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Algorithms in Discrete Convex Analysis

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Abstract

This is a survey of algorithmic results in the theory of "discrete convex analysis" for integer-valued functions defined on integer lattice points. The theory parallels the ordinary convex analysis, covering discrete analogues of the fundamental concepts such as conjugacy, the Fenchel min-max duality, and separation theorems. The technical development is based on matroid-theoretic concepts, in particular, submodular functions and exchange axioms.

Keywords: discrete convex analysis, matroid, L-convex function, M-convex function

1 Introduction

In the field of nonlinear programming (in continuous variables) convex analysis plays a pivotal role both in theory and in practice (e.g., [36]). An analogous theory for discrete optimization (nonlinear integer programming), called "discrete convex analysis," has been advocated by the author [29, 30]. The theory is developed by adapting the ideas in convex analysis and generalizing the results in matroid theory. Symbolically,

Discrete Convex Analysis
= Matroid Theory + Convex Analysis.

This theory puts emphasis on discrete conjugacy and discrete duality, in contrast to the previous works [7, 14, 23] which focus on greedy algorithms.

In the theory of matroids and submodular functions (cf., [12]), the relationship between submodular functions and convex functions was made clear in the eighties through the works of A. Frank, S. Fujishige, and L. Lovász. In particular, Lovász [22] pointed out that a set function is submodular if and only if the so-called

Table 1: Results in matroid theory related to convexity

Year (ca.)	Author	Result
1935	Whitney [39]	Matroid axioms
1965	Edmonds [5]	Polymatroid intersection theorem
1975		Weighted intersection
	Edmonds [6]	
	Lawler [21]	
	Iri–Tomizawa [17]	potential char.
	Frank [8]	weight splitting
1982		Relation to convexity
	Frank [9]	discrete separation
	Fujishige [11]	Fenchel duality
	Lovász [22]	Lovász extension
1990	Dress–Wenzel [3, 4]	Valuated matroid
	Favati–Tardella [7]	Submod. integral convexity
1995	Murota [26, 29, 30]	Valuated matroid intersect. M/L-convexity

Lovász extension of that function is convex. Relevant results in matroid theory are summarized in Table 1.

Those developments in the eighties are not sufficient for us to fully understand the relationship between convexity and submodularity in nonlinear discrete optimization. In this connection we mention that the minimum-cost flow problem with a separable convex objective function was treated successfully in [24], followed by an extension to the submodular flow problem [12], and the minimization of a separable convex function over a base polyhedron was solved in [12, 14, 16, 20, 40].

Independent of the developments in the theory of submodular functions, Dress and Wenzel [3, 4] introduced the concept of valuated matroids, which turned out to provide a nice combinatorial framework to which the optimization algorithms for matroids can be generalized. Variants of greedy algorithms work for optimizing a matroid valuation [3, 25], and the weighted matroid intersection algorithm can be extended to the valuated matroid intersection problem [26, 27].

Combining the idea of valuated matroids and the convexity argument for submodular functions, the present author proposed the concepts of "M-convexity" and "L-convexity" for functions defined on integer lattice points [28, 29, 30]. They are generalizations of base polyhedra and submodular set functions, respectively, and are general enough to cover the nonlinear discrete optimization problems treated

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successfully in the literature. Furthermore, the submodular flow problem can be generalized to the M-convex submodular flow problem.

The M-convex/L-convex functions are amenable to efficient algorithms. The minimization of an M-convex function can be done by purely combinatorial algorithms; a greedy-type algorithm of Dress–Wenzel [3] for valuated matroids and a domain reduction-type polynomial-time algorithm of Shioura [38] for M-convex functions. The minimum of an L-convex function can be found by a polynomial-time algorithm of Favati–Tardella [7] that relies on the ellipsoid method. The valuated matroid intersection algorithms of Murota [27] can be extended for the M-convex submodular flow problem; a primal (cycle-cancelling) algorithm of Murota [28] and a polynomial-time primal-dual algorithm with conjugate-scaling of Iwata–Shigeno [19].

The objective of this paper is to describe the fundamental results on M- and L-convex functions with special emphasis on algorithmic aspects.

2 Discrete Convex Functions

Let V be a nonempty finite set and $\mathbf{Z}$ be the set of integers. For a function $f : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ define $\mathrm{dom}\, f = \{x \in \mathbf{Z}^V \mid f(x) < +\infty\}$, called the effective domain of f.

A function $f : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ with $\mathrm{dom}\, f \neq \emptyset$ is called M-convex if it satisfies

(M-EXC) For $x, y \in \mathrm{dom}\, f$ and $u \in \mathrm{supp}^+(x - y)$, there exists $v \in \mathrm{supp}^-(x - y)$ such that

$$f(x) + f(y) \geq f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v),$$

where, for any $u \in V$, χ_u is the characteristic vector of u (defined by: $\chi_u(v) = 1$ if $v = u$, and $= 0$ otherwise), and

$$\begin{aligned} \mathrm{supp}^+(z) &= \{v \in V \mid z(v) > 0\} \quad (z \in \mathbf{Z}^V), \\ \mathrm{supp}^-(z) &= \{v \in V \mid z(v) < 0\} \quad (z \in \mathbf{Z}^V). \end{aligned}$$

A function f with $\mathrm{dom}\, f \subseteq \{0, 1\}^V$ is M-convex if and only if $-f$ is a valuated matroid in the sense of [3, 4].

A set $B \subseteq \mathbf{Z}^V$ is said to be an M-convex set if its indicator function (defined by: $\delta_B(x) = 0$ if $x \in B$, and $= +\infty$ otherwise) is an M-convex function, or equivalently, if B satisfies

(B-EXC) For $x, y \in B$ and for $u \in \mathrm{supp}^+(x - y)$, there exists $v \in \mathrm{supp}^-(x - y)$ such that $x - \chi_u + \chi_v \in B$ and $y + \chi_u - \chi_v \in B$.

This means that an M-convex set is the same as the set of integer points of the base polyhedron of an integral submodular system (see [12] for submodular systems). The effective domain of an M-convex function is an M-convex set.

A function $g : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ with $\operatorname{dom} g \neq \emptyset$ is called L-convex if
(L1) $g(p) + g(q) \geq g(p \vee q) + g(p \wedge q) \quad (p, q \in \mathbf{Z}^V)$,
(L2) $\exists r \in \mathbf{Z} : \ g(p + \mathbf{1}) = g(p) + r \quad (p \in \mathbf{Z}^V)$,
where

$$\begin{aligned} p \vee q &= (\max(p(v), q(v)) \mid v \in V) \in \mathbf{Z}^V, \\ p \wedge q &= (\min(p(v), q(v)) \mid v \in V) \in \mathbf{Z}^V, \end{aligned}$$

and $\mathbf{1}$ is the vector in $\mathbf{Z}^V$ with all of its components being equal to 1.

A set $D \subseteq \mathbf{Z}^V$ is said to be an L-convex set if its indicator function δ_D is an L-convex function, i.e., if (i) $D \neq \emptyset$, (ii) $p, q \in D \Rightarrow p \vee q, p \wedge q \in D$, and (iii) $p \in D \Rightarrow p \pm \mathbf{1} \in D$. The effective domain of an L-convex function is an L-convex set.

M-convexity and L-convexity are conjugate to each other under the integral Fenchel-Legendre transformation $f \mapsto f^\bullet$ defined by

$$f^\bullet(p) = \sup\{\langle p, x \rangle - f(x) \mid x \in \mathbf{Z}^V\} \quad (p \in \mathbf{Z}^V),$$

where $\langle p, x \rangle = \sum_{v \in V} p(v)x(v)$.

