Discrete Convex Analysis — Survey of the Present Status —

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Abstract: The present knowledge about L-convex and M-convex functions are surveyed on the basis of [27].

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Introduction

In the field of nonlinear programming (in continuous variables) convex analysis [30, 31] plays a pivotal role both in theory and in practice. An analogous theory for discrete optimization (nonlinear integer programming), called "discrete convex analysis" [24, 25], is developed for L-convex and M-convex functions by adapting the ideas in convex analysis and generalizing the results in matroid theory. The L- and M-convex functions are introduced in [24] and [22, 23], respectively. A polyhedral theory on L- and M-convex functions is developed recently in [29].

Definitions of L-convexity and M-convexity

Let V be a nonempty finite set and $\mathbf{Z}$ be the set of integers. For any function $g : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ define $\mathrm{dom}\, g = \{p \in \mathbf{Z}^V \mid g(p) < +\infty\}$, called the effective domain of g.

A function $g : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ with $\mathrm{dom}\, g \neq \emptyset$ is called L-convex if

$$g(p) + g(q) \geq g(p \vee q) + g(p \wedge q) \quad (p, q \in \mathbf{Z}^V),$$

$$\exists r \in \mathbf{Z} : \ g(p + \mathbf{1}) = g(p) + r \quad (p \in \mathbf{Z}^V),$$

where $p \vee q = (\max(p(v), q(v)) \mid v \in V) \in \mathbf{Z}^V$, $p \wedge q = (\min(p(v), q(v)) \mid v \in V) \in \mathbf{Z}^V$, and $\mathbf{1}$ is the vector in $\mathbf{Z}^V$ with all of its components being equal to 1.

A set $D \subseteq \mathbf{Z}^V$ is said to be an L-convex set if its indicator function δ_D (defined by: $\delta_D(p) = 0$ if $p \in D$, and $= +\infty$ otherwise) is an L-convex function, i.e., if (i) $D \neq \emptyset$, (ii) $p, q \in D \Rightarrow p \vee q, p \wedge q \in D$, and (iii) $p \in D \Rightarrow p \pm \mathbf{1} \in D$.

A function $f : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ with $\mathrm{dom}\, f \neq \emptyset$ is called M-convex if it satisfies

(M-EXC) For $x, y \in \mathrm{dom}\, f$ and $u \in \mathrm{supp}^+(x - y)$, there exists $v \in \mathrm{supp}^-(x - y)$ such that

$$f(x) + f(y) \geq f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v)$$

where, for any $u \in V$, χ_u is the characteristic vector of u (defined by: $\chi_u(v) = 1$ if $v = u$, and $= 0$ otherwise), and

$$\begin{aligned} \mathrm{supp}^+(z) &= \{v \in V \mid z(v) > 0\} \quad (z \in \mathbf{Z}^V), \\ \mathrm{supp}^-(z) &= \{v \in V \mid z(v) < 0\} \quad (z \in \mathbf{Z}^V). \end{aligned}$$

A set $B \subseteq \mathbf{Z}^V$ is said to be an M-convex set if its indicator function is an M-convex function, i.e., if B satisfies

(B-EXC) For $x, y \in B$ and for $u \in \mathrm{supp}^+(x - y)$, there exists $v \in \mathrm{supp}^-(x - y)$ such that $x - \chi_u + \chi_v \in B$ and $y + \chi_u - \chi_v \in B$.

This means that an M-convex set is the same as the set of integer points of the base polyhedron of an integral submodular system (see [10] for submodular systems).

L-convexity and M-convexity are conjugate to each other under the integral Fenchel-Legendre transformation $f \mapsto f^\bullet$ defined by

$$f^\bullet(p) = \sup\{\langle p, x \rangle - f(x) \mid x \in \mathbf{Z}^V\} \quad (p \in \mathbf{Z}^V),$$

where $\langle p, x\rangle = \sum_{v\in V} p(v)x(v)$. That is, for L-convex function g and M-convex function f, it holds [24] that $g^\bullet$ is M-convex, $f^\bullet$ is L-convex, $g^{\bullet\bullet} = g$, and $f^{\bullet\bullet} = f$.

Example 1: Minimum cost flow problem

L-convexity and M-convexity are inherent in the integer minimum-cost flow problem, as pointed out in [23, 24]. Let $G = (V, A)$ be a graph with vertex set V and arc set A, and let $T \subseteq V$ be given. For $\xi : A \to \mathbf{Z}$ its boundary $\partial\xi : V \to \mathbf{Z}$ is defined by

$$\begin{aligned}\partial\xi(v) &= \sum\{\xi(a) \mid a \in \delta^+ v\} \\ &\quad - \sum\{\xi(a) \mid a \in \delta^- v\} \quad (v \in V),\end{aligned}$$

where $\delta^+ v$ and $\delta^- v$ denote the sets of out-going and in-coming arcs incident to v, respectively. For $\tilde{p} : V \to \mathbf{Z}$ its coboundary $\delta\tilde{p} : A \to \mathbf{Z}$ is defined by

$$\delta\tilde{p}(a) = \tilde{p}(\partial^+ a) - \tilde{p}(\partial^- a) \quad (a \in A),$$

where $\partial^+ a$ and $\partial^- a$ mean the initial and terminal vertices of a, respectively. Denote the class of one-dimensional discrete convex functions by

$$\begin{aligned}\mathcal{C}_1 = \{\varphi : \mathbf{Z} \to \mathbf{Z} \cup \{+\infty\} \mid \operatorname{dom}\varphi \neq \emptyset, \\ \varphi(t-1) + \varphi(t+1) \geq 2\varphi(t)\ (t \in \mathbf{Z})\}.\end{aligned}$$

For $\varphi_a \in \mathcal{C}_1$ $(a \in A)$, representing the arc-cost in terms of flow, the total cost function $f : \mathbf{Z}^T \to \mathbf{Z} \cup \{+\infty\}$ defined by

$$\begin{aligned}f(x) = \inf_\xi\{\sum_{a\in A} \varphi_a(\xi(a)) \mid \\ \partial\xi(v) = x(v)\ (v \in T), \\ \partial\xi(v) = 0\ (v \in V \setminus T)\} \quad (x \in \mathbf{Z}^T)\end{aligned}$$

is M-convex, provided that $f > -\infty$ (i.e., f does not take the value of $-\infty$). For $\psi_a \in \mathcal{C}_1$ $(a \in A)$, representing the arc-cost in terms of tension, the total cost function $g : \mathbf{Z}^T \to \mathbf{Z} \cup \{+\infty\}$ defined by

$$\begin{aligned}g(p) = \inf_{\tilde{p}}\{\sum_{a\in A} \psi_a(\delta\tilde{p}(a)) \mid \\ \tilde{p}(v) = p(v)\ (v \in T)\} \quad (p \in \mathbf{Z}^T)\end{aligned}$$

is L-convex, provided that $g > -\infty$. The two cost functions $f(x)$ and $g(p)$ are conjugate to each other in the sense that, if $\psi_a = \varphi_a{}^\bullet$ $(a \in A)$, then $g = f^\bullet$.

Example 2: Polynomial matrix

Let $A(s)$ be an $m \times n$ matrix of rank m with each entry being a polynomial in a variable s, and let $\mathcal{B} \subseteq 2^V$ be the family of bases of $A(s)$ with respect to linear independence of the column vectors; namely, $J \subseteq V$ belongs to $\mathcal{B}$ if and only if $|J| = m$ and the column vectors with indices in J are linearly independent. Then $f : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ defined by

$$f(x) = \begin{cases} -\deg_s \det A[J] & (x = \chi_J, J \in \mathcal{B}) \\ +\infty & (\text{otherwise}) \end{cases}$$

is M-convex, where $\chi_J \in \{0,1\}^V$ is the characteristic vector of J (defined by: $\chi_J(v) = 1$ if $v \in J$, and $= 0$ otherwise), $A[J]$ denotes the $m \times m$ submatrix with column indices in $J \in \mathcal{B}$, and $\deg_s(\cdot)$ means the degree as a polynomial in s. The Grassmann-Plücker identity implies the exchange property of f. This example was the motivation of valuated matroids in [2, 3], which in turn can be identified with the negative of M-convex functions f with $\operatorname{dom} f \subseteq \{0,1\}^V$.

For $p = (p(v) \mid v \in V) \in \mathbf{Z}^V$ denote by $D(p)$ the diagonal matrix of order $n = |V|$ with diagonal elements $s^{p(v)}$ $(v \in V)$. Then the function $g : \mathbf{Z}^V \to \mathbf{Z}$ defined by

$$g(p) = \max\{\deg_s \det(A \cdot D(p))[J] \mid J \in \mathcal{B}\}$$

is L-convex [25], where $(A \cdot D(p))[J]$ means the $m \times m$ submatrix of $A \cdot D(p)$ with column indices in J. We have $g = f^\bullet$.

L-convex sets

An L-convex set $D \subseteq \mathbf{Z}^V$ has "no holes" in the sense that $D = \overline{D} \cap \mathbf{Z}^V$, where $\overline{D}$ denotes the convex hull of D in $\mathbf{R}^V$. Hence it is natural to consider the polyhedral description of $\overline{D}$, "L-convex polyhedron" (see [24, 25]). For any function $\gamma : V \times V \to \mathbf{Z} \cup \{+\infty\}$ with $\gamma(v, v) = 0$ $(v \in V)$, define

$$\begin{aligned}\mathbf{D}(\gamma) = \{p \in \mathbf{R}^V \mid \\ p(v) - p(u) \leq \gamma(u, v)\ (\forall u, v \in V)\}.\end{aligned}$$

If $\mathbf{D}(\gamma) \neq \emptyset$, $\mathbf{D}(\gamma)$ is an integral polyhedron and $D = \mathbf{D}(\gamma) \cap \mathbf{Z}^V$ is an L-convex set. If γ satisfies triangle inequality:

$$\gamma(u, v) + \gamma(v, w) \geq \gamma(u, w) \quad (u, v, w \in V),$$

then $\mathbf{D}(\gamma) \neq \emptyset$ and

$$\gamma(u,v) = \sup\{p(v) - p(u) \mid p \in \mathbf{D}(\gamma)\} \quad (u,v \in V).$$

Conversely, for any nonempty $D \subseteq \mathbf{Z}^V$,

$$\gamma(u,v) = \sup\{p(v) - p(u) \mid p \in D\} \quad (u,v \in V)$$

satisfies triangle inequality as well as $\gamma(v,v) = 0$ $(v \in V)$, and if D is L-convex, then $\overline{D} = \mathbf{D}(\gamma)$. Thus there is a one-to-one correspondence between L-convex set D and function γ satisfying triangle inequality. In particular, $D \subseteq \mathbf{Z}^V$ is L-convex if and only if $D = \mathbf{D}(\gamma) \cap \mathbf{Z}^V$ for some γ satisfying triangle inequality. For L-convex sets $D_1, D_2 \subseteq \mathbf{Z}^V$, it holds that $D_1 + D_2 = \overline{D_1 + D_2} \cap \mathbf{Z}^V$ and $\overline{D_1} \cap \overline{D_2} = \overline{D_1 \cap D_2}$.

It is also true that a function γ satisfying triangle inequality corresponds one-to-one to a positively homogeneous M-convex function f (i.e, $f(\lambda x) = \lambda f(x)$ for $x \in \mathbf{Z}^V$ and $0 \leq \lambda \in \mathbf{Z}$). The correspondence $f \mapsto \gamma$ is given by

$$\gamma(u,v) = f(\chi_v - \chi_u) \quad (u,v \in V),$$

whereas $\gamma \mapsto f$ by

$$f(x) = \inf_{\lambda}\{ \sum_{u,v \in V} \lambda_{uv}\gamma(u,v) \mid \sum_{u,v \in V} \lambda_{uv}(\chi_v - \chi_u) = x, \; 0 \leq \lambda_{uv} \in \mathbf{Z} \; (u,v \in V)\} \quad (x \in \mathbf{Z}^V).$$

The correspondence between L-convex sets and positively homogeneous M-convex functions via functions with triangle inequality is a special case of the conjugacy relationship between L- and M-convex functions.

M-convex sets

An M-convex set $B \subseteq \mathbf{Z}^V$ has "no holes" in the sense that $B = \overline{B} \cap \mathbf{Z}^V$. Hence it is natural to consider the polyhedral description of $\overline{B}$, "M-convex polyhedron." A set function $\rho : 2^V \to \mathbf{Z} \cup \{+\infty\}$ is said to be submodular if

$$\rho(X) + \rho(Y) \geq \rho(X \cup Y) + \rho(X \cap Y) \quad (X, Y \subseteq V),$$

where the inequality is satisfied if $\rho(X)$ or $\rho(Y)$ is equal to $+\infty$. It is assumed throughout that $\rho(\emptyset) = 0$ and $\rho(V) < +\infty$ for any set function $\rho : 2^V \to \mathbf{Z} \cup \{+\infty\}$. For a set function ρ, define

$$\mathbf{B}(\rho) = \{x \in \mathbf{R}^V \mid x(X) \leq \rho(X) \; (\forall X \subset V), x(V) = \rho(V)\},$$

where $x(X) = \sum\{x(v) \mid v \in X\}$. If ρ is submodular, $\mathbf{B}(\rho)$ is a nonempty integral polyhedron, $B = \mathbf{B}(\rho) \cap \mathbf{Z}^V$ is an M-convex set, and

$$\rho(X) = \sup\{x(X) \mid x \in \mathbf{B}(\rho)\} \quad (X \subseteq V).$$

Conversely, for any nonempty $B \subseteq \mathbf{Z}^V$, define a set function ρ by

$$\rho(X) = \sup\{x(X) \mid x \in B\} \quad (X \subseteq V).$$

If B is M-convex, then ρ is submodular and $\overline{B} = \mathbf{B}(\rho)$. Thus there is a one-to-one correspondence between M-convex set B and submodular set function ρ. In particular, $B \subseteq \mathbf{Z}^V$ is M-convex if and only if $B = \mathbf{B}(\rho) \cap \mathbf{Z}^V$ for some submodular ρ. The correspondence $B \leftrightarrow \rho$ is a restatement of a well-known fact [4, 10]. For M-convex sets $B_1, B_2 \subseteq \mathbf{Z}^V$, it holds that $B_1 + B_2 = \overline{B_1 + B_2} \cap \mathbf{Z}^V$ and $\overline{B_1} \cap \overline{B_2} = \overline{B_1 \cap B_2}$.

It is also true that a submodular set function ρ corresponds one-to-one to a positively homogeneous L-convex function g. The correspondence $g \mapsto \rho$ is given by the restriction

$$\rho(X) = g(\chi_X) \quad (X \subseteq V)$$

(χ_X is the characteristic vector of X), whereas $\rho \mapsto g$ by the Lovász extension (explained below). The correspondence between M-convex sets and positively homogeneous L-convex functions via submodular set functions is a special case of the conjugacy relationship between M- and L-convex functions.

For a set function $\rho : 2^V \to \mathbf{Z} \cup \{+\infty\}$, the Lovász extension [16] of ρ is a function $\hat{\rho} : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ defined by

$$\hat{\rho}(p) = \sum_{j=1}^{n} (p_j - p_{j+1})\rho(V_j) \quad (p \in \mathbf{R}^V),$$

where, for each $p \in \mathbf{R}^V$, the elements of V are indexed as $\{v_1, v_2, \cdots, v_n\}$ (with $n = |V|$) in such a way that $p(v_1) \geq p(v_2) \geq \cdots \geq p(v_n)$; $p_j = p(v_j)$,

$V_j = \{v_1, v_2, \cdots, v_j\}$ for $j = 1, \cdots, n$, and $p_{n+1} = 0$. The right-hand side of the above expression is equal to $+\infty$ if and only if $p_j - p_{j+1} > 0$ and $\rho(V_j) = +\infty$ for some j with $1 \leq j \leq n-1$. The Lovász extension $\hat{\rho}$ is indeed an extension of ρ, since $\hat{\rho}(\chi_X) = \rho(X)$ for $X \subseteq V$.

The relationship between submodularity and convexity is revealed by the statement [16] that a set function ρ is submodular if and only if its Lovász extension $\hat{\rho}$ is convex.

The restriction to $\mathbf{Z}^V$ of the Lovász extension of a submodular set function is a positively homogeneous L-convex function, and any positively homogeneous L-convex function can be obtained in this way [24].

Properties of L-convex functions

For any $g : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ and $x \in \mathbf{R}^V$, define $g[-x] : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ by

$$g[-x](p) = g(p) - \langle p, x \rangle \quad (p \in \mathbf{Z}^V).$$

The set of the minimizers of $g[-x]$ is denoted as $\operatorname{argmin}(g[-x])$.

Let $g : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ be L-convex. Then $\operatorname{dom} g$ is an L-convex set. For each $p \in \operatorname{dom} g$,

$$\rho_p(X) = g(p + \chi_X) - g(p) \quad (X \subseteq V)$$

is a submodular set function with $\rho_p(\emptyset) = 0$ and $\rho_p(V) < +\infty$.

An L-convex function g can be extended to a convex function $\overline{g} : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$ through the Lovász extension of the submodular set functions ρ_p for $p \in \operatorname{dom} g$. Namely, for $p \in \operatorname{dom} g$ and $q \in [0,1]^V$, it holds [24] that

$$\overline{g}(p+q) = g(p) + \sum_{j=1}^{n} (q_j - q_{j+1})(g(p + \chi_{V_j}) - g(p)),$$

where, for each q, the elements of V are indexed as $\{v_1, v_2, \cdots, v_n\}$ (with $n = |V|$) in such a way that $q(v_1) \geq q(v_2) \geq \cdots \geq q(v_n)$; $q_j = q(v_j)$, $V_j = \{v_1, v_2, \cdots, v_j\}$ for $j = 1, \cdots, n$, and $q_{n+1} = 0$. The expression of $\overline{g}$ shows that an L-convex function is an integrally convex function in the sense of [5].

An L-convex function g enjoys "discrete midpoint convexity":

$$g(p) + g(q) \geq g\left(\left\lceil \frac{p+q}{2} \right\rceil\right) + g\left(\left\lfloor \frac{p+q}{2} \right\rfloor\right)$$

for $p, q \in \mathbf{Z}^V$, where $\lceil p \rceil$ (or $\lfloor p \rfloor$) for any $p \in \mathbf{R}^V$ denotes the vector obtained by rounding up (or down) the components of p to the nearest integers.

The minimum of an L-convex function g is characterized by the local minimality in the sense that, for $p \in \operatorname{dom} g$, $g(p) \leq g(q)$ for all $q \in \mathbf{Z}^V$ if and only if $g(p+\mathbf{1}) = g(p) \leq g(p+\chi_X)$ for all $X \subseteq V$.

The minimizers of an L-convex function, if nonempty, forms an L-convex set. For any $x \in \mathbf{R}^V$, $\operatorname{argmin}(g[-x])$, if nonempty, is an L-convex set. Conversely, this property characterizes L-convex functions, under an auxiliary assumption that the function can be extented to a convex function over $\mathbf{R}^V$ (see [29]).

A number of operations can be defined for L-convex functions [24, 25]. For $x \in \mathbf{Z}^V$, $g[-x]$ is an L-convex function. For $a \in \mathbf{Z}^V$ and $\beta \in \mathbf{Z}$, $g(a + \beta p)$ is L-convex in p. For $U \subseteq V$, the projection of g to U:

$$g^U(p') = \inf\{g(p', p'') \mid p'' \in \mathbf{Z}^{V \setminus U}\} \quad (p' \in \mathbf{Z}^U)$$

is L-convex in p', provided that $g^U > -\infty$. For $\psi_v \in \mathcal{C}_1$ $(v \in V)$,

$$\tilde{g}(p) = \inf_{q \in \mathbf{Z}^V} \left[g(q) + \sum_{v \in V} \psi_v(p(v) - q(v)) \right]$$

is L-convex in $p \in \mathbf{Z}^V$, provided that $\tilde{g} > -\infty$. The sum of two (or more) L-convex functions is L-convex, provided that its effective domain is nonempty.

Properties of M-convex functions

Let $f : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ be M-convex. Then $\operatorname{dom} f$ is an M-convex set. For each $x \in \operatorname{dom} f$,

$$\gamma_x(u, v) = f(x - \chi_u + \chi_v) - f(x) \quad (u, v \in V)$$

satisfies [25] triangle inequality.

An M-convex function f can be extended to a convex function $\overline{f} : \mathbf{R}^V \to \mathbf{R} \cup \{+\infty\}$, and the value of $\overline{f}(x)$ for $x \in \mathbf{R}^V$ is determined by

$\{f(y) \mid y \in \mathbf{Z}^V, \lfloor x \rfloor \leq y \leq \lceil x \rceil\}$. That is, an M-convex function is an integrally convex function in the sense of [5].

The minimum of an M-convex function f is characterized by the local minimality in the sense that for $x \in \operatorname{dom} f$, $f(x) \leq f(y)$ for all $y \in \mathbf{Z}^V$ if and only if $f(x) \leq f(x - \chi_u + \chi_v)$ for all $u, v \in V$ [17, 23, 24].

The minimizers of an M-convex function, if nonempty, forms an M-convex set. Moreover, for any $p \in \mathbf{R}^V$, $\operatorname{argmin}(f[-p])$, if nonempty, is an M-convex set. Conversely, this property characterizes M-convex functions, under an auxiliary assumption that the effective domain is bounded or the function can be extented to a convex function over $\mathbf{R}^V$ (see [23, 24]).

The level set of an M-convex function is not necessarily an M-convex set, but enjoys a weaker exchange property. Namely, for any $p \in \mathbf{R}^V$ and $\alpha \in \mathbf{R}$, $S = \{x \in \mathbf{Z}^V \mid f[-p](x) \leq \alpha\}$ (the level set of $f[-p]$) satisfies: For $x, y \in S$ and for $u \in \operatorname{supp}^+(x - y)$, there exists $v \in \operatorname{supp}^-(x - y)$ such that either $x - \chi_u + \chi_v \in S$ or $y + \chi_u - \chi_v \in S$. Conversely, this property characterizes M-convex functions [35].

A number of operations can be defined for M-convex functions [24, 25]. For $p \in \mathbf{Z}^V$, $f[-p]$ is an M-convex function. For $a \in \mathbf{Z}^V$, $f(a - x)$ and $f(a + x)$ are M-convex in x. For $U \subseteq V$, the restriction of f to U:

$$f_U(x') = f(x', \mathbf{0}_{V \setminus U}) \quad (x' \in \mathbf{Z}^U)$$

(where $\mathbf{0}_{V \setminus U}$ is the zero vector in $\mathbf{Z}^{V \setminus U}$) is M-convex in x', provided that $\operatorname{dom} f_U \neq \emptyset$. For $\varphi_v \in \mathcal{C}_1$ $(v \in V)$,

$$\tilde{f}(x) = f(x) + \sum_{v \in V} \varphi_v(x(v)) \quad (x \in \mathbf{Z}^V)$$

is M-convex, provided that $\operatorname{dom} \tilde{f} \neq \emptyset$. In particular, a separable convex function $\tilde{f}(x) = \sum_{v \in V} \varphi_v(x(v))$ with $\operatorname{dom} \tilde{f}$ being an M-convex set is an M-convex function. For two M-convex functions f_1 and f_2, the integral convolution

$$(f_1 \square f_2)(x) = \inf\{f_1(x_1) + f_2(x_2) \mid x = x_1 + x_2; x_1, x_2 \in \mathbf{Z}^V\} \quad (x \in \mathbf{Z}^V)$$

is either M-convex or else $(f_1 \square f_2)(x) = \pm\infty$ for all $x \in \mathbf{Z}^V$.

An M-convex function can be transformed into another M-convex function through a network ([23, 25]; see also [33]). Let $G = (V, A; V^+, V^-)$ be a directed graph with vertex set V, arc set A, set V^+ of entrances and set V^- of exits such that $V^+, V^- \subseteq V$ and $V^+ \cap V^- = \emptyset$. For $\varphi_a \in \mathcal{C}_1$ $(a \in A)$ and $f : \mathbf{Z}^{V^+} \to \mathbf{Z} \cup \{+\infty\}$, define $\tilde{f} : \mathbf{Z}^{V^-} \to \mathbf{Z} \cup \{\pm\infty\}$ by

$$\tilde{f}(x) = \inf_{\xi}\{f((\partial\xi)^+) + \sum_{a \in A} \varphi_a(\xi(a)) \mid \partial\xi(v) = x(v) \ (v \in V^-), \ \partial\xi(v) = 0 \ (v \in V \setminus (V^+ \cup V^-))\} \quad (x \in \mathbf{Z}^{V^-}),$$

where $\xi : A \to \mathbf{Z}$ and $(\partial\xi)^+$ (resp. $(\partial\xi)^-$) denotes the restriction of $\partial\xi$ to V^+ (resp. V^-). If f is M-convex, then $\tilde{f}$ is M-convex, provided that $\tilde{f} > -\infty$.

Sum of two M-convex functions is not necessarily M-convex; such function with nonempty effective domain is called M_2-convex. Convolution of two L-convex functions is not necessarily L-convex; such function with nonempty effective domain is called L_2-convex. M_2- and L_2-convex functions are in one-to-one correspondence through the integral Fenchel-Legendre transformation.

$\mathrm{L}^\natural$- and $\mathrm{M}^\natural$-convexity

$\mathrm{L}^\natural$- and $\mathrm{M}^\natural$-convexity are variants of, and essentially equivalent to, L- and M-convexity, respectively. $\mathrm{L}^\natural$- and $\mathrm{M}^\natural$-convex functions are introduced in [11] and [28], respectively.

Let v_0 be a new element not in V and define $\tilde{V} = \{v_0\} \cup V$. A function $g : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ with $\operatorname{dom} g \neq \emptyset$ is called $\mathrm{L}^\natural$-convex if it is expressed in terms of an L-convex function $\tilde{g} : \mathbf{Z}^{\tilde{V}} \to \mathbf{Z} \cup \{+\infty\}$ as $g(p) = \tilde{g}(0, p)$. Namely, an $\mathrm{L}^\natural$-convex function is a function obtained as the restriction of an L-convex function. Conversely, an $\mathrm{L}^\natural$-convex function determines the corresponding L-convex function up to the constant r in the definition of L-convex function.

An $\mathrm{L}^\natural$-convex function is essentially the same as a submodular integrally convex function of [5], and hence is characterized by "discrete midpoint convexity" [11]. An L-convex function, enjoy-

ing "discrete midpoint convexity," is an $L^\natural$-convex function.

Quadratic function

$$g(p) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}p_ip_j \quad (p \in \mathbf{Z}^n)$$

with $a_{ij} = a_{ji} \in \mathbf{Z}$ is $L^\natural$-convex if and only if $a_{ij} \leq 0$ $(i \neq j)$ and $\sum_{j=1}^{n} a_{ij} \geq 0$ $(i = 1, \cdots, n)$. For $\{\psi_i \in \mathcal{C}_1 \mid i = 1, \cdots, n\}$, a separable convex function

$$g(p) = \sum_{i=1}^{n} \psi_i(p_i) \quad (p \in \mathbf{Z}^n)$$

is $L^\natural$-convex.

The properties of L-convex functions mentioned above are carried over, mutatis mutandis, to $L^\natural$-convex functions. In addition, the restriction of an $L^\natural$-convex function g to $U \subseteq V$, denoted g_U, is $L^\natural$-convex.

A subset of $\mathbf{Z}^V$ is called an $L^\natural$-convex set if its indicator function is an $L^\natural$-convex function. A set $E \subseteq \mathbf{Z}^V$ is an $L^\natural$-convex set if and only if

$$p, q \in E \implies \left\lceil \frac{p+q}{2} \right\rceil, \left\lfloor \frac{p+q}{2} \right\rfloor \in E.$$

A function $f : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ with $\operatorname{dom} f \neq \emptyset$ is called $M^\natural$-convex if it is expressed in terms of an M-convex function $\tilde{f} : \mathbf{Z}^{\tilde{V}} \to \mathbf{Z} \cup \{+\infty\}$ as

$$\tilde{f}(x_0, x) = \begin{cases} f(x) & \text{if } x_0 + \sum_{u \in V} x(u) = 0 \\ +\infty & \text{otherwise.} \end{cases}$$

Namely, an $M^\natural$-convex function is a function obtained as the projection of an M-convex function. Conversely, an $M^\natural$-convex function determines the corresponding M-convex function up to a translation of $\operatorname{dom} f$ in the direction of v_0. A function $f : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ with $\operatorname{dom} f \neq \emptyset$ is $M^\natural$-convex if and only if (see [28]) it satisfies

($M^\natural$-EXC) For $x, y \in \operatorname{dom} f$ and $u \in \operatorname{supp}^+(x-y)$,

$$\begin{aligned} &f(x) + f(y) \\ &\geq \min[\, f(x - \chi_u) + f(y + \chi_u), \\ &\qquad \min_{v \in \operatorname{supp}^-(x-y)} \{f(x - \chi_u + \chi_v) \\ &\qquad\qquad + f(y + \chi_u - \chi_v)\} \,]. \end{aligned}$$

Since (M-EXC) implies ($M^\natural$-EXC), an M-convex function is an $M^\natural$-convex function.

Quadratic function

$$f(x) = \sum_{i=1}^{n} a_i {x_i}^2 + b\sum_{i<j} x_ix_j \qquad (x \in \mathbf{Z}^n)$$

with $a_i \in \mathbf{Z}$ $(1 \leq i \leq n)$, $b \in \mathbf{Z}$ is $M^\natural$-convex if $0 \leq b \leq 2\min_{1 \leq i \leq n} a_i$ (cf. [28]). For $\{\varphi_i \in \mathcal{C}_1 \mid i = 0, 1, \cdots, n\}$, a function of the form

$$f(x) = \varphi_0(\sum_{i=1}^{n} x_i) + \sum_{i=1}^{n} \varphi_i(x_i) \quad (x \in \mathbf{Z}^n)$$

is $M^\natural$-convex [28]; a separable convex function is a special case of this (with $\varphi_0 = 0$. More generally, for $\{\varphi_X \in \mathcal{C}_1 \mid X \in \mathcal{T}\}$ indexed by a laminar family $\mathcal{T} \subseteq 2^V$, the function

$$f(x) = \sum_{X \in \mathcal{T}} \varphi_X(x(X)) \quad (x \in \mathbf{Z}^V)$$

is $M^\natural$-convex [1], where $\mathcal{T}$ is called laminar if for any $X, Y \in \mathcal{T}$, at least one of $X \cap Y$, $X \setminus Y$, $Y \setminus X$ is empty.

The properties of M-convex functions mentioned above are carried over, mutatis mutandis, to $M^\natural$-convex functions. In addition, the projection of an $M^\natural$-convex function f to $U \subseteq V$, denoted f^U, is $M^\natural$-convex.

A subset of $\mathbf{Z}^V$ is called an $M^\natural$-convex set if its indicator function is an $M^\natural$-convex function. A set $Q \subseteq \mathbf{Z}^V$ is an $M^\natural$-convex set if and only if Q is the set of integer points of an integral generalized polymatroid (cf. [7] for generalized polymatroids).

As a consequence of the conjugacy between L- and M-convexity, $L^\natural$-convex functions and $M^\natural$-convex functions are conjugate to each other under the integral Fenchel-Legendre transformation.

Duality

Discrete duality theorems hold true for L-convex/concave and M-convex/concave functions. A function $g : \mathbf{Z}^V \to \mathbf{Z} \cup \{-\infty\}$ is called L-concave (resp., $L^\natural$-, M-, or $M^\natural$-concave) if $-g$ is L-convex (resp., $L^\natural$-, M-, or $M^\natural$-convex); $\operatorname{dom} g$ means the effective domain of $-g$. The concave counterpart of the discrete Fenchel-Legendre transform is defined as

$$g^\circ(p) = \inf\{\langle p, x\rangle - g(x) \mid x \in \mathbf{Z}^V\} \quad (p \in \mathbf{Z}^V).$$

A discrete separation theorem for L-convex/ L-concave functions, named L-separation theorem [24] (see also [12]), reads as follows. Let $f : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ be an $L^\natural$-convex function and $g : \mathbf{Z}^V \to \mathbf{Z} \cup \{-\infty\}$ be an $L^\natural$-concave function such that $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$ or $\operatorname{dom} f^\bullet \cap \operatorname{dom} g^\circ \neq \emptyset$. If $f(p) \geq g(p)$ $(p \in \mathbf{Z}^V)$, there exist $\beta^* \in \mathbf{Z}$ and $x^* \in \mathbf{Z}^V$ such that

$$f(p) \geq \beta^* + \langle p, x^* \rangle \geq g(p) \quad (p \in \mathbf{Z}^V).$$

Since a submodular set function can be identified with a positively homogeneous L-convex function, the L-separation theorem implies Frank's discrete separation theorem for a pair of sub/supermodular functions [6], which reads as follows. Let $\rho : 2^V \to \mathbf{Z} \cup \{+\infty\}$ and $\mu : 2^V \to \mathbf{Z} \cup \{-\infty\}$ be submodular and supermodular functions, respectively, with $\rho(\emptyset) = \mu(\emptyset) = 0$, $\rho(V) < +\infty$, $\mu(V) > -\infty$, where μ is called supermodular if $-\mu$ is submodular. If $\rho(X) \geq \mu(X)$ $(X \subseteq V)$, there exists $x^* \in \mathbf{Z}^V$ such that

$$\rho(X) \geq x^*(X) \geq \mu(X) \quad (X \subseteq V).$$

Another discrete separation theorem, M-separation theorem [23, 24] (see also [12]), holds true for M-convex/concave functions. Namely, let $f : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ be an $M^\natural$-convex function and $g : \mathbf{Z}^V \to \mathbf{Z} \cup \{-\infty\}$ be an $M^\natural$-concave function such that $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$ or $\operatorname{dom} f^\bullet \cap \operatorname{dom} g^\circ \neq \emptyset$. If $f(x) \geq g(x)$ $(x \in \mathbf{Z}^V)$, there exist $\alpha^* \in \mathbf{Z}$ and $p^* \in \mathbf{Z}^V$ such that

$$f(x) \geq \alpha^* + \langle p^*, x \rangle \geq g(x) \quad (x \in \mathbf{Z}^V).$$

The L- and M-separation theorems are conjugate to each other, while a self-conjugate statement can be made in the form of the Fenchel-type duality [23, 24], as follows. Let $f : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ be an $L^\natural$-convex function and $g : \mathbf{Z}^V \to \mathbf{Z} \cup \{-\infty\}$ be an $L^\natural$-concave function such that $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$ or $\operatorname{dom} f^\bullet \cap \operatorname{dom} g^\circ \neq \emptyset$. Then it holds that

$$\begin{aligned} &\inf\{f(p) - g(p) \mid p \in \mathbf{Z}^V\} \\ &= \sup\{g^\circ(x) - f^\bullet(x) \mid x \in \mathbf{Z}^V\}. \end{aligned}$$

Moreover, if this common value is finite, the infimum is attained by some $p \in \operatorname{dom} f \cap \operatorname{dom} g$ and the supremum is attained by some $x \in \operatorname{dom} f^\bullet \cap \operatorname{dom} g^\circ$.

Here is a simple example to illustrate the non-triviality of discrete separation for discrete functions. Functions $f : \mathbf{Z}^2 \to \mathbf{Z}$ and $g : \mathbf{Z}^2 \to \mathbf{Z}$ defined by $f(x_1, x_2) = \max(0, x_1 + x_2)$ and $g(x_1, x_2) = \min(x_1, x_2)$ can be extended respectively to a convex function $\overline{f} : \mathbf{R}^2 \to \mathbf{R}$ and a concave function $\overline{g} : \mathbf{R}^2 \to \mathbf{R}$ according to the defining expressions. With $\overline{p} = \left(\frac{1}{2}, \frac{1}{2}\right)$, we have $\overline{f}(x) \geq \langle \overline{p}, x \rangle \geq \overline{g}(x)$ for all $x \in \mathbf{R}^2$, and a fortiori, $f(x) \geq \langle \overline{p}, x \rangle \geq g(x)$ for all $x \in \mathbf{Z}^2$. However, there exists no integral vector $p \in \mathbf{Z}^2$ such that $f(x) \geq \langle p, x \rangle \geq g(x)$ for all $x \in \mathbf{Z}^2$. Note also that f is $M^\natural$-convex and g is L-concave.

Subdifferentials

The subdifferential of $f : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ at $x \in \operatorname{dom} f$ is defined by $\{p \in \mathbf{R}^V \mid f(y) - f(x) \geq \langle p, y - x \rangle \ (\forall y \in \mathbf{Z}^V)\}$. The subdifferential of an L_2- or M_2-convex function forms an integral polyhedron. More specifically:

- The subdifferential of an L-convex function is an integral base polyhedron (an M-convex polyhedron).
- The subdifferential of an L_2-convex function is the intersection of two integral base polyhedra (M-convex polyhedra).
- The subdifferential of an M-convex function is an L-convex polyhedron.
- The subdifferential of an M_2-convex function is the Minkowski sum of two L-convex polyhedra.

Similar statements hold true with L and M replaced respectively by $L^\natural$ and $M^\natural$.

Algorithms

On the basis of the equivalence of $L^\natural$-convex functions and submodular integrally convex functions, the minimization of an L-convex function can be done by the algorithm of [5], which relies on the ellipsoid method. The minimization of an M-convex function can be done by purely combinatorial algorithms; a greedy-type algorithm [2] for valuated matroids and a domain reduction-type polynomial-time algorithm [34] for M-convex

functions. Algorithms for duality of M-convex functions (in other words, for M_2-convex functions) are also developed; polynomial algorithms [22, 32] for valuated matroids, and a finite primal algorithm [17] and a polynomial-time conjugate-scaling algorithm [13] for the submodular flow problem.

Applications

A discrete analogue of the conjugate duality framework [31] for nonlinear optimization is developed in [24]. An application of M-convex functions to engineering system analysis and matrix theory is in [19, 26]. M-convex functions find applications also in mathematical economics [1] and in resource allocation problems [14].

References

[1] V. Danilov, G. Koshevoy and K. Murota, Equilibria in economies with indivisible goods and money, RIMS Preprint 1204, Kyoto University (May 1998).

[2] A. W. M. Dress and W. Wenzel, Valuated matroid: A new look at the greedy algorithm, *Applied Mathematics Letters* 3 (1990) 33–35.

[3] A. W. M. Dress and W. Wenzel, Valuated matroids, *Advances in Mathematics* 93 (1992) 214–250.

[4] J. Edmonds, Submodular functions, matroids and certain polyhedