It is shown in [33, 35, 36] that $f_{\mathbf{Z}}$ satisfies (M-EXC[$\mathbf{Z}$]) and $g_{\mathbf{Z}}$ satisfies (LF1[$\mathbf{Z}$]) and (LF2[$\mathbf{Z}$]), i.e., $f_{\mathbf{Z}}$ are $g_{\mathbf{Z}}$ are M-convex and L-convex, respectively.

These results indicate that the polyhedral convex functions $f$ and $g$ defined by (1.2) and (1.4) must have nice combinatorial properties like M-convexity and L-convexity, respectively. As is shown later in Example 2.4, $f$ satisfies the property (M-EXC):

**(M-EXC)** $\forall x, y \in \mathrm{dom}\, f$, $\forall u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$, $\exists \alpha_0 > 0$ such that

$$f(x) + f(y) \geq f(x - \alpha(\chi_u - \chi_v)) + f(y + \alpha(\chi_u - \chi_v)) \qquad (0 \leq \forall \alpha \leq \alpha_0),$$

which is a generalization of (M-EXC[$\mathbf{Z}$]), and $g$ satisfies (LF1) and (LF2):

**(LF1)** $\quad g(p) + g(q) \geq g(p \wedge q) + g(p \vee q) \quad (\forall p, q \in \mathrm{dom}\, g)$,

**(LF2)** $\quad \exists r \in \mathbf{R}$ such that $g(p + \lambda \mathbf{1}) = g(p) + \lambda r \quad (\forall p \in \mathrm{dom}\, g, \forall \lambda \in \mathbf{R})$,

which can be obtained by generalizing (LF1[$\mathbf{Z}$]) and (LF2[$\mathbf{Z}$]), where $\mathrm{dom}\, f = \{x \in \mathbf{R}^V \mid -\infty < f(x) < +\infty\}$, $\mathrm{dom}\, g = \{p \in \mathbf{R}^V \mid -\infty < g(p) < +\infty\}$.

The observation above indicates the possibility of extending the concepts of M-convexity and L-convexity to polyhedral convex functions. This can be done in the following way. For a polyhedral convex function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$, we call $f$ M-convex if $\mathrm{dom}\, f \neq \emptyset$ and $f$ satisfies the property (M-EXC). Similarly, for a polyhedral convex function $g : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ we call $g$ L-convex if $\mathrm{dom}\, g \neq \emptyset$ and $g$ satisfies (LF1) and (LF2).

The aim of this paper is to investigate the structures of polyhedral M-convex and L-convex functions from the dual viewpoint of analysis and combinatorics, and to provide a nice framework for well-solvable nonlinear combinatorial optimization problems in real variable. The organization of this paper is as follows.

Section 2 provides some natural classes of polyhedral M/L-convex functions. We also prove the polyhedral M-convexity and L-convexity of the functions $f$ and $g$ defined in (1.2) and (1.4).

To investigate polyhedral M/L-convex functions, we need to consider the set version of M/L-convexity. A polyhedron $B \subseteq \mathbf{R}^V$ is called M-convex if it is not empty and satisfies (B-EXC):

**(B-EXC)** $\forall x, y \in B$, $\forall u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$, $\exists \alpha_0 > 0$ such that

$$x - \alpha(\chi_u - \chi_v) \in B,\ y + \alpha(\chi_u - \chi_v) \in B \qquad (0 \leq \forall \alpha \leq \alpha_0).$$

As is explained later in Theorem 3.3, an M-convex polyhedron is nothing but the base polyhedron of a submodular system [17]. Similarly, a polyhedron $D \subseteq \mathbf{R}^V$ is called L-convex if it is not empty and satisfies (LS1) and (LS2):

**(LS1)** $p \wedge q,\ p \vee q \in D\ (\forall p,\ q \in D)$, **(LS2)** $p \in D \Longrightarrow p + \lambda\mathbf{1} \in D\ (\forall \lambda \in \mathbf{R})$.

We investigate the structure of M/L-convex polyhedra in Section 3.

Section 4 shows fundamental properties of polyhedral M/L-convex functions. We present various equivalent axioms for polyhedral M/L-convex functions, and give some properties on local structure of polyhedral M/L-convex functions such as directional derivatives, subdifferentials, minimizers, etc. In Section 4, we also investigate positively homogeneous polyhedral M/L-convex functions, which are important subclasses of polyhedral M/L-convex functions. It is shown that positively homogeneous polyhedral M/L-convex functions have one-to-one correspondences with certain set functions, and also with L/M-convex polyhedra.

For a function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$, its conjugate function $f^\bullet : \mathbf{R}^V \to \mathbf{R} \cup \{\pm\infty\}$ is defined by

$$f^\bullet(p) = \sup_{x \in \mathbf{R}^V} \{\langle p, x\rangle - f(x)\} \qquad (p \in \mathbf{R}^V),$$

where $\langle p, x\rangle = \sum\{p(v)x(v) \mid v \in V\}$. It is shown in [33, 36] that there is a conjugacy relationship between M/L-convex functions over the integer lattice. In Section 5, we show that the conjugacy relationship also exists for polyhedral M/L-convex functions. Section 5 also provides various characterization of polyhedral M/L-convex functions by local structures such as directional derivative, the set of minimizers, and subdifferentials.

As is mentioned above, the concepts of M/L-convexity were originally introduced for functions defined over the integer lattice [30, 31, 33]. In Section 6, we clarify the relationship of M/L-convexity over the integer lattice and polyhedral M/L-convexity discussed in this paper.

Finally, some duality theorems on polyhedral M/L-convex functions are presented in Section 7. Although duality theorems in this section can be obtained by application of the existing theorems in convex analysis to polyhedral M/L-convex functions, they are worth stating in their own right in connection to integrality properties. We also explain a more general and stronger result on the transformation of polyhedral M/L-convex function called "network induction." The validity of network induction is proved by using the duality for polyhedral M/L-convex functions.

In this paper, we focus on polyhedral convex functions with M/L-convexity. The concepts of M/L-convexity can be further extended to general (not necessarily polyhedral) convex functions by considering the property (M-EXC) and properties (LF1), (LF2), respectively. The results in this direction will be reported in a forthcoming paper.

## 2 Examples of Polyhedral M/L-convex Functions

Polyhedral M-convex and L-convex functions have various examples. Let $V$ be a nonempty finite set.

**Example 2.1 (affine functions)** Let $B \subseteq \mathbf{R}^V$ be an M-convex polyhedron, $p_0 \in \mathbf{R}^V$, and $\alpha \in \mathbf{R}$. Then, the function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ such that

$$\mathrm{dom}\, f = B, \qquad f(x) = \langle p_0, x \rangle + \alpha \quad (x \in B)$$

is polyhedral M-convex with equality in (M-EXC), which is an immediate consequence of (B-EXC) for $B$.

Let $D \subseteq \mathbf{R}^V$ be an L-convex polyhedron, $x_0 \in \mathbf{R}^V$, and $\lambda \in \mathbf{R}$. Then, the function $g : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ such that

$$\mathrm{dom}\, g = D, \qquad g(p) = \langle p, x_0 \rangle + \lambda \quad (p \in D)$$

is polyhedral L-convex with equality in the submodular inequality (LF1) and $r = \langle \mathbf{1}, x_0 \rangle$ in (LF2). □

We denote by $\mathcal{C}^1$ the class of one-dimensional piecewise-linear convex functions with nonempty effective domain, i.e.,

$$\mathcal{C}^1 = \{\varphi \mid \varphi : \mathbf{R} \to \mathbf{R} \cup \{+\infty\},\ \text{piecewise-linear convex},\ \mathrm{dom}\, \varphi \neq \emptyset\}. \tag{2.1}$$

A piecewise-linear convex function (in this paper) is nothing but a one-dimensional polyhedral convex function. Note that the effective domain $\mathrm{dom}\, \varphi$ of a function $\varphi \in \mathcal{C}^1$ is a closed interval on $\mathbf{R}$.

**Example 2.2** For $\varphi \in \mathcal{C}^1$, the function $f : \mathbf{R}^2 \to \mathbf{R} \cup \{+\infty\}$ defined by

$$f(x_1, x_2) = \begin{cases} \varphi(x_1) & (x_1 + x_2 = 0), \\ +\infty & (\text{otherwise}) \end{cases}$$

is polyhedral M-convex. For $\psi \in \mathcal{C}^1$, the function $g : \mathbf{R}^2 \to \mathbf{R} \cup \{+\infty\}$ defined by

$$g(p_1, p_2) = \psi(p_1 - p_2) \quad ((p_1, p_2) \in \mathbf{R}^2)$$

is polyhedral L-convex. Furthermore, $f$ and $g$ are conjugate to each other if $\varphi$ and $\psi$ are conjugate to each other. □

**Example 2.3 (separable-convex functions)** Let $B \subseteq \mathbf{R}^V$ be an M-convex polyhedron. For a family of functions $f_w \in \mathcal{C}^1$ $(w \in V)$, the function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ defined by

$$f(x) = \begin{cases} \sum_{w \in V} f_w(x(w)) & (x \in B), \\ +\infty & (x \notin B) \end{cases}$$

is polyhedral M-convex if $\mathrm{dom}\, f \neq \emptyset$.

Let $D \subseteq \mathbf{R}^V$ be an L-convex polyhedron. For any functions $g_{uv} \in \mathcal{C}^1$ indexed by $u, v \in V$, the function $g : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ defined by

$$g(p) = \begin{cases} \sum_{u,v \in V} g_{uv}(p(v) - p(u)) & (p \in D), \\ +\infty & (p \notin D) \end{cases}$$

is polyhedral L-convex with $r = 0$ in (LF2) if $\mathrm{dom}\, g \neq \emptyset$. □

**Example 2.4 (minimum cost flow/tension problems)** In Introduction, we have defined the cost functions $f$ and $g$ of the minimum cost flow/tension problems by (1.2) and (1.4), respectively. We here state the following properties of $f$ and $g$:

(i) $f$ is polyhedral M-convex if $f(x_0)$ is finite for some $x_0 \in \mathbf{R}^T$.
(ii) $g$ is polyhedral L-convex if $g(p_0)$ is finite for some $p_0 \in \mathbf{R}^T$.
(iii) Suppose that $f_a$ and $g_a$ are conjugate to each other for all $a \in A$. Then, $f$ and $g$ are conjugate to each other if one of the following conditions holds:

(a) $f(x_0)$ is finite (i.e., $-\infty < f(x_0) < +\infty$) for some $x_0 \in \mathbf{R}^T$,
(b) $g(p_0)$ is finite (i.e., $-\infty < g(p_0) < +\infty$) for some $p_0 \in \mathbf{R}^T$,
(c) $f(x_0) < +\infty$, $g(p_0) < +\infty$ for some $x_0 \in \mathbf{R}^T$, $p_0 \in \mathbf{R}^T$.

In the following, we prove (M-EXC) for $f$ and (LF1), (LF2) for $g$. We omit the proofs of (iii) and the following properties:

if $f(x_0)$ is finite for some $x_0 \in \mathbf{R}^T$, then $\mathrm{dom}\, f \neq \emptyset$ and $f > -\infty$,
if $g(p_0)$ is finite for some $p_0 \in \mathbf{R}^T$, then $\mathrm{dom}\, g \neq \emptyset$ and $g > -\infty$,

since their proofs are just a straightforward extension of the results in [21, 41] for the case $|T| = 2$.
[(M-EXC) for $f$] Let $x, y \in \mathrm{dom}\, f$ and $u \in \mathrm{supp}^+(x - y)$. Also, let $\xi, \eta \in \mathbf{R}^A$ be flows such that

$$\partial\xi(w) = -x(w) \ (w \in T), \quad \partial\xi(w) = 0 \ (w \in V \setminus T), \quad f(x) = \sum_{a \in A} f_a(\xi(a)),$$
$$\partial\eta(w) = -y(w) \ (w \in T), \quad \partial\eta(w) = 0 \ (w \in V \setminus T), \quad f(y) = \sum_{a \in A} f_a(\eta(a)).$$

By a standard augmenting path argument we see that there exist $\pi : A \to \{0, \pm 1\}$ and $v \in \mathrm{supp}^-(x - y)$ $(\subseteq T)$ such that

$$\mathrm{supp}^+(\pi) \subseteq \mathrm{supp}^+(\xi - \eta), \ \mathrm{supp}^-(\pi) \subseteq \mathrm{supp}^-(\xi - \eta), \ \partial\pi = \chi_v - \chi_u.$$

Put $\alpha_0 = \min\{|\xi(a) - \eta(a)| \mid a \in A, \ \pi(a) \neq 0\} > 0$. Let $\alpha$ be any value with $0 \leq \alpha \leq \alpha_0$. Then, we have

$$\partial(\xi - \alpha\pi) = -x + \alpha(\chi_u - \chi_v), \quad \partial(\eta + \alpha\pi) = -y - \alpha(\chi_u - \chi_v),$$
$$f_a(\xi(a) - \alpha\pi(a)) + f_a(\eta(a) + \alpha\pi(a)) \leq f_a(\xi(a)) + f_a(\eta(a)) \quad (a \in A),$$

which implies the inequality:

$$\begin{aligned} f(x - \alpha(\chi_u - \chi_v)) + f(y + \alpha(\chi_u - \chi_v)) &\leq \sum_{a \in A} \{f_a(\xi(a) - \alpha\pi(a)) + f_a(\eta(a) + \alpha\pi(a))\} \\ &\leq \sum_{a \in A} \{f_a(\xi(a)) + f_a(\eta(a))\} \ = \ f(x) + f(y). \end{aligned}$$

Thus, $f$ satisfies (M-EXC).
[(LF1) for $g$] Let $p', q' \in \mathrm{dom}\, g$, and $p, q \in \mathbf{R}^V$ be vectors with

$$p(w) = p'(w) \ (w \in T), \ \sum_{a \in A} g_a(-\delta p(a)) = g(p'), \quad q(w) = q'(w) \ (w \in T), \ \sum_{a \in A} g_a(-\delta q(a)) = g(q').$$

For $a = (u, v) \in A$, we have

$$g_a(-\delta(p \wedge q)(a)) + g_a(-\delta(p \vee q)(a)) \leq g_a(-\delta p(a)) + g_a(-\delta q(a))$$

by the convexity of $g_a$. It is obvious that $(p \wedge q)(w) = (p' \wedge q')(w)$, $(p \vee q)(w) = (p' \vee q')(w)$ $(w \in T)$. Hence, we have

$$\begin{aligned} g(p' \wedge q') + g(p' \vee q') &\leq \sum_{a\in A} g_a(-\delta(p \wedge q)(a)) + \sum_{a\in A} g_a(-\delta(p \vee q)(a)) \\ &\leq \sum_{a\in A} g_a(-\delta p(a)) + \sum_{a\in A} g_a(-\delta q(a)) = g(p') + g(q'). \end{aligned}$$

[(LF2) for $g$] Let $\mathbf{1}_T$ (resp. $\mathbf{1}_V$) be the vector in $\mathbf{R}^T$ (resp. $\mathbf{R}^V$) with each component being equal to one. For $p' \in \mathbf{R}^T$ and $\lambda \in \mathbf{R}$, we have

$$\begin{aligned} g(p' + \lambda \mathbf{1}_T) &= \inf\{\sum_{a\in A} g_a(-\delta\tilde{p}(a)) \mid \tilde{p} \in \mathbf{R}^V,\ \tilde{p}(w) = p'(w) + \lambda\ (w \in T)\} \\ &= \inf\{\sum_{a\in A} g_a(-\delta(p + \lambda\mathbf{1}_V)(a)) \mid p \in \mathbf{R}^V,\ p(w) + \lambda = p'(w) + \lambda\ (w \in T)\} \\ &= \inf\{\sum_{a\in A} g_a(-\delta p(a)) \mid p \in \mathbf{R}^V,\ p(w) = p'(w)\ \ (w \in T)\} \ = \ g(p'). \end{aligned}$$

□

# 3 M-convex and L-convex Polyhedra

In this section, we introduce two classes of polyhedra, called M-convex polyhedra and L-convex polyhedra. In fact, M-convex and L-convex polyhedra are familiar objects in combinatorial optimization, and this section is a recapitulation of almost known facts on these polyhedra from our point of view. The results in this section will be the basis of argument on polyhedral M-convex and L-convex functions beginning in Section 4.

## 3.1 Definitions and Notation

We denote by $\mathbf{R}$ the set of reals, and by $\mathbf{Z}$ the set of integers. Also, denote by $\mathbf{R}_+$ the set of nonnegative reals. Throughout this paper, we assume that $V$ is a nonempty finite set. For any finite set $X$, its cardinality is denoted by $|X|$. The characteristic vector of a subset $X \subseteq V$ is denoted by $\chi_X$ $(\in \{0,1\}^V)$, i.e.,

$$\chi_X(w) = \begin{cases} 1 & (w \in X), \\ 0 & (w \in V \setminus X). \end{cases}$$

In particular, we use the notation $\mathbf{0} = \chi_\emptyset$, $\mathbf{1} = \chi_V$. We sometimes write $\mathbf{0}_V$ or $\mathbf{1}_V$ to indicate that it is a vector in $\mathbf{R}^V$.

Let $x = (x(w) \mid w \in V) \in \mathbf{R}^V$. We define

$$\mathrm{supp}(x) = \{v \in V \mid x(v) \neq 0\}, \quad \mathrm{supp}^+(x) = \{v \in V \mid x(v) > 0\}, \quad \mathrm{supp}^-(x) = \{v \in V \mid x(v) < 0\},$$
$$\|x\|_1 = \sum_{v \in V} |x(v)|, \qquad \|x\|_\infty = \max_{v \in V} |x(v)|,$$
$$\langle p, x\rangle = \sum_{v \in V} p(v)x(v) \ (p \in \mathbf{R}^V), \qquad x(X) = \sum_{v \in X} x(v) \ (X \subseteq V).$$

For $\varepsilon > 0$, we define

$$\mathrm{N}_\infty(p, \varepsilon) = \{q \in \mathbf{R}^V \mid \|q - p\|_\infty < \varepsilon\}.$$

For any $p, q \in \mathbf{R}^V$, $p \wedge q$ and $p \vee q$ denote the vectors in $\mathbf{R}^V$ such that $(p \wedge q)(w) = \min\{p(w), q(w)\}$, $(p \vee q)(w) = \max\{p(w), q(w)\}$ $(w \in V)$.

For $a : V \to \mathbf{R} \cup \{-\infty\}$ and $b : V \to \mathbf{R} \cup \{+\infty\}$ with $a(v) \leq b(v)$ $(v \in V)$, we define the interval $[a, b]$ $(\subseteq \mathbf{R}^V)$ by

$$[a, b] = \{x \in \mathbf{R}^V \mid a(v) \leq x(v) \leq b(v) \ (v \in V)\}.$$

For any two vectors $p, q \in \mathbf{R}^V$, we denote $\mathrm{Box}[p, q] = [p \wedge q, p \vee q]$.

Let $S \subseteq \mathbf{R}^V$. The set $S$ is called *convex* if $(1 - \alpha)x + \alpha y \in S$ holds for any $x, y \in S$ and any $\alpha \in [0, 1]$, and called *conic* if $\alpha x \in S$ holds for any $x \in S$ and any $\alpha \geq 0$. A conic set is also called a *cone*. The *convex hull* of $S$, denoted by $\mathrm{conv}(S)$, is the smallest convex set containing $S$; that is,

$$\mathrm{conv}(S) = \left\{ \sum_{i=1}^k \lambda_i x_i \,\middle|\, k \geq 1, \ x_i \in S, \ \lambda_i \geq 0, \ \sum_{i=1}^k \lambda_i = 1 \right\}.$$

Note that $S$ is convex if and only if $S = \mathrm{conv}(S)$. For any set $S \subseteq \mathbf{R}^V$, the *convex closure* of $S$, denoted by $\overline{S}$, is the smallest closed convex set containing $S$. Hence, $\overline{\mathrm{N}_\infty(x, \varepsilon)}$ denotes a closed neighborhood $\{y \in \mathbf{R}^V \mid \|y - x\|_\infty \leq \varepsilon\}$.

A set $S \subseteq \mathbf{R}^V$ is called *polyhedral* if it is represented as the intersection of a finite number of closed half-spaces, i.e., there exist some $\{p_i\}_{i=1}^k$ $(\subseteq \mathbf{R}^V)$ and $\{\alpha_i\}_{i=1}^k$ $(\subseteq \mathbf{R})$ $(k \geq 0)$ such that

$$S = \{x \in \mathbf{R}^V \mid \langle p_i, x\rangle \leq \alpha_i \ (\forall i = 1, \cdots, k)\}. \tag{3.1}$$

A polyhedral set is also called a *polyhedron*. Obviously, any polyhedron is a closed convex set.

We use the following convention involving $\pm\infty$:

$$\inf\{\alpha \mid \alpha \in \emptyset\} = +\infty, \qquad \sup\{\alpha \mid \alpha \in \emptyset\} = -\infty, \qquad +\infty \leq +\infty, \qquad -\infty \leq -\infty,$$
$$0 \times (+\infty) = (+\infty) \times 0 = 0 \times (-\infty) = (-\infty) \times 0 = 0.$$

In this paper, we never use expressions such as $+\infty - \infty$ or $-\infty + \infty$.

## 3.2 M-convex Polyhedra

A polyhedron $B \subseteq \mathbf{R}^V$ is called *M-convex* if it is not empty and satisfies (B-EXC):

**(B-EXC)** $\forall x, y \in B$, $\forall u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$, $\exists \alpha_0 > 0$ such that

$$x - \alpha(\chi_u - \chi_v) \in B, \quad y + \alpha(\chi_u - \chi_v) \in B \qquad (0 \leq \forall \alpha \leq \alpha_0).$$

We denote by $\mathcal{M}_0$ the family of M-convex polyhedra, i.e.,

$$\mathcal{M}_0 = \{B \subseteq \mathbf{R}^V \mid B: \text{ M-convex polyhedron}\}.$$

It is well-known as a folklore that what we call an "M-convex polyhedron" is nothing but the base polyhedron of a submodular system [17] (see also Theorem 3.3). We use the term "M-convex polyhedron" for denotational symmetry to "L-convex polyhedron."

### 3.2.1 Properties of M-convex Polyhedra

For any $S \subseteq \mathbf{R}^V$ and $x \in S$, the *exchange capacity* $\tilde{c}_S(x, v, u)$ $(u, v \in V)$ is defined by

$$\tilde{c}_S(x, v, u) = \sup\{\alpha \mid \alpha \in \mathbf{R}_+, \ x + \alpha(\chi_v - \chi_u) \in S\},$$

where the subscript $S$ to $\tilde{c}$ may be omitted when there is no ambiguity. The property (B-EXC) can be rewritten in terms of exchange capacity, as follows:

**(B-EXC′)** $\forall x, y \in B$, $\forall u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$ such that $\tilde{c}_B(x, v, u) > 0$, $\tilde{c}_B(y, u, v) > 0$.

We also consider a one-sided exchange property:

**(B-EXC$_-$)** $\forall x, y \in B$, $\forall u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$ such that $\tilde{c}_B(x, v, u) > 0$.

The exchange property (B-EXC$_-$), apparently weaker than (B-EXC), is in fact equivalent to (B-EXC).

**Theorem 3.1** *For a polyhedron $B \subseteq \mathbf{R}^V$,* (B-EXC) $\iff$ (B-EXC$_-$).

**Proof.** Proof is given in Section 3.2.2. □

Fundamental properties of M-convex polyhedra are shown below. An M-convex polyhedron is contained in a hyperplane $\{x \in \mathbf{R}^V \mid x(V) = r\}$ for some $r \in \mathbf{R}$.

**Theorem 3.2** *Let $B \subseteq \mathbf{R}^V$ be a polyhedron with* (B-EXC$_-$)*. Then,* $x(V) = y(V)$ $(\forall x, y \in B)$.

**Proof.** Assume, to the contrary, that $x_0(V) > y_0(V)$ holds for some $x_0, y_0 \in B$. Let $x_* \in B$ be a vector with

$$\|x_* - y_0\|_1 = \inf\{\|x - y_0\|_1 \mid x \in B, x(V) = x_0(V)\}.$$

Note that $\mathrm{supp}^+(x_* - y_0) \neq \emptyset$. Let $u \in \mathrm{supp}^+(x_* - y_0)$. By (B-EXC$_-$), there exist some $v \in \mathrm{supp}^-(x_* - y_0)$ and a sufficiently small $\alpha > 0$ such that $x' = x_* - \alpha(\chi_u - \chi_v) \in B$. It, however, is a contradiction to the choice of $x_*$ since $x'(V) = x_0(V)$ and $\|x' - y_0\|_1 = \|x_* - y_0\|_1 - 2\alpha$. □

We shall show that an M-convex polyhedron is described by a submodular set function. Let $\rho : 2^V \to \mathbf{R} \cup \{+\infty\}$ and denote $\mathrm{dom}\,\rho = \{X \subseteq V \mid \rho(X) < +\infty\}$. A function $\rho$ is said to be *submodular* if it satisfies

$$\rho(X) + \rho(Y) \geq \rho(X \cap Y) + \rho(X \cup Y) \quad (\forall X, Y \subseteq V). \tag{3.2}$$

We denote the class of (normalized) submodular set functions by

$$\mathcal{S} \;=\; \{\rho : 2^V \to \mathbf{R} \cup \{+\infty\} \mid \rho : \text{submodular},\ \rho(\emptyset) = 0,\ \rho(V) < +\infty\}.$$

A set function $\mu : 2^V \to \mathbf{R} \cup \{-\infty\}$ is called *supermodular* if $-\mu$ is submodular. For any nonempty $B \subseteq \mathbf{R}^V$, we define $\rho_B : 2^V \to \mathbf{R} \cup \{+\infty\}$ by

$$\rho_B(X) = \sup_{x \in B} x(X) \quad (X \subseteq V). \tag{3.3}$$

For a set function $\rho : 2^V \to \mathbf{R} \cup \{+\infty\}$, we define $\mathrm{B}(\rho) \subseteq \mathbf{R}^V$ by

$$\mathrm{B}(\rho) = \{x \in \mathbf{R}^V \mid x(X) \leq \rho(X)\ (X \subseteq V),\ x(V) = \rho(V)\}. \tag{3.4}$$

Note that the exchange capacity associated with $x \in \mathrm{B}(\rho)$ can be written as follows [17]:

$$\tilde{c}_{\mathrm{B}(\rho)}(x, v, u) = \min\{\rho(X) - x(X) \mid X \subseteq V,\ u \notin X,\ v \in X\} \quad (u, v \in V), \tag{3.5}$$

which is independent of submodularity of $\rho$. The following fact has been known to experts (cf. [4], [9], [44, Chapter 18]), but the precise statement cannot be found in the literature.

**Theorem 3.3**
(i) *For $B \in \mathcal{M}_0$, we have $\rho_B \in \mathcal{S}$ and* $\mathrm{B}(\rho_B) = B$. (ii) *For $\rho \in \mathcal{S}$, we have* $\mathrm{B}(\rho) \in \mathcal{M}_0$ *and* $\rho_{\mathrm{B}(\rho)} = \rho$. (iii) *The mappings $B \mapsto \rho_B$ $(B \in \mathcal{M}_0)$ and $\rho \mapsto \mathrm{B}(\rho)$ $(\rho \in \mathcal{S})$ provide one-to-one correspondences between $\mathcal{M}_0$ and $\mathcal{S}$, and are the inverse of each other.*

**Proof.** Proof is given later in Section 3.2.2. □

For any sets $S_1, S_2 \subseteq \mathbf{R}^V$, we denote by $S_1 + S_2$ $(\subseteq \mathbf{R}^V)$ the *Minkowski-sum* of $S_1$ and $S_2$, i.e.,

$$S_1 + S_2 = \{p_1 + p_2 \mid p_i \in S_i\ (i = 1, 2)\}.$$

**Theorem 3.4 ([17])** *For $B_1, B_2 \in \mathcal{M}_0$, the set $B_1 + B_2$ is M-convex with $B_1 + B_2 = \mathrm{B}(\rho_{B_1} + \rho_{B_2})$.*

**Remark 3.5** The intersection of two M-convex polyhedra is not necessarily an M-convex polyhedron, as shown in the example below.

Let $V = \{a, b, c, d\}$, and define $B_1, B_2 \subseteq \mathbf{R}^V$ as

$$B_1 = \mathrm{conv}\{(1,1,0,0), (1,0,1,0), (0,0,1,1), (0,1,0,1)\},$$
$$B_2 = \mathrm{conv}\{(1,1,0,0), (1,0,1,0), (0,0,1,1), (0,1,1,0), (1,0,0,1)\}.$$

Then, $B_1$ and $B_2$ are M-convex polyhedra, and the submodular functions $\rho_i = \rho_{B_i}$ $(i = 1, 2)$ defined by (3.3) are

$$\rho_i(\emptyset) = 0,\ \rho_i(\{v\}) = 1 \text{ for } v \in V,\ \rho_i(X) = 2 \text{ for } X \subseteq V \text{ with } |X| \geq 3\ (i = 1, 2),$$
$$\rho_1(\{a, d\}) = \rho_1(\{b, c\}) = 1,\ \rho_1(\{a, b\}) = \rho_1(\{c, d\}) = \rho_1(\{a, c\}) = \rho_1(\{b, d\}) = 2,$$
$$\rho_2(\{b, d\}) = 1,\ \rho_2(\{a, b\}) = \rho_2(\{a, c\}) = \rho_2(\{a, d\}) = \rho_2(\{b, c\}) = \rho_2(\{c, d\}) = 2.$$

We see that

$$B_1 \cap B_2 = \mathrm{conv}\{(1,1,0,0), (1,0,1,0), (0,0,1,1)\},$$

which is not an M-convex polyhedron. Note that the associated set function $\rho = \rho_{B_1 \cap B_2}$ in (3.3) is

$$\rho(\{a, d\}) = \rho(\{b, c\}) = \rho(\{b, d\}) = 1,\ \rho(\{a, b\}) = \rho(\{c, d\}) = \rho(\{a, c\}) = 2,$$
$$\rho(X) = \rho_1(X) = \rho_2(X) \text{ if } |X| \neq 2,$$

which is not submodular. □

Separation theorems of the following form hold for submodular functions and M-convex polyhedra. The latter half of Theorem 3.7 is already shown in [35, Theorem 3.6].

**Theorem 3.6 (Frank [12])** *Let $\rho : 2^V \to \mathbf{R} \cup \{+\infty\}$, $\mu : 2^V \to \mathbf{R} \cup \{-\infty\}$ be a pair of functions with $\rho, -\mu \in \mathcal{S}$. If $\mu(X) \leq \rho(X)$ $(\forall X \subseteq V)$, then there exists $x_* \in \mathbf{R}^V$ such that*

$$\mu(X) \leq x_*(X) \leq \rho(X) \qquad (\forall X \subseteq V).$$

*Moreover, if $\rho$ and $\mu$ are integer-valued, then there exists such $x_* \in \mathbf{Z}^V$.*

**Theorem 3.7** *Let $B_1, B_2 \in \mathcal{M}_0$. If $B_1 \cap B_2 = \emptyset$, then there exists $p_* \in \{0, \pm 1\}^V$ such that*

$$\inf_{x \in B_1} \langle p_*, x \rangle - \sup_{x \in B_2} \langle p_*, x \rangle > 0. \tag{3.6}$$

*Moreover, if both of $B_1$ and $B_2$ are integral polyhedra, i.e., $B_i = \overline{B_i \cap \mathbf{Z}^V}$ $(i = 1, 2)$, then "$> 0$" in (3.6) can be replaced by "$\geq 1$."*

**Proof.** Put $\rho_i = \rho_{B_i}$ $(i = 1, 2)$. If $\rho_1(V) \neq \rho_2(V)$, then we can choose $p_* = \chi_V$ or $-\chi_V$. Hence, we may assume that $\rho_1(V) = \rho_2(V)$. Define the functions $\rho$ and $\mu$ by

$$\rho(X) = \rho_1(X),\ \mu(X) = \rho_2(V) - \rho_2(V \setminus X) \qquad (X \subseteq V).$$

Then, $\rho, -\mu \in \mathcal{S}$ and there is no $x \in \mathbf{R}^V$ with $\mu(X) \leq x(X) \leq \rho(X)$ $(\forall X \subseteq V)$. By Theorem 3.6, there exists some $X_0 \subseteq V$ such that $\mu(X_0) > \rho(X_0)$. Since

$$\sup_{x \in B_1} \langle \chi_{X_0}, x \rangle = \rho(X_0) < \mu(X_0) = \inf_{x \in B_2} \langle \chi_{X_0}, x \rangle,$$

we can choose $p_* = -\chi_{X_0}$. The second claim follows from the integrality of $B_1, B_2$. □

**Remark 3.8** Theorem 3.2 shows that an M-convex polyhedron is contained in a hyperplane of the form $\{x \in \mathbf{R}^V \mid x(V) = r\}$ for some $r \in \mathbf{R}$. Therefore, any M-convex polyhedron loses no information by the projection onto a $(|V|-1)$-dimensional space. We call a polyhedron $Q \subseteq \mathbf{R}^V$ *M*$^\natural$*-convex* if the set $\tilde{Q}$ $(\subseteq \mathbf{R}^{\tilde{V}})$ defined by

$$\tilde{Q} = \{(x_0, x) \in \mathbf{R}^{\tilde{V}} \mid x \in Q,\ x_0 = -x(V)\}$$

is an M-convex polyhedron, where $\tilde{V} = \{v_0\} \cup V$. In fact, what we name M$^\natural$-convex polyhedron here is nothing but a *generalized polymatroid* in [13, 14]. M$^\natural$-convex polyhedron is an essentially equivalent concept to M-convex polyhedron, while the class of M$^\natural$-convex polyhedra properly contains that of M-convex polyhedra. Every property of M-convex polyhedra can be restated in terms of M$^\natural$-convex polyhedra, and vice versa. The following property seems more natural when described in terms of M$^\natural$-convex polyhedron.

**Theorem 3.9** *Let $Q \subseteq \mathbf{R}^V$ be an M$^\natural$-convex polyhedron. Then, $[x, y] \subseteq Q$ holds for any $x, y \in Q$ with $x \leq y$.*

**Proof.** An M$^\natural$-convex polyhedron can be represented by a pair of submodular and supermodular functions $\rho : 2^V \to \mathbf{R} \cup \{+\infty\}$, $\mu : 2^V \to \mathbf{R} \cup \{-\infty\}$ as

$$Q = \{x' \in \mathbf{R}^V \mid \mu(X) \leq x'(X) \leq \rho(X)\ (\forall X \subseteq V)\}$$

(see Theorem 3.3 or [15, 17]). Then, for any $x' \in [x, y]$, we have $\mu(X) \leq x(X) \leq x'(X) \leq y(X) \leq \rho(X)$ $(\forall X \subseteq V)$, from which the claim follows. □

### 3.2.2 Proofs of Theorems 3.1 and 3.3

This section provides the proofs of Theorems 3.1 and 3.3.

We use the following results on submodular set functions. A family $\mathcal{D} \subseteq 2^V$ is called a *distributive lattice* if $X \cap Y, X \cup Y \in \mathcal{D}$ holds for any $X, Y \in \mathcal{D}$. Note that $\mathrm{dom}\,\rho$ of a function $\rho \in \mathcal{S}$ is a distributive lattice.

**Theorem 3.10** *Let $\rho \in \mathcal{S}$. For any $x \in \mathrm{B}(\rho)$, the family $\mathcal{D}(x) = \{X \subseteq V \mid x(X) = \rho(X)\}$ is a distributive lattice with $\{\emptyset, V\} \subseteq \mathcal{D}(x)$.*

We first prove a slightly stronger claim than Theorem 3.3 (i). A sequence $\{X_i\}_{i=0}^k$ $(k \geq 0)$ of distinct elements in $2^V$ with $X_0 \subseteq X_1 \subseteq \cdots \subseteq X_k$ is called a *chain*.

**Lemma 3.11** *Let $B \subseteq \mathbf{R}^V$ be a nonempty polyhedron with* (B-EXC$_-$)*, and $\{X_i\}_{i=0}^k$ $(k \geq 0)$ be a chain such that $\{X_i\}_{i=0}^k \subseteq \mathrm{dom}\,\rho_B$. Then, there exists $x_* \in B$ with $x_*(X_i) = \rho_B(X_i)$ $(\forall i = 0, 1, \cdots, k)$.*

**Proof.** The claim is shown by induction on the integer $k$. Let $x_k \in B$ be a vector with $x_k(X_k) = \rho_B(X_k)$. By the inductive hypothesis, there exists a vector $x \in B$ satisfying $x(X_i) = \rho_B(X_i)$ $(i = 0, 1, \cdots, k-1)$. We may assume that $x$ minimizes the value $\|x - x_k\|_1$ of all such vectors. Suppose that $x(X_k) < \rho_B(X_k)$. By (B-EXC$_-$), there exist $u \in \mathrm{supp}^+(x - x_k) \setminus X_k$ and $v \in \mathrm{supp}^-(x - x_k)$ such that $x' = x - \alpha(\chi_u - \chi_v) \in B$ with a sufficiently small $\alpha > 0$. Here $v \in V \setminus X_{k-1}$ holds since $u \in V \setminus X_{k-1}$ and $x'(X_{k-1}) \leq \rho_B(X_{k-1}) = x(X_{k-1})$. The vector $x'$ satisfies $x'(X_i) = \rho_B(X_i)$ $(\forall i = 0, 1, \cdots, k-1)$ and $\|x' - x_k\|_1 = \|x - x_k\|_1 - 2\alpha$, a contradiction. Hence $x(X_k) = \rho_B(X_k)$. □

For any polyhedron $S \subseteq \mathbf{R}^V$, a vector $x \in S$ is called an *extreme point* of $S$ if there are no $y_1, y_2 \in S \setminus \{x\}$ and $\alpha \in \mathbf{R}$ with $0 < \alpha < 1$ such that $x = \alpha y_1 + (1 - \alpha) y_2$.

**Theorem 3.12 ([17, Theorem 3.22])** *Let $\rho \in \mathcal{S}$. Then, $x \in \mathbf{R}^V$ is an extreme point of $\mathrm{B}(\rho)$ if and only if there exists a chain $\{X_i\}_{i=0}^{|V|}$ such that $\{X_i\}_{i=0}^{|V|} \subseteq \mathrm{dom}\,\rho$, $X_0 = \emptyset$, $X_{|V|} = V$, and $x(X_i) = \rho(X_i)$ $(\forall i = 0, 1, \cdots, |V|)$.*

**Lemma 3.13** *For any nonempty polyhedron $B \subseteq \mathbf{R}^V$ with (B-EXC$_-$), we have $\rho_B \in \mathcal{S}$ and $B = \mathrm{B}(\rho_B)$.*

**Proof.** The outline of the proof is similar to that for [4, Lemma 5.2].

We first consider the case when $B$ is bounded. Let $X, Y \in \mathrm{dom}\,\rho_B$. From Lemma 3.11, there exists $x_* \in B$ with $x_*(X \cap Y) = \rho_B(X \cap Y)$ and $x_*(X \cup Y) = \rho_B(X \cup Y)$, which implies the inequality

$$\rho_B(X) + \rho_B(Y) \geq x_*(X) + x_*(Y) = x_*(X \cap Y) + x_*(X \cup Y) = \rho_B(X \cap Y) + \rho_B(X \cup Y).$$

Therefore, $\rho_B \in \mathcal{S}$. Lemma 3.11 and Theorem 3.12 yield that any extreme point of $\mathrm{B}(\rho_B)$ is contained in $B$, which implies that $\mathrm{B}(\rho_B) \subseteq B$. The reverse inclusion is obvious.

Next, assume that $B$ is unbounded. Let $x_0$ be any vector in $B$. For each nonnegative integer $k$, put $B_k = \{x \in B \mid \|x - x_0\|_\infty \leq k\}$ and $\rho_k = \rho_{B_k}$. Since each $B_k$ is a bounded M-convex polyhedron, it holds that $\rho_k \in \mathcal{S}$ and $B_k = \mathrm{B}(\rho_k)$. Hence, we have $\mathrm{B}(\rho_B) = \bigcup_{k=1}^\infty \mathrm{B}(\rho_k) = \bigcup_{k=1}^\infty B_k = B$, and

$$\begin{aligned}\rho_B(X) + \rho_B(Y) &= \lim_{k\to\infty} \{\rho_k(X) + \rho_k(Y)\} \\ &\geq \lim_{k\to\infty} \{\rho_k(X \cap Y) + \rho_k(X \cup Y)\} \;=\; \rho_B(X \cap Y) + \rho_B(X \cup Y)\end{aligned}$$

for any $X, Y \in \mathrm{dom}\,\rho_B$, i.e., $\rho_B \in \mathcal{S}$. □

Next, we prove Theorem 3.3 (ii).

**Theorem 3.14 ([17])** *For any $\rho \in \mathcal{S}$, we have $\mathrm{B}(\rho) \neq \emptyset$ and $\rho_{B(\rho)} = \rho$.*

**Lemma 3.15** *For any $\rho \in \mathcal{S}$, the set $\mathrm{B}(\rho)$ satisfies (B-EXC$_-$).*

**Proof.** Let $x, y \in \mathrm{B}(\rho)$ with $x \neq y$ and $u \in \mathrm{supp}^+(x - y)$. To the contrary assume $\tilde{c}(x, v, u) = 0$ for all $v \in \mathrm{supp}^-(x - y)$. By (3.5), for $v \in \mathrm{supp}^-(x - y)$ there exists $S_v \subseteq V$ such that $u \notin S_v$, $v \in S_v$ and $x(S_v) = \rho(S_v)$. The set $S = \bigcup\{S_v \mid v \in \mathrm{supp}^-(x - y)\}$ satisfies $u \notin S$ and $\mathrm{supp}^-(x - y) \subseteq S$, which implies $x(S) < y(S)$. However, we have $x(S) = \rho(S) \geq y(S)$ by Theorem 3.10, a contradiction. □

**Lemma 3.16** *For any $\rho \in \mathcal{S}$, the set $\mathrm{B}(\rho)$ satisfies (B-EXC).*

**Proof.** Define

$$\begin{aligned}\mathcal{P} = \{(x, y) \mid x, y \in \mathrm{B}(\rho),\ \exists u_* \in \mathrm{supp}^+(x - y),\ \forall v \in \mathrm{supp}^-(x - y) :& \\ \text{either } \tilde{c}(x, v, u_*) = 0 \text{ or } \tilde{c}(y, u_*, v) = 0\}.&\end{aligned}$$

Assume to the contrary that $\mathcal{P} \neq \emptyset$. Suppose $(x_*, y_0) \in \mathcal{P}$ minimizes the cardinality of $\mathrm{supp}(x_* - y_0)$ among all pairs in $\mathcal{P}$, and let $u_* \in \mathrm{supp}^+(x_* - y_0)$ satisfy the condition for $(x_*, y_0)$ to be in $\mathcal{P}$.

Put

$$S^+ = \mathrm{supp}^+(x_* - y_0), \quad S^- = \mathrm{supp}^-(x_* - y_0), \quad G = \{v \in S^- \mid \tilde{c}(x_*, v, u_*) > 0\}.$$

From Lemma 3.15 follows $G \neq \emptyset$. We define

$$\mathcal{Y} = \{y \in \mathrm{B}(\rho) \mid \mathrm{supp}^+(x_* - y) = S^+,\ \mathrm{supp}^-(x_* - y) = S^-,\ \tilde{c}(y, u_*, v) = 0\ (\forall v \in G)\}.$$

Clearly, $y_0 \in \mathcal{Y}$, and $(x_*, y) \in \mathcal{P}$ for any $y \in \mathcal{Y}$. Let $v_* \in G$, and suppose that $y_* \in \mathcal{Y}$ maximizes the cardinality of the set $N(y_*) = \{u \in S^+ \mid \tilde{c}(y_*, u, v_*) = 0\}$ among all vectors in $\mathcal{Y}$. From Lemma 3.15, $\tilde{c}(y_*, u', v_*) > 0$ holds for some $u' \in \mathrm{supp}^-(y_* - x_*) \setminus \{u_*\}$. Put

$$\alpha = \min\{\tilde{c}(y_*, u', v_*), x_*(u') - y_*(u'), y_*(v_*) - x_*(v_*)\}, \qquad y' = y_* + \alpha(\chi_{u'} - \chi_{v_*}) \ (\in \mathrm{B}(\rho)).$$

Then, we claim that

$$\tilde{c}(y', u_*, v) = 0 \ (\forall v \in G), \qquad \tilde{c}(y', u, v_*) = 0 \ (\forall u \in N(y_*)). \tag{3.7}$$

First we show the former in (3.7). Since $y_* \in \mathcal{Y}$, we have $\tilde{c}(y_*, u_*, v) = 0$ $(\forall v \in G)$. Hence, for $v \in G$ there exists $X_v \subseteq V$ such that $u_* \in X_v,\ v \notin X_v$, and $y_*(X_v) = \rho(X_v)$. Putting $X = \bigcap_{v \in G} X_v$, we have

$$u_* \in X,\ v_* \in G \subseteq V \setminus X,\ y'(X) \geq y_*(X) = \rho(X).$$

Thus, it holds that $\tilde{c}(y', u_*, v) \leq \rho(X) - y'(X) \leq 0$ $(\forall v \in G)$. The proof for the latter in (3.7) is quite similar and omitted.

Suppose that $\alpha = \tilde{c}(y_*, u', v_*) < \min\{x_*(u') - y_*(u'),\ y_*(v_*) - x_*(v_*)\}$. Then, we have $\tilde{c}(y', u', v_*) = 0$, $\mathrm{supp}^+(x_* - y') = S^+$, and $\mathrm{supp}^-(x_* - y') = S^-$. From (3.7) follows $y' \in \mathcal{Y}$ and $N(y') \supseteq N(y_*) \cup \{u'\}$, which contradicts the choice of $y_* \in \mathcal{Y}$. If $\alpha = \min\{x_*(u') - y_*(u'), y_*(v_*) - x_*(v_*)\} \leq \tilde{c}(y_*, u', v_*)$ then $\mathrm{supp}(x_* - y')$ is a proper subset of $\mathrm{supp}(x_* - y_*) = \mathrm{supp}(x_* - y_0)$, a contradiction since $(x_*, y') \in \mathcal{P}$ by (3.7). □

Theorem 3.3 (iii) is an immediate corollary of Lemmas 3.13, 3.16, and Theorem 3.14.

Finally, we give the proof of Theorem 3.1.

**Proof of Theorem 3.1** The implication (B-EXC) ⇒ (B-EXC$_-$) is obvious. Let $B \subseteq \mathbf{R}^V$ be a nonempty polyhedron with (B-EXC$_-$). Then, we have $\rho_B \in \mathcal{S}$ and $B = \mathrm{B}(\rho_B)$, as shown in Lemma 3.13. Hence, Lemma 3.16 implies (B-EXC) for $B$. □

## 3.3 L-convex Polyhedra

A polyhedron $D \subseteq \mathbf{R}^V$ is called *L-convex* if it is not empty and satisfies (LS1) and (LS2):

**(LS1)** $p,\ q \in D \Longrightarrow p \wedge q,\ p \vee q \in D$, **(LS2)** $p \in D \Longrightarrow p + \lambda \mathbf{1} \in D \ (\forall \lambda \in \mathbf{R})$.

We denote by $\mathcal{L}_0$ the family of L-convex polyhedra, i.e.,

$$\mathcal{L}_0 = \{D \subseteq \mathbf{R}^V \mid D: \text{ L-convex polyhedron}\}.$$

The properties (LS1) and (LS2) can be characterized by local properties:

**(LS1$_{\mathbf{loc}}$-a)** $\forall p \in D$, $\exists \varepsilon > 0$ such that if $p_1, p_2 \in D \cap \overline{\mathrm{N}_\infty(p, \varepsilon)}$ then $p_1 \wedge p_2,\ p_1 \vee p_2 \in D$,

**(LS1$_{\mathbf{loc}}$-b)** $\forall p, q \in D$, $0 < \exists \varepsilon \leq 1$ such that $(1-\varepsilon)p + \varepsilon(p \wedge q) \in D$, $(1-\varepsilon)p + \varepsilon(p \vee q) \in D$,

**(LS2$_{\mathbf{loc}}$)** $\forall p \in D$, $\exists \varepsilon > 0$ such that

$$q \in \overline{\mathrm{N}_\infty(p, \varepsilon)} \cap D,\ q + \lambda \mathbf{1} \in \overline{\mathrm{N}_\infty(p, \varepsilon)} \ (\lambda \in \mathbf{R}) \Longrightarrow q + \lambda \mathbf{1} \in D.$$

**Theorem 3.17** *Let $D \subseteq \mathbf{R}^V$ be a closed set.*
(i) *If $D$ is convex, then* (LS1) $\iff$ (LS1$_{\rm loc}$-a) $\iff$ (LS1$_{\rm loc}$-b). (ii) (LS2) $\iff$ (LS2$_{\rm loc}$).

**Proof.** (i): Since (LS1) $\Rightarrow$ (LS1$_{\rm loc}$-a) is obvious, we show (LS1$_{\rm loc}$-a) $\Rightarrow$ (LS1$_{\rm loc}$-b) $\Rightarrow$ (LS1).

Assume (LS1$_{\rm loc}$-a). Let $p, q \in D$, and $\varepsilon$ be the value in (LS1$_{\rm loc}$-a) associated with $p$. The convexity of $D$ implies $(1-\varepsilon')p + \varepsilon' q \in D \cap \overline{\mathrm{N}_\infty(p, \varepsilon)}$ for a sufficiently small $\varepsilon' > 0$. Hence, we have

$$p \wedge \{(1-\varepsilon')p + \varepsilon' q\} = (1-\varepsilon')p + \varepsilon'(p \wedge q) \in D, \quad p \vee \{(1-\varepsilon')p + \varepsilon' q\} = (1-\varepsilon')p + \varepsilon'(p \vee q) \in D,$$

i.e., (LS1$_{\rm loc}$-b) holds.

Next assume (LS1$_{\rm loc}$-b) for $D$ to prove (LS1). Let $p, q \in D$. We show $p \wedge q \in D$ only, since $p \vee q \in D$ can be shown similarly. Set

$$\alpha_* = \sup\{\alpha \mid 0 \le \alpha \le 1,\ (1-\alpha)p + \alpha(p \wedge q) \in D\}, \quad p_* = (1-\alpha_*)p + \alpha_*(p \wedge q)\ (\in D).$$

Assume $\alpha_* < 1$. By (LS1$_{\rm loc}$-b), there exists some $\varepsilon$ with $0 < \varepsilon \le 1$ such that

$$(1-\varepsilon)p_* + \varepsilon(p_* \wedge q) = (1 - \varepsilon - \alpha_* + \varepsilon\alpha_*)p + (\varepsilon + \alpha_* - \varepsilon\alpha_*)(p \wedge q) \in D,$$

a contradiction to the definition of $\alpha_*$. Hence $\alpha_* = 1$ and $p \wedge q \in D$ hold.

(ii): We show the direction (LS2$_{\rm loc}$) $\Longrightarrow$ (LS2) only. Let $q \in D$ and $\alpha_* = \sup\{\alpha \mid \alpha \in \mathbf{R}_+,\ q + \alpha\mathbf{1} \in D\}$. To the contrary assume that $\alpha_* < +\infty$. Then, we have $q_* = q + \alpha_*\mathbf{1} \in D$. However, (LS2$_{\rm loc}$) implies $q_* + \varepsilon\mathbf{1} \in D$ for some $\varepsilon > 0$, a contradiction to the definition of $\alpha_*$. Thus we have $q + \alpha\mathbf{1} \in D$ for all $\alpha \ge 0$. Similarly, we have $q + \alpha\mathbf{1} \in D$ for all $\alpha \le 0$. □

We show the system of inequalities which describes the polyhedral structure of L-convex polyhedra.

A function $\gamma : V \times V \to \mathbf{R} \cup \{+\infty\}$ with $\gamma(v, v) = 0$ $(\forall v \in V)$ is called a *distance function*. For a distance function $\gamma$ we define the set $\mathrm{D}(\gamma) \subseteq \mathbf{R}^V$ by

$$\mathrm{D}(\gamma) = \{p \in \mathbf{R}^V \mid p(v) - p(u) \le \gamma(u, v)\ (u, v \in V)\}. \tag{3.8}$$

Given a nonempty set $D \subseteq \mathbf{R}^V$, the function $\gamma_D : V \times V \to \mathbf{R} \cup \{+\infty\}$ is defined by

$$\gamma_D(u, v) = \sup_{p \in D}\{p(v) - p(u)\}. \tag{3.9}$$

Note that $\gamma_D$ is a distance function, and $D \subseteq \mathrm{D}(\gamma_D)$ holds in general.

Given a distance function $\gamma$, we denote by $G_\gamma$ the directed graph with the vertex set $V$ and the arc set $A_\gamma = \{(u, v) \mid u, v \in V,\ u \ne v,\ \gamma(u, v) < +\infty\}$, where the value $\gamma(u, v)$ is regarded as the length of the arc $(u, v) \in A_\gamma$. For $u, v \in V$, let $\overline{\gamma}(u, v)$ be the length of a shortest path from $u$ to $v$ in $G_\gamma$, i.e.,

$$\overline{\gamma}(u, v) = \inf\{\sum_{(u', v') \in P} \gamma(u', v') \mid P \subseteq A_\gamma,\ P: \text{ path from } u \text{ to } v\} \quad (u, v \in V).$$

It is well-known that $G_\gamma$ has no negative cycle if and only if $\overline{\gamma} > -\infty$.

We consider the *triangle inequality*

$$\gamma(v_1, v_2) + \gamma(v_2, v_3) \ge \gamma(v_1, v_3) \quad (\forall v_1, v_2, v_3 \in V) \tag{3.10}$$

for distance functions, and let $\mathcal{T}$ be the family of distance functions with triangle inequality, i.e.,

$$\mathcal{T} = \{\gamma \mid \gamma : V \times V \to \mathbf{R} \cup \{+\infty\},\ \gamma(v, v) = 0\ (v \in V),\ \gamma \text{ satisfies } (3.10)\}.$$

Note that for any distance function $\gamma : V \times V \to \mathbf{R} \cup \{+\infty\}$, if $\overline{\gamma} > -\infty$ then $\overline{\gamma} \in \mathcal{T}$, and if $\gamma \in \mathcal{T}$ then $\overline{\gamma} = \gamma$.

**Lemma 3.18** *Let $\gamma : V \times V \to \mathbf{R} \cup \{+\infty\}$ be a distance function.*
(i) $\mathrm{D}(\gamma) \neq \emptyset \iff G_\gamma$ *has no negative cycle.* (ii) $\mathrm{D}(\gamma) \neq \emptyset \Rightarrow \mathrm{D}(\gamma) = \mathrm{D}(\overline{\gamma})$.
(iii) $\mathrm{D}(\gamma) \neq \emptyset \Rightarrow \mathrm{D}(\gamma) \in \mathcal{L}_0$. (iv) *If $\gamma \in \mathcal{T}$, then $\mathrm{D}(\gamma) \neq \emptyset$ and $\gamma = \gamma_{\mathrm{D}(\gamma)}$.*
(v) *If $\gamma(u,v) \in \mathbf{Z} \cup \{+\infty\}$ $(\forall u, v \in V)$, then $\mathrm{D}(\gamma)$ is an integral polyhedron, i.e.,* $\mathrm{D}(\gamma) = \overline{\mathrm{D}(\gamma) \cap \mathbf{Z}^V}$.

**Proof.** (i), (ii): These are well-known facts. See, e.g., [21, 24].

(iii): (LS2) for $\mathrm{D}(\gamma)$ is obvious. To show (LS1) for $\mathrm{D}(\gamma)$, we take $p, q \in \mathrm{D}(\gamma)$. Then, it is easy show that $(p \wedge q)(v) - (p \wedge q)(u) \leq \gamma(u,v)$, $(p \vee q)(v) - (p \vee q)(u) \leq \gamma(u,v)$ $(\forall u, v \in V)$, i.e., $p \wedge q, p \vee q \in D$.

(iv): Since $\gamma \in \mathcal{T}$, the graph $G_\gamma$ has no negative cycle and therefore $\mathrm{D}(\gamma) \neq \emptyset$ by (i). We have

$$\gamma(u,v) = \overline{\gamma}(u,v) = \sup_{p \in \mathrm{D}(\gamma)} \{p(v) - p(u)\} = \gamma_{\mathrm{D}(\gamma)}(u,v),$$

where the second equality is by a fundamental result concerning the shortest path problem.

(v): The claim is obvious from the total unimodularity of the coefficient matrix of the system of inequalities in (3.8). □

**Lemma 3.19** *For any nonempty polyhedron $D \subseteq \mathbf{R}^V$, we have $\gamma_D \in \mathcal{T}$. Moreover, if $D \in \mathcal{L}_0$, then* $\mathrm{D}(\gamma_D) = D$.

**Proof.** The first claim is clear from the following inequality:

$$\begin{aligned} \gamma_D(v_1,v_2) + \gamma_D(v_2,v_3) &= \sup_{p \in D}\{p(v_2) - p(v_1)\} + \sup_{p \in D}\{p(v_3) - p(v_2)\} \\ &\geq \sup_{p \in D}\{p(v_3) - p(v_1)\} = \gamma_D(v_1,v_3) \qquad (\forall v_1, v_2, v_3 \in V). \end{aligned}$$

Suppose $D \in \mathcal{L}_0$. We show that $\mathrm{D}(\gamma_D) \subseteq D$. Let $q \in \mathrm{D}(\gamma_D)$. For any $u, v \in V$ there exists $p_{uv} \in D$ with $p_{uv}(v) - p_{uv}(u) \geq q(v) - q(u)$, where we may assume that $p_{uv}(u) = q(u)$ and $p_{uv}(v) \geq q(v)$ by (LS2). For each $u \in V$, the vector $p_u = \bigvee_{v \in V} p_{uv}$ $(\in D)$ satisfies $p_u(u) = q(u)$, $p_u(v) \geq q(v)$ $(\forall v \in V)$. Therefore, $q = \bigwedge_{u \in V} p_u \in D$. □

**Theorem 3.20**
(i) *For $D \in \mathcal{L}_0$, we have $\gamma_D \in \mathcal{T}$ and $\mathrm{D}(\gamma_D) = D$.* (ii) *For $\gamma \in \mathcal{T}$, we have $\mathrm{D}(\gamma) \in \mathcal{L}_0$ and $\gamma_{\mathrm{D}(\gamma)} = \gamma$.*
(iii) *The mappings $D \mapsto \gamma_D$ $(D \in \mathcal{L}_0)$ and $\gamma \mapsto \mathrm{D}(\gamma)$ $(\gamma \in \mathcal{T})$ provide a one-to-one correspondence between $\mathcal{L}_0$ and $\mathcal{T}$, and are the inverse of each other.*

**Proof.** Immediate from Lemma 3.18 (iii), (iv), and Lemma 3.19. □

As a special case of L-convex polyhedra, we investigate the class of *L-convex cones*, which are conic L-convex polyhedra. This amounts to restricting the function values of $\gamma$ in the above discussion to $\{0, +\infty\}$. We show below the relationship of L-convex cones with "transitive" directed graphs, and with distributive lattices.

A directed graph $G = (V, A)$ is called *transitive* if for any $(u,v), (v,w) \in A$ we have $(u,w) \in A$. Given a set of vectors $D \subseteq \mathbf{R}^V$, we define the directed graph $G_D = (V, A_D)$ by

$$A_D = \{(u,v) \mid u, v \in V,\ p(u) \geq p(v)\ (\forall p \in D)\}.$$

For a directed graph $G = (V, A)$, the set $D_G$ $(\subseteq \mathbf{R}^V)$ is defined by

$$D_G = \{p \in \mathbf{R}^V \mid p(u) \geq p(v)\ (\forall (u,v) \in A)\}.$$

The next theorem is an immediate consequence of Theorem 3.20.

**Theorem 3.21**
(i) *For any L-convex cone* $D \subseteq \mathbf{R}^V$, $G_D = (V, A_D)$ *is transitive and* $D_{G_D} = D$.
(ii) *For any transitive directed graph* $G = (V, A)$, $D_G$ $(\subseteq \mathbf{R}^V)$ *is an L-convex cone and* $G_{D_G} = G$.
(iii) *The mappings* $D \mapsto G_D$ *and* $G \mapsto D_G$ *provide a one-to-one correspondence between L-convex cones* $D$ $(\subseteq \mathbf{R}^V)$ *and transitive directed graphs* $G = (V, A)$, *and are the inverse of each other.*

On the other hand, Birkhoff's representation theorem [2] yields a one-to-one correspondence between transitive directed graphs $G = (V, A)$ and distributive lattices $\mathcal{F}$ $(\subseteq 2^V)$ with $\{\emptyset, V\} \subseteq \mathcal{F}$. This fact, together with Theorem 3.21, provides a one-to-one correspondence between L-convex cones and distributive lattices, as follows.

Given a set of vectors $D \subseteq \mathbf{R}^V$, we define $\mathcal{F}_D \subseteq 2^V$ by

$$\mathcal{F}_D = \{X \subseteq V \mid \chi_X \in D\}.$$

For any family of subsets $\mathcal{F} \subseteq 2^V$ with $\{\emptyset, V\} \subseteq \mathcal{F}$, we define $D_\mathcal{F} \subseteq \mathbf{R}^V$ by

$$D_\mathcal{F} = \{\sum_{X \in \mathcal{F}} \lambda_X \chi_X \mid \lambda_X \geq 0 \ (X \in \mathcal{F} \setminus \{V\})\}.$$

**Theorem 3.22**
(i) *For an L-convex cone* $D \subseteq \mathbf{R}^V$, $\mathcal{F}_D$ *is a distributive lattice with* $\{\emptyset, V\} \subseteq \mathcal{F}_D$, *and* $D_{\mathcal{F}_D} = D$.
(ii) *For a distributive lattice* $\mathcal{F} \subseteq 2^V$ *with* $\{\emptyset, V\} \subseteq \mathcal{F}$, $D_\mathcal{F}$ *is an L-convex cone and* $\mathcal{F}_{D_\mathcal{F}} = \mathcal{F}$. *Moreover, for any* $p \in D$ *there exists a chain* $\mathcal{F}_0$ *with* $\{\emptyset, V\} \subseteq \mathcal{F}_0 \subseteq \mathcal{F}$ *and* $\lambda_X \in \mathbf{R}$ $(X \in \mathcal{F}_0)$ *such that* $\lambda_X \geq 0$ $(\forall X \in \mathcal{F}_0 \setminus \{V\})$ *and* $p = \sum_{X \in \mathcal{F}_0} \lambda_X \chi_X$.
(iii) *The mappings* $D \mapsto \mathcal{F}_D$ *and* $\mathcal{F} \mapsto D_\mathcal{F}$ *provide a one-to-one correspondence between L-convex cones* $D$ $(\subseteq \mathbf{R}^V)$ *and distributive lattices* $\mathcal{F}$ $(\subseteq 2^V)$ *with* $\{\emptyset, V\} \subseteq \mathcal{F}$, *and are the inverse of each other.*

The intersection of L-convex polyhedra is also L-convex.

**Theorem 3.23** *Let* $D_1, D_2 \in \mathcal{L}_0$. *Define* $\gamma : V \times V \to \mathbf{R} \cup \{+\infty\}$ *by* $\gamma(u, v) = \min\{\gamma_{D_1}(u, v), \gamma_{D_2}(u, v)\}$ $(u, v \in V)$. *Then,* (i) $D_1 \cap D_2 = \mathrm{D}(\gamma)$, (ii) $D_1 \cap D_2 \neq \emptyset \iff G_\gamma$ *has no negative cycle,* (iii) $D_1 \cap D_2 \neq \emptyset \Rightarrow D_1 \cap D_2 \in \mathcal{L}_0$.

**Proof.** The claim (i) follows from $D_1 \cap D_2 = \mathrm{D}(\gamma_{D_1}) \cap \mathrm{D}(\gamma_{D_2}) = \mathrm{D}(\gamma)$. The claims (ii) and (iii) are by (i) and (iii) of Lemma 3.18, respectively. □

**Remark 3.24** The Minkowski-sum of two L-convex polyhedra is not necessarily an L-convex polyhedron, as shown below.

Put $V = \{a, b, c, d\}$. Let $\gamma_1, \gamma_2 : V \times V \to \{0, 1\}$ be such that

$$\gamma_1(u, v) = 1 \iff (u, v) \in \{(c, a), (c, b), (d, a), (d, b), (d, c)\},$$
$$\gamma_2(u, v) = 1 \iff (u, v) \in \{(b, a), (b, c), (d, a), (d, b), (d, c)\}.$$

Since $\gamma_1, \gamma_2 \in \mathcal{T}$, we have $\mathrm{D}(\gamma_1), \mathrm{D}(\gamma_2) \in \mathcal{L}_0$ by Lemma 3.18. However, $D = \mathrm{D}(\gamma_1) + \mathrm{D}(\gamma_2)$ is not L-convex. Let

$$p_1 = p_2 = (0, 0, 0, 0), \ q_1 = (1, 1, 0, 0), \ q_2 = (1, 0, 1, 0).$$

Then, it holds that $p_i, q_i \in \mathrm{D}(\gamma_i)$ $(i=1,2)$. Hence, we have $p_1+p_2, q_1+q_2 \in D$, but $(p_1+p_2)\wedge(q_1+q_2) = (1,0,0,0) \not\in D$. Note that

$$\{p \in \mathrm{D}(\gamma_i) \mid p(d)=0\} = \mathrm{conv}(S_i) \ (i=1,2), \qquad \{p \in D \mid p(d)=0\} = \mathrm{conv}(S_1+S_2),$$

where $S_1 = \{(0,0,0,0),(1,1,0,0),(1,1,1,0)\}$, $S_2 = \{(0,0,0,0),(1,0,1,0),(1,1,1,0)\}$. □

Separation theorems of the following form hold for distance functions with triangle inequality and for L-convex polyhedra:

**Theorem 3.25** *Let $\gamma : V\times V \to \mathbf{R}\cup\{+\infty\}$, $\nu : V\times V \to \mathbf{R}\cup\{-\infty\}$ be functions such that $\gamma, -\nu \in \mathcal{T}$. Suppose that $\sum_{i=1}^{k}\nu(v_{2i+1},v_{2i}) \le \sum_{i=1}^{k}\gamma(v_{2i-1},v_{2i})$ holds for any distinct elements $\{v_i\}_{i=1}^{2k} \subseteq V$ with $v_{2k+1}=v_1$. Then, there exists $p_* \in \mathbf{R}^V$ with*

$$\nu(u,v) \le p_*(v)-p_*(u) \le \gamma(u,v) \qquad (\forall u,v \in V). \tag{3.11}$$

*Moreover, if $\gamma$ and $\nu$ are integer-valued, then there exists such $p_* \in \mathbf{Z}^V$.*

**Proof.** The existence of $p_* \in \mathbf{R}^V$ with (3.11) is equivalent to $\mathrm{D}(\nu^\sharp)\cap\mathrm{D}(\gamma) \neq \emptyset$, where $\nu^\sharp : V\times V \to \mathbf{R}\cup\{+\infty\}$ is defined by $\nu^\sharp(u,v) = -\nu(v,u)$ $(u,v\in V)$. Define $\eta : V\times V \to \mathbf{R}\cup\{+\infty\}$ by $\eta(u,v) = \min\{\gamma(u,v),\nu^\sharp(u,v)\}$ $(u,v\in V)$. By Theorem 3.23 (ii), it suffices to show that $G_\eta$ has no negative cycle. Suppose, to the contrary, that $\{v_1,v_2,\cdots,v_h\} \subseteq V$ $(h\ge 1)$ is a negative cycle. Since $\gamma$ and $\nu^\sharp$ satisfy the triangle inequality, we may assume that $h=2k$ $(k\ge 1)$ and

$$\eta(v_{2i-1},v_{2i}) = \gamma(v_{2i-1},v_{2i}),\ \eta(v_{2i},v_{2i+1}) = -\nu(v_{2i+1},v_{2i}), \quad (i=1,2,\cdots,k),$$

where $v_{2k+1}=v_1$. This implies that $\sum_{i=1}^{k}\nu(v_{2i+1},v_{2i}) > \sum_{i=1}^{k}\gamma(v_{2i-1},v_{2i})$, which is a contradiction. The integrality assertion follows from the argument above and Lemma 3.18 (v). □

**Remark 3.26** The assumption in Theorem 3.25 cannot be replaced by the weaker condition $\nu(u,v) \le \gamma(u,v)$ $(\forall u,v \in V)$, as shown in the following example.

Put $V=\{a,b,c,d\}$, and define $\gamma : V\times V \to \mathbf{R}$ and $\nu : V\times V \to \mathbf{R}$ as follow:

| $\gamma$ | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | 0 | 0 | 1 | −1 |
| $b$ | 2 | 0 | 1 | 1 |
| $c$ | 1 | −1 | 0 | 0 |
| $d$ | 1 | 1 | 2 | 0 |

| $\nu$ | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | 0 | 0 | −1 | −1 |
| $b$ | 0 | 0 | −1 | −1 |
| $c$ | −1 | −1 | 0 | 0 |
| $d$ | −1 | −1 | 0 | 0 |

where it reads as $\gamma(a,d)=-1$, $\gamma(b,a)=2$, etc. We can easily check that $\gamma,-\nu\in\mathcal{T}$ and $\nu(u,v)\le\gamma(u,v)$ $(\forall u,v\in V)$. However, there is no $p_*\in\mathbf{R}^V$ satisfying (3.11). Suppose, to the contrary, that such $p_*\in\mathbf{R}^V$ exists. Then, we have

$$\begin{aligned} &p_*(a)-p_*(b) \le \min\{\gamma(b,a),-\nu(a,b)\} = 0, \quad p_*(b)-p_*(c) \le \min\{\gamma(c,b),-\nu(b,c)\} = -1,\\ &p_*(c)-p_*(d) \le \min\{\gamma(d,c),-\nu(c,d)\} = 0, \quad p_*(d)-p_*(a) \le \min\{\gamma(a,d),-\nu(d,a)\} = -1. \end{aligned}$$

Addition of these inequalities yields $0 \le -2$, a contradiction. Note that the assumption of Theorem 3.25 is not satisfied since $\nu(a,b)+\nu(c,d) = 0 > -2 = \gamma(c,b)+\gamma(a,d)$. □

The latter part of the following theorem is already shown in [35, Theorem 5.6].

**Theorem 3.27** *Let $D_1, D_2 \in \mathcal{L}_0$. If $D_1 \cap D_2 = \emptyset$, then there exists $x_* \in \{0, \pm 1\}^V$ such that*

$$\inf_{p \in D_1} \langle p, x_* \rangle - \sup_{p \in D_2} \langle p, x_* \rangle > 0. \tag{3.12}$$

*Moreover, if $D_1$ and $D_2$ are integral polyhedra, i.e., $D_i = \overline{D_i} \cap \mathbf{Z}^V$ $(i = 1, 2)$, then "$> 0$" in (3.12) can be replaced by "$\geq 1$".*

**Proof.** Put $\gamma_i = \gamma_{D_i}$ $(i = 1, 2)$. Since $\mathrm{D}(\gamma_1) \cap \mathrm{D}(\gamma_2) = \emptyset$, it follows from Theorem 3.25 that there exists a sequence of distinct elements $\{v_i\}_{i=1}^{2k} \subseteq V$ $(k \geq 1)$ such that $\sum_{i=1}^{k} \gamma_1(v_{2i-1}, v_{2i}) + \sum_{i=1}^{k} \gamma_2(v_{2i}, v_{2i+1}) < 0$, where $v_{2k+1} = v_1$. Let $x_* \in \{0, \pm 1\}^V$ be a vector such that $x_*(v_{2i}) = -1$, $x_*(v_{2i-1}) = 1$ $(i = 1, \cdots, k)$, and $x_*(w) = 0$ otherwise. Then, we have

$$\begin{aligned} \inf_{p \in D_1} \langle p, x_* \rangle - \sup_{p \in D_2} \langle p, x_* \rangle &= \inf_{p \in D_1} \sum_{i=1}^{k} \{p(v_{2i-1}) - p(v_{2i})\} - \sup_{p \in D_2} \sum_{i=1}^{k} \{p(v_{2i+1}) - p(v_{2i})\} \\ &\geq -\sum_{i=1}^{k} \gamma_1(v_{2i-1}, v_{2i}) - \sum_{i=1}^{k} \gamma_2(v_{2i}, v_{2i+1}) \ > \ 0. \end{aligned}$$

The integrality assertion is obvious from the inequality above. □

**Remark 3.28** Due to the property (LS2), an L-convex polyhedron loses no information when restricted to a hyperplane $\{p \in \mathbf{R}^V \mid p(v) = 0\}$ for any $v \in V$. We call a polyhedron $P \subseteq \mathbf{R}^V$ *$\mathrm{L}^\natural$-convex* if the set $\tilde{P} \subseteq \mathbf{R}^{\tilde{V}}$ defined by

$$\tilde{P} = \{(\lambda, p + \lambda \mathbf{1}) \mid p \in P,\ \lambda \in \mathbf{R}\}$$

is an L-convex polyhedron, where $\tilde{V} = \{v_0\} \cup V$. We see that $\mathrm{L}^\natural$-convex polyhedra are essentially the same as L-convex polyhedra, while the class of $\mathrm{L}^\natural$-convex polyhedra properly contains that of L-convex polyhedra. □

# 4 Polyhedral M-convex and L-convex Functions

## 4.1 Recapitulation of Fundamental Results on Polyhedral Convex Functions

This section is devoted to a summary of the relevant results on polyhedral convex functions. See [40], [43] for more accounts.

Let $f : \mathbf{R}^V \to \mathbf{R} \cup \{\pm\infty\}$ be a function. The *epigraph* of $f$, denoted by epi$f$, is the set

$$\mathrm{epi} f = \{(x, \alpha) \mid x \in \mathbf{R}^V,\ \alpha \in \mathbf{R},\ \alpha \geq f(x)\}\ (\subseteq \mathbf{R}^V \times \mathbf{R}).$$

The effective domain of $f$, denoted by $\mathrm{dom}\, f$, is the set $\mathrm{dom}\, f = \{x \in \mathbf{R}^V \mid -\infty < f(x) < +\infty\}$. We call $f$ *convex* if epi$f$ is a convex set in $\mathbf{R}^V \times \mathbf{R}$. A function $f$ is called *concave* if $-f$ is convex. When $f > -\infty$, $f$ is convex if and only if

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y) \quad (\forall x, y \in \mathbf{R}^V,\ 0 \leq \forall \alpha \leq 1).$$

A convex function is said to be *polyhedral* if its epigraph is polyhedral.

Let $f : \mathbf{R}^V \to \mathbf{R} \cup \{\pm\infty\}$. The *(convex) conjugate* $f^\bullet : \mathbf{R}^V \to \mathbf{R} \cup \{\pm\infty\}$ of $f$ is defined by

$$f^\bullet(p) = \sup_{x \in \mathbf{R}^V} \{\langle p, x\rangle - f(x)\} \qquad (p \in \mathbf{R}^V). \tag{4.1}$$

**Theorem 4.1 ([40, Theorem 19.2])**
(i) *For a polyhedral convex function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ with $\mathrm{dom}\, f \neq \emptyset$, its conjugate function $f^\bullet$ is also polyhedral convex with $\mathrm{dom}\, f^\bullet \neq \emptyset$ and $f^\bullet > -\infty$, and $f^{\bullet\bullet} = f$.*
(ii) *The mapping $f \mapsto f^\bullet$ induces a symmetric one-to-one correspondence in the class of polyhedral convex functions $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ with $\mathrm{dom}\, f \neq \emptyset$.*

A function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ is called *positively homogeneous* if $f(\alpha x) = \alpha f(x)$ for any $x \in \mathbf{R}^V$ and any $\alpha \geq 0$. Note that $f(\mathbf{0}) = 0$ from the definition.

**Theorem 4.2 ([40, Corollary 4.7.1])** *If $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ is a positively homogeneous convex function, then $f(\sum_{i=1}^k \alpha_i x_i) \leq \sum_{i=1}^k \alpha_i f(x_i)$ for any $x_i \in \mathbf{R}^V$ and $\alpha_i \geq 0$ $(i = 1, \cdots, k)$.*

For any nonempty set $S \subseteq \mathbf{R}^V$, we define the *support function* $\delta_S^* : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ of $S$ by

$$\delta_S^*(p) = \sup_{x \in S} \langle p, x\rangle \qquad (p \in \mathbf{R}^V). \tag{4.2}$$

For any positively homogeneous function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$, we define the set $S_f \subseteq \mathbf{R}^V$ by

$$S_f = \{x \in \mathbf{R}^V \mid \langle p, x\rangle \leq f(p)\ (\forall p \in \mathbf{R}^V)\}. \tag{4.3}$$

**Theorem 4.3 ([40, Section 13])**
(i) *For any nonempty polyhedron $S$ $(\subseteq \mathbf{R}^V)$, $\delta_S^*$ is a positively homogeneous polyhedral convex function such that $\mathrm{dom}\, \delta_S^* \neq \emptyset$, $\delta_S^* > -\infty$, and $S = S_{\delta_S^*}$.*
(ii) *For any positively homogeneous polyhedral convex function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ with $\mathrm{dom}\, f \neq \emptyset$, $S_f$ is a nonempty polyhedron with $S_f \neq \emptyset$, and $f = \delta_{S_f}^*$.*
(iii) *The mappings $S \mapsto \delta_S^*$ and $f \mapsto S_f$ provide a one-to-one correspondence between nonempty polyhedra $S$ $(\subseteq \mathbf{R}^V)$ and positively homogeneous polyhedral convex functions $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ with $\mathrm{dom}\, f \neq \emptyset$, and are the inverse of each other.*

Let $f : \mathbf{R}^V \to \mathbf{R} \cup \{\pm\infty\}$. For any $x \in \mathrm{dom}\, f$ and $y \in \mathbf{R}^V$, we define the *(one-sided) directional derivative* of $f$ at $x$ with respect to $y$ by

$$f'(x;y) = \lim_{\alpha \downarrow 0} \{f(x + \alpha y) - f(x)\}/\alpha$$

if the limit exists. Note that if $f$ is polyhedral convex, then $f'(x;y)$ is well-defined for any $x \in \mathrm{dom}\, f$ and $y \in \mathbf{R}^V$ and $f'(x;y) = \{f(x+\alpha y) - f(x)\}/\alpha$ for a sufficiently small $\alpha > 0$. The *subdifferential* $\partial f(x)$ of $f$ at $x$ is defined by

$$\partial f(x) = \{p \in \mathbf{R}^V \mid f(y) \geq f(x) + \langle p, y - x\rangle \ (\forall y \in \mathbf{R}^V)\},$$

and each vector in $\partial f(x)$ is called a *subgradient* of $f$ at $x$.

**Theorem 4.4 ([40, Theorem 23.10])** *Let $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ be a polyhedral convex function with* $\mathrm{dom}\, f \neq \emptyset$ *and* $x \in \mathrm{dom}\, f$. *Then, $\partial f(x)$ is a nonempty polyhedral convex set with $\partial f(x) \neq \emptyset$, $f'(x;\cdot)$ is a positively homogeneous polyhedral convex function such that* $\mathrm{dom}\, f'(x;\cdot) \neq \emptyset$, *$f'(x;\cdot) > -\infty$, and $f'(x;\cdot) = \delta^*_{\partial f(x)}$.*

Let $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ be any function. We denote by $\arg\min f$ the set of minimizers of $f$, i.e.,

$$\arg\min f = \{x \in \mathbf{R}^V \mid f(x) \leq f(y) \ (\forall y \in \mathbf{R}^V)\}.$$

Note that $\arg\min f$ can be empty. For any $p \in \mathbf{R}^V$, the function $f[p] : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ is defined by

$$f[p](x) = f(x) + \langle p, x\rangle \qquad (x \in \mathbf{R}^V). \tag{4.4}$$

**Theorem 4.5 ([40, Theorem 23.5])** *Let $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ be a polyhedral convex function with* $\mathrm{dom}\, f \neq \emptyset$. *For any $x \in \mathbf{R}^V$ and $p \in \mathbf{R}^V$, we have*

$$p \in \partial f(x) \iff x \in \arg\min f[-p] \iff x \in \partial f^\bullet(p) \iff p \in \arg\min f^\bullet[-x].$$

We consider the special case of one-dimensional functions. A function $f : \mathbf{R} \to \mathbf{R} \cup \{\pm\infty\}$ is called *piecewise-linear* if $\mathrm{dom}\, f$ is a closed set and can be divided into a finite number of subintervals on each of which $f$ is linear. We denote by $\mathcal{C}^1$ the class of piecewise-linear convex functions (cf. (2.1)). Note that a piecewise-linear convex function is nothing but a one-dimensional polyhedral convex function. For $f : \mathbf{R} \to \mathbf{R} \cup \{\pm\infty\}$ and $x \in \mathrm{dom}\, f$, we denote $f'_+(x) = f'(x;1)$, $f'_-(x) = -f'(x;-1)$. If $f$ is piecewise-linear, then the value $f'_+(x)$ and $f'_-(x)$ are well-defined for any $x \in \mathrm{dom}\, f$ and

$$\lim_{y \downarrow x} f'_+(y) = \lim_{y \downarrow x} f'_-(y) = f'_+(x), \qquad \lim_{y \uparrow x} f'_+(y) = \lim_{y \uparrow x} f'_-(y) = f'_-(x).$$

## 4.2 Polyhedral M-convex Functions

A polyhedral convex function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ is called *M-convex* if $\mathrm{dom}\, f \neq \emptyset$ and $f$ satisfies (M-EXC):

**(M-EXC)** $\forall x, y \in \mathrm{dom}\, f$, $\forall u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$, $\exists \alpha_0 > 0$ such that

$$f(x) + f(y) \geq f(x - \alpha(\chi_u - \chi_v)) + f(y + \alpha(\chi_u - \chi_v)) \qquad (0 \leq \forall \alpha \leq \alpha_0).$$

We denote

$$\begin{aligned} \mathcal{M} &= \{f \mid f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\},\ \text{polyhedral M-convex}\}, \\ {}_0\mathcal{M} &= \{f \mid f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\},\ \text{positively homogeneous polyhedral M-convex}\}. \end{aligned}$$

It may be obvious from the definition that polyhedral M-convex functions are quantitative extension of M-convex polyhedra.

**Theorem 4.6**
(i) *For a function* $f : \mathbf{R}^V \to \{0, +\infty\}$, *we have* $f \in \mathcal{M} \iff \mathrm{dom}\, f \in \mathcal{M}_0$. (ii) *For* $f \in \mathcal{M}$, *we have* $\mathrm{dom}\, f \in \mathcal{M}_0$.

### 4.2.1 Axioms for Polyhedral M-convex Functions

We consider two slightly different exchange axioms, where the former is weaker and the latter is stronger than (M-EXC).

**(M-EXC$_\mathrm{w}$)** $\forall x, y \in \mathrm{dom}\, f$, $\forall u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$, $\exists \alpha > 0$ such that

$$f(x) + f(y) \geq f(x - \alpha(\chi_u - \chi_v)) + f(y + \alpha(\chi_u - \chi_v)).$$

**(M-EXC$_\mathrm{s}$)** $\forall x, y \in \mathrm{dom}\, f$, $\forall u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$ such that

$$f(x) + f(y) \geq f(x - \alpha(\chi_u - \chi_v)) + f(y + \alpha(\chi_u - \chi_v)) \quad (0 \leq \forall \alpha \leq \{x(u) - y(u)\}/2k),$$

where $k = |\mathrm{supp}^-(x-y)|$.

**Theorem 4.7** *For a polyhedral convex function* $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ *with* $\mathrm{dom}\, f \neq \emptyset$, (M-EXC) $\iff$ (M-EXC$_\mathrm{w}$).

**Proof.** We show (M-EXC$_\mathrm{w}$) $\Rightarrow$ (M-EXC) only. Let $x, y \in \mathrm{dom}\, f$ and $u \in \mathrm{supp}^+(x-y)$. Then, there exists $v \in \mathrm{supp}^-(x-y)$ and $\alpha_0 > 0$ such that

$$f(x) + f(y) \geq f(x - \alpha(\chi_u - \chi_v)) + f(y + \alpha(\chi_u - \chi_v)) \tag{4.5}$$

for $\alpha = \alpha_0$. The convexity of $f$ implies the inequality (4.5) for any $\alpha \in [0, \alpha_0]$. □

**Lemma 4.8** *Let* $f \in \mathcal{M}$, $x, y \in \mathrm{dom}\, f$ *be distinct vectors,* $u \in \mathrm{supp}^+(x-y)$, *and* $\mathrm{supp}^-(x-y) = \{v_1, v_2, \cdots, v_k\}$ $(k = |\mathrm{supp}^-(x-y)|)$. *Then, there exist sequences* $\{x_i\}_{i=0}^k$, $\{y_i\}_{i=0}^k$ $(\subseteq \mathrm{dom}\, f)$ *and* $\{\alpha_i\}_{i=0}^k$ $(\subseteq \mathbf{R}_+)$ *satisfying* $\sum_{i=1}^k \alpha_i = \{x(u) - y(u)\}/2$ *and the following conditions:*

$$x_0 = x,\ y_0 = y,\ x_i = x_{i-1} - \alpha_i(\chi_u - \chi_{v_i}),\ y_i = y_{i-1} + \alpha_i(\chi_u - \chi_{v_i})\ (i = 1, \cdots, k), \tag{4.6}$$

$$f(x_{i-1}) + f(y_{i-1}) \geq f(x_i) + f(y_i) \quad (i = 1, \cdots, k). \tag{4.7}$$

**Proof.** Put $x_0 = x$, $y_0 = y$. For $i = 1, 2, \cdots, k$, we define $\alpha_i \in \mathbf{R}$ and $x_i, y_i \in \mathbf{R}^V$ iteratively by the following equations:

$$\begin{aligned} \alpha_i \ &= \ \sup\{\alpha \in \mathbf{R}_+ \mid \alpha \leq (1/2)\min\{x_{i-1}(u) - y_{i-1}(u), y_{i-1}(v_i) - x_{i-1}(v_i)\}, \\ &\qquad\qquad f(x_{i-1} - \alpha(\chi_u - \chi_{v_i})) + f(y_{i-1} + \alpha(\chi_u - \chi_{v_i})) \leq f(x_{i-1}) + f(y_{i-1})\}, \\ x_i &= x_{i-1} - \alpha_i(\chi_u - \chi_{v_i}), \quad y_i = y_{i-1} + \alpha_i(\chi_u - \chi_{v_i}). \end{aligned}$$

Then, the sequences $\{x_i\}_{i=0}^k$, $\{y_i\}_{i=0}^k$, and $\{\alpha_i\}_{i=0}^k$ satisfy the conditions (4.6) and (4.7). In the following, we show that $\sum_{i=1}^k \alpha_i = \{x(u) - y(u)\}/2$ holds.

Assume, to the contrary, that $\sum_{i=1}^k \alpha_i < \{x(u) - y(u)\}/2$. Since $u \in \mathrm{supp}^+(x_k - y_k)$, there exist some $v_i \in \mathrm{supp}^-(x_k - y_k) \subseteq \mathrm{supp}^-(x - y)$ and $\alpha_0 > 0$ such that

$$f(x_k) + f(y_k) \geq f(x_k - \alpha(\chi_u - \chi_{v_i})) + f(y_k + \alpha(\chi_u - \chi_{v_i})) \quad (0 \leq \forall\alpha \leq \alpha_0). \tag{4.8}$$

Here, $v_i \neq v_k$ holds by the choice of $x_k$ and $y_k$. By Theorem 4.13 to be shown later, we have

$$\begin{aligned} &\{f(x_i - \alpha(\chi_u - \chi_{v_i})) - f(x_i)\} + \{f(y_i + \alpha(\chi_u - \chi_{v_i})) - f(y_i)\} \\ &\qquad \leq \ \{f(x_k - \alpha(\chi_u - \chi_{v_i})) - f(x_k)\} + \{f(y_k + \alpha(\chi_u - \chi_{v_i})) - f(y_k)\} \ \leq \ 0 \end{aligned}$$

for any $\alpha \in [0, \alpha_0]$, where the second inequality is by (4.8). Hence, it holds that

$$f(x_i) + f(y_i) \geq f(x_i - \alpha(\chi_u - \chi_{v_i})) + f(y_i + \alpha(\chi_u - \chi_{v_i})) \quad (0 \leq \forall\alpha \leq \alpha_0),$$

which contradicts the definitions of $x_i$ and $y_i$. □

**Theorem 4.9** *For a polyhedral convex function* $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ *with* $\mathrm{dom} f \neq \emptyset$, (M-EXC) $\iff$ (M-EXC$_\mathrm{s}$).

**Proof.** It suffices to prove (M-EXC) $\Rightarrow$ (M-EXC$_\mathrm{s}$). Assume (M-EXC) for $f$. Let $x, y \in \mathrm{dom} f$ be distinct vectors, and $u \in \mathrm{supp}^+(x - y)$. By Lemma 4.8, there exist sequences $\{x_i\}_{i=0}^k$, $\{y_i\}_{i=0}^k$ ($\subseteq \mathrm{dom} f$) and $\{\alpha_i\}_{i=0}^k$ ($\subseteq \mathbf{R}_+$) satisfying $\sum_{i=1}^k \alpha_i = \{x(u) - y(u)\}/2$, (4.6), and (4.7). Let $i_*$ ($1 \leq i_* \leq k$) be such that $\alpha_{i_*} = \max_{1 \leq i \leq k} \alpha_i$ ($\geq \{x(u) - y(u)\}/2k$). From Theorem 4.13 to be shown later, we obtain the inequalities:

$$\begin{aligned} &\{f(x - \alpha_{i_*}(\chi_u - \chi_{v_{i_*}})) - f(x)\} + \{f(y + \alpha_{i_*}(\chi_u - \chi_{v_{i_*}})) - f(y)\} \\ &\qquad\qquad \leq \{f(x_{i_*}) - f(x_{i_*-1})\} + \{f(y_{i_*}) - f(y_{i_*-1})\} \leq 0, \end{aligned}$$

where the second inequality is by (4.7). It follows from the convexity of $f$ that

$$f(x) + f(y) \geq f(x - \alpha(\chi_u - \chi_{v_{i_*}})) + f(y + \alpha(\chi_u - \chi_{v_{i_*}})) \qquad (0 \leq \forall\alpha \leq \alpha_{i_*}).$$

□

We can rewrite (M-EXC) in terms of directional derivative. For a polyhedral convex function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ with $\mathrm{dom} f \neq \emptyset$ and $x \in \mathrm{dom} f$, define $f'(x; \cdot, \cdot) : V \times V \to \mathbf{R} \cup \{+\infty\}$ by

$$f'(x; v, u) = f'(x; \chi_v - \chi_u) \qquad (u, v \in V).$$

Note that $f(x - \alpha(\chi_u - \chi_v)) - f(x) = \alpha f'(x; v, u)$ for a sufficiently small $\alpha > 0$. We consider (M-EXC$'$):

**(M-EXC$'$)** $\forall x, y \in \mathrm{dom} f$, $\forall u \in \mathrm{supp}^+(x - y)$, $\exists v \in \mathrm{supp}^-(x - y)$ such that

$$f'(x, v, u) + f'(y, u, v) \leq 0.$$

**Theorem 4.10** *For a polyhedral convex function* $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ *with* $\mathrm{dom} f \neq \emptyset$, (M-EXC) $\iff$ (M-EXC$'$).

### 4.2.2 Fundamental Properties of Polyhedral M-convex Functions

In this section, we show various properties of polyhedral M-convex functions.

Global optimality of a polyhedral M-convex function is characterized by local optimality.

**Theorem 4.11** *Let $f \in \mathcal{M}$ and $x \in \operatorname{dom} f$. Then, $f(x) \le f(y)$ $(\forall y \in \mathbf{R}^V)$ $\iff$ $f'(x; v, u) \ge 0$ $(\forall u, v \in V)$.*

**Proof.** We show the "$\Longleftarrow$" direction only. Assume, to the contrary, that $f(x_0) < f(x)$ holds for some $x_0 \in \mathbf{R}^V$. Put $S = \{y \in \mathbf{R}^V \mid f(y) \le f(x_0)\}$. Let $x_* \in S$ be a vector such that $\|x_* - x\|_1 = \inf_{y \in S} \|y - x\|_1$. By (M-EXC) applied to $x, x_*$ and some $u \in \operatorname{supp}^+(x - x_*)$, there exists $v \in \operatorname{supp}^-(x - x_*)$ and a sufficiently small $\alpha > 0$ such that

$$f(x_*) - f(x_* + \alpha(\chi_u - \chi_v)) \ge f(x - \alpha(\chi_u - \chi_v)) - f(x) \ge \alpha f'(x; v, u) \ge 0.$$

Hence, we have $f(x_* + \alpha(\chi_u - \chi_v)) \le f(x_*) \le f(x_0)$, which contradicts the choice of $x_*$ since $\|(x_* + \alpha(\chi_u - \chi_v)) - x\|_1 = \|x_* - x\|_1 - 2\alpha$. □

A polyhedral M-convex function has supermodularity when projected onto the hyperplane $\{x \in \mathbf{R}^V \mid x(w_0) = 0\}$ $(w_0 \in V)$, as stated in Theorem 4.13.

**Lemma 4.12** *Let $f \in \mathcal{M}$, $w_0 \in V$, and $x \in \mathbf{R}^V$. Then, for any $u, v \in V$ and $\lambda, \mu \ge 0$ we have*

$$f(x + \lambda(\chi_u - \chi_{w_0})) + f(x + \mu(\chi_v - \chi_{w_0})) \le f(x) + f(x + \lambda(\chi_u - \chi_{w_0}) + \mu(\chi_v - \chi_{w_0})). \tag{4.9}$$

**Proof.** We fix $u, v \in V$ and $\lambda, \mu \ge 0$. For $\lambda',\ \mu' \in \mathbf{R}_+$, we denote $x(\lambda', \mu') = x + \lambda'(\chi_u - \chi_{w_0}) + \mu'(\chi_v - \chi_{w_0})$. We may assume that $x(0,0), x(\lambda, \mu) \in \operatorname{dom} f$. Then, it follows from Theorem 3.9 that $x(\lambda', 0),\ x(\lambda', \mu) \in \operatorname{dom} f$ $(0 \le \forall \lambda' \le \lambda)$. Define the functions $f_1, f_2 : [0, \lambda] \to \mathbf{R}$ by

$$f_1(\lambda') = f(x(\lambda', 0)), \quad f_2(\lambda') = f(x(\lambda', \mu)) \qquad (0 \le \lambda' \le \lambda).$$

Let $\gamma$ be any value with $0 < \gamma \le \lambda$. For any $\beta \in [0, \gamma)$, (M-EXC$'$) yields $(f_1)'_+(\beta) - (f_2)'_-(\gamma) \le 0$ since $\operatorname{supp}^+(x(\gamma, \mu) - x(\beta, 0)) = \{u, v\}$ and $\operatorname{supp}^-(x(\gamma, \mu) - x(\beta, 0)) = \{w_0\}$. This implies

$$(f_2 - f_1)'_-(\gamma) = (f_2)'_-(\gamma) - (f_1)'_-(\gamma) = (f_2)'_-(\gamma) - \lim_{\beta \uparrow \gamma} (f_1)'_+(\beta) \ge 0 \qquad (0 < \forall \gamma \le \lambda).$$

Therefore, the piecewise-linear function $f_2 - f_1$ is nondecreasing on $[0, \lambda]$, from which we have

$$f(x(\lambda, \mu)) - f(x(\lambda, 0)) = f_2(\lambda) - f_1(\lambda) \ge f_2(0) - f_1(0) = f(x(0, \mu)) - f(x(0, 0)),$$

i.e., the inequality (4.9) holds. □

**Theorem 4.13** *Let $f \in \mathcal{M}$, $w_0 \in V$, and $x \in \mathbf{R}^V$. For any $X, Y \subseteq V \setminus \{w_0\}$ with $X \cap Y = \emptyset$ and $\lambda_w \ge 0$ $(w \in X \cup Y)$, we have*

$$f(x + \sum_{w \in X} \lambda_w(\chi_w - \chi_{w_0})) + f(x + \sum_{w \in Y} \lambda_w(\chi_w - \chi_{w_0})) \le f(x) + f(x + \sum_{w \in X \cup Y} \lambda_w(\chi_w - \chi_{w_0})). \tag{4.10}$$

**Proof.** We show the inequality (4.10) by induction on the cardinality of the set $X \cup Y$. We consider the case when $X \ne \emptyset$ and $Y \ne \emptyset$ only, since otherwise the inequality (4.10) follows immediately.

We may assume that $x \in \mathrm{dom}\, f$ and $x + \sum_{w \in X \cup Y} \lambda_w(\chi_w - \chi_{w_0}) \in \mathrm{dom}\, f$. Let $v \in Y$. We have $x' = x + \lambda_v(\chi_v - \chi_{w_0}) \in \mathrm{dom}\, f$ by Theorem 3.9. Hence, the inductive assumption yields that

$$\begin{aligned} f(x + \sum_{w \in X} \lambda_w(\chi_w - \chi_{w_0})) - f(x) &\leq f(x + \sum_{w \in X \cup \{v\}} \lambda_w(\chi_w - \chi_{w_0})) - f(x + \lambda_v(\chi_v - \chi_{w_0})) \\ &\leq f(x + \sum_{w \in X \cup Y} \lambda_w(\chi_w - \chi_{w_0})) - f(x + \sum_{w \in Y} \lambda_w(\chi_w - \chi_{w_0})). \end{aligned}$$

□

Directional derivative functions and subdifferentials of a polyhedral M-convex function have nice structures such as M/L-convexity, and they can be explicitly described by certain distance functions with triangle inequality (cf. Theorem 4.14 (i)).

For any distance function $\gamma : V \times V \to \mathbf{R} \cup \{+\infty\}$ (i.e., $\gamma(v, v) = 0$ for all $v \in V$), we define $f_\gamma : \mathbf{R}^V \to \mathbf{R} \cup \{\pm\infty\}$ by

$$f_\gamma(x) = \inf\{ \sum_{u,v \in V} \lambda_{uv}\gamma(u, v) \mid \sum_{u,v \in V} \lambda_{uv}(\chi_v - \chi_u) = x,\ \lambda_{uv} \geq 0\ (u, v \in V)\} \quad (x \in \mathbf{R}^V). \tag{4.11}$$

**Theorem 4.14** *Let $f \in \mathcal{M}$ and $x \in \mathrm{dom}\, f$.*
(i) *The function $\gamma_x : V \times V \to \mathbf{R} \cup \{+\infty\}$ defined by*

$$\gamma_x(u, v) = f'(x; v, u) \qquad (u, v \in V) \tag{4.12}$$

*satisfies $\gamma_x(v, v) = 0$ ($\forall v \in V$) and the triangle inequality (3.10), i.e., $\gamma_x \in \mathcal{T}$.*
(ii) *We have $f'(x; \cdot) = f_{\gamma_x}$ and $f'(x; \cdot) \in {}_0\mathcal{M}$.*

**Proof.** (i): For $u, v, w \in V$ and a sufficiently small $\alpha > 0$, we have

$$\begin{aligned} \alpha\gamma_x(u, v) + \alpha\gamma_x(v, w) &= \{f(x + \alpha(\chi_v - \chi_u)) - f(x)\} + \{f(x + \alpha(\chi_w - \chi_v)) - f(x)\} \\ &\geq f(x + \alpha(\chi_w - \chi_u)) - f(x) \;=\; \alpha\gamma_x(u, w), \end{aligned}$$

where the inequality is by Lemma 4.12. Obviously, $\gamma_x$ is a distance function.
(ii) The proof is postponed after Theorem 4.18. □

L-convexity appears in subdifferentials of a polyhedral M-convex function.

**Theorem 4.15** *Let $f \in \mathcal{M}$ and $x \in \mathrm{dom}\, f$.*
(i) *$\partial f(x) \in \mathcal{L}_0$ and $\partial f(x)$ is represented as*

$$\partial f(x) = \mathrm{D}(\gamma_x) = \{p \in \mathbf{R}^V \mid p(v) - p(u) \leq f'(x; v, u)\ (u, v \in V)\}, \tag{4.13}$$

*where $\gamma_x : V \times V \to \mathbf{R} \cup \{+\infty\}$ is given by (4.12).*
(ii) *For any $y \in \mathbf{R}^V$ we have $f(y) - f(x) \geq \sup_{p \in \partial f(x)} \langle p, y - x \rangle = f_{\gamma_x}(y - x)$.*

**Proof.** The equation (4.13) follows from Theorems 4.5 and 4.11. The L-convexity of $\partial f(x)$ is immediate from (4.13) and Theorem 4.14 (i). The claim (ii) is immediate from (i) and the linear programming duality. □

The next theorem shows that each face of the epigraph of a polyhedral M-convex function is an M-convex polyhedron when it is projected to $\mathbf{R}^V$. The proof is obvious and therefore omitted.

**Theorem 4.16** *For $f \in \mathcal{M}$ and $p \in \mathbf{R}^V$, we have* $\arg\min f[-p] \in \mathcal{M}_0$ *if* $\inf f[-p] > -\infty$.

The class of polyhedral M-convex functions is closed under various fundamental operations. Let $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ be a function. For any subset $U \subseteq V$, we define $f_U : \mathbf{R}^U \to \mathbf{R} \cup \{+\infty\}$ by

$$f_U(y) = f(y, \mathbf{0}_{V \setminus U}) \qquad (y \in \mathbf{R}^U). \tag{4.14}$$

For $S \subseteq \mathbf{R}^V$, the *restriction* $f_S : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ of $f$ to $S$ is defined by

$$f_S(x) = \begin{cases} f(x) & (x \in S), \\ +\infty & (x \notin S). \end{cases} \tag{4.15}$$

**Theorem 4.17** *Let $f, f_1, f_2 \in \mathcal{M}$.*
(1) *For $p \in \mathbf{R}^V$, the function $f[-p] : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ given by (4.4) is polyhedral M-convex.*
(2) *For $a \in \mathbf{R}^V$, the functions $f(a - x)$ and $f(a + x)$ are polyhedral M-convex in $x$.*
(3) *For any $U \subseteq V$, the function $f_U : \mathbf{R}^U \to \mathbf{R} \cup \{+\infty\}$ is polyhedral M-convex if* $\mathrm{dom}\, f_U \neq \emptyset$.
(4) *For $\varphi_v \in \mathcal{C}^1$ $(v \in V)$, the function $\tilde{f} : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ defined by*

$$\tilde{f}(x) = f(x) + \sum_{v \in V} \varphi_v(x(v)) \quad (x \in \mathbf{R}^V)$$

*is polyhedral M-convex if* $\mathrm{dom}\, \tilde{f} \neq \emptyset$.
(5) *For any $a : V \to \mathbf{R} \cup \{-\infty\}$ and $b : V \to \mathbf{R} \cup \{+\infty\}$ with $a \leq b$, the restriction $f_{[a,b]}$ of $f$ is polyhedral M-convex if* $\mathrm{dom}\, f \cap [a, b] \neq \emptyset$.

A more general and stronger transformation, called "network induction," is explained in Section 7.3.

### 4.2.3 Positively Homogeneous Polyhedral M-convex Functions

This section clarifies the relationship of positively homogeneous polyhedral M-convex functions to distance functions with triangle inequalities, and also to L-convex polyhedra.

For a positively homogeneous polyhedral convex function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ with $\mathbf{0} \in \mathrm{dom}\, f$, define $\gamma_f : V \times V \to \mathbf{R} \cup \{+\infty\}$ by

$$\gamma_f(u, v) = f'(\mathbf{0}; v, u) \ (= f(\chi_v - \chi_u)) \qquad (u, v \in V). \tag{4.16}$$

Recall the definition of $f_\gamma$ in (4.11).

**Theorem 4.18**
(i) *For $f \in {}_0\mathcal{M}$, we have $\gamma_f \in \mathcal{T}$ and $f_{\gamma_f} = f$.* (ii) *For $\gamma \in \mathcal{T}$, we have $f_\gamma \in {}_0\mathcal{M}$ and $\gamma_{f_\gamma} = \gamma$.*
(iii) *The mappings $f \mapsto \gamma_f$ $(f \in {}_0\mathcal{M})$ and $\gamma \mapsto f_\gamma$ $(\gamma \in \mathcal{T})$ provide a one-to-one correspondence between ${}_0\mathcal{M}$ and $\mathcal{T}$, and are the inverse of each other.*

**Proof.** (i): By Theorem 4.14 (i) and Theorem 4.15 (ii), we have $\gamma_f \in \mathcal{T}$ and $f \geq f_{\gamma_f}$. For any $x \in \mathbf{R}^V$ and $\{\lambda_{uv}\}_{u,v \in V}$ satisfying $\sum_{u,v \in V} \lambda_{uv}(\chi_v - \chi_u) = x$ and $\lambda_{uv} \geq 0$ $(u, v \in V)$, Theorem 4.2 implies that

$$\sum_{u,v \in V} \lambda_{uv} \gamma_f(u, v) \ = \ \sum_{u,v \in V} \lambda_{uv} f(\chi_v - \chi_u) \ \geq \ f(\sum_{u,v \in V} \lambda_{uv}(\chi_v - \chi_u)) \ = \ f(x).$$

Hence, we have $f_{\gamma_f} = f$.

(ii): Polyhedral M-convexity of $f_\gamma$ can be shown as a special case of Example 2.4. To be more precise, we have $f_\gamma(x) = f(-x)$ $(\forall x \in \mathbf{R}^V)$ for the function $f$ in Example 2.4, where $T = V$, $A = \{(u,v) \mid u,v \in V,\ u \neq v\}$, and

$$f_{(u,v)}(\xi) = \begin{cases} \gamma(u,v) & (\xi \geq 0), \\ +\infty & (\xi < 0), \end{cases} \qquad ((u,v) \in A).$$

For any $u, v \in V$, we have $\gamma_{f_\gamma}(u,v) = f_\gamma(\chi_v - \chi_u)$, which is equal to the shortest path distance from $u$ to $v$ in the directed graph $G = (V, A)$, where $A = V \times V$ and the length of an arc $(u,v) \in A$ is given by $\gamma(u,v)$. Since $\gamma \in \mathcal{T}$, it holds $\gamma_{f_\gamma}(u,v) = \gamma(u,v)$ $(\forall u, v \in V)$.

(iii): Clear from (i) and (ii). □

We now prove the polyhedral M-convexity of the directional derivative function $f'(x;\cdot)$.

**Proof of Theorem 4.14 (ii)** By Theorems 4.14 (i) and 4.18, it suffices to show $f'(x;\cdot) = f_{\gamma_x}$, which can be done in the same way as the proof of Theorem 4.18 (i). □

As immediate consequences of Theorems 3.20, 4.3, and 4.18, we obtain a one-to-one correspondence between positively homogeneous polyhedral M-convex functions and L-convex polyhedra. Recall the notation $S_f$ and $\delta^*_S$ in (4.2) and (4.3).

**Theorem 4.19**
(i) *For $f \in {}_0\mathcal{M}$, we have $S_f \in \mathcal{L}_0$ and $\delta^*_{S_f} = f$.* (ii) *For $D \in \mathcal{L}_0$, we have $\delta^*_D \in {}_0\mathcal{M}$ and $S_{\delta^*_D} = D$.*
(iii) *The mappings $f \mapsto S_f$ $(f \in {}_0\mathcal{M})$ and $D \mapsto \delta^*_D$ $(D \in \mathcal{L}_0)$ provide a one-to-one correspondence between ${}_0\mathcal{M}$ and $\mathcal{L}_0$, are the inverse of each other.*

### 4.2.4 Polyhedral M$^\natural$-convex Functions

We introduce a variant of polyhedral M-convex functions, called polyhedral M$^\natural$-convex functions. From Theorem 3.2 and 4.6 (ii), we see that the effective domain of a polyhedral M-convex function is contained in a hyperplane $\{x \in \mathbf{R}^V \mid x(V) = r\}$ for some $r \in \mathbf{R}$. Therefore, no information is lost when a polyhedral M-convex function is projected onto a $(|V|-1)$-dimensional space. A function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ is called *polyhedral M$^\natural$-convex* if the function $\tilde{f} : \mathbf{R}^{\tilde{V}} \to \mathbf{R} \cup \{+\infty\}$ defined by

$$\tilde{f}(x_0, x) = \begin{cases} f(x) & (x_0 = -x(V)), \\ +\infty & (x_0 \neq -x(V)), \end{cases} \qquad ((x_0, x) \in \mathbf{R}^{\tilde{V}})$$

is a polyhedral M-convex function, where $\tilde{V} = \{v_0\} \cup V$. Polyhedral M$^\natural$-convex functions are essentially equivalent to polyhedral M-convex functions, whereas the class of polyhedral M$^\natural$-convex functions properly contains that of polyhedral M-convex functions. M$^\natural$-convexity was originally introduced in [37] as a concept for functions defined over the integer lattice (see also [18]).

The exchange property (M-EXC) for $\tilde{f}$ is translated as follows (cf. [37]):

**(M$^\natural$-EXC)** $\forall x, y \in \operatorname{dom} f$, $\forall u \in \operatorname{supp}^+(x-y)$, either (i) or (ii) (or both) holds:
(i) $\exists v \in \operatorname{supp}^-(x-y)$, $\exists \alpha_0 > 0$ such that

$$f(x) + f(y) \geq f(x - \alpha(\chi_u - \chi_v)) + f(y + \alpha(\chi_u - \chi_v)) \qquad (0 \leq \forall \alpha \leq \alpha_0),$$

(ii) $\exists \alpha_0 > 0$ such that

$$f(x) + f(y) \geq f(x - \alpha\chi_u) + f(y + \alpha\chi_u) \qquad (0 \leq \forall \alpha \leq \alpha_0).$$

Therefore, a polyhedral convex function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ is M$^\natural$-convex if and only if $\mathrm{dom}\, f \neq \emptyset$ and $f$ satisfies (M$^\natural$-EXC).

**Example 4.20 (separable-convex functions)** Let $\mathcal{A} \subseteq 2^V$ be a laminar family, i.e., $\mathcal{A}$ is a family of subsets of $V$ such that for any $X, Y \in \mathcal{A}$ at least one of $X \setminus Y$, $Y \setminus X$ and $X \cap Y$ is empty. For $f_X \in \mathcal{C}^1$ $(X \in \mathcal{A})$, the function $f_{\mathcal{A}} : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ given by

$$f_{\mathcal{A}}(x) = \sum_{X \in \mathcal{A}} f_X(x(X)) \qquad (x \in \mathbf{R}^V)$$

is polyhedral M$^\natural$-convex if $\mathrm{dom}\, f_{\mathcal{A}} \neq \emptyset$ [6]. This can be proved similarly to Example 2.4.

Let $B \subseteq \mathbf{R}^V$ be an M$^\natural$-convex polyhedron. For $f_0 \in \mathcal{C}^1$ and $f_v \in \mathcal{C}^1$ $(v \in V)$, the function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ defined by

$$f(x) = \begin{cases} f_0(x(V)) + \sum_{v \in V} f_v(x(v)) & (x \in B), \\ +\infty & (x \notin B) \end{cases}$$

is polyhedral M$^\natural$-convex if $\mathrm{dom}\, f \neq \emptyset$. □

Another class of polyhedral M$^\natural$-convex functions arises from the minimum cost flow problem, as in Example 2.4.

From the definition of polyhedral M$^\natural$-convex functions, every property of polyhedral M-convex functions can be restated in terms of polyhedral M$^\natural$-convex functions, and vice versa. We present some properties below, which seem more natural when restated in terms of polyhedral M$^\natural$-convex functions.

For any function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ and any subset $U \subseteq V$, we define the *projection* $f^U : \mathbf{R}^U \to \mathbf{R} \cup \{\pm\infty\}$ of $f$ to $U$ as

$$f^U(y) = \inf_{z \in \mathbf{R}^{V \setminus U}} f(y, z) \qquad (y \in \mathbf{R}^U). \tag{4.17}$$

**Theorem 4.21** *For any polyhedral M$^\natural$-convex function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ and $U \subseteq V$, $f^U$ is polyhedral M$^\natural$-convex if $f^U > -\infty$.*

**Theorem 4.22** *If $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ is polyhedral M$^\natural$-convex, then it satisfies the supermodular inequality:*

$$f(x) + f(y) \leq f(x \wedge y) + f(x \vee y) \qquad (\forall x, y \in \mathbf{R}^V).$$

**Proof.** Immediate from Theorem 4.13. □

## 4.3 Polyhedral L-convex Functions

A polyhedral convex function $g : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ is said to be *L-convex* if $\mathrm{dom}\, g \neq \emptyset$ and $g$ satisfies (LF1) and (LF2):

**(LF1)** $g(p) + g(q) \geq g(p \wedge q) + g(p \vee q) \qquad (\forall p, q \in \mathrm{dom}\, g)$,
**(LF2)** $\exists r \in \mathbf{R}$ such that $g(p + \lambda \mathbf{1}) = g(p) + \lambda r \qquad (\forall p \in \mathrm{dom}\, g,\ \forall \lambda \in \mathbf{R})$.

We denote

$$\begin{aligned} \mathcal{L} &= \{g \mid g: \mathbf{R}^V \to \mathbf{R}\cup\{+\infty\},\ \text{polyhedral L-convex}\}, \\ {}_0\mathcal{L} &= \{g \mid g: \mathbf{R}^V \to \mathbf{R}\cup\{+\infty\},\ \text{positively homogeneous polyhedral L-convex}\}. \end{aligned}$$

As is obvious from the definition, polyhedral L-convex functions are quantitative generalization of L-convex polyhedra.

**Theorem 4.23**
(i) *For a function* $g : \mathbf{R}^V \to \{0, +\infty\}$, $g \in \mathcal{L} \iff \mathrm{dom}\, g \in \mathcal{L}_0$. (ii) *For* $g \in \mathcal{L}$, *we have* $\mathrm{dom}\, g \in \mathcal{L}_0$.

### 4.3.1 Axioms for Polyhedral L-convex Functions

We first show that (LF1) and (LF2) are equivalent to local properties ($\mathrm{LF1_{loc}}$) and ($\mathrm{LF2_{loc}}$):

**($\mathbf{LF1_{loc}}$)** $\forall p \in \mathrm{dom}\, g$, $\exists \varepsilon > 0$ such that

$$g(q_1) + g(q_2) \geq g(q_1 \wedge q_2) + g(q_1 \vee q_2) \quad (\forall q_1, q_2 \in \overline{\mathrm{N}_\infty(p,\varepsilon)}).$$

**($\mathbf{LF2_{loc}}$)** $\forall p \in \mathrm{dom}\, g$, $\exists \varepsilon > 0$, $\exists r \in \mathbf{R}$ such that

$$q \in \overline{\mathrm{N}_\infty(p,\varepsilon)},\ q + \lambda\mathbf{1} \in \overline{\mathrm{N}_\infty(p,\varepsilon)}\ (\lambda \in \mathbf{R}) \Longrightarrow g(q + \lambda\mathbf{1}) = g(q) + \lambda r.$$

**Theorem 4.24** *Let* $g : \mathbf{R}^V \to \mathbf{R}\cup\{+\infty\}$ *be a polyhedral convex function with* $\mathrm{dom}\, g \neq \emptyset$. *Then,*
(i) (LF1) $\iff$ ($\mathrm{LF1_{loc}}$). (ii) (LF2) $\iff$ ($\mathrm{LF2_{loc}}$).

**Proof.** (i): We show ($\mathrm{LF1_{loc}}$) $\Rightarrow$ (LF1) only. Assume ($\mathrm{LF1_{loc}}$) for $g$. Then, Theorem 3.17 (i) implies (LS1) for $\mathrm{dom}\, g$. For $p \in \mathrm{dom}\, g$, we denote by $\varepsilon(p)$ the value $\varepsilon$ in ($\mathrm{LF1_{loc}}$) associated with $p$.

Let $p, p' \in \mathrm{dom}\, g$, and recall the definition of $\mathrm{Box}[p,q]$ in Section 3.1. Since $\mathrm{Box}[p,p'] \cap \mathrm{dom}\, g$ is a compact set, there exist $q_1, q_2, \cdots, q_m \in \mathrm{Box}[p,p'] \cap \mathrm{dom}\, g$ ($m \geq 1$) such that $\mathrm{Box}[p,p'] \cap \mathrm{dom}\, g \subseteq \bigcup_{i=1}^m \mathrm{N}_\infty(q_i, \varepsilon(q_i))$. We show the submodular inequality

$$g(p) + g(p') \geq g(p \wedge p') + g(p \vee p') \tag{4.18}$$

by induction on the integer $m$. We may assume $m \geq 2$, since otherwise (4.18) holds immediately.

For $\lambda, \mu \in [0,1]$, put $p(\lambda,\mu) = p + \lambda\{(p \vee p') - p\} + \mu\{(p \wedge p') - p\}$. Then, for $w \in V$ we have

$$p(\lambda,\mu)(w) = \begin{cases} (1-\lambda)p(w) + \lambda p'(w) & (w \in \mathrm{supp}^+(p'-p)), \\ p(w) & (w \in V \setminus \mathrm{supp}(p'-p)), \\ (1-\mu)p(w) + \mu p'(w) & (w \in \mathrm{supp}^-(p'-p)). \end{cases} \tag{4.19}$$

Furthermore,

$$p(\lambda,\mu) \in \mathrm{conv}\{p,\ p',\ p \wedge p',\ p \vee p'\} \subseteq \mathrm{Box}[p,p'] \cap \mathrm{dom}\, g \qquad (\forall \lambda, \mu \in [0,1]).$$

We may assume that $p \in \mathrm{N}_\infty(q_1, \varepsilon(q_1))$. Put $p_* = p(\lambda_*, \mu_*)$, where

$$\lambda_* = \sup\{\lambda \in [0,1] \mid p(\lambda,0) \in \mathrm{N}_\infty(q_1,\varepsilon(q_1))\}, \qquad \mu_* = \sup\{\mu \in [0,1] \mid p(0,\mu) \in \mathrm{N}_\infty(q_1,\varepsilon(q_1))\}.$$

Then, we have $p_* \in \overline{\mathrm{N}_\infty(q_1,\varepsilon(q_1))}$, implying

$$g(p) + g(p_*) \geq g(p \wedge p_*) + g(p \vee p_*). \tag{4.20}$$

In the following, we assume $\lambda_* < 1$ and $\mu_* < 1$, since the other cases can be shown similarly and more easily.

We claim that $\mathrm{Box}[p \vee p_*, p' \vee p_*] \cap \mathrm{dom}\, g \subseteq \bigcup_{i=2}^m \mathrm{N}_\infty(q_i, \varepsilon(q_i))$. Note that $\mathrm{Box}[p \vee p_*, p' \vee p_*] = [p_*, p \vee p']$. Let $\tilde{q} \in [p_*, p \vee p'] \cap \mathrm{dom}\, g$. Since $\lambda_* < 1$, there exists some $v_* \in \mathrm{supp}^+(p' - p)$ such that

$$\varepsilon(q_1) = (1 - \lambda_*)p(v_*) + \lambda_* p'(v_*) - q_1(v_*) = p_*(v_*) - q_1(v_*) \leq \tilde{q}(v_*) - q_1(v_*).$$

This implies $\tilde{q} \notin \mathrm{N}_\infty(q_1, \varepsilon(q_1))$, from which the claim follows.

In the similar way, we can show that

$$\mathrm{Box}[p \wedge p_*, p' \wedge p_*] \cap \mathrm{dom}\, g \subseteq \bigcup_{i=2}^m \mathrm{N}_\infty(q_i, \varepsilon(q_i)), \qquad \mathrm{Box}[p_*, p'] \cap \mathrm{dom}\, g \subseteq \bigcup_{i=2}^m \mathrm{N}_\infty(q_i, \varepsilon(q_i)).$$

Hence, the inductive hypothesis implies that

$$\begin{aligned} &g(p \vee p_*) + g(p' \vee p_*) \geq g(p_*) + g(p \vee p'), \\ &g(p \wedge p_*) + g(p' \wedge p_*) \geq g(p \wedge p') + g(p_*), \\ &g(p_*) + g(p') \geq g(p_* \wedge p') + g(p_* \vee p'), \end{aligned}$$

where we note that $(p \wedge p_*) \wedge (p' \wedge p_*) = p \wedge p'$, $(p \wedge p_*) \vee (p' \wedge p_*) = p_*$. Combining the above inequalities with (4.20), we obtain the submodular inequality (4.18).

(ii) We show (LF2$_{\mathrm{loc}}$) $\Rightarrow$ (LF2) only. Assume (LF2$_{\mathrm{loc}}$) for $g$. Then, $\mathrm{dom}\, g$ satisfies (LS2) by Theorem 3.17 (ii). Let $x_i \in \mathbf{R}^V$ and $\lambda_i \in \mathbf{R}$ $(i = 1, \cdots, k)$ be such that $g(p) = \max_{1 \leq i \leq k}\{\langle p, x_i\rangle + \lambda_i\}$ $(p \in \mathrm{dom}\, g)$. As is shown later, the value $\langle \mathbf{1}, x_i\rangle$ is the same for all $i$. Hence, we have (LF2) for $g$.

We now prove that $\langle \mathbf{1}, x_i\rangle = \langle \mathbf{1}, x_j\rangle$ for any $i, j$. Let $p_0$ be any vector in $\mathrm{dom}\, g$, and define $\psi : \mathbf{R} \to \mathbf{R}$ by $\psi(\mu) = g(p_0 + \mu\mathbf{1})$ $(\mu \in \mathbf{R})$. Then, $\psi$ is piecewise-linear convex. Moreover, we have $\psi'_+(\mu_+) = \max_{1 \leq i \leq k}\langle \mathbf{1}, x_i\rangle$ for a sufficiently large value $\mu_+$, and $\psi'_+(\mu_-) = \min_{1 \leq i \leq k}\langle \mathbf{1}, x_i\rangle$ for a sufficiently small value $\mu_-$. By (LF2$_{\mathrm{loc}}$), we see that $\psi$ has no breaking point, i.e., $\psi$ is a linear function. Hence, we have $\max_{1 \leq i \leq k}\langle \mathbf{1}, x_i\rangle = \psi'_+(\mu_+) = \psi'_+(\mu_-) = \min_{1 \leq i \leq k}\langle \mathbf{1}, x_i\rangle$. □

**Theorem 4.25** *Let $a : V \to \mathbf{R} \cup \{-\infty\}$ and $b : V \to \mathbf{R} \cup \{+\infty\}$ be such that $a \leq b$. Also, let $g : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ be a function with $\mathrm{dom}\, g = [a, b]$.*
(i) *$g$ is submodular if and only if for any $p \in \mathbf{R}^V$ and $u, v \in V$ $(u \neq v)$, we have*

$$g(p + \lambda\chi_u) + g(p + \mu\chi_v) \geq \ g(p) + g(p + \lambda\chi_u + \mu\chi_v) \quad (\forall\lambda, \mu \geq 0). \tag{4.21}$$

(ii) *The condition (4.21) can be replaced by the following:*

$$\begin{aligned} g(p + \lambda\chi_u) + g(p + \mu\chi_v) \ &\geq \ g(p) + g(p + \lambda\chi_u + \mu\chi_v) \\ &(0 \leq \forall\lambda \leq p_{l-1} - p_l, \ 0 \leq \mu \leq p_{m-1} - p_m), \end{aligned} \tag{4.22}$$

*where $p_1 > p_2 > \cdots > p_k$ are distinct values in $\{p(v)\}_{v \in V}$, and $l$, $m$ are such that $p(u) = p_l$, $p(v) = p_m$.*

**Proof.** (i): Let $p, q \in [a, b]$. Note that $p \wedge q, p \vee q \in [a, b]$. We show the submodular inequality for $p, q$ by induction on the cardinality of $\mathrm{supp}(p - q)$. Assume that $\mathrm{supp}^+(p - q) \neq \emptyset$ and $\mathrm{supp}^-(p - q) \neq \emptyset$, since otherwise the submodular inequality holds obviously. Let $v \in \mathrm{supp}^-(p - q)$, $\lambda = q(v) - p(v)$. Since $p \wedge q + \lambda\chi_v \in [a, b]$, the inductive assumption and (4.21) imply that

$$g(p) - g(p \wedge q) \geq g(p + \lambda\chi_v) - g(p \wedge q + \lambda\chi_v) \geq g(p \vee q) - g(q).$$

(ii): Assume that (4.22) holds. We fix $\lambda, \mu \geq 0$. Let $l'$, $m'$ $(1 \leq l' \leq l, 1 \leq m' \leq m)$ be such that $p_{l'-1} \geq p_l + \lambda \geq p_{l'}$ and $p_{m'-1} \geq p_m + \mu \geq p_{m'}$. Put $g(\lambda', \mu') = g(p + \lambda'\chi_u + \mu'\chi_v)$ $(\lambda', \mu' \in \mathbf{R}_+)$, and

$$\lambda_i = p_i - p_l \quad (i = l', l'+1, \cdots, l), \qquad \mu_j = p_j - p_m \quad (j = m', m'+1, \cdots, m).$$

Then, (4.22) implies that

$$\begin{array}{rcll} g(\lambda_{i-1}, \mu_j) + g(\lambda_i, \mu_{j-1}) & \geq & g(\lambda_{i-1}, \mu_{j-1}) + g(\lambda_i, \mu_j) & (i = l'+1, \cdots, l,\ j = m'+1, \cdots, m), \\ g(\lambda_{i-1}, \mu) + g(\lambda_i, \mu_{m'}) & \geq & g(\lambda_{i-1}, \mu_{m'}) + g(\lambda_i, \mu) & (i = l'+1, \cdots, l), \\ g(\lambda_{l'}, \mu_j) + g(\lambda, \mu_{j-1}) & \geq & g(\lambda_{l'}, \mu_{j-1}) + g(\lambda, \mu_j) & (j = m'+1, \cdots, m), \\ g(\lambda_{l'}, \mu) + g(\lambda, \mu_{m'}) & \geq & g(\lambda_{l'}, \mu_{m'}) + g(\lambda, \mu), & \end{array}$$

from which (4.21) follows. □

### 4.3.2 Fundamental Properties of Polyhedral L-convex Functions

In this section, we show various properties of polyhedral L-convex functions.

**Lemma 4.26** *Let $g \in \mathcal{L}$. Then,*

$$g(p) + g(q) \geq g(p \vee (q - \lambda\mathbf{1})) + g((p + \lambda\mathbf{1}) \wedge q) \quad (\forall p, q \in \mathrm{dom}\, g,\ \forall\lambda \in \mathbf{R}).$$

*In particular, we have*

$$g(p) + g(q) \geq g(p + \lambda\chi_X) + g(q - \lambda\chi_X) \qquad (\forall p, q \in \mathrm{dom}\, g,\ 0 \leq \forall\lambda \leq \lambda_1 - \lambda_2),$$

*where*

$$\lambda_1 = \max_{v \in V}\{q(v) - p(v)\}, \quad X = \{v \in V \mid q(v) - p(v) = \lambda_1\}, \quad \lambda_2 = \max_{v \in V \setminus X}\{q(v) - p(v)\}. \tag{4.23}$$

**Proof.** The first inequality can be obtained as follows:

$$\begin{array}{rcl} \text{LHS} = g(p) + g(q - \lambda\mathbf{1}) + \lambda r & \geq & g(p \vee (q - \lambda\mathbf{1})) + g(p \wedge (q - \lambda\mathbf{1})) + \lambda r \\ & = & g(p \vee (q - \lambda\mathbf{1})) + g(\{p \wedge (q - \lambda\mathbf{1})\} + \lambda\mathbf{1}) \ = \ \text{RHS}. \end{array}$$

The latter is obvious from this inequality since $p \vee \{q - (\lambda_1 - \lambda)\mathbf{1}\} = p + \lambda\chi_X$, $(p + (\lambda_1 - \lambda)\mathbf{1}) \wedge q = q - \lambda\chi_X$ for any $\lambda$ with $0 \leq \lambda \leq \lambda_1 - \lambda_2$. □

Global optimality of a polyhedral L-convex function is characterized by local optimality.

**Theorem 4.27** *Let $g \in \mathcal{L}$ and $p \in \mathrm{dom}\, g$. Then, $g(p) \leq g(q)$ $(\forall q \in \mathbf{R}^V)$ if and only if $g'(p; \chi_X) \geq 0$ $(\forall X \subseteq V)$ and $g'(p; \chi_V) = r = 0$, where $r$ is in* (LF2)*.*

**Proof.** We show the sufficiency by contradiction. Suppose that there exists some $q \in \mathbf{R}^V$ with $g(q) < g(p)$, and assume that $q$ minimizes the number of distinct values in $\{p(v) - q(v)\}_{v \in V}$ of all such vectors. Define $\lambda_1, \lambda_2$, and $X$ $(\subseteq V)$ by (4.23). Since $g'(p; \chi_V) = r = 0$, we have $X \neq V$ and $\lambda_2 > -\infty$. By Lemma 4.26, we obtain

$$g(p) + g(q) \geq g(p + (\lambda_1 - \lambda_2)\chi_X) + g(q - (\lambda_1 - \lambda_2)\chi_X). \tag{4.24}$$

Put $q' = q - (\lambda_1 - \lambda_2)\chi_X$. Since $g'(p; \chi_X) \geq 0$, (4.24) implies $g(q') \leq g(q) < g(p)$. This inequality, however, is a contradiction since the number of distinct values in $\{p(v) - q'(v)\}_{v \in V}$ is less than that of $\{p(v) - q(v)\}_{v \in V}$. □

Given a set function $\rho : 2^V \to \mathbf{R} \cup \{+\infty\}$, we define $g_\rho : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ by

$$g_\rho(p) = \sum_{j=1}^{k-1} (p_j - p_{j+1})\rho(V_j) + p_k \rho(V_k), \tag{4.25}$$

where $p_1 > p_2 > \cdots > p_k$ are distinct values in $\{p(v)\}_{v \in V}$, and $V_j = \{v \in V \mid p(v) \geq p_j\}$ $(j = 1, \cdots, k)$. The function $g_\rho$ is called the *Lovász extension* of $\rho$.

**Theorem 4.28** *If $\rho \in \mathcal{S}$, then $g_\rho(p) = \sup\{\langle p, x\rangle \mid x \in \mathrm{B}(\rho)\}$ $(\forall p \in \mathbf{R}^V)$.*

**Proof.** See [17]. □

**Theorem 4.29 (Lovász [25])** *Let $\rho : 2^V \to \mathbf{R} \cup \{+\infty\}$ be a function such that $\rho(\emptyset) = 0$ and $\rho(V) < +\infty$. Then, $\rho \in \mathcal{S} \iff g_\rho$ is convex.*

Directional derivative functions and subdifferentials of a polyhedral L-convex function have nice structures such as M/L-convexity, and they can be explicitly described by certain submodular functions (cf. Theorem 4.30 (i)).

**Theorem 4.30** *Let $g \in \mathcal{L}$ and $p \in \mathrm{dom}\, g$.*
(i) *The function $\rho_p : 2^V \to \mathbf{R} \cup \{+\infty\}$ defined by*

$$\rho_p(X) = g'(p; \chi_X) \qquad (X \subseteq V) \tag{4.26}$$

*satisfies $\rho_p(\emptyset) = 0$, $-\infty < \rho_p(V) < +\infty$, and the submodular inequality (3.2), i.e., $\rho_p \in \mathcal{S}$.*
(ii) *We have $g'(p; \cdot) = g_{\rho_p}$ and $g'(p; \cdot) \in {}_0\mathcal{L}$.*

**Proof.** We show $g'(p; \cdot) \in {}_0\mathcal{L}$ only. The claim (i) is an immediate corollary of (ii), and the equation $g'(p; \cdot) = g_{\rho_p}$ will be shown later in Theorem 4.34.

Let $p \in \mathrm{dom}\, g$. For $q_1, q_2 \in \mathbf{R}^V$ and a sufficiently small $\mu > 0$, we have

$$\begin{aligned} g'(p; q_1) + g'(p; q_2) &= \{g(p + \mu q_1) - g(p)\}/\mu + \{g(p + \mu q_2) - g(p)\}/\mu \\ &\geq \{g(p + \mu(q_1 \wedge q_2)) - g(p)\}/\mu + \{g(p + \mu(q_1 \vee q_2)) - g(p)\}/\mu \\ &= g'(p; q_1 \wedge q_2) + g'(p; q_1 \vee q_2). \end{aligned}$$

Hence, (LF1) holds for $g'(p; \cdot)$. (LF2) for $g'(p; \cdot)$ can be shown similarly and the proof is omitted. □

M-convexity appears in subdifferentials of a polyhedral L-convex function.

**Theorem 4.31** *Let $g \in \mathcal{L}$ and $p \in \mathrm{dom}\, g$.*
(i) *$\partial g(p) \in \mathcal{M}_0$ and $\partial g(p)$ is represented as*

$$\partial g(p) = \mathrm{B}(\rho_p) = \{x \in \mathbf{R}^V \mid x(X) \leq g'(p; \chi_X)\ (\forall X \subseteq V),\ x(V) = g'(p; \chi_V)\}, \tag{4.27}$$

*where $\rho_p : 2^V \to \mathbf{R} \cup \{+\infty\}$ is defined by (4.26).*
(ii) *For any $q \in \mathbf{R}^V$ we have $g(p + q) - g(p) \geq \sup\{\langle q, x\rangle \mid x \in \partial g(p)\} = g_{\rho_p}(q)$.*

**Proof.** The equation (4.27) follows from Theorems 4.5 and 4.27. The second claim is immediate from (4.27) and Theorem 4.30 (i). The claim (ii) is immediate from (i) and Theorem 4.28. □

The next theorem shows that each face of the epigraph of a polyhedral L-convex function is an L-convex polyhedron when it is projected to $\mathbf{R}^V$. The proof is obvious and therefore omitted.

**Theorem 4.32** *For $g \in \mathcal{L}$ and $x \in \mathbf{R}^V$, we have* $\arg\min g[-x] \in \mathcal{L}_0$ *if* $\inf g[-x] > -\infty$.

The class of polyhedral L-convex functions are closed under various fundamental operations.

**Theorem 4.33** *Let $g, g_1, g_2 \in \mathcal{L}$.*
(1) *For $x \in \mathbf{R}^V$, the function $g[-x] : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ given by (4.4) is polyhedral L-convex.*
(2) *For $a \in \mathbf{R}^V$ and $\beta \in \mathbf{R}$, the function $g(a + \beta p)$ is polyhedral L-convex in $p$.*
(3) *For any $U \subseteq V$, the function $g^U : \mathbf{R}^U \to \mathbf{R} \cup \{\pm\infty\}$ given by (4.17) is polyhedral L-convex if $g^U > -\infty$.*
(4) *For $\psi_v \in \mathcal{C}^1$ $(v \in V)$, the function $\tilde{g} : \mathbf{R}^V \to \mathbf{R} \cup \{\pm\infty\}$ defined by*

$$\tilde{g}(p) = \inf_{q \in \mathbf{R}^V} \{g(q) + \sum_{v \in V} \psi_v(p(v) - q(v))\} \quad (p \in \mathbf{R}^V)$$

*is polyhedral L-convex if $\tilde{g} > -\infty$ and* $\mathrm{dom}\,\tilde{g} \neq \emptyset$.
(5) *For any $\gamma \in \mathcal{T}$, the restriction $g_{\mathrm{D}(\gamma)} : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ of $g$ to $\mathrm{D}(\gamma)$ is polyhedral L-convex if* $\mathrm{dom}\, g \cap \mathrm{D}(\gamma) \neq \emptyset$.
(6) $g_1 + g_2 \in \mathcal{L}$ *if* $\mathrm{dom}\, g_1 \cap \mathrm{dom}\, g_2 \neq \emptyset$.

A more general and stronger transformation, called "network induction," is explained in Section 7.3.

### 4.3.3 Positively Homogeneous Polyhedral L-convex Functions

This section clarifies the relationship of positively homogeneous polyhedral L-convex functions with submodular functions, and with M-convex polyhedra.

For a positively homogeneous polyhedral convex function $g : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ with $\mathbf{0} \in \mathrm{dom}\, g$, define a set function $\rho_g : 2^V \to \mathbf{R} \cup \{+\infty\}$ by

$$\rho_g(X) = g'(\mathbf{0}; \chi_X) \ (= g(\chi_X)) \qquad (X \subseteq V). \tag{4.28}$$

Recall the definition of $g_\rho$ in (4.25).

**Theorem 4.34**
(i) *For $g \in {}_0\mathcal{L}$, we have $\rho_g \in \mathcal{S}$ and $g_{\rho_g} = g$.* (ii) *For $\rho \in \mathcal{S}$, we have $g_\rho \in {}_0\mathcal{L}$ and $\rho_{g_\rho} = \rho$.*
(iii) *The mappings $g \mapsto \rho_g$ $(g \in {}_0\mathcal{L})$ and $\rho \mapsto g_\rho$ $(\rho \in \mathcal{S})$ provide a one-to-one correspondence between ${}_0\mathcal{L}$ and $\mathcal{S}$, and are the inverse of each other.*

**Proof.** (i): (LF1) for $g$ yields $\rho_g \in \mathcal{S}$. We see from Theorem 4.31 (ii) that $g \geq g_{\rho_g}$. On the other hand, it follows from Theorem 4.2 that

$$g_{\rho_g}(p) = \sum_{j=1}^{k-1} (p_j - p_{j+1}) g(\chi_{V_j}) + p_k g(\chi_{V_k}) \geq g(\sum_{j=1}^{k-1} (p_j - p_{j+1}) \chi_{V_j} + p_k \chi_{V_k}) = g(p),$$

where $p_j$ and $V_j$ $(j = 1, \cdots, k)$ are defined as in the Lovász extension (4.25). Hence, we have $g = g_{\rho_g}$.

(ii): We show (LF1) for $g_\rho$ only. The other claims are obvious.

First assume that $\rho < +\infty$. By Theorem 4.25, it suffices to show that (4.22) holds for any $p \in \mathbf{R}^V$ and $u, v \in V$ $(u \neq v)$, where $g = g_\rho$. Define $p_j$ and $V_j$ $(j = 1, \cdots, k)$ as in the Lovász extension (4.25).

Let $l$, $m$ be such that $p(u) = p_l$, $p(v) = p_m$, and $\lambda \in [0, p_{l-1} - p_l]$, $\mu \in [0, p_{m-1} - p_m]$. From the definition of $g_\rho$ we have

$$\begin{aligned} g_\rho(p + \lambda\chi_u) &= g_\rho(p) + \lambda\{\rho(V_{l-1} \cup \{u\}) - \rho(V_{l-1})\}, \\ g_\rho(p + \mu\chi_v) &= g_\rho(p) + \mu\{\rho(V_{m-1} \cup \{v\}) - \rho(V_{m-1})\}. \end{aligned}$$

(Case 1: $l \neq m$): We may assume that $l < m$. Then, (4.22) holds with equality:

$$\begin{aligned} g_\rho(p + \lambda\chi_u + \mu\chi_v) &= g_\rho(p) + \lambda\{\rho(V_{l-1} \cup \{u\}) - \rho(V_{l-1})\} + \mu\{\rho(V_{m-1} \cup \{v\}) - \rho(V_{m-1})\} \\ &= g_\rho(p + \lambda\chi_u) + g_\rho(p + \mu\chi_v) - g_\rho(p). \end{aligned}$$

(Case 2: $l = m$): We may assume that $\lambda \geq \mu$. Then,

$$\begin{aligned} g_\rho(p + \lambda\chi_u + \mu\chi_v) &= g_\rho(p) + \lambda\{\rho(V_{l-1} \cup \{u\}) - \rho(V_{l-1})\} + \mu\{\rho(V_{l-1} \cup \{u, v\}) - \rho(V_{l-1} \cup \{u\})\} \\ &\leq g_\rho(p) + \lambda\{\rho(V_{l-1} \cup \{u\}) - \rho(V_{l-1})\} + \mu\{\rho(V_{l-1} \cup \{v\}) - \rho(V_{l-1})\} \\ &= g_\rho(p + \lambda\chi_u) + g_\rho(p + \mu\chi_v) - g_\rho(p), \end{aligned}$$

where the inequality is by the submodularity of $\rho$. Hence, (4.22) follows.

Next, we consider the general case where $\rho$ may take the value $+\infty$. Let $x \in \mathrm{B}(\rho)$. For each positive integer $k$, we define $\rho_k : 2^V \to \mathbf{R}$ by

$$\rho_k(X) = \min_{Y \subseteq X}\{\rho(X \setminus Y) + x(Y) + k|Y|\} \quad (X \subseteq V).$$

Note that $\rho_k$ is the submodular function associated with $\{y \in \mathrm{B}(\rho) \mid y(v) \leq x(v) + k \ (v \in V)\}$, which is a bounded M-convex polyhedron. Hence, we have $\rho_k < +\infty$ and $g_{\rho_k}$ satisfies (LF1). Since $\rho(X) = \lim_{k\to\infty} \rho_k(X)$ $(\forall X \subseteq V)$, it holds that $g_\rho(p) = \lim_{k\to\infty} g_{\rho_k}(p)$ $(p \in \mathbf{R}^V)$ and therefore $g_\rho$ also satisfies (LF1).

(iii): Clear from (i) and (ii). □

From Theorem 4.34, we see that a polyhedral convex function is positively homogeneous polyhedral L-convex if and only if it is the Lovász extension of a submodular set function.

**Corollary 4.35** ${}_0\mathcal{L} = \{g_\rho \mid \rho \in \mathcal{S}\}$.

As immediate consequences of Theorems 3.3, 4.3, and 4.34, we obtain a one-to-one correspondense between positively homogeneous polyhedral L-convex functions and M-convex polyhedra. Recall the notation $S_g$ and $\delta_S^*$ in (4.2) and (4.3).

**Theorem 4.36**
(i) *For* $g \in {}_0\mathcal{L}$, *we have* $S_g \in \mathcal{M}_0$ *and* $\delta_{S_g}^* = g$. (ii) *For* $B \in \mathcal{M}_0$, *we have* $\delta_B^* \in {}_0\mathcal{L}$ *and* $S_{\delta_B^*} = B$.
(iii) *The mappings* $g \mapsto S_g$ $(g \in {}_0\mathcal{L})$ *and* $B \mapsto \delta_B^*$ $(B \in \mathcal{M}_0)$ *provide a one-to-one correspondence between* ${}_0\mathcal{L}$ *and* $\mathcal{M}_0$, *and are the inverse of each other.*

### 4.3.4 Polyhedral L$^\natural$-convex Functions

Due to the property (LF2), a polyhedral L-convex function loses no information when restricted to a hyperplane $\{p \in \mathbf{R}^V \mid p(v) = 0\}$ for any $v \in V$. We call a polyhedral convex function $g : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ *L$^\natural$-convex* if the function $\tilde{g} : \mathbf{R}^{\tilde{V}} \to \mathbf{R} \cup \{+\infty\}$ defined by

$$\tilde{g}(p_0, p) = g(p - p_0\mathbf{1}) \quad ((p_0, p) \in \mathbf{R}^{\tilde{V}})$$

is polyhedral L-convex, where $\tilde{V} = \{v_0\} \cup V$. We see that polyhedral L$^\natural$-convex functions are essentially the same as polyhedral L-convex functions, while the class of polyhedral L$^\natural$-convex functions properly contains that of polyhedral L-convex functions. The concept of L$^\natural$-convexity was originally introduced in [18] as a concept for functions defined over the integer lattice.

**Example 4.37 (separable-convex functions)** Let $D \subseteq \mathbf{R}^V$ be an L$^\natural$-convex polyhedron. For any convex functions $g_v \in \mathcal{C}^1$ $(v \in V)$, the function $g : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ defined by

$$g(p) = \begin{cases} \sum_{v \in V} g_v(p(v)) & (p \in D), \\ +\infty & (p \notin D) \end{cases}$$

is polyhedral L$^\natural$-convex if $\mathrm{dom}\, g \neq \emptyset$. □

**Remark 4.38** There exists no function which is both polyhedral M-convex and L-convex, i.e., $\mathcal{M} \cap \mathcal{L} = \emptyset$ (Proof: Theorems 3.2 and 4.6 (ii) imply that $x(V) = y(V)$ for any $x, y \in \mathrm{dom}\, f$ and $f \in \mathcal{M}$, whereas (LF2) implies that $x + \lambda \mathbf{1} \in \mathrm{dom}\, f$ for any $\lambda \in \mathbf{R}$.) On the other hand, the classes of polyhedral M$^\natural$/L$^\natural$-convex functions have a nonempty intersection. Indeed, Examples 4.20 and 4.37 show that any separable-convex function $g(p) = \sum_{v \in V} g_v(p(v))$ $(g_v \in \mathcal{C}^1)$ is polyhedral M$^\natural$-convex as well as polyhedral L$^\natural$-convex. □

From the definition of polyhedral L$^\natural$-convex functions, every property of polyhedral L-convex functions can be restated in terms of polyhedral L$^\natural$-convex functions, and vice versa. We show two properties below, which seem more natural when restated in terms of polyhedral L$^\natural$-convex functions.

**Theorem 4.39** *Let $g : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ be a polyhedral L$^\natural$-convex function.*
(i) *For any $a : V \to \mathbf{R} \cup \{-\infty\}$ and $b : V \to \mathbf{R} \cup \{+\infty\}$, the restriction $g_{[a,b]}$ of $g$ to $[a, b]$ is polyhedral L$^\natural$-convex if* $\mathrm{dom}\, g \cap [a, b] \neq \emptyset$.
(ii) *For any $U \subseteq V$, the function $g_U : \mathbf{R}^U \to \mathbf{R} \cup \{+\infty\}$ given by (4.14) is polyhedral L$^\natural$-convex if* $\mathrm{dom}\, g_U \neq \emptyset$.

**Proof.** (i) is immediate from the translation of Theorem 4.33 (5), and (ii) is a corollary of (i). □

# 5 Conjugacy and Characterizations

Polyhedral M-convex and L-convex functions are conjugate to each other.

**Theorem 5.1** *For $f \in \mathcal{M}$ and $g \in \mathcal{L}$, we have $f^\bullet \in \mathcal{L}$ and $g^\bullet \in \mathcal{M}$. More specifically, the mappings $f \mapsto f^\bullet$ $(f \in \mathcal{M})$ and $g \mapsto g^\bullet$ $(g \in \mathcal{L})$ provide a one-to-one correspondence between $\mathcal{M}$ and $\mathcal{L}$, and are the inverse of each other.*

Polyhedral M/L-convex functions are characterized by local polyhedral structures such as directional derivative functions, subdifferentials, and the sets of minimizers.

**Theorem 5.2** *Let $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ be a polyhedral convex function with* $\mathrm{dom}\, f \neq \emptyset$. *Then,*

$$\begin{aligned} \text{(i) } f \in \mathcal{M} \quad &\Longleftrightarrow \quad \text{(ii) } f'(x;\cdot) \in {}_0\mathcal{M} \ (\forall x \in \mathrm{dom}\, f) \quad \Longleftrightarrow \quad \text{(iii) } \partial f(x) \in \mathcal{L}_0 \ (\forall x \in \mathrm{dom}\, f) \\ &\Longleftrightarrow \quad \text{(iv) } \arg\min f[-p] \in \mathcal{M}_0 \ (\forall p \in \mathbf{R}^V \text{ with } \inf f[-p] > -\infty). \end{aligned}$$

**Theorem 5.3** *Let $g : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ be a polyhedral convex function with* $\mathrm{dom}\, g \neq \emptyset$. *Then,*

$$\begin{aligned} \text{(i) } g \in \mathcal{L} \quad &\Longleftrightarrow \quad \text{(ii) } g'(p;\cdot) \in {}_0\mathcal{L} \ (\forall p \in \mathrm{dom}\, g) \quad \Longleftrightarrow \quad \text{(iii) } \partial g(p) \in \mathcal{M}_0 \ (\forall p \in \mathrm{dom}\, g) \\ &\Longleftrightarrow \quad \text{(iv) } \arg\min g[-x] \in \mathcal{L}_0 \ (\forall x \in \mathbf{R}^V \text{ with } \inf g[-x] > -\infty). \end{aligned}$$

In the following, we first prove Theorem 5.3, then Theorem 5.1, and finally Theorem 5.2.

**Proof of Theorem 5.3** The implication (i) $\Rightarrow$ (ii) is by Theorems 4.4 and 4.30 (ii). To show the reverse implication, we assume (ii) for $g$. For any $p \in \mathrm{dom}\, g$, there exists $\varepsilon > 0$ such that $g(q) - g(p) = g'(p; q-p)$ $(\forall q \in \overline{\mathrm{N}_\infty(p, \varepsilon)})$. Hence, $g$ satisfies ($\mathrm{LF1_{loc}}$) and ($\mathrm{LF2_{loc}}$), yielding (i) by Theorem 4.24.

The equivalence (ii) $\Longleftrightarrow$ (iii) is by Theorems 4.4 and 4.36. The implication (i) $\Rightarrow$ (iv) follows from Theorem 4.32. We conclude the proof of Theorem 5.3 by showing that (iv) $\Rightarrow$ (ii) holds.

**Lemma 5.4** *Let $g : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ be positively homogeneous polyhedral convex with* $\mathrm{dom}\, g \neq \emptyset$. *Suppose that* $\arg\min g[-x]$ *is an L-convex cone for any $x \in \mathbf{R}^V$ with* $\inf g[-x] > -\infty$. *Then, $g \in {}_0\mathcal{L}$.*

**Proof.** Define $\rho_g : 2^V \to \mathbf{R} \cup \{+\infty\}$ by (4.28). Since $\mathbf{0} \in \mathrm{dom}\, g$, there exists some $x \in \mathbf{R}^V$ with $\mathbf{0} \in \arg\min g[-x]$. By (LS2) for $\arg\min g[-x]$, we have $\mathbf{1} \in \arg\min g[-x] \subseteq \mathrm{dom}\, g$, i.e., $\rho_g(V) = g(\mathbf{1}) < +\infty$. As is shown below, we have $g = g_{\rho_g}$, where $g_{\rho_g}$ is defined by (4.25) with $\rho = \rho_g$. Thus, we have $\rho_g \in \mathcal{S}$ by Theorem 4.29, which, together with Corollary 4.35, implies $g \in {}_0\mathcal{L}$.

We now prove $g = g_{\rho_g}$. Note that $g(p) \le g_{\rho_g}(p)$ $(p \in \mathbf{R}^V)$ by Theorem 4.2. This implies that $g(p) = g_{\rho_g}(p) = +\infty$ if $p \notin \mathrm{dom}\, g$. Therefore, we may assume that $p \in \mathrm{dom}\, g$. Then, there exists some $x \in \mathbf{R}^V$ such that $p \in \arg\min g[-x]$. By Theorem 3.22 (ii), there exists some chain $\mathcal{F}_0$ $(\subseteq 2^V)$ and $(\lambda_X \mid X \in \mathcal{F}_0)$ $(\subseteq \mathbf{R})$ such that $\mathcal{F}_0 \subseteq \{X \subseteq V \mid \chi_X \in \arg\min g[-x]\}$, $\lambda_X \ge 0$ $(\forall X \in \mathcal{F}_0 \setminus \{V\})$ and $p = \sum_{X \in \mathcal{F}_0} \lambda_X \chi_X$. By the linearity of $g$ over $\arg\min g[-x]$ and by (4.25), we have $g(p) = \sum_{X \in \mathcal{F}_0} \lambda_X g(\chi_X) = g_{\rho_g}(p)$. □

We now prove (iv) $\Rightarrow$ (ii). For any $p \in \mathrm{dom}\, g$, we denote $g'_p = g'(p;\cdot)$ for notational simplicity. By Lemma 5.4, we have only to prove that $\arg\min g'_p[-x]$ is an L-convex cone for any $p \in \mathrm{dom}\, g$ and any $x \in \mathbf{R}^V$ with $\inf g'_p[-x] > -\infty$.

Let $p \in \mathrm{dom}\, g$, and $x \in \mathbf{R}^V$ be such that $\inf g'_p[-x] > -\infty$. Then, we have $\inf g[-x] > -\infty$ and $\arg\min g[-x] \in \mathcal{L}_0$. Let $\gamma \in \mathcal{T}$ be such that $\mathrm{D}(\gamma) = \arg\min g[-x]$. Since $\arg\min g'_p[-x]$ is the tangent cone of $\arg\min g[-x]$ at $p$, it can be represented as

$$\arg\min g'_p[-x] = \{q \in \mathbf{R}^V \mid q(v) - q(u) \leq 0 \ (\forall (u,v) \in A_p)\},$$

where $A_p = \{(u,v) \mid u, v \in V,\ p(v) - p(u) = \gamma(u,v)\}$. Since $A_p$ is transitive, $\arg\min g'_p[-x]$ is an L-convex cone by Theorem 3.21. Thus, Theorem 5.3 is proven.

The following lemma is needed for the proof of Theorem 5.1.

**Lemma 5.5** *Let $g \in \mathcal{L}$, and $x, y \in \mathbf{R}^V$ be such that $\inf g[-x] > -\infty$ and $\inf g[-y] > -\infty$. Then, for any $u \in \mathrm{supp}^+(x-y)$, there exists $v \in \mathrm{supp}^-(x-y)$ such that*

$$p(v) - p(u) \leq q(v) - q(u) \quad (\forall p \in \arg\min g[-x], \forall q \in \arg\min g[-y]). \tag{5.1}$$

**Proof.** First we note that $x(V) = y(V) = r$, where $r \in \mathbf{R}$ is the value in (LF2) for $g$. Let $u \in \mathrm{supp}^+(x-y)$. We have $\arg\min g[-x], \arg\min g[-y] \in \mathcal{L}_0$ by Theorem 4.32. Hence, it suffices to show the following: there exists some $v \in \mathrm{supp}^-(x-y)$ such that $p(v) \leq q(v)$ ($\forall p \in D_x,\ \forall q \in D_y$), where

$$D_x = \{p \mid p \in \arg\min g[-x],\ p(u) = 0\}, \quad D_y = \{q \mid q \in \arg\min g[-y],\ q(u) = 0\}.$$

Assume, to the contrary, that for any $v \in \mathrm{supp}^-(x-y)$, there exists a pair of vectors $p_v \in D_x$, $q_v \in D_y$ such that $p_v(v) > q_v(v)$. Set $p_* = \bigvee\{p_v \mid v \in \mathrm{supp}^-(x-y)\}$, $q_* = \bigwedge\{q_v \mid v \in \mathrm{supp}^-(x-y)\}$. Then, we have $p_* \in D_x$, $q_* \in D_y$, and $p_*(v) > q_*(v)$ ($\forall v \in \mathrm{supp}^-(x-y)$). Let $\lambda_0 = \min\{p_*(v) - q_*(v) \mid v \in \mathrm{supp}^+(p_* - q_*)\}$, and $\lambda \in [0, \lambda_0]$. Putting

$$\begin{aligned} p' &= (p_* - \lambda\mathbf{1}) \vee q_* = \begin{cases} p_*(v) - \lambda & (v \in \mathrm{supp}^+(p_* - q_*)), \\ q_*(v) & (v \in V \setminus \mathrm{supp}^+(p_* - q_*)), \end{cases} \\ q' &= p_* \wedge (q_* + \lambda\mathbf{1}) = \begin{cases} q_*(v) + \lambda & (v \in \mathrm{supp}^+(p_* - q_*)), \\ p_*(v) & (v \in V \setminus \mathrm{supp}^+(p_* - q_*)), \end{cases} \end{aligned}$$

we have

$$g(p_*) + g(q_*) \geq g(p') + g(q') \tag{5.2}$$

by Lemma 4.26. Since $\mathrm{supp}^-(x-y) \subseteq \mathrm{supp}^+(p_* - q_*)$, we obtain

$$\begin{aligned} \langle p', x\rangle + \langle q', y\rangle - \langle p_*, x\rangle - \langle q_*, y\rangle &= \lambda \sum\{y(v) - x(v) \mid v \in \mathrm{supp}^+(p_* - q_*)\} \\ &\quad + \sum\{(q_*(v) - p_*(v))(x(v) - y(v)) \mid v \in V \setminus \mathrm{supp}^+(p_* - q_*)\} \\ &\geq \lambda \sum\{y(v) - x(v) \mid v \in V \setminus \{u\}\} \ = \ \lambda\{x(u) - y(u)\} \ > \ 0. \end{aligned}$$

Combining this inequality with (5.2), we have $g[-x](p') + g[-y](q') < g[-x](p_*) + g[-y](q_*)$, which is a contradiction since $p_* \in \arg\min g[-x]$, $q_* \in \arg\min g[-y]$. □

**Proof of Theorem 5.1** First recall Theorem 4.1.

[$g \in \mathcal{L} \Rightarrow g^\bullet \in \mathcal{M}$] Let $x, y \in \mathrm{dom}\, g^\bullet$. Then, $\inf g[-x] > -\infty$ and $\inf g[-y] > -\infty$. By Lemma 5.5, for any $u \in \mathrm{supp}^+(x-y)$ there exists $v \in \mathrm{supp}^-(x-y)$ satisfying the inequality (5.1), which implies

$$\begin{aligned} &(g^\bullet)'(x; v, u) + (g^\bullet)'(y; u, v) \\ &\quad = \sup\{p(v) - p(u) \mid p \in \arg\min g[-x]\} + \sup\{q(u) - q(v) \mid q \in \arg\min g[-y]\} \ \leq \ 0, \end{aligned}$$

where the equality is by Theorem 4.4. Hence, we have (M-EXC$'$) for $g^\bullet$ and therefore (M-EXC) by Theorem 4.10.

[$f \in \mathcal{M} \Rightarrow f^\bullet \in \mathcal{L}$] If $f \in \mathcal{M}$, then Theorems 4.5 and 4.16 imply $\partial f^\bullet(p) = \arg\min f[-p] \in \mathcal{M}_0$ ($\forall p \in \mathrm{dom}\, f^\bullet$), which, together with Theorem 5.3, yields $f^\bullet \in \mathcal{L}$. $\square$

**Proof of Theorem 5.2** The equivalence (ii) $\iff$ (iii) is by Theorems 4.4 and 4.19. By Theorems 4.5 and 5.1, the equivalence (i) $\iff$ (iii) $\iff$ (iv) is rewritten as follows in terms of $f^\bullet$:

$$\begin{aligned} f^\bullet \in \mathcal{L} \quad &\iff \quad \arg\min f^\bullet[-x] \in \mathcal{L}_0 \ (\forall x \in \mathbf{R}^V \text{ with } \inf f^\bullet[-x] > -\infty) \\ &\iff \quad \partial f^\bullet(p) \in \mathcal{M}_0 \ (\forall p \in \mathrm{dom}\, f^\bullet), \end{aligned}$$

which is already shown in Theorem 5.3. $\square$

## 6 Relationship with M/L-convex Functions over the Integer Lattice

The concepts of M-convexity and L-convexity were originally introduced for functions defined over the integer lattice [30, 31, 33]. In this section, we present the relationship of M/L-convexity over the integer lattice and polyhedral M/L-convexity.

A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is called *discrete M-convex* if $\mathrm{dom}_{\mathbf{Z}} f \neq \emptyset$ and it satisfies

**(M-EXC[Z])** $\forall x, y \in \mathrm{dom}_{\mathbf{Z}} f$, $\forall u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$ such that

$$f(x) + f(y) \geq f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v),$$

where $\mathrm{dom}_{\mathbf{Z}} f = \{x \in \mathbf{Z}^V \mid -\infty < f(x) < +\infty\}$. On the other hand, a function $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is called *discrete L-convex* if $\mathrm{dom}_{\mathbf{Z}} g \neq \emptyset$ and it satisfies

**(LF1[Z])** $\quad g(p) + g(q) \geq g(p \wedge q) + g(p \vee q) \qquad (\forall p, q \in \mathrm{dom}_{\mathbf{Z}} g)$,

**(LF2[Z])** $\quad \exists r \in \mathbf{R}$ such that $g(p + \lambda \mathbf{1}) = g(p) + \lambda r \quad (\forall p \in \mathrm{dom}_{\mathbf{Z}} g,\ \forall \lambda \in \mathbf{Z})$.

We denote by $\mathcal{M}[\mathbf{Z}]$ and $\mathcal{L}[\mathbf{Z}]$, respectively, the classes of discrete M-convex and L-convex functions:

$$\begin{aligned} \mathcal{M}[\mathbf{Z}] &= \{f \mid f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\},\ \text{discrete M-convex}\}, \\ \mathcal{L}[\mathbf{Z}] &= \{g \mid g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\},\ \text{discrete L-convex}\}. \end{aligned}$$

In the following, we sometimes regard $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ as a function defined over $\mathbf{R}^V$. In such a case, we have $\mathrm{dom}\, f \subseteq \mathbf{Z}^V$.

**Remark 6.1** The concept of M-convex function over the integer lattice was first defined in [30, 31], where the effective domain is assumed to be bounded. The paper [33] discusses the case when the effective domain is not necessarily bounded, but assumes that functions take only integer values.

The concept of L-convex function over the integer lattice appeared first in [33], where function values are assumed be integral. Hence, the definition of L-convex functions above is a slight extension of the original one. □

We also define the set version of discrete M/L-convexity as follows. A set $B \subseteq \mathbf{Z}^V$ is called *discrete M-convex* if $B \neq \emptyset$ and it satisfies

**(B-EXC[Z])** $\forall x, y \in B$, $\forall u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$ such that $x - \chi_u + \chi_v \in B$, $y + \chi_u - \chi_v \in B$.

We call $D \subseteq \mathbf{Z}^V$ *discrete L-convex* if $D \neq \emptyset$ and it satisfies

**(LS1[Z])** $p \wedge q,\ p \vee q \in D\ (\forall p,\ q \in D)$, $\qquad$ **(LS2[Z])** $p \in D \Longrightarrow p + \lambda \mathbf{1} \in D \quad (\forall \lambda \in \mathbf{Z})$.

We denote

$$\mathcal{M}_0[\mathbf{Z}] = \{B \mid B \subseteq \mathbf{Z}^V,\ \text{discrete M-convex}\}, \qquad \mathcal{L}_0[\mathbf{Z}] = \{D \mid D \subseteq \mathbf{Z}^V,\ \text{discrete L-convex}\}.$$

In this section, we use some known results on discrete M/L-convexity. See [30, 31, 33, 36] as references.

Discrete M/L-convex functions can be extended to ordinary convex functions by taking their convex closure. Moreover, convex extensions of discrete M/L-convex functions coincide with "local" convex extensions. For any integer $k$ $(\geq 0)$, we define the set

$$\mathrm{HC}_k(x) = \{y \in \mathbf{Z}^V \mid \lfloor x(v) \rfloor - k \leq y(v) \leq \lceil x(v) \rceil + k\ (\forall v \in V)\}.$$

**Theorem 6.2** *For $B \in \mathcal{M}_0[\mathbf{Z}]$, we have $\overline{B} \in \mathcal{M}_0$ and*

$$\overline{B \cap \mathrm{HC}_0(x)} = \{y \in \overline{B} \mid \lfloor x(v) \rfloor \leq y(v) \leq \lceil x(v) \rceil \ (\forall v \in V)\} \qquad (x \in \mathbf{R}^V). \tag{6.1}$$

**Proof.** Let $\rho : 2^V \to \mathbf{Z} \cup \{+\infty\}$ be the submodular function associated with $B$, i.e., $\rho$ is such that $B = \mathrm{B}(\rho) \cap \mathbf{Z}^V$. Since $\mathrm{B}(\rho)$ is an integral polyhedron, i.e., $\mathrm{B}(\rho) = \overline{B}$ (cf. [17]), we have $\overline{B} \in \mathcal{M}_0$, implying (6.1). □

**Theorem 6.3** *For $D \in \mathcal{L}_0[\mathbf{Z}]$, we have $\overline{D} \in \mathcal{L}_0$ and*

$$\overline{D \cap \mathrm{HC}_0(p)} = \{q \in \overline{D} \mid \lfloor p(v) \rfloor \leq q(v) \leq \lceil p(v) \rceil \ (\forall v \in V)\} \qquad (p \in \mathbf{R}^V). \tag{6.2}$$

**Proof.** Let $\gamma : V \times V \to \mathbf{Z} \cup \{+\infty\}$ be the distance function with triangle inequality which is associated with $D$ (see [33, Theorem 4.20], [36, Theorem 2.28]), i.e., $\gamma$ is such that $D = \mathrm{D}(\gamma) \cap \mathbf{Z}^V$. Lemma 3.18 (v) implies that $\overline{D} = \mathrm{D}(\gamma) \in \mathcal{L}_0$. Hence, (6.2) follows. □

Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$. The *convex closure* $\bar{f} : \mathbf{R}^V \to \mathbf{R} \cup \{\pm\infty\}$ of $f$ is given by

$$\bar{f}(x) = \sup_{p \in \mathbf{R}^V, \alpha \in \mathbf{R}} \{\langle p, x \rangle + \alpha \mid \langle p, y \rangle + \alpha \leq f(y) \ (\forall y \in \mathbf{Z}^V)\} \qquad (x \in \mathbf{R}^V). \tag{6.3}$$

We also define a function $\tilde{f} : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ by

$$\tilde{f}(x) = \sup_{p \in \mathbf{R}^V, \alpha \in \mathbf{R}} \{\langle p, x \rangle + \alpha \mid \langle p, y \rangle + \alpha \leq f(y) \ (\forall y \in \mathrm{HC}_0(x))\} \qquad (x \in \mathbf{R}^V). \tag{6.4}$$

Note that $\tilde{f}$ is the local convex extension of $f$, i.e., the convex closure of the restriction of $f$ to the integral points around $x$. It admits an alternative expression

$$\tilde{f}(x) = \inf\{\sum_{y \in \mathrm{HC}_0(x)} \lambda_y f(y) \mid \sum_{y \in \mathrm{HC}_0(x)} \lambda_y y = x, \ \sum_{y \in \mathrm{HC}_0(x)} \lambda_y = 1, \ \lambda_y \geq 0 \ (y \in \mathrm{HC}_0(x))\} \quad (x \in \mathbf{R}^V) \tag{6.5}$$

by LP duality. From the definition, we have

$$\tilde{f}(x) \geq \bar{f}(x) \ (\forall x \in \mathbf{R}^V), \qquad \tilde{f}(x) = f(x) \ (\forall x \in \mathbf{Z}^V). \tag{6.6}$$

Theorems 6.4 and 6.6 show that the inequality in (6.6) holds with equality if $f$ is discrete M/L-convex.

**Theorem 6.4** *For $f \in \mathcal{M}[\mathbf{Z}]$, we have $\bar{f}(x) = \tilde{f}(x)$ $(\forall x \in \mathbf{R}^V)$ and $\bar{f}(x) = f(x)$ $(\forall x \in \mathbf{Z}^V)$.*

**Proof.** We prove the former only. The latter follows from the former and the equation in (6.6).

We first consider the case when $x \notin \overline{\mathrm{dom}_{\mathbf{Z}} f}$. Define the conjugate $f^\bullet$ of $f$ by (4.1). Let $\rho : 2^V \to \mathbf{Z} \cup \{+\infty\}$ be the submodular function associated with the discrete M-convex set $\mathrm{dom}_{\mathbf{Z}} f \in \mathcal{M}_0[\mathbf{Z}]$, and $p \in \mathrm{dom}\, f^\bullet$. If $x(V) \neq \rho(V)$, then

$$\begin{aligned} \bar{f}(x) &\geq \sup_{\alpha \in \mathbf{R}} \left[ \langle p + \alpha\chi_V, x \rangle - \sup_{y \in \mathbf{R}^V} \{\langle p + \alpha\chi_V, y \rangle - f(y)\} \right] \\ &= \langle p, x \rangle - f^\bullet(p) + \sup_{\alpha \in \mathbf{R}} \alpha\{x(V) - \rho(V)\} \ = \ +\infty. \end{aligned}$$

In the similar way, we can show that $\bar{f}(x) = +\infty$ when $x(X) > \rho(X)$ $(\exists X \subset V)$. By the inequality in (6.6), we have $\tilde{f}(x) = \bar{f}(x) = +\infty$

To show $\bar{f}(x) = \tilde{f}(x)$ for $x \in \overline{\mathrm{dom}_{\mathbf{Z}} f}$, we consider the following dual pair of LP problems:

(LP1) Maximize $\langle p,x\rangle+\alpha$ subject to $\langle p,y\rangle+\alpha\le f(y)$ $(y\in \mathrm{HC}_1(x))$, $p\in\mathbf{R}^V$, $\alpha\in\mathbf{R}$,

(LP2) Minimize $\displaystyle\sum_{y\in \mathrm{HC}_1(x)}\lambda_y f(y)$ subject to $\displaystyle\sum_{y\in \mathrm{HC}_1(x)}\lambda_y y=x$, $\displaystyle\sum_{y\in \mathrm{HC}_1(x)}\lambda_y=1$, $\lambda_y\ge 0$ $(y\in \mathrm{HC}_1(x))$.

(LP2) has a feasible solution since $x\in\overline{\mathrm{HC}_1(x)\cap \mathrm{dom}_{\mathbf{Z}} f}$. Let $(p^*,\alpha^*)$ $(\in\mathbf{R}^V\times\mathbf{R})$ and $\lambda^*=(\lambda^*_y\mid y\in \mathrm{HC}_1(x))$ be optimal solutions of (LP1) and (LP2), respectively. Then, it holds that

$$\bar f(x)\le\langle p^*,x\rangle+\alpha^*=\sum_{y\in \mathrm{HC}_1(x)}\lambda^*_y f(y)\le\tilde f(x). \tag{6.7}$$

We will show that both of the inequalities hold with equality.

Put

$$B=\{y\in \mathrm{HC}_1(x)\mid\langle p^*,y\rangle+\alpha^*=f(y)\}=\arg\min_{y\in \mathrm{HC}_1(x)} f[-p^*](y)\ (\in\mathcal{M}_0[\mathbf{Z}]).$$

The complementary slackness condition yields that $\{y\in \mathrm{HC}_1(x)\mid\lambda^*_y>0\}\subseteq B$, which implies $x\in\overline{B}$. In particular, we have $x\in\overline{B\cap \mathrm{HC}_0(f,x)}$ by Theorem 6.2 since $B\in\mathcal{M}_0[\mathbf{Z}]$. Hence, there is another optimal solution $\tilde\lambda=(\tilde\lambda_y\mid y\in \mathrm{HC}_1(x))$ of (LP2) such that if $\tilde\lambda_y>0$ then $y\in B\cap \mathrm{HC}_0(x)$. Since

$$\sum_{y\in \mathrm{HC}_1(x)}\lambda^*_y f(y)=\sum_{y\in \mathrm{HC}_1(x)}\tilde\lambda_y f(y)=\sum_{y\in \mathrm{HC}_0(x)}\tilde\lambda_y f(y)\ge\tilde f(x),$$

the second inequality in (6.7) holds with equality.

Let $y_0\in B\cap \mathrm{HC}_0(x)$. Then, we have $f[-p^*](y_0-\chi_u+\chi_v)\ge f[-p^*](y_0)$ for any $u,v\in V$. Since local optimality means global optimality for discrete M-convex functions [33, Theorem 4.6], we have $\alpha^*=f[-p^*](y_0)\le f[-p^*](y)=-\langle p^*,y\rangle+f(y)$ $(\forall y\in \mathrm{dom}_{\mathbf{Z}} f)$, i.e., $\langle p^*,y\rangle+\alpha^*\le f(y)$ $(\forall y\in \mathrm{dom}_{\mathbf{Z}} f)$. By the equation (6.3) for $\bar f$, the first inequality in (6.7) holds with equality. $\square$

The following theorem claims that we can choose a common optimal $\lambda$ in (6.5) for two discrete M-convex functions. The implication of this fact is discussed later in Corollary 6.8.

**Theorem 6.5** *For any $f,g\in\mathcal{M}[\mathbf{Z}]$ and $x\in\mathbf{R}^V$, there exists $\lambda=(\lambda_y\mid y\in \mathrm{HC}_0(x))$ such that*

$$\sum_{y\in \mathrm{HC}_0(x)}\lambda_y y=x,\qquad\sum_{y\in \mathrm{HC}_0(x)}\lambda_y=1,\quad\lambda_y\ge 0\ (y\in \mathrm{HC}_0(x)), \tag{6.8}$$

*and*

$$\bar f(x)=\tilde f(x)=\sum_{y\in \mathrm{HC}_0(x)}\lambda_y f(y),\qquad\bar g(x)=\tilde g(x)=\sum_{y\in \mathrm{HC}_0(x)}\lambda_y g(y). \tag{6.9}$$

**Proof.** We may assume that $x\in\overline{\mathrm{dom}_{\mathbf{Z}} f}\cap\overline{\mathrm{dom}_{\mathbf{Z}} g}$, which implies both $\tilde f(x)$ and $\tilde g(x)$ are finite. By (6.4), there exist $(p,\alpha),(q,\beta)\in\mathbf{R}^V\times\mathbf{R}$ such that

$$\begin{aligned}&\langle p,y\rangle+\alpha\le f(y)\ (y\in \mathrm{HC}_0(x)),\ \langle p,x\rangle+\alpha=\tilde f(x),\\&\langle q,y\rangle+\beta\le g(y)\ (y\in \mathrm{HC}_0(x)),\ \langle q,x\rangle+\beta=\tilde g(x).\end{aligned}$$

Put

$$B_f=\{y\in \mathrm{HC}_0(x)\mid\langle p,y\rangle+\alpha=f(y)\}\ (\in\mathcal{M}_0[\mathbf{Z}]),\quad B_g=\{y\in \mathrm{HC}_0(x)\mid\langle q,y\rangle+\beta=g(y)\}\ (\in\mathcal{M}_0[\mathbf{Z}]).$$

Then, we have $x\in\overline{B_f}\cap\overline{B_g}=\overline{B_f\cap B_g}$. Therefore, there exists $\lambda=(\lambda_y\mid y\in \mathrm{HC}_0(x))$ satisfying (6.8) and $\lambda_y=0$ $(y\notin B_f\cap B_g)$. Such $\lambda$ satisfies (6.9) by LP-duality. $\square$

**Theorem 6.6 ([33, Theorem 4.18])** *Let $g \in \mathcal{L}[\mathbf{Z}]$.*
(1) *For $q \in \mathrm{dom}_{\mathbf{Z}}\, g$ and $p \in [0,1]^V$, we have*

$$\bar{g}(q+p) = \tilde{g}(q+p) = g(q) + \sum_{j=1}^{k-1}(p_j - p_{j+1})\{g(q+\chi_{V_j}) - g(q)\} + p_k g(q+\chi_{V_k}),$$

*where $p_j$ and $V_j$ $(j = 1, \cdots, k)$ are as in the definition of the Lovász extension (4.25).*
(2) $\bar{g}(p) = \tilde{g}(p) = g(p)$ $(\forall p \in \mathbf{Z}^V)$.
(3) $\bar{g}(p + \lambda\mathbf{1}) = \bar{g}(p) + \lambda r$ $(\forall p \in \mathbf{R}^V,\ \lambda \in \mathbf{R})$, *where $r$ is in* (LF2[$\mathbf{Z}$]).
(4) *For $q \in \mathrm{dom}_{\mathbf{Z}}\, g$ and $p \in \mathbf{R}^V$ with $\max_{v\in V} p(v) - \min_{v \in V} p(v) \le 1$, we have*

$$\bar{g}(q+p) = g(q) + \sum_{j=1}^{k-1}(p_j - p_{j+1})\{g(q+\chi_{V_j}) - g(q)\} + p_k g(q+\chi_{V_k}),$$

*where the notation is the same as in* (1).

**Remark 6.7** Favati–Tardella [11] investigated the class of functions $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ for which $\bar{f} = \tilde{f}$ holds. They call such a function *integrally convex.* From Theorems 6.4 and 6.6, we see that discrete M/L-convex functions are integrally convex. □

As a corollary of Theorems 6.5 and 6.6, we obtain the following properties. For a function $g : \mathbf{Z}^V \to \mathbf{R} \cup \{-\infty\}$, we define the *concave closure* $\hat{g} : \mathbf{R}^V \to \mathbf{R} \cup \{\pm\infty\}$ of $g$ by

$$\hat{g}(x) = \inf_{p \in \mathbf{R}^V, \alpha \in \mathbf{R}} \{\langle p, x\rangle + \alpha \mid \langle p, y \rangle + \alpha \ge g(y)\ (\forall y \in \mathbf{Z}^V)\} \qquad (x \in \mathbf{R}^V).$$

Note that $\hat{g} = -\overline{(-g)}$.

**Corollary 6.8**
(i) *Let $f$, $g$ be functions such that $f, -g \in \mathcal{M}[\mathbf{Z}]$. If $f(x) \ge g(x)$ for any $x \in \mathbf{Z}^V$, then we have $\bar{f}(x) \ge \hat{g}(x)$ for any $x \in \mathbf{R}^V$.*
(ii) *Let $f$, $g$ be functions such that $f, -g \in \mathcal{L}[\mathbf{Z}]$. If $f(p) \ge g(p)$ for any $p \in \mathbf{Z}^V$, then we have $\bar{f}(p) \ge \hat{g}(p)$ for any $p \in \mathbf{R}^V$.*

The following theorems show that the convex extension of discrete M/L-convex functions are closely related to polyhedral M/L-convexity. Recall the definition of the restriction of a function in (4.15).

**Theorem 6.9** *Let $f \in \mathcal{M}[\mathbf{Z}]$. Then, for any $a, b \in \mathbf{Z}^V$ with $a \le b$, we have $\overline{(f_{[a,b]})} = (\bar{f})_{[a,b]} \in \mathcal{M}$ if $[a,b] \cap \mathrm{dom}_{\mathbf{Z}}\, f \neq \emptyset$. In particular, $\bar{f} \in \mathcal{M}$ if $\mathrm{dom}_{\mathbf{Z}}\, f$ is bounded.*

**Proof.** Put $h = f_{[a,b]}$ $(\in \mathcal{M}[\mathbf{Z}])$. Theorem 6.4 yields that $\bar{h} = (\bar{f})_{[a,b]}$. Since $\mathrm{dom}_{\mathbf{Z}}\, h$ is bounded, we have $\overline{\arg\min h[-p]} = \arg\min \bar{h}[-p] \in \mathcal{M}_0$ for any $p \in \mathbf{R}^V$ (see [31, Theorem 4.4]). From this property and Theorem 5.2 follows $\bar{h} \in \mathcal{M}$. □

**Theorem 6.10** *Let $g \in \mathcal{L}[\mathbf{Z}]$. Then, for any $a, b \in \mathbf{Z}^V$ with $a \le b$, we have $\overline{(g_{[a,b]})} = (\bar{g})_{[a,b]} \in \mathcal{L}$ if $[a,b] \cap \mathrm{dom}_{\mathbf{Z}}\, g \neq \emptyset$. In particular, $\bar{g} \in \mathcal{L}$ if $\mathrm{dom}_{\mathbf{Z}}\, g$ is bounded.*

**Proof.** Put $h = g_{[a,b]}$ $(\in \mathcal{L}[\mathbf{Z}])$. By Theorem 6.6 (1), we have $\bar{h} = (\bar{g})_{[a,b]}$. Since $\mathrm{dom}_{\mathbf{Z}}\, h$ is bounded, we have $\overline{\arg\min h[-x]} = \arg\min \bar{h}[-x] \in \mathcal{L}_0$ for any $x \in \mathbf{R}^V$. From this property and Theorem 5.3 follows $\bar{h} \in \mathcal{L}$. □

**Remark 6.11** For $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$, if $\mathrm{dom}_{\mathbf{Z}} f$ is bounded then its convex closure $\bar{f}$ is polyhedral convex, but in general $\bar{f}$ is not necessarily polyhedral convex, even if $f$ is discrete M/L-convex.

For example, consider the function $f : \mathbf{Z}^2 \to \mathbf{Z} \cup \{+\infty\}$ defined by

$$f(x,y) = \begin{cases} x^2 & ((x,y) \in \mathbf{Z}^2,\ x+y=0), \\ +\infty & (\text{otherwise}). \end{cases}$$

It is easy to see that $f$ is discrete M-convex. However, the convex closure $\bar{f}$ of $f$ has infinite number of linear-pieces, and is not a polyhedral convex function. □

We now introduce two new classes of M/L-convexity. We call a function *integral M-convex* (resp. *integral L-convex*) if it is represented as the convex closure of some integer-valued discrete M-convex (resp. discrete L-convex) function. We denote by $\mathcal{M}_{\mathrm{int}}$ and $\mathcal{L}_{\mathrm{int}}$, respectively, the classes of integral M-convex and L-convex functions:

$$\begin{aligned} \mathcal{M}_{\mathrm{int}} &= \{f \mid f = \overline{f_{\mathbf{Z}}} \text{ for some } f_{\mathbf{Z}} : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\},\ f_{\mathbf{Z}} \in \mathcal{M}[\mathbf{Z}]\}, \\ \mathcal{L}_{\mathrm{int}} &= \{g \mid g = \overline{g_{\mathbf{Z}}} \text{ for some } g_{\mathbf{Z}} : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\},\ g_{\mathbf{Z}} \in \mathcal{L}[\mathbf{Z}]\}. \end{aligned}$$

It was shown in [33] that integer-valued discrete M-convex and L-convex functions are conjugate to each other under the integer Fenchel transformation.

**Theorem 6.12 ([33, Theorem 4.24])**
(i) *For an integer-valued discrete M-convex function $f : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$, the function $g : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ defined by $g(p) = \sup_{x \in \mathbf{Z}^V}\{\langle p, x\rangle - f(x)\}$ $(p \in \mathbf{Z}^V)$ is discrete L-convex.*
(ii) *For an integer-valued discrete L-convex function $g : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$, the function $f : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ defined by $f(x) = \sup_{p \in \mathbf{Z}^V}\{\langle p, x\rangle - g(p)\}$ $(x \in \mathbf{Z}^V)$ is discrete M-convex.*
(iii) *The mappings $f \mapsto g$ and $g \mapsto f$ defined in* (i) *and* (ii)*, respectively, provide a one-to-one correspondence between the classes of integer-valued discrete M-convex and L-convex functions.*

Using this conjugacy relationship, we can show that integral M-convex and L-convex functions are conjugate to each other. For any $f : \mathbf{R}^V \to \mathbf{R} \cup \{\pm\infty\}$, we define $f_{\mathbf{Z}} : \mathbf{Z}^V \to \mathbf{R} \cup \{\pm\infty\}$ by

$$f_{\mathbf{Z}}(x) = \begin{cases} f(x) & (x \in \mathbf{Z}^V), \\ +\infty & (x \notin \mathbf{Z}^V). \end{cases} \tag{6.10}$$

**Theorem 6.13**
(i) *For $f \in \mathcal{M}_{\mathrm{int}}$, it holds that $f^\bullet \in \mathcal{L}_{\mathrm{int}}$ and*

$$f^\bullet(p) = \sup_{x \in \mathbf{Z}^V}\{\langle p, x\rangle - f(x)\} \qquad (p \in \mathbf{R}^V). \tag{6.11}$$

(ii) *For $g \in \mathcal{L}_{\mathrm{int}}$, it holds that $g^\bullet \in \mathcal{M}_{\mathrm{int}}$ and*

$$g^\bullet(x) = \sup_{p \in \mathbf{Z}^V}\{\langle p, x\rangle - g(p)\} \qquad (x \in \mathbf{R}^V). \tag{6.12}$$

(iii) *The mappings $f \mapsto f^\bullet$ $(f \in \mathcal{M}_{\mathrm{int}})$ and $g \mapsto g^\bullet$ $(g \in \mathcal{L}_{\mathrm{int}})$ provide a one-to-one correspondence between $\mathcal{M}_{\mathrm{int}}$ and $\mathcal{L}_{\mathrm{int}}$, are the inverse of each other.*

**Proof.** We prove the claim (i) only, since the claim (ii) can be shown similarly to (i), and (iii) is an immediate corollary to (i), (ii), and Theorem 4.1.

The equation (6.11) follows from Theorem 6.4. We denote $(f^\bullet)_{\mathbf{Z}}$ by $f^\bullet_{\mathbf{Z}}$ for simplicity. Theorem 6.12 and the equation (6.11) yield $f^\bullet_{\mathbf{Z}} \in \mathcal{L}[\mathbf{Z}]$. Therefore, it suffices to show $f^\bullet(p) = \overline{f^\bullet_{\mathbf{Z}}}(p)$ $(p \in \mathbf{R}^V)$.

We have $f^\bullet(p) \leq \overline{f^\bullet_{\mathbf{Z}}}(p)$ for any $p \in \mathbf{R}^V$. Hence, we may assume $f^\bullet(p) < +\infty$, since otherwise $f^\bullet(p) = \overline{f^\bullet_{\mathbf{Z}}}(p) = +\infty$ holds. By (6.11), there exists some $x_0 \in \mathrm{dom}_{\mathbf{Z}} f_{\mathbf{Z}}$ such that $f^\bullet(p) = \langle p, x_0 \rangle - f_{\mathbf{Z}}(x_0)$. Put

$$D = \{q \in \mathbf{R}^V \mid f^\bullet(q) = \langle q, x_0 \rangle - f_{\mathbf{Z}}(x_0)\}.$$

Then, we have $D = \partial f_{\mathbf{Z}}(x_0) = \overline{\partial f_{\mathbf{Z}}(x_0) \cap \mathbf{Z}^V}$ and $\partial f_{\mathbf{Z}}(x_0) \cap \mathbf{Z}^V \in \mathcal{L}_0[\mathbf{Z}]$ (see, e.g., [33, Theorem 4.7], [36, Theorem 2.41]). Thus, there exist some $\{q_i\}_{i=1}^k \subseteq \partial f_{\mathbf{Z}}(x_0) \cap \mathbf{Z}^V$ and $\{\lambda_i^*\}_{i=1}^k \subseteq \mathbf{R}$ $(k \geq 1)$ such that $\lambda_i^* \geq 0$ $(i = 1, \cdots, k)$, $\sum_{i=1}^k \lambda_i^* = 1$, and $\sum_{i=1}^k \lambda_i^* q_i = p$, which implies that

$$\begin{aligned}
\overline{f^\bullet_{\mathbf{Z}}}(p) &\leq \sup\{\langle p, x \rangle + \alpha \mid \langle q_i, x \rangle + \alpha \leq f^\bullet_{\mathbf{Z}}(q_i)\ (i = 1, \cdots, k)\} \\
&\leq \inf\{\sum_{i=1}^k \lambda_i f^\bullet_{\mathbf{Z}}(q_i) \mid \lambda_i \geq 0,\ \sum_{i=1}^k \lambda_i = 1,\ \sum_{i=1}^k \lambda_i q_i = p\} \\
&\leq \sum_{i=1}^k \lambda_i^* f^\bullet_{\mathbf{Z}}(q_i) = \sum_{i=1}^k \lambda_i^* \{\langle q_i, x_0 \rangle - f_{\mathbf{Z}}(x_0)\} = \langle p, x_0 \rangle - f_{\mathbf{Z}}(x_0) = f^\bullet(p).
\end{aligned}$$

□

# 7 Duality

## 7.1 Duality Theorems

In this section, we discuss duality theorems for polyhedral M/L-convex functions.

We first state the Fenchel duality for convex/concave functions. For a convex function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ and a concave function $g : \mathbf{R}^V \to \mathbf{R} \cup \{-\infty\}$ with $\operatorname{dom} f \neq \emptyset$, $\operatorname{dom} g \neq \emptyset$, we have

$$\inf_{x \in \mathbf{Z}^V} \{f(x) - g(x)\} \geq \inf_{x \in \mathbf{R}^V} \{f(x) - g(x)\} \geq \sup_{p \in \mathbf{R}^V} \{g^\circ(p) - f^\bullet(p)\} \geq \sup_{p \in \mathbf{Z}^V} \{g^\circ(p) - f^\bullet(p)\}, \qquad (7.1)$$

where the *concave conjugate* $g^\circ : \mathbf{R}^V \to \mathbf{R} \cup \{\pm\infty\}$ is defined by

$$g^\circ(p) = \inf_{x \in \mathbf{R}^V} \{\langle p, x\rangle - g(x)\} \qquad (p \in \mathbf{R}^V).$$

The Fenchel duality (7.2) below shows that the second inequality in (7.1) is satisfied with equality under a certain assumption. This is just an application of the existing result in convex analysis to polyhedral convex/concave functions, and it stands independently of M-convexity/M-concavity or L-convexity/L-concavity of functions. On the other hand, the claim (ii) of Theorem 7.1 below shows that all inequalities in (7.1) are satisfied with equality if $f$ and $g$ are integral M-convex/M-concave. This integrality result is not obtained from the convexity/concavity alone, but from the combination of the convexity/concavity and the combinatorial properties of $f$ and $g$.

**Theorem 7.1** (i) *Let* $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ *be a polyhedral convex function with* $\operatorname{dom} f \neq \emptyset$, *and* $g : \mathbf{R}^V \to \mathbf{R} \cup \{-\infty\}$ *be a polyhedral concave function with* $\operatorname{dom} g \neq \emptyset$. *If either of* (a) $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$ *and* (b) $\operatorname{dom} f^\bullet \cap \operatorname{dom} g^\circ \neq \emptyset$ *holds, then we have*

$$\inf_{x \in \mathbf{R}^V} \{f(x) - g(x)\} = \sup_{p \in \mathbf{R}^V} \{g^\circ(p) - f^\bullet(p)\}. \qquad (7.2)$$

*The supremum in (7.2) is attained at some* $p_* \in \mathbf{R}^V$ *if* (a) *is satisfied, and the infimum in (7.2) is attained at some* $x_* \in \mathbf{R}^V$ *if* (b) *is satisfied.*
(ii) *If* $f, -g \in \mathcal{M}_{\rm int}$ *and either of the condition* (a) *and* (b) *is fulfilled, then we have (7.2). The supremum in (7.2) is attained at some* $p_* \in \mathbf{Z}^V$ *if* (a) *is satisfied, and the infimum in (7.2) is attained at some* $x_* \in \mathbf{Z}^V$ *if* (b) *is satisfied.*

**Proof.** The claim (i) is already shown in [40, 43]. Hence, we show (ii) only.

We define the function $f_{\mathbf{Z}} : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ by (6.10), and $g_{\mathbf{Z}} : \mathbf{Z}^V \to \mathbf{Z} \cup \{-\infty\}$ by

$$g_{\mathbf{Z}}(p) = \begin{cases} g(p) & (p \in \mathbf{Z}^V), \\ -\infty & (p \notin \mathbf{Z}^V). \end{cases}$$

We also define $f_{\mathbf{Z}}^\bullet : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ and $g_{\mathbf{Z}}^\circ : \mathbf{Z}^V \to \mathbf{Z} \cup \{-\infty\}$ in the similar way, where it is noted that (cf. (6.11), (6.12))

$$f_{\mathbf{Z}}^\bullet(p) = \sup_{x \in \mathbf{Z}^V} \{\langle p, x\rangle - f_{\mathbf{Z}}(x)\}, \qquad g_{\mathbf{Z}}^\circ(p) = \inf_{x \in \mathbf{Z}^V} \{\langle p, x\rangle - g_{\mathbf{Z}}(x)\} \qquad (p \in \mathbf{Z}^V).$$

We have $\operatorname{dom} f = \overline{\operatorname{dom}_{\mathbf{Z}} f_{\mathbf{Z}}}$ and $\operatorname{dom} g = \overline{\operatorname{dom}_{\mathbf{Z}} g_{\mathbf{Z}}}$, and from Theorem 6.13 follows $\operatorname{dom} f^\bullet = \overline{\operatorname{dom}_{\mathbf{Z}} f_{\mathbf{Z}}^\bullet}$ and $\operatorname{dom} g^\circ = \overline{\operatorname{dom}_{\mathbf{Z}} g_{\mathbf{Z}}^\circ}$. These properties imply that the condition (a) is equivalent to $\operatorname{dom}_{\mathbf{Z}} f_{\mathbf{Z}} \cap \operatorname{dom}_{\mathbf{Z}} g_{\mathbf{Z}} \neq \emptyset$, and (b) is equivalent to $\operatorname{dom}_{\mathbf{Z}} f_{\mathbf{Z}}^\bullet \cap \operatorname{dom}_{\mathbf{Z}} g_{\mathbf{Z}}^\circ \neq \emptyset$. Thus, the first and the last term in (7.1) are equal by the Fenchel-type duality for discrete M-convex/M-concave functions [33, Theorem 5.2], and each inequality in (7.1) holds with equality. $\square$

When $f$ and $g$ are polyhedral M-convex and M-concave, respectively, in (7.2), then $f^\bullet$ and $g^\circ$ are polyhedral L-convex and L-concave, respectively, by Theorem 5.1. Hence, Fenchel duality can be read that the minimization of the sum of two polyhedral M-convex functions is equivalent to the minimization of the sum of their conjugate polyhedral L-convex functions. Note that the sum of polyhedral L-convex functions is also polyhedral L-convex, whereas the sum of two polyhedral M-convex functions is not necessarily polyhedral M-convex (cf. Remark 3.5). As a corollary of Theorem 7.1, we obtain an optimality criterion of the minimization of the sum of two polyhedral convex functions.

**Corollary 7.2** (i) *Let $f_i : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ be a polyhedral convex function with $\mathrm{dom}\, f_i \neq \emptyset$ for $i = 1, 2$. If $\mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2 \neq \emptyset$, then there exists $p_* \in \mathbf{R}^V$ such that*

$$\inf_{x \in \mathbf{R}^V} \{f_1(x) + f_2(x)\} = \inf_{x \in \mathbf{R}^V} f_1[-p_*](x) + \inf_{x \in \mathbf{R}^V} f_2[p_*](x). \tag{7.3}$$

(ii) *If $f_1, f_2 \in \mathcal{M}_{\mathrm{int}}$ and $\mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2 \neq \emptyset$, then we can choose an integral $p_* \in \mathbf{Z}^V$ with (7.3).*

**Remark 7.3** The conditions (a) and (b) in Theorem 7.1 cannot be removed, as shown in the following example.

Let $f : \mathbf{R}^2 \to \mathbf{R} \cup \{+\infty\}$ and $g : \mathbf{R}^2 \to \mathbf{R} \cup \{-\infty\}$ be defined by

$$f(x_1, x_2) = \begin{cases} x_1 & (x_1 + x_2 = 1) \\ +\infty & (x_1 + x_2 \neq 1) \end{cases} \qquad g(x_1, x_2) = \begin{cases} -x_1 & (x_1 + x_2 = -1) \\ -\infty & (x_1 + x_2 \neq -1) \end{cases} \tag{7.4}$$

Then, we have $f, -g \in \mathcal{M} \cap \mathcal{M}_{\mathrm{int}}$, $\mathrm{dom}\, f \cap \mathrm{dom}\, g = \emptyset$, and $\inf_{(x_1,x_2) \in \mathbf{R}^2} \{f(x_1, x_2) - g(x_1, x_2)\} = +\infty$.

On the other hand, it holds that

$$f^\bullet(p_1, p_2) = \begin{cases} p_2 & (p_1 - p_2 = 1) \\ +\infty & (p_1 - p_2 \neq 1) \end{cases} \qquad g^\circ(p_1, p_2) = \begin{cases} -p_2 & (p_1 - p_2 = -1) \\ -\infty & (p_1 - p_2 \neq -1). \end{cases} \tag{7.5}$$

We have $f^\bullet, -g^\bullet \in \mathcal{L} \cap \mathcal{L}_{\mathrm{int}}$, $\mathrm{dom}\, f^\bullet \cap \mathrm{dom}\, g^\circ = \emptyset$, and $\sup_{(p_1,p_2) \in \mathbf{R}^2} \{g^\circ(p_1, p_2) - f^\bullet(p_1, p_2)\} = -\infty$. Hence, neither (a) nor (b) is fulfilled, and the equation (7.2) does not hold. □

We state separation theorems for polyhedral convex/concave functions. Though the claim (i) of Theorems 7.4 below is just the separation theorem in convex analysis, the claims (ii), (iii) are based on the combinatorial properties of M/L-convexity. See also [19] for the proof of the claims (ii), (iii).

**Theorem 7.4** *Let $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ and $g : \mathbf{R}^V \to \mathbf{R} \cup \{-\infty\}$ be functions such that $\mathrm{dom}\, f \cap \mathrm{dom}\, g \neq \emptyset$. Suppose that $f(x) \geq g(x)$ $(\forall x \in \mathbf{R}^V)$ holds.*
(i) *If $f$ and $g$ are polyhedral convex and concave, respectively, then there exist $p_* \in \mathbf{R}^V$ and $\alpha_* \in \mathbf{R}$ such that*

$$f(x) \geq \langle p_*, x \rangle + \alpha_* \geq g(x) \qquad (x \in \mathbf{R}^V). \tag{7.6}$$

(ii) *If $f, -g \in \mathcal{M}_{\mathrm{int}}$, then we can take integral $p_* \in \mathbf{Z}^V$ and $\alpha_* \in \mathbf{Z}$ with (7.6).*
(iii) *If $f, -g \in \mathcal{L}_{\mathrm{int}}$, then we can take integral $p_* \in \mathbf{Z}^V$ and $\alpha_* \in \mathbf{Z}$ with (7.6).*

## 7.2 Submodular Flow Problem with M-convex Cost Function

We consider a generalization of the ordinary submodular flow problem and provide two optimality criteria, one of which is by negative cycles and the other by potentials. See, e.g., [3, 10, 12, 17, 39] for the ordinary submodular flow problem.

Let $G = (V, A)$ be a directed graph. A *flow* is a function $\xi : A \to \mathbf{R}$. Suppose that we are given a flow cost function $f_a : \mathbf{R} \to \mathbf{R} \cup \{+\infty\}$ for each $a \in A$ and a boundary cost function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$. Then, the *generalized submodular flow problem* is formulated as follows:

$$\text{(GSF)} \quad \text{Minimize} \quad F(\xi) = f(\partial\xi) + \sum_{a \in A} f_a(\xi(a)) \quad \text{subject to} \quad \xi \in \mathbf{R}^A,$$

where $\partial\xi \in \mathbf{R}^V$ is the boundary of $\xi$ defined by (1.1). The problem (GSF) coincides with the ordinary (linear) submodular flow problem if each $f_a$ is a linear function in a nonempty closed interval and $f$ is the indicater function of some M-convex set, i.e., $f$ is a polyhedral M-convex function with $f : \mathbf{R}^V \to \{0, +\infty\}$. We call a flow $\xi : A \to \mathbf{R}$ *feasible* if $F(\xi) < +\infty$. Till the end of this section, we assume that the problem (GSF) has a feasible flow, i.e.,

$$\operatorname{dom} f \cap \{\partial\xi \mid \xi \in \mathbf{R}^A,\ \xi(a) \in \operatorname{dom} f_a\ (a \in A)\} \neq \emptyset.$$

To state the first optimality criterion, we define the *auxiliary network* $(G_\xi = (V, A_\xi), w_\xi)$ associated with a feasible flow $\xi \in \mathbf{R}^A$. The underlying graph $G_\xi$ has the vertex set $V$ and the arc set $A_\xi$ consisting of three disjoint parts: $A_\xi = F_\xi \cup B_\xi \cup J_\xi$, where

$$\begin{aligned}
F_\xi &= \{a \mid a \in A,\ f'_a(\xi(a); +1) < +\infty\}, \\
B_\xi &= \{\bar{a} \mid a \in A,\ f'_a(\xi(a); -1) < +\infty\} \qquad (\bar{a} : \text{ reorientation of } a), \\
J_\xi &= \{(u, v) \mid u, v \in V,\ u \neq v,\ \tilde{c}_{\operatorname{dom} f}(\partial\xi, v, u) > 0\}.
\end{aligned}$$

The *weight* function $w_\xi : A_\xi \to \mathbf{R}$ is defined by

$$w_\xi(a) = \begin{cases} f'_a(\xi(a); +1) & (a \in F_\xi), \\ f'_{\bar{a}}(\xi(\bar{a}); -1) & (a \in B_\xi), \\ f'(\partial\xi; v, u) & (a \in J_\xi). \end{cases}$$

For any cycle $C$ ($\subseteq A_\xi$) in $G_\xi$, we define the *weight* of $C$ by $w_\xi(C) = \sum\{w_\xi(a) \mid a \in C\}$, and call $C$ a *negative cycle* if $w_\xi(C) < 0$.

**Theorem 7.5** *Let $f$ be a polyhedral convex function, and $\xi \in \mathbf{R}^A$ be a feasible flow.*
(i) *There is no negative cycle in $(G_\xi, w_\xi)$ if $\xi$ is optimal for* (GSF)*.*
(ii) *Suppose $f \in \mathcal{M}$. Then, $\xi$ is optimal for* (GSF) *if there is no negative cycle in $(G_\xi, w_\xi)$.*

**Proof.** (i): Suppose that there exists a negative cycle $C$ ($\subseteq A_\xi$) in $G_\xi$, and put $k = |C|$ ($\geq 1$). We show that $F(\xi_*) < F(\xi)$ holds for some flow $\xi_* \in \mathbf{R}^A$.

Put $C_F = C \cap F_\xi$, $C_B = C \cap B_\xi$, $C_J = C \cap J_\xi$, and $y = \sum\{\chi_v - \chi_u \mid (u, v) \in C_J\}$. Then, there exists a sufficiently small $\alpha > 0$ such that

$$f(\partial\xi + \alpha y) - f(\partial\xi) = \alpha f'(\partial\xi; y) \leq \alpha \sum_{(u,v) \in C_J} w_\xi(u, v), \tag{7.7}$$

$$f_a(\xi(a) + \alpha) - f_a(\xi(a)) = \alpha w_\xi(a) \qquad (a \in C_F), \tag{7.8}$$

$$f_{\bar{a}}(\xi(\bar{a}) - \alpha) - f_{\bar{a}}(\xi(\bar{a})) = \alpha w_\xi(\bar{a}) \qquad (a \in C_B), \tag{7.9}$$

where the inequality in (7.7) is from Theorems 4.2 and 4.4. We define a flow $\xi_* \in \mathbf{R}^A$ by

$$\xi_*(a) = \begin{cases} \xi(a) + \alpha & (a \in C_F), \\ \xi(\bar{a}) - \alpha & (a \in C_B), \\ \xi(a) & (\text{otherwise}). \end{cases}$$

Combining the inequalities (7.7), (7.8), and (7.9), we obtain

$$F(\xi_*) - F(\xi) = \{f(\partial\xi + \alpha y) - f(\partial\xi)\} + \sum_{a\in C_F} \{f_a(\xi(a)+\alpha) - f_a(\xi(a))\} + \sum_{a\in C_B} \{f_{\bar a}(\xi(\bar a)-\alpha) - f_{\bar a}(\xi(\bar a))\} \le \alpha w_\xi(C) < 0.$$

(ii): Suppose that there exists a flow $\xi_* \in \mathbf{R}^V$ with $F(\xi_*) < F(\xi)$. To prove the existence of a negative cycle in $(G_\xi, w_\xi)$, it suffices to show the existence of a "circulation" in $(G_\xi, w_\xi)$ with negative cost w.r.t. $w_\xi$, i.e., the existence of a function $\zeta : A_\xi \to \mathbf{R}$ satisfying $\zeta \ge 0$, $\sum_{a\in A_\xi} w_\xi(a)\zeta(a) < 0$ and

$$\sum\{\zeta(a) \mid a\,(\in A_\xi) \text{ leaves } v\} = \sum\{\zeta(a) \mid a\,(\in A_\xi) \text{ enters } v\} \quad (\forall v \in V). \tag{7.10}$$

We have

$$f_a(\xi_*(a)) - f_a(\xi(a)) \ \ge\ \{\xi_*(a) - \xi(a)\} f'_a(\xi(a); +1) \quad (\forall a \in \mathrm{supp}^+(\xi_* - \xi)), \tag{7.11}$$

$$f_a(\xi_*(a)) - f_a(\xi(a)) \ \ge\ \{\xi(a) - \xi_*(a)\} f'_a(\xi(a); -1) \quad (\forall a \in \mathrm{supp}^-(\xi_* - \xi)), \tag{7.12}$$

$$f(\partial\xi_*) - f(\partial\xi) \ \ge\ f'(\partial\xi; \partial\xi_* - \partial\xi). \tag{7.13}$$

By Theorem 4.14 (ii), there exists some $\lambda = (\lambda_{uv} \mid (u,v) \in J_\xi)$ such that

$$\sum_{(u,v)\in J_\xi} \lambda_{uv}(\chi_v - \chi_u) = \partial\xi_* - \partial\xi, \qquad \lambda_{uv} \ge 0 \quad ((u,v) \in J_\xi),$$

$$\sum_{(u,v)\in J_\xi} \lambda_{uv} f'(\partial\xi; v, u) = f'(\partial\xi; \partial\xi_* - \partial\xi). \tag{7.14}$$

We now define a function $\zeta : A_\xi \to \mathbf{R}$ as follows:

$$\zeta(a) = \begin{cases} \max\{\xi_*(a) - \xi(a), 0\} & (a \in F_\xi), \\ \max\{\xi(\bar a) - \xi_*(\bar a), 0\} & (a \in B_\xi), \\ \lambda_{uv} & (a = (u,v) \in J_\xi). \end{cases}$$

It is easy to see that $\zeta \ge 0$ and $\zeta$ satisfies (7.10). Moreover, (7.11), (7.12), and (7.14) imply

$$\sum_{a\in A_\xi} w_\xi \zeta(a) \ =\ \sum_{a\in \mathrm{supp}^+(\xi_*-\xi)} \{\xi_*(a) - \xi(a)\} f'_a(\xi(a); +1) + \sum_{a\in \mathrm{supp}^-(\xi_*-\xi)} \{\xi(a) - \xi_*(a)\} f'_a(\xi(a); -1) + \sum_{u,v\in V} \lambda_{uv} f'(\partial\xi; v, u) \ \le\ F(\xi_*) - F(\xi) \ <\ 0.$$

Hence, $\zeta$ is a circulation in $(G_\xi, w_\xi)$ with negative cost. □

**Remark 7.6** The claim (ii) of Theorem 7.5 does not hold in general when $f \notin \mathcal{M}$.

Suppose that we are given a directed graph $G = (V, A)$ with $V = \{u, v, w\}$ and $A = \{(u,w), (v,w)\}$, a family of flow cost functions $f_a : \mathbf{R} \to \mathbf{R} \cup \{+\infty\}$ such that

$$f_a(\alpha) = \begin{cases} 0 & (0 \le \alpha \le 1), \\ +\infty & (\text{otherwise}), \end{cases}$$

for all $a \in A$, and a boundary cost function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ such that

$$f(x) = \begin{cases} -\min\{x(u), x(v)\} & (x(u) \ge 0,\ x(v) \ge 0,\ x(u) + x(v) + x(w) = 0), \\ +\infty & (\text{otherwise}). \end{cases}$$

It is easy to see that $f$ is polyhedral convex, but not polyhedral M-convex.

For a feasible flow $\xi = (0,0) \in \mathbf{R}^A$, the auxiliary network $(G_\xi = (V, A_\xi), w_\xi)$ is such that

$$A_\xi = \{(u,w),(v,w),(w,u),(w,v)\}, \qquad w_\xi(a) = 0 \ \ (a \in A_\xi).$$

There is no negative cycle in $(G_\xi, w_\xi)$, but $\xi$ is not optimal; a feasible flow $\xi_* \in \mathbf{R}^A$ is optimal if and only if either $\xi_*(u,w) = 1$ or $\xi_*(v,w) = 1$ holds. □

We next state an optimality criterion by potentials. Recall the definition of the coboundary $\delta p$ of a potential $p \in \mathbf{R}^V$ in (1.3).

**Theorem 7.7** *Let $\xi_* \in \mathbf{R}^A$ be a feasible flow.*
(i) *If the condition*

$$\partial\xi_* \in \arg\min f[-p_*], \qquad \xi_*(a) \in \arg\min f_a[-\eta(a)] \quad (\forall a \in A) \tag{7.15}$$

*holds for some $p_* \in \mathbf{R}^V$ and $\eta = -\delta p_*$, then $\xi_*$ is optimal for* (GSF).
(ii) *Suppose that $f$ is polyhedral convex with* $\mathrm{dom}\, f \neq \emptyset$*, and $f_a \in \mathcal{C}^1$ $(a \in A)$. Then, $\xi_*$ is optimal for* (GSF) *if and only if (7.15) holds for some $p_* \in \mathbf{R}^V$ and $\eta = -\delta p_*$.*
(iii) *If $p_* \in \mathbf{R}^V$ and $\eta = -\delta p_*$ satisfies (7.15), then we have*

$$p_*(v) - p_*(u) \le w_\xi(u,v) \qquad (\forall (u,v) \in A_{\xi_*}). \tag{7.16}$$

(iv) *Suppose $f \in \mathcal{M}$ and $f_a \in \mathcal{C}^1$ $(a \in A)$. Then, $\xi_*$ is optimal for* (GSF) *if and only if (7.16) holds for some $p_* \in \mathbf{R}^V$.*

**Proof.** Note that $F(\xi) = f[-p_*](\partial\xi) + \sum_{a\in A} f_a[-\eta(a)](\xi(a))$ for any $p_* \in \mathbf{R}^V$ and $\eta = -\delta p_*$. Hence, the claim (i) follows immediately. The claim (iii) is also immediate from the definition of $w_\xi$.

(ii): Suppose $\xi_*$ is optimal for (GSF). Define $f_i : \mathbf{R}^V \to \mathbf{R} \cup \{\pm\infty\}$ $(i = 1,2)$ by

$$f_1(x) = f(x), \qquad f_2(x) = \inf\{\sum_{a\in A} f_a(\xi(a)) \mid \xi \in \mathbf{R}^A,\ \partial\xi = x\} \quad (x \in \mathbf{R}^V). \tag{7.17}$$

Then, Corollary 7.2 implies the existence of $p_* \in \mathbf{R}^V$ satisfying (7.3). Note that

$$\inf_{x\in\mathbf{R}^V} f_2[p_*](x) = \inf_{\xi\in\mathbf{R}^A}\{\langle p_*, \partial\xi\rangle_V + \sum_{a\in A} f_a(\xi(a))\} = \sum_{a\in A} \inf_{\alpha\in\mathbf{R}}\{\delta p_*(a)\alpha + f_a(\alpha)\},$$

where the second equality is by $\langle p_*, \partial\xi\rangle_V = \langle \delta p_*, \xi\rangle_A$. Hence, we have (7.15).

(iv): By Theorem 4.11 and the convexity of $f_a$ $(a \in A)$, the condition (7.15) is equivalent to (7.16). Hence, the claim follows from (ii). □

**Remark 7.8** The following claim is easy to show: for a feasible flow $\xi \in \mathbf{R}^A$, if there exists $p_* \in \mathbf{R}^V$ with (7.16), then there is no negative cycle in the auxiliary network $(G_\xi, w_\xi)$. Indeed, from (7.16) we have

$$w_\xi(C) = \sum_{(u,v)\in C} w_\xi(u,v) \ge \sum_{(u,v)\in C} \{p_*(v) - p_*(u)\} = 0$$

for any cycle $C$ in $(G_\xi, w_\xi)$. The converse is also a well-known fact in graph theory. □

Summarizing the above results when $f \in \mathcal{M}$, we obtain the following statement.

**Corollary 7.9** *Suppose $f \in \mathcal{M}$. For any feasible flow $\xi \in \mathbf{R}^A$, the following three conditions are equivalent:*
(OPT) *$\xi$ is optimal.* (NNC) *There is no negative cycle in the auxiliary network $(G_\xi, w_\xi)$.*
(POT) *There exists a potential $p \in \mathbf{R}^V$ with (7.16).*

### 7.3 Network Induction

As shown in Section 4 (Theorems 4.17 and 4.33 in particular), there are various operations for polyhedral M/L-convex functions. In this section, we explain a more powerful operation called *network induction*. Network induction is a transformation by using networks, and includes many operations such as translation, restriction, projection, etc., as its special cases. We show that polyhedral M/L-convex functions can be transformed into polyhedral M/L-convex functions by network induction.

Let $G = (V, A; S, T)$ be a directed graph with vertex set $V$, arc set $A$, and two disjoint vertex subsets $S$ and $T$ called entrance set and exit set, respectively. For $f : \mathbf{R}^S \to \mathbf{R} \cup \{+\infty\}$ and $f_a : \mathbf{R} \to \mathbf{R} \cup \{+\infty\}$ $(a \in A)$, define $\tilde{f} : \mathbf{R}^T \to \mathbf{R} \cup \{\pm\infty\}$ by

$$\begin{aligned}\tilde{f}(y) = \inf_{\xi, x}\{f(x) + \sum_{a \in A} f_a(\xi(a)) \mid \partial\xi = (x, -y, \mathbf{0}),\\ \xi \in \mathbf{R}^A,\ (x, -y, \mathbf{0}) \in \mathbf{R}^S \times \mathbf{R}^T \times \mathbf{R}^{V \setminus (S \cup T)}\} \quad (y \in \mathbf{R}^T).\end{aligned}$$

For $g : \mathbf{R}^S \to \mathbf{R} \cup \{+\infty\}$ and $g_a : \mathbf{R} \to \mathbf{R} \cup \{+\infty\}$ $(a \in A)$, define $\tilde{g} : \mathbf{R}^T \to \mathbf{R} \cup \{\pm\infty\}$ by

$$\begin{aligned}\tilde{g}(q) = \inf_{\eta, p, r}\{g(p) + \sum_{a \in A} g_a(\eta(a)) \mid \\ \eta = -\delta(p, q, r) \in \mathbf{R}^A,\ (p, q, r) \in \mathbf{R}^S \times \mathbf{R}^T \times \mathbf{R}^{V \setminus (S \cup T)}\} \quad (q \in \mathbf{R}^T).\end{aligned}$$

**Theorem 7.10** *Let $f$, $g$ be polyhedral M-convex and L-convex, respectively, such that $g = f^\bullet$, and $f_a, g_a \in \mathcal{C}^1$ with $g_a = f_a^\bullet$ $(a \in A)$. Suppose that $\tilde{f}(y_0) < +\infty$, $\tilde{g}(q_0) < +\infty$ for some $y_0 \in \mathbf{R}^T$, $q_0 \in \mathbf{R}^T$. Then, $\tilde{f}$ is polyhedral M-convex, $\tilde{g}$ is polyhedral L-convex, and $\tilde{g} = \tilde{f}^\bullet$.*

**Proof.** We first prove the conjugacy between $\tilde{f}$ and $\tilde{g}$. For any $\xi \in \mathbf{R}^A, x \in \mathbf{R}^S, \eta \in \mathbf{R}^A, (p, q, r) \in \mathbf{R}^V$ satisfying $\partial\xi = (x, -y, \mathbf{0}) \in \mathbf{R}^V$ and $\eta = -\delta(p, q, r)$, it holds that

$$\langle \eta, \xi \rangle_A = \langle -\delta(p, q, r), \xi \rangle_A = -\langle (p, q, r), \partial\xi \rangle_V = -\langle p, x \rangle_S + \langle q, y \rangle_T,$$

whereas

$$f(x) + g(p) \geq \langle p, x \rangle_S, \qquad f_a(\xi(a)) + g_a(\eta(a)) \geq \eta(a)\xi(a) \ (a \in A),$$

since $g = f^\bullet$ and $g_a = f_a^\bullet$ $(a \in A)$. Therefore,

$$\left\{f(x) + \sum_{a \in A} f_a(\xi(a))\right\} + \left\{g(p) + \sum_{a \in A} g_a(\eta(a))\right\} \geq \langle q, y \rangle_T. \tag{7.18}$$

Taking the infimum of the left-hand side, we obtain

$$\tilde{f}(y) + \tilde{g}(q) \geq \langle q, y \rangle_T. \tag{7.19}$$

In particular, we have

$$\tilde{f}(y) \geq \langle q_0, y \rangle_T - \tilde{g}(q_0) > -\infty \ \ (y \in \mathbf{R}^T), \qquad \tilde{g}(q) \geq \langle q, y_0 \rangle_T - \tilde{f}(y_0) > -\infty \ \ (q \in \mathbf{R}^T).$$

Hence, it holds that $\tilde{f} > -\infty$, $\mathrm{dom}\, \tilde{f} \neq \emptyset$, $\tilde{g} > -\infty$, and $\mathrm{dom}\, \tilde{g} \neq \emptyset$.

For the proof of $\tilde{g} = \tilde{f}^\bullet$, it remains to be shown that for each $y \in \mathrm{dom}\, \tilde{f}$, the inequality (7.19) turns into an equality for some $q$. Theorem 7.7 (iv) implies that there exists $(p, q, r) \in \mathbf{R}^S \times \mathbf{R}^T \times \mathbf{R}^{V \setminus (S \cup T)}$ such that

$$\tilde{f}(y) = \langle q, y \rangle_S + \inf f[-p] + \sum_{a \in A} \inf f_a[-\eta(a)]$$

for $\eta = -\delta(p, q, r)$. Hence,

$$\tilde{f}(y) = \langle q, y\rangle_S - f^\bullet(p) - \sum_a f_a^\bullet(\eta(a)) = \langle q, y\rangle_S - g(p) - \sum_a g_a(\eta(a)) \leq \langle q, y\rangle_S - \tilde{g}(q).$$

This yields an equality in (7.19).

We next show the polyhedral L-convexity of $\tilde{g}$. Define $r_0 = \{g(p + \lambda\mathbf{1}) - g(p)\}/\lambda$, which is independent of $p \in \mathbf{R}^S$ and $\lambda \in \mathbf{R}$. Since $\delta(p + \lambda\mathbf{1}, q + \lambda\mathbf{1}, r + \lambda\mathbf{1}) = \delta(p, q, r)$ $(\forall\lambda \in \mathbf{R})$, we have

$$\begin{aligned}\tilde{g}(q + \lambda\mathbf{1}) &= \inf_{\eta, p', r'}\{g(p') + \sum_{a\in A} g_a(\eta(a)) \mid \eta = -\delta(p', q + \lambda\mathbf{1}, r')\} \\ &= \inf_{\eta, p, r}\{g(p + \lambda\mathbf{1}) + \sum_{a\in A} g_a(\eta(a)) \mid \eta = -\delta(p, q, r)\} \;=\; \tilde{g}(q) + \lambda r_0.\end{aligned}$$

For $q_1, q_2 \in \operatorname{dom}\tilde{g}$, we show that $\tilde{g}(q_1) + \tilde{g}(q_2) \geq \tilde{g}(q_1 \wedge q_2) + \tilde{g}(q_1 \vee q_2)$. For $i = 1, 2$, let $(p_i, q_i, r_i) \in \mathbf{R}^V$ be a vector such that $\tilde{g}(q_i) = g(p_i) + \sum_{a\in A} g_a(\eta_i(a))$, where $\eta_i = -\delta(p_i, q_i, r_i)$. Putting

$$\eta_\wedge = -\delta(p_1 \wedge p_2, q_1 \wedge q_2, r_1 \wedge r_2), \quad \eta_\vee = -\delta(p_1 \vee p_2, q_1 \vee q_2, r_1 \vee r_2),$$

we obtain

$$g_a(\eta_1(a)) + g_a(\eta_2(a)) \geq g_a(\eta_\wedge(a)) + g_a(\eta_\vee(a)) \quad (a \in A)$$

by the convexity of $g_a$, while

$$g(p_1) + g(p_2) \geq g(p_1 \wedge p_2) + g(p_1 \vee p_2)$$

holds by the submodularity of $g$. Hence follows

$$\tilde{g}(q_1) + \tilde{g}(q_2) \geq g(p_1 \wedge p_2) + g(p_1 \vee p_2) + \sum_{a\in A}\{g_a(\eta_\wedge(a)) + g_a(\eta_\vee(a))\} \geq \tilde{g}(q_1 \wedge q_2) + \tilde{g}(q_1 \vee q_2).$$

The polyhedral M-convexity of $\tilde{f}$ follows from the above facts and Theorem 5.1. □

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