Theorem 1 ([30]) *For an M-convex function f and an L-convex function g, it holds that $f^\bullet$ is L-convex, $g^\bullet$ is M-convex, $f^{\bullet\bullet} = f$, and $g^{\bullet\bullet} = g$.* □

The concepts of M$^\natural$- and L$^\natural$-convexity are variants of M- and L-convexity, respectively. Let v_0 be a new element not in V and define $\tilde{V} = \{v_0\} \cup V$. A function $f : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ with $\operatorname{dom} f \neq \emptyset$ is called M$^\natural$-convex [33] if it is expressed in terms of an M-convex function $\tilde{f} : \mathbf{Z}^{\tilde{V}} \to \mathbf{Z} \cup \{+\infty\}$ as

$$\tilde{f}(x_0, x) = \begin{cases} f(x) & \text{if } x_0 + \sum_{u \in V} x(u) = 0 \\ +\infty & \text{otherwise.} \end{cases}$$

Namely, an M$^\natural$-convex function is a function obtained as the projection of an M-convex function. Conversely, an M$^\natural$-convex function determines the corresponding M-convex function up to a translation of $\operatorname{dom} f$ in the direction of v_0. A function $f : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ with $\operatorname{dom} f \neq \emptyset$ is M$^\natural$-convex if and only if it satisfies

(M$^\natural$-EXC) For $x, y \in \operatorname{dom} f$ and $u \in \operatorname{supp}^+(x - y)$,

$$\begin{aligned} & f(x) + f(y) \\ & \geq \min[\ f(x - \chi_u) + f(y + \chi_u), \\ & \quad \min_{v \in \operatorname{supp}^-(x-y)} \{f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v)\}\ ]. \end{aligned}$$

A function $g : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ with $\operatorname{dom} g \neq \emptyset$ is called L$^\natural$-convex [13] if it is expressed in terms of an L-convex function $\tilde{g} : \mathbf{Z}^{\tilde{V}} \to \mathbf{Z} \cup \{+\infty\}$ as $g(p) = \tilde{g}(0, p)$.

Namely, an $\mathrm{L}^\natural$-convex function is a function obtained as the restriction of an L-convex function. Conversely, an $\mathrm{L}^\natural$-convex function determines the corresponding L-convex function up to the constant r in the definition of L-convex function. An $\mathrm{L}^\natural$-convex function is characterized by "discrete midpoint convexity":

$$g(p) + g(q) \geq g\left(\left\lceil \frac{p+q}{2} \right\rceil\right) + g\left(\left\lfloor \frac{p+q}{2} \right\rfloor\right) \tag{1}$$

for $p, q \in \mathbf{Z}^V$, where $\lceil p \rceil$ (or $\lfloor p \rfloor$) for any $p \in \mathbf{R}^V$ denotes the vector obtained by rounding up (or down) the components of p to the nearest integers. It is pointed out in [13] that a function is $\mathrm{L}^\natural$-convex if and only if it is submodular integrally convex in the sense of [7].

Whereas the concept of $\mathrm{M}^\natural$-convexity is equivalent to that of M-convexity as above, it is known that the class of $\mathrm{M}^\natural$-convex functions properly contains that of M-convex functions. A similar statement holds true for $\mathrm{L}^\natural$- and L-convexity. We denote this symbolically as

$$\mathcal{M} \subseteq \mathcal{M}^\natural \simeq \mathcal{M}, \quad \mathcal{L} \subseteq \mathcal{L}^\natural \simeq \mathcal{L},$$

where $\mathcal{M}$, etc., denote the class of M-convex functions, etc. As a consequence of Theorem 1, $\mathrm{M}^\natural$-convex functions and $\mathrm{L}^\natural$-convex functions are conjugate to each other under the integral Fenchel-Legendre transformation. It is known [35] that $\mathcal{M}^\natural \cap \mathcal{L}^\natural$ coincides with the class of separable convex functions, while $\mathcal{M} \cap \mathcal{L} = \emptyset$.

3 Duality Theorems

Discrete duality theorems hold true for a pair of M-convex and M-concave functions, and also for a pair of L-convex and L-concave functions. A function $h : \mathbf{Z}^V \to \mathbf{Z} \cup \{-\infty\}$ is called M-concave if $-h$ is M-convex; $\operatorname{dom} h$ means the effective domain of $-h$. The concave counterpart of the discrete Fenchel-Legendre transform is defined as

$$h^\circ(p) = \inf\{\langle p, x\rangle - h(x) \mid x \in \mathbf{Z}^V\} \quad (p \in \mathbf{Z}^V).$$

A discrete separation theorem, called M-separation theorem, holds true for M-convex/concave functions [28, 29, 30]. Namely, let $f : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ be an $\mathrm{M}^\natural$-convex function and $h : \mathbf{Z}^V \to \mathbf{Z} \cup \{-\infty\}$ be an $\mathrm{M}^\natural$-concave function such that $\operatorname{dom} f \cap \operatorname{dom} h \neq \emptyset$ or $\operatorname{dom} f^\bullet \cap \operatorname{dom} h^\circ \neq \emptyset$. If $f(x) \geq h(x)$ $(x \in \mathbf{Z}^V)$, there exist $\alpha^* \in \mathbf{Z}$ and $p^* \in \mathbf{Z}^V$ such that

$$f(x) \geq \alpha^* + \langle p^*, x\rangle \geq h(x) \quad (x \in \mathbf{Z}^V).$$

Another discrete separation theorem, named L-separation theorem, is known [30]. Let $g : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ be an $\mathrm{L}^\natural$-convex function and $k : \mathbf{Z}^V \to \mathbf{Z} \cup \{-\infty\}$ be an $\mathrm{L}^\natural$-concave function such that $\operatorname{dom} g \cap \operatorname{dom} k \neq \emptyset$ or $\operatorname{dom} g^\bullet \cap \operatorname{dom} k^\circ \neq \emptyset$. If $g(p) \geq k(p)$ $(p \in \mathbf{Z}^V)$, there exist $\beta^* \in \mathbf{Z}$ and $x^* \in \mathbf{Z}^V$ such that

$$g(p) \geq \beta^* + \langle p, x^*\rangle \geq k(p) \quad (p \in \mathbf{Z}^V).$$

Since a submodular set function can be identified with a positively homogeneous L-convex function, the L-separation theorem is a generalization of Frank's discrete separation theorem for a pair of sub/supermodular functions [9], which reads as follows. Let $\rho : 2^V \to \mathbf{Z} \cup \{+\infty\}$ and $\mu : 2^V \to \mathbf{Z} \cup \{-\infty\}$ be submodular and supermodular functions, respectively, with $\rho(\emptyset) = \mu(\emptyset) = 0$, $\rho(V) < +\infty$, $\mu(V) > -\infty$, where μ is called supermodular if $-\mu$ is submodular. If $\rho(X) \geq \mu(X)$ $(X \subseteq V)$, there exists $x^* \in \mathbf{Z}^V$ such that

$$\rho(X) \geq x^*(X) \geq \mu(X) \quad (X \subseteq V).$$

The L- and M-separation theorems are conjugate to each other, while a self-conjugate statement can be made in the form of Fenchel-type duality [29, 30]. Let $f : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ be an $\mathrm{M}^\natural$-convex function and $h : \mathbf{Z}^V \to \mathbf{Z} \cup \{-\infty\}$ be an $\mathrm{M}^\natural$-concave function such that $\operatorname{dom} f \cap \operatorname{dom} h \neq \emptyset$ or $\operatorname{dom} f^\bullet \cap \operatorname{dom} h^\circ \neq \emptyset$. Then it holds that

$$\begin{aligned} &\inf\{f(x) - h(x) \mid x \in \mathbf{Z}^V\} \\ &= \sup\{h^\circ(p) - f^\bullet(p) \mid p \in \mathbf{Z}^V\}. \end{aligned} \tag{2}$$

Moreover, if this common value is finite, the infimum is attained by some $x \in \operatorname{dom} f \cap \operatorname{dom} h$ and the supremum is attained by some $p \in \operatorname{dom} f^\bullet \cap \operatorname{dom} h^\circ$.

4 M/L-convexity and Network Flow

M-convexity and L-convexity prevail in combinatorial optimization. Here we illustrate this for the minimum-cost flow problem.

Let $G = (V, A; V^+, V^-)$ be a directed graph with vertex set V, arc set A, set V^+ of entrances and set V^- of exits such that $V^+, V^- \subseteq V$ and $V^+ \cap V^- = \emptyset$. For a flow $\xi : A \to \mathbf{Z}$, its boundary $\partial\xi : V \to \mathbf{Z}$ is defined by

$$\begin{aligned} \partial\xi(v) \;=\; &\sum\{\xi(a) \mid a \in \delta^+ v\} \\ &- \sum\{\xi(a) \mid a \in \delta^- v\} \quad (v \in V), \end{aligned} \tag{3}$$

where $\delta^+ v$ and $\delta^- v$ denote the sets of leaving and entering arcs incident to $v \in V$, respectively. For a potential $\bar{p} : V \to \mathbf{Z}$, its coboundary $\delta\bar{p} : A \to \mathbf{Z}$ is defined by

$$\delta\bar{p}(a) = \bar{p}(\partial^+ a) - \bar{p}(\partial^- a) \quad (a \in A),$$

where $\partial^+ a$ and $\partial^- a$ mean the initial and terminal vertices of a, respectively. Denote the class of one-dimensional discrete convex functions by

$$\begin{aligned} \mathcal{C}_1 = \{&\varphi : \mathbf{Z} \to \mathbf{Z} \cup \{+\infty\} \mid \operatorname{dom}\varphi \neq \emptyset, \\ &\varphi(t-1) + \varphi(t+1) \geq 2\varphi(t) \ (t \in \mathbf{Z})\}. \end{aligned}$$

A key construction is the transformation of a conjugate pair of M-convex and L-convex functions through a network. For $f : \mathbf{Z}^{V^+} \to \mathbf{Z} \cup \{+\infty\}$ and $\varphi_a : \mathbf{Z} \to \mathbf{Z} \cup \{+\infty\}$ $(a \in A)$, define $\tilde{f} : \mathbf{Z}^{V^-} \to \mathbf{Z} \cup \{\pm\infty\}$ by

$$\begin{aligned}\tilde{f}(y) &= \inf_{\xi,x}\{f(x) + \sum_{a\in A} \varphi_a(\xi(a)) \mid \partial\xi = (x, -y, 0),\\ &\qquad \xi \in \mathbf{Z}^A, (x, -y, 0) \in \mathbf{Z}^{V^+} \times \mathbf{Z}^{V^-} \times \mathbf{Z}^{V\setminus(V^+\cup V^-)}\}\\ &\qquad\qquad (y \in \mathbf{Z}^{V^-}).\end{aligned}$$

For $g : \mathbf{Z}^{V^+} \to \mathbf{Z} \cup \{+\infty\}$ and $\psi_a : \mathbf{Z} \to \mathbf{Z} \cup \{+\infty\}$ $(a \in A)$, define $\tilde{g} : \mathbf{Z}^{V^-} \to \mathbf{Z} \cup \{\pm\infty\}$ by

$$\begin{aligned}\tilde{g}(q) &= \inf_{\eta,p,r}\{g(p) + \sum_{a\in A} \psi_a(\eta(a)) \mid \eta = -\delta(p, q, r),\\ &\qquad \eta \in \mathbf{Z}^A, (p, q, r) \in \mathbf{Z}^{V^+} \times \mathbf{Z}^{V^-} \times \mathbf{Z}^{V\setminus(V^+\cup V^-)}\}\\ &\qquad\qquad (q \in \mathbf{Z}^{V^-}).\end{aligned}$$

Theorem 2 *Assume that f is M-convex, g is L-convex, $\varphi_a \in \mathcal{C}_1$ $(a \in A)$ and $\psi_a \in \mathcal{C}_1$ $(a \in A)$. Assume further that $\tilde{f} > -\infty$, $\mathrm{dom}\,\tilde{f} \neq \emptyset$, and $\tilde{g} > -\infty$, $\mathrm{dom}\,\tilde{g} \neq \emptyset$. Then, (a) $\tilde{f}$ is M-convex, (b) $\tilde{g}$ is L-convex, and (c) if $g = f^\bullet$ and $\psi_a = \varphi_a{}^\bullet$ $(a \in A)$, then $\tilde{g} = \tilde{f}^\bullet$.* □

The conjugacy in Theorem 2 (c) is equivalent to the duality theorems in §3. The duality nature of the statement in Theorem 2(c) manifests itself in the special case of $(V^+, V^-) = (V, \emptyset)$. Define

$$\begin{aligned}\Phi(x, \xi) &= f(x) + \sum_{a\in A} \varphi_a(\xi(a)) \quad (x \in \mathbf{Z}^V, \xi \in \mathbf{Z}^A),\\ \Psi(p, \eta) &= -g(p) - \sum_{a\in A} \psi_a(\eta(a)) \quad (p \in \mathbf{Z}^V, \eta \in \mathbf{Z}^A).\end{aligned}$$

Then Theorem 2 (c) reduces to the following duality statement of Rockafellar-type [37]:

$$\begin{aligned}&\inf\{\Phi(x, \xi) \mid \partial\xi = x, x \in \mathbf{Z}^V, \xi \in \mathbf{Z}^A\}\\ &= \sup\{\Psi(p, \eta) \mid \eta = -\delta p, p \in \mathbf{Z}^V, \eta \in \mathbf{Z}^A\}.\end{aligned} \tag{4}$$

Another special case with $(V^+, V^-) = (V \setminus T, T)$, $f(x) = 0$ for $x = 0 \in \mathbf{Z}^{V\setminus T}$ and $= +\infty$ for $x \neq 0$, and $g(p) = 0$ for all $p \in \mathbf{Z}^{V\setminus T}$ yields

$$\begin{aligned}\tilde{f}(y) &= \inf_{\xi}\{\sum_{a\in A} \varphi_a(\xi(a)) \mid\\ &\qquad \partial\xi = (0, -y) \in \mathbf{Z}^{V\setminus T} \times \mathbf{Z}^T, \xi \in \mathbf{Z}^A\} \quad (y \in \mathbf{Z}^T),\\ \tilde{g}(q) &= \inf_{\eta,p}\{\sum_{a\in A} \psi_a(\eta(a)) \mid \eta = -\delta(p, q),\\ &\qquad p \in \mathbf{Z}^{V\setminus T}, \eta \in \mathbf{Z}^A\} \qquad (q \in \mathbf{Z}^T).\end{aligned}$$

Then the statements of Theorem 2 agree with the observations made in [31, Example 2.4] with reference to an electrical network of nonlinear resistors.

5 Minimizing M/L-convex Functions

The minimum of an M-convex function f is characterized by the local minimality in the sense that for $x \in \mathrm{dom}\, f$, we have $f(x) \leq f(y)$ for all $y \in \mathbf{Z}^V$ if and only if $f(x) \leq f(x - \chi_u + \chi_v)$ for all $u, v \in V$. This implies that the minimum of an M-convex function can be found by the following algorithm of steepest descent-type:

Steepest descent algorithm (f: M-convex)

S0: Let x be any vector in $\mathrm{dom}\, f$.

S1: If $f(x) = \min_{s,t \in V} f(x - \chi_s + \chi_t)$, then stop
[x is a minimizer of f].

S2: Find $u, v \in V$ such that
$f(x - \chi_u + \chi_v) = \min_{s,t \in V} f(x - \chi_s + \chi_t)$.

S3: Set $x := x - \chi_u + \chi_v$ and go to S1.

If f has the minimum (i.e., is not unbounded below), the number of function evaluations needed in this descent algorithm is finite, though it is not bounded by a polynomial in $|V|$.

Shioura's domain-reduction algorithm [38] finds the minimum of an M-convex function in polynomial time, provided that its effective domain is bounded. This algorithm searches for the minimum by generating a sequence of nested subsets of the domain on the basis of the fact that if $x \in \mathrm{dom}\, f$ is not a minimizer of an M-convex function f and $u, v \in V$ are such that $f(x - \chi_u + \chi_v) = \min_{s,t \in V} f(x - \chi_s + \chi_t)$, then there exits a minimizer x^* satisfying $x^*(u) \leq x(u) - 1$ and $x^*(v) \geq x(v) + 1$. For a bounded nonempty M-convex set B, define $l_B, u_B, \tilde{l}_B, \tilde{u}_B \in \mathbf{Z}^V$ and $B^\diamond \subseteq \mathbf{Z}^V$ by

$$
\begin{aligned}
& l_B(v) = \min_{y \in B} y(v), \quad u_B(v) = \max_{y \in B} y(v) \qquad (v \in V), \\
& \tilde{l}_B = \left\lfloor (1 - \frac{1}{n}) l_B + \frac{1}{n} u_B \right\rfloor, \tilde{u}_B = \left\lceil \frac{1}{n} l_B + (1 - \frac{1}{n}) u_B \right\rceil, \\
& B^\diamond = \{ y \in B \mid \tilde{l}_B \leq y \leq \tilde{u}_B \},
\end{aligned}
$$

where $n = |V|$. It can be shown that $B^\diamond$ is nonempty.

Domain-reduction algorithm [38] (f: M-convex)

S0: Set $B := \mathrm{dom}\, f$.

S1: Find a vector $x \in B^\diamond$.

S2: If $f(x) = \min_{s,t \in V} f(x - \chi_s + \chi_t)$, then stop
[x is a minimizer of f].

S3: Find $u, v \in V$ such that
$f(x - \chi_u + \chi_v) = \min_{s,t \in V} f(x - \chi_s + \chi_t)$.

S4: Set $B := B \cap \{ y \in \mathbf{Z}^V \mid$
$y(u) \leq x(u) - 1, y(v) \geq x(v) + 1 \}$
and go to S1.

Since $u_B(w) - l_B(w)$ for $w = u, v$ decreases by the factor $(1 - \frac{1}{n})$, the algorithm terminates in $O(n^2 \log L)$ iterations, where $L = \max\{u_{B_1}(v) - l_{B_1}(v) \mid v \in V\}$ for $B_1 = \mathrm{dom}\, f$. Once a vector in B is given, a vector in $B^\diamond$ can be generated with at most n^2 evaluations of the exchange capacity $\tilde{c}_B(x, v, u) = \max\{\alpha \mid x + \alpha(\chi_v - \chi_u) \in B\}$. Each evaluation of the exchange capacity can be done by a binary search with $\log L$ evaluations of f. Hence, the domain-reduction algorithm, given a vector in $\mathrm{dom}\, f$ and L, finds a minimizer of f with $O(n^4(\log L)^2)$ evaluations of f.

For an M-convex function f with $\mathrm{dom}\, f \subseteq \{0,1\}^V$, the greedy algorithm of Dress–Wenzel [3] finds the minimum of f with a polynomial number of function evaluations. We identify $x \in \{0,1\}^V$ with a subset X of V such that $x = \chi_X$ (characteristic vector of X), and define $\bar{f}(X) = f(\chi_X)$. Put $r = |X|$ for (any) X with $\chi_X \in \mathrm{dom}\, f$.

Greedy algorithm [3] (f: M-convex, $\mathrm{dom}\, f \subseteq \{0,1\}^V$)

S0: Let $X_0 = \{u_1, u_2, \cdots, u_r\}$ be any subset
such that $\bar{f}(X_0) < +\infty$, where the elements
are indexed arbitrarily.

S1: Repeats the following for $k = 1, 2, \cdots, r$:
Find $v_k \in (V \setminus X_{k-1}) \cup \{u_k\}$
$= V \setminus \{v_1, \cdots, v_{k-1}, u_{k+1}, \cdots, u_r\}$ such that
$\bar{f}(X_{k-1} - u_k + v_k) \leq \bar{f}(X_{k-1} - u_k + v)$
for all $v \in (V \setminus X_{k-1}) \cup \{u_k\}$ and
put $X_k := X_{k-1} - u_k + v_k$.

S2: Output $x^* = \chi_{X_r}$ as a minimizer of f.

In this way a minimizer of f can be found with $r(|V| - r) + 1$ function evaluations of f.

The minimum of an L-convex function g is characterized by the local minimality in the sense that, for $p \in \mathrm{dom}\, g$, $g(p) \leq g(q)$ for all $q \in \mathbf{Z}^V$ if and only if $g(p + \mathbf{1}) = g(p) \leq g(p + \chi_X)$ for all $X \subseteq V$. We assume that $g(p + \mathbf{1}) = g(p)$, since, otherwise, g has no minimum.

The local characterization above implies that the minimum of an L-convex function can be found by the following algorithm of steepest descent-type.

Steepest-descent algorithm (g: L-convex)

S0: Let p be any vector in $\mathrm{dom}\, g$.

S1: If $g(p) = \min_{Y \subseteq V} g(p + \chi_Y)$, then stop
[p is a minimizer of g].

S2: Find $X \subseteq V$ such that
$g(p + \chi_X) = \min_{Y \subseteq V} g(p + \chi_Y)$.

S3: Set $p := p + \chi_X$ and go to S1.

The operation needed in S2, as well as in S1, amounts to minimizing a submodular set function $\rho(Y) = g(p + \chi_Y) - g(p)$, which can be done by way of the ellipsoid

method [15] (in polynomial time), or by the minimum-norm point algorithm of Fujishige [12, §7.1]. If g has the minimum (i.e., is not unbounded below), the number of function evaluations of this descent algorithm is finite, though it is not bounded by a polynomial in $|V|$.

An L-convex function g can be extended to a piecewise-linear convex function $\overline{g}$ over $\mathbf{R}^V$ according to the Lovász extension in each unit hypercube (see [30, Theorem 4.18]). Namely, for $p \in \operatorname{dom} g$ and $q \in [0,1]^n$ we have

$$\overline{g}(p+q) = g(p) + \sum_{j=1}^{m} (\hat{q}_j - \hat{q}_{j+1})(g(p+\chi_{V_j}) - g(p)),$$

where $\hat{q}_1 > \hat{q}_2 > \cdots > \hat{q}_m$ are the distinct values of $\{q(v) \mid v \in V\}$ with $\hat{q}_{m+1} = 0$, and $V_j = \{v \in V \mid q(v) \geq \hat{q}_j\}$ for $j = 1, \cdots, m$. This expression makes it possible to evaluate the function $\overline{g}$ efficiently.

When $\{p \in \operatorname{dom} g \mid p(v_0) = 0\}$ is a bounded rectangle (where v_0 is any element of V), $\overline{g}$ is a polyhedral convex function, and a general result ([15, §6.5]) for the minimization of a polyhedral convex function implies the existence of a polynomial algorithm to find $q \in \mathbf{R}^V$ that minimizes $\overline{g}$. Then $p = \lfloor q \rfloor$ is a minimizer of g. This is the algorithm of Favati–Tardella [7] (applied to L-convex functions).

6 M-convex Submodular Flow Problem

A generalization of the submodular flow problem with an M-convex cost function can be solved efficiently. The optimality criterion for the submodular flow problem, to be stated in Theorem 3 below, is equivalent to the duality theorems for M/L-convex functions in §3, and thus the algorithms for the generalized submodular flow problem can be used to solve the pair of primal/dual problems in the Fenchel-type duality (2) and in the network duality (4).

Let $G = (V, A)$ be a graph with a vertex set V and an arc set A. We are given an upper capacity function $\overline{c} : A \to \mathbf{Z} \cup \{+\infty\}$, a lower capacity function $\underline{c} : A \to \mathbf{Z} \cup \{-\infty\}$, a cost function $\gamma : A \to \mathbf{Z}$, and an M-convex function $f : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ such that $x(V) = 0$ for (any) $x \in \operatorname{dom} f$.

We consider the following generalization of the submodular flow problem, in which the cost of a flow ξ involves a term $f(\partial\xi)$ for the boundary (cf. (3)) in addition to the usual arc cost.

M-convex submodular flow problem [28]

Minimize $\Gamma(\xi) = \sum_{a \in A} \gamma(a)\xi(a) + f(\partial\xi)$

subject to

$$\underline{c}(a) \leq \xi(a) \leq \overline{c}(a) \qquad (a \in A), \tag{5}$$

$$\partial\xi \in \operatorname{dom} f, \tag{6}$$

$$\xi(a) \in \mathbf{Z} \qquad (a \in A). \tag{7}$$

The ordinary submodular flow problem, for which the reader is referred to [1, 18, 9, 10, 12], is a special case where f is the indicator function of an M-convex set.

The valuated matroid intersection problem [26, 27] is another special case. A more general problem with a separable convex arc cost $\sum_{a\in A}\varphi_a(\xi(a))$ in place of the linear arc cost $\sum_{a\in A}\gamma(a)\xi(a)$ can be formulated [28] as an M-convex submodular flow problem.

The optimality criteria refer to an auxiliary network and a potential function. The auxiliary network (G_ξ, ℓ_ξ) associated with a feasible flow ξ is defined as follows. The underlying graph $G_\xi = (V, A_\xi)$ has the vertex set V and the arc set A_ξ consisting of three disjoint parts: $A_\xi = A^*_\xi \cup B^*_\xi \cup C_\xi$, where

$$\begin{aligned} A^*_\xi &= \{a \mid a \in A, \xi(a) < \overline{c}(a)\}, \\ B^*_\xi &= \{\overline{a} \mid a \in A, \underline{c}(a) < \xi(a)\} \ (\overline{a}\text{: reorientation of } a), \\ C_\xi &= \{(u,v) \mid u, v \in V, u \neq v, \partial\xi - \chi_u + \chi_v \in \mathrm{dom}\, f\}. \end{aligned}$$

The length function $\ell_\xi : A_\xi \to \mathbf{Z}$ is defined by

$$\ell_\xi(a) = \begin{cases} \gamma(a) & (a \in A^*_\xi) \\ -\gamma(\overline{a}) & (a \in B^*_\xi, \overline{a} \in A) \\ f(\partial\xi, u, v) & (a = (u,v) \in C_\xi) \end{cases}$$

where $f(\partial\xi, u, v) = f(\partial\xi - \chi_u + \chi_v) - f(\partial\xi)$. We call a directed cycle of negative length a negative cycle.

By a potential function we mean a function $p : V \to \mathbf{Z}$ (or a vector $p \in \mathbf{Z}^V$). We define $\gamma_p : A \to \mathbf{Z}$ and $f[-p] : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ by

$$\begin{aligned} \gamma_p(a) &= \gamma(a) + p(\partial^+ a) - p(\partial^- a) \qquad (a \in A), \\ f[-p](x) &= f(x) - \sum_{v\in V} p(v)x(v) \qquad (x \in \mathbf{Z}^V). \end{aligned}$$

Theorem 3 *(1) The following three conditions are equivalent for a feasible flow $\xi : A \to \mathbf{Z}$ (satisfying (5) and (6)):*
(OPT) *ξ is optimal.*
(NNC) *There is no negative cycle in the auxiliary network (G_ξ, ℓ_ξ).*
(POT) *There exists a potential $p : V \to \mathbf{Z}$ such that*
(i) for each $a \in A$,

$$\gamma_p(a) > 0 \implies \xi(a) = \underline{c}(a), \tag{8}$$
$$\gamma_p(a) < 0 \implies \xi(a) = \overline{c}(a), \tag{9}$$

(ii) $\partial\xi$ minimizes $f[-p]$.
(2) Let p be a potential that satisfies (i)–(ii) above for some (optimal) flow ξ. A feasible flow ξ' is optimal if and only if it satisfies (i)–(ii) (with ξ replaced by ξ'). □

Let $g : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ be the L-convex function conjugate to the M-convex function f. It is convenient to introduce a notation $B_p(g) = \mathrm{argmin}\, f[-p]$ to

indicate the fact that $B_p(g)$ coincides with the set of the integer points of the base polyhedron defined by the submodular set function $\rho(X) = g(p + \chi_X) - g(p)$, i.e.,

$$B_p(g) = \{x \in \mathbf{Z}^V \mid x(X) \leq \rho(X)\ (X \subset V); x(V) = 0\}.$$

With this notation the condition (ii) in (POT) can be expressed as

$$\partial\xi \in B_p(g). \tag{10}$$

The optimality criterion (NNC) leads to the following cycle-cancelling algorithm. When (G_ξ, ℓ_ξ) has a negative cycle, let Q $(\subseteq A_\xi)$ be the arc set of a negative cycle having the smallest number of arcs, and define $\xi' : A \to \mathbf{Z}$ by

$$\xi'(a) = \begin{cases} \xi(a) + 1 & (a \in Q \cap A^*_\xi) \\ \xi(a) - 1 & (\overline{a} \in Q \cap B^*_\xi) \\ \xi(a) & (\text{otherwise}). \end{cases} \tag{11}$$

Then it can be shown that ξ' is a feasible flow with a strictly smaller cost: $\Gamma(\xi') \leq \Gamma(\xi) + \ell_\xi(Q) < \Gamma(\xi)$. See [12, 18] for algorithms to find an initial feasible flow.

Cycle-cancelling algorithm [28]

S0: Let ξ be a feasible flow.
S1: If (G_ξ, ℓ_ξ) has no negative cycle, then stop
[ξ is the optimal flow].
S2: Let Q be the arc set of a negative cycle having
the smallest number of arcs.
S3: Set $\xi := \xi'$ of (11) and go to S1.

If Γ has the minimum (i.e., is not unbounded below), the number of iterations in this algorithm is finite, though it is not bounded by a polynomial in $|V|$. In the case of the valuated matroid intersection problem, some variants of the cycle-cancelling algorithm are known [27] to be strongly polynomial.

The other optimality criterion (POT) leads to primal-dual algorithms of Murota [27] (for the special case of the valuated matroid intersection problem) and of Iwata–Shigeno [19] (a natural extension of the idea of [2] for the ordinary submodular flow problem). We shall describe Iwata–Shigeno's algorithm below.

The algorithm maintains a feasible flow ξ and a potential p that satisfy the condition (10) and repeat modifying (ξ, p) to increase the set of in-kilter arcs, where an arc is said to be in kilter if it satisfies (8) and (9), and out of kilter otherwise. An initial (ξ, p) can be found, for example, by taking any feasible flow ξ and finding an integral vector p such that $p(v) - p(u) \leq f(\partial\xi, u, v)$ for all $u, v \in V$, where such p can be found by solving a shortest-path problem on the graph $G = (V, A)$ with respect to the arc length $f(\partial\xi, u, v)$ for arc (u, v).

For $v \in V$ define

$$\begin{aligned} D^+_\xi(v) &= \{a \in \delta^+ v \mid \gamma_p(a) < 0, \xi(a) < \overline{c}(a)\}, \\ D^-_\xi(v) &= \{a \in \delta^- v \mid \gamma_p(a) > 0, \xi(a) > \underline{c}(a)\}, \\ D_\xi(v) &= D^+_\xi(v) \cup D^-_\xi(v), \end{aligned}$$

where the dependence on p is implicit in the notation. Note that $\{D_\xi(v) \mid v \in V\}$ gives a partition of the set of out-of-kilter arcs.

If $D_\xi(v) = \emptyset$ for all $v \in V$, then the condition (i) of (POT) is satisfied and the current (ξ, p) is optimal. Otherwise, the algorithm picks up any $v^\star \in V$ with nonempty $D_\xi(v^\star)$, and tries to meet the conditions (8) and (9) for $a \in D_\xi^-(v^\star)$ and $a \in D_\xi^+(v^\star)$, respectively, by changing the flow to ξ'. The flow ξ' is determined by solving the following maximum submodular flow problem on $G = (V, A)$ with a more restrictive capacity constraint defined by

$$(\ \underline{c}^\star(a), \overline{c}^\star(a)\ ) = \begin{cases} (\ \underline{c}(a), \xi(a)\ ) & (\gamma_p(a) > 0) \\ (\ \underline{c}(a), \overline{c}(a)\ ) & (\gamma_p(a) = 0) \\ (\ \xi(a), \overline{c}(a)\ ) & (\gamma_p(a) < 0). \end{cases}$$

Maximum submodular flow problem (MSF)

Maximize $\quad \xi'(D_\xi^+(v^\star)) - \xi'(D_\xi^-(v^\star))$

$$\left(= \sum_{a \in D_\xi^+(v^\star)} \xi'(a) - \sum_{a \in D_\xi^-(v^\star)} \xi'(a)\right)$$

subject to

$$\underline{c}^\star(a) \le \xi'(a) \le \overline{c}^\star(a) \qquad (a \in A), \tag{12}$$

$$\partial\xi' \in B_p(g), \tag{13}$$

$$\xi'(a) \in \mathbf{Z} \qquad (a \in A). \tag{14}$$

Note that the maximum submodular flow problem is a special case of the submodular flow problem and a number of efficient algorithms are available for it (see [12, §5.5(b)]).

If $D_{\xi'}(v^\star)$ is empty, the algorithm updates ξ to ξ' without changing p. The condition (10) is maintained because of (13). If $D_{\xi'}(v^\star)$ is nonempty, the algorithm finds a "minimum cut" $W \subseteq V$ (explained later) for (MSF) and updates p to

$$p' = p + \chi_W \tag{15}$$

as well as ξ to ξ'. The condition $\partial\xi' \in B_{p'}(g)$ is guaranteed by the discrete midpoint convexity of the L-convex function g.

In the course of iteration,

- the set of out-of-kilter arcs is nonincreasing,
- for each arc a, $|\gamma_p(a)|$ is nonincreasing as long as a is out of kilter, and
- each time the potential p is changed, $\max_{a \in D_\xi(v^\star)} |\gamma_p(a)|$ decreases at least by one.

As a consequence, the sum of $\max_{a \in D_\xi(v)} |\gamma_p(a)|$ over all $v \in V$ with $D_\xi(v) \neq \emptyset$ strictly decreases in each update of (ξ, p), and therefore, the algorithm terminates after a finite number of iterations of the above procedure.

Primal-dual algorithm [19]

S0: Find (ξ, p) satisfying (5) and (10).
S1: If $D_\xi(v) = \emptyset$ for all $v \in V$, then stop
[(ξ, p) is optimal].
S2: Take any $v^\star \in V$ with $D_\xi(v^\star) \neq \emptyset$ and
solve (MSF) to obtain ξ'.
S3: If $D_{\xi'}(v^\star) \neq \emptyset$, then find a "minimum cut" W
and set $p := p + \chi_W$.
S4: Set $\xi := \xi'$ and go to S1.

It remains to explain about the "minimum cut" for (MSF). For $W \subseteq V$ we denote by $\Delta^+ W$ and $\Delta^- W$ the sets of arcs leaving W and entering W, respectively. The max-flow min-cut type theorem for (MSF) (cf. [12, Theorem 5.11]) states that the flow value $\xi'(D_\xi^+(v^\star)) - \xi'(D_\xi^-(v^\star))$ is bounded by the cut capacity

$$\begin{aligned}
&\overline{c}^\star(\Delta^- W \setminus D_\xi^-(v^\star)) + \overline{c}^\star(D_\xi^+(v^\star) \setminus \Delta^+ W) \\
&-\underline{c}^\star(\Delta^+ W \setminus D_\xi^+(v^\star)) - \underline{c}^\star(D_\xi^-(v^\star) \setminus \Delta^- W) \\
&+g(p + \chi_W) - g(p)
\end{aligned}$$

for any $W \subseteq V$ that contains $v^\star$, and that this bound is tight for some W, which is referred to as a "minimum cut" in the above.

When the potential is changed to $p' = p + \chi_W$ for a minimum cut W, $\gamma_p(a)$ changes only for $a \in \Delta^+ W \cup \Delta^- W$. We have

$$\xi'(a) = \underline{c}^\star(a) = \begin{cases} \underline{c}(a) & (\gamma_p(a) \geq 0) \\ \xi(a) & (\gamma_p(a) < 0) \end{cases}$$

for $a \in (\Delta^+ W \setminus D_\xi^+(v^\star)) \cup (D_\xi^-(v^\star) \setminus \Delta^- W)$, and $\gamma_{p'}(a) = \gamma_p(a) + 1$ for $a \in \Delta^+ W \setminus D_\xi^+(v^\star)$. Therefore, in-kilter arcs in $\Delta^+ W \setminus D_\xi^+(v^\star)$ as well as all arcs in $D_\xi^-(v^\star) \setminus \Delta^- W$ are in kilter with respect to (ξ', p'). Similarly for arcs in $\Delta^- W \setminus D_\xi^-(v^\star)$ and $D_\xi^+(v^\star) \setminus \Delta^+ W$, respectively. For $a \in (\Delta^+ W \cap D_\xi^+(v^\star)) \cup (\Delta^- W \cap D_\xi^-(v^\star))$ we have $|\gamma_{p'}(a)| = |\gamma_p(a)| - 1$. Hence follows the claim that $\max_{a \in D_\xi(v^\star)} |\gamma_p(a)|$ decreases at least by one. Finally, the claim $\partial\xi' \in B_{p'}(g)$ follows from

$$\begin{aligned}
\partial\xi'(X) &= \partial\xi'(X \cup W) + \partial\xi'(X \cap W) - \partial\xi'(W) \\
&\leq g(p + \chi_{X \cup W}) + g(p + \chi_{X \cap W}) - g(p) - g(p + \chi_W) \\
&\leq g(p + \chi_W + \chi_X) - g(p + \chi_W),
\end{aligned}$$

where the first inequality is due to (13) and $\partial\xi'(W) = g(p + \chi_W) - g(p)$, and the second is by the discrete midpoint convexity (1).

The primal-dual algorithm can be polished up to a (weakly) polynomial algorithm by means of a cost scaling technique. The standard cost scaling technique for the linear arc cost employs $\sum_{a \in A} \lceil \gamma(a)/\alpha \rceil \xi(a)$ as the scaled cost function, where α is a positive integer. A similar straightforward scaling $\lceil f(\partial\xi)/\alpha \rceil$ for the boundary cost does not seem to work, as it does not preserve M-convexity. Instead we utilize

a novel scaling technique, called the conjugate scaling, for the M-convex function f to obtain the polynomial time algorithm, called the conjugate-scaling algorithm. For its description it is convenient to consider a continuous generalization of the M-convex submodular flow problem, as follows.

The concepts of M-convexity and L-convexity can be defined for real-valued functions on $\mathbf{Z}^V$ in an obvious manner in terms of (M-EXC) for M-convexity and (L1) and (L2) for L-convexity, where $r \in \mathbf{R}$ in (L2). Furthermore, those concepts can be extended to polyhedral convex functions [34]. A polyhedral convex function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ is said to be M-convex if $\mathrm{dom}\, f \neq \emptyset$ and f satisfies

(M-EXC[$\mathbf{R}$]) For $x, y \in \mathrm{dom}\, f$ and $u \in \mathrm{supp}^+(x-y)$, there exists $v \in \mathrm{supp}^-(x-y)$ and $\alpha_0 > 0$ such that

$$f(x) + f(y) \geq f(x - \alpha(\chi_u - \chi_v)) + f(y + \alpha(\chi_u - \chi_v)),$$

for all α with $0 \leq \alpha \leq \alpha_0$. It should be clear that $\mathrm{dom}\, f = \{x \in \mathbf{R}^V \mid f(x) < +\infty\}$. A polyhedral convex function $g : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ is defined to be L-convex if $\mathrm{dom}\, g \neq \emptyset$ and g satisfies

(L1[$\mathbf{R}$]) $g(p) + g(q) \geq g(p \vee q) + g(p \wedge q) \qquad (p, q \in \mathbf{R}^V)$,

(L2[$\mathbf{R}$]) $\exists r \in \mathbf{R} : g(p + \lambda \mathbf{1}) = g(p) + \lambda r \qquad (p \in \mathbf{R}^V, \lambda \in \mathbf{R})$.

The M- and L-convex functions are conjugate to each other under the (ordinary) Fenchel-Legendre transformation.

In the continuous version of the M-convex submodular flow problem, we consider a real-valued flow ξ for real-valued data: an upper capacity function $\overline{c} : A \to \mathbf{R} \cup \{+\infty\}$, a lower capacity function $\underline{c} : A \to \mathbf{R} \cup \{-\infty\}$, a cost function $\gamma : A \to \mathbf{R}$, and a polyhedral M-convex function $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ such that $x(V) = 0$ for (any) $x \in \mathrm{dom}\, f$. Accordingly, the constraint (7) is replaced by

$$\xi(a) \in \mathbf{R} \qquad (a \in A). \tag{16}$$

The optimality criteria (Theorem 3) can be generalized with obvious modifications (see [34, Corollary 7.9]). We discuss here the integrality of the optimal flow (primal integrality) and that of the potential (dual integrality). In the ordinary submodular flow problem (with $\{0, +\infty\}$-valued f), the primal integrality is guaranteed if the capacity functions $\overline{c}$ and $\underline{c}$ are integer-valued and $\mathrm{dom}\, f$ is an integral base polyhedron, whereas the dual integrality holds in the case of an integer-valued cost function γ. For the M-convex submodular flow problem, this can be generalized as follows (where the statement (2) is in [19]).

Theorem 4 *Assume that the (continuous) M-convex submodular flow problem has an optimal flow.*
(1) [Primal integrality] There exists an integral optimal flow $\xi : A \to \mathbf{Z}$ if $\overline{c}$ and $\underline{c}$ are integer-valued and $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ is obtained as the convex closure of an M-convex function $f^I : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$.
(2) [Dual integrality] There exists an integral optimal potential $p : V \to \mathbf{Z}$ if γ is integer-valued and $f^\bullet : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ is obtained as the convex closure of an L-convex function $g^I : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$. □

The cycle-cancelling algorithm with the flow update (11) is applicable to problems with primal integrality and the primal-dual algorithm with the potential update (15) works for problems with dual integrality, where "$\mathbf{Z}$" in (14) in (MSF) is replaced by "$\mathbf{R}$".

The conjugate scaling algorithm repeatedly applies the primal-dual algorithm to a sequence of problems with dual integrality. Let $f : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ be an M-convex function with dual integrality, which means

$$f(x) = \sup\{\langle p, x\rangle - g^I(p) \mid p \in \mathbf{Z}^V\} \quad (x \in \mathbf{R}^V) \tag{17}$$

for an L-convex function $g^I : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$. For a positive integer α and $q \in \mathbf{Z}^V$, the function $g^I_{\alpha q}(p) = g^I(\alpha p + q)/\alpha$ is a real-valued L-convex function on $\mathbf{Z}^V$, provided that $\operatorname{dom} g^I_{\alpha q} \neq \emptyset$. Denote the conjugate of $g^I_{\alpha q}$ in the sense of (17) by $f_{\alpha q} : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$, which is an M-convex function with dual integrality. The conjugate scaling algorithm employs $f_{\alpha q}$ as the scaled boundary cost function.

The algorithm described below solves the (continuous) M-convex submodular flow problem with dual integrality, as long as $\operatorname{dom} g^I_{\alpha q} \neq \emptyset$ in the course of computation. Note that the boundedness of $\operatorname{dom} f$ implies $\operatorname{dom} f^\bullet = \mathbf{Z}^V$, which ensures $\operatorname{dom} g^I_{\alpha q} \neq \emptyset$ for all α and q. Recall the notation $n = |V|$.

Conjugate-scaling algorithm [19]

S0: Find $(\xi, p) \in \mathbf{R}^A \times \mathbf{Z}^V$ satisfying (5) and (10),
and set $q := p$, $p^* := 0$, $K := \max_{a \in A} |\gamma(a)|$,
and $\alpha := 2^{\lceil \log_2 K \rceil}$.

S1: If $\alpha = 1$, then stop [(ξ, p) is optimal].

S2: Find $p \in \mathbf{Z}^V$ that minimizes $g^I_{\alpha q}(p) - \langle p, \partial\xi\rangle$
subject to $2p^* \leq p \leq 2p^* + n\mathbf{1}$.

S3: Solve the M-convex submodular flow problem
for $(\gamma, f) = (\lceil \gamma/\alpha \rceil, f_{\alpha q})$ by the primal-dual
algorithm starting from (ξ, p) to obtain
an optimal $(\xi^*, p^*) \in \mathbf{R}^A \times \mathbf{Z}^V$.

S4: Set $\xi := \xi^*$, $\alpha := \alpha/2$, and go to S1.

It can be proven that the vector p in S2 minimizes $g^I_{\alpha q}(p) - \langle p, \partial\xi\rangle$ over $\mathbf{Z}^V$, which implies $\partial\xi \in B_p(g^I_{\alpha q}) = \operatorname{argmin} f_{\alpha q}[-p]$. The conjugate scaling algorithm solves the M-convex submodular flow problem in time polynomial in n and $\log K$.

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References

[1] R. E. Bixby and W. H. Cunningham, "Matroid optimization and algorithms," in Handbook of Combinatorics, vol. I, eds. R. L. Graham, M. Grötschel, and L. Lovász, Chapter 11, pp.551–609, Elsevier Science B. V., Amsterdam, 1995.

[2] W. H. Cunningham and A. Frank, "A primal-dual algorithm for submodular flows," Math. Operations Research, vol.10, pp.251–262, 1985.

[3] A. W. M. Dress and W. Wenzel, "Valuated matroid: A new look at the greedy algorithm," Appl. Math. Lett., vol.3, pp.33–35, 1990.

[4] A. W. M. Dress and W. Wenzel, "Valuated matroids," Advances in Math., vol.93, pp.214–250, 1992.

[5] J. Edmonds, "Submodular functions, matroids and certain polyhedra," in Combinatorial Structures and Their Applications, eds. R. Guy, H. Hanani, N. Sauer, and J. Schönheim, pp.69–87, Gordon and Breach, New York, 1970.

[6] J. Edmonds, "Matroid intersection," Annals of Discrete Math., vol.14, pp.39–49, 1979.

[7] P. Favati and F. Tardella, "Convexity in nonlinear integer programming," Ricerca Operativa, vol.53, pp.3–44, 1990.

[8] A. Frank, "A weighted matroid intersection algorithm," J. Algorithms, vol.2, pp.328–336, 1981.

[9] A. Frank, "An algorithm for submodular functions on graphs," Annals of Discrete Math., vol.16, pp.97–120, 1982.

[10] A. Frank and É. Tardos, "Generalized polymatroids and submodular flows," Math. Programming, vol.42, pp.489–563, 1988.

[11] S. Fujishige, "Theory of submodular programs: A Fenchel-type min-max theorem and subgradients of submodular functions," Math. Programming, vol.29, pp.142-155, 1984.

[12] S. Fujishige, "Submodular Functions and Optimization," Annals of Discrete Math., vol.47, North-Holland, Amsterdam, 1991.

[13] S. Fujishige and K. Murota, "Notes on L-/M-convex functions and the separation theorems," Math. Programming, to appear.

[14] H. Groenevelt, "Two algorithms for maximizing a separable concave function over a polymatroidal feasible region," European J. Operations Research, vol.54, pp.227–236, 1991.

[15] M. Grötschel, L. Lovász, and A. Schrijver, "Geometric Algorithms and Combinatorial Optimization (2nd. ed.)," Springer-Verlag, Berlin, 1993.

[16] D. S. Hochbaum, "Lower and upper bounds for the allocation problem and other nonlinear optimization problems," Math. Operations Research, vol.19, pp.390–409, 1994.

[17] M. Iri and N. Tomizawa, "An algorithm for finding an optimal 'independent assignment'," J. Operations Research Soc. Japan, vol.19, pp.32–57, 1976.

[18] S. Iwata: "The submodular flow problem (in Japanese)," in Discrete Structures and Algorithms, VI, ed. S. Fujishige, pp.127–170, Kindai-Kagaku-sha, Tokyo, 1999.

[19] S. Iwata and M. Shigeno: "Conjugate scaling technique for Fenchel-type duality in discrete convex optimization," Information Processing Society of Japan, SIG Notes 98-AL-65, pp.33–40, 1998.

[20] N. Katoh and T. Ibaraki, "Resource allocation problems," in Handbook of Combinatorial Optimization, vol.2, eds. D. -Z. Du and P. M. Pardalos, Kluwer Academic Publishers, Boston/Dordrecht/London, pp.159–260, 1998.

[21] E. L. Lawler, "Matroid intersection algorithms," Math. Programming, vol.9, pp.31–56, 1975.

[22] L. Lovász, "Submodular functions and convexity," in Mathematical Programming — The State of the Art, eds. A. Bachem, M. Grötschel, and B. Korte, pp.235–257, Springer-Verlag, Berlin, 1983.

[23] B. L. Miller, "On minimizing nonseparable functions defined on the integers with an inventory application," SIAM J. Appl. Math., vol.21, pp.166–185, 1971.

[24] M. Minoux, "Solving integer minimum cost flows with separable convex objective polynomially," Math. Programming, vol.26, pp.237–239, 1986.

[25] K. Murota, "Finding optimal minors of valuated bimatroids," Appl. Math. Lett., vol.8, pp.37–42, 1995.

[26] K. Murota, "Valuated matroid intersection, I: optimality criteria," SIAM J. Discrete Math., vol.9, pp.545–561, 1996.

[27] K. Murota, "Valuated matroid intersection, II: algorithms," SIAM J. Discrete Math., vol.9, pp.562–576, 1996.

[28] K. Murota, "Submodular flow problem with a nonseparable cost function," Combinatorica, vol.19, pp.87–109, 1999.

[29] K. Murota, "Convexity and Steinitz's exchange property," Advances in Math., vol.124, pp.272–311, 1996.

[30] K. Murota, "Discrete convex analysis," Math. Programming, vol.83, pp.313–371, 1998.

[31] K. Murota, "Discrete convex analysis (in Japanese)," in Discrete Structures and Algorithms, V, ed. S. Fujishige, pp.51–100, Kindai-Kagaku-sha, Tokyo, 1998.

[32] K. Murota, "Matrices and Matroids for Systems Analysis," Springer-Verlag, Berlin, 1999.

[33] K. Murota and A. Shioura, "M-convex function on generalized polymatroid," Math. Operations Research, vol.24, pp.95–105, 1999.

[34] K. Murota and A. Shioura, "Extension of M-convexity and L-convexity to polyhedral convex functions," RIMS Preprint 1227, Kyoto University, March 1999.

[35] K. Murota and A. Shioura, "Relationship of M-/L-convex functions with discrete convex functions by Miller and by Favati–Tardella," RIMS Preprint 1239, Kyoto University, June 1999.

[36] R. T. Rockafellar, "Convex Analysis," Princeton University Press, Princeton, NJ, 1970.

[37] R. T. Rockafellar, "Network Flows and Monotropic Optimization," John Wiley and Sons, New York, 1984.

[38] A. Shioura, "Minimization of an M-convex function," Discrete Appl. Math., vol.84, pp.215–220, 1998.

[39] H. Whitney, "On the abstract properties of linear dependence," American J. Math., vol.57, pp.509–533, 1935.

[40] U. Zimmermann, "Minimization of some nonlinear functions over polymatroidal network flows," Annals of Discrete Math., vol.16, pp.287–309, 1982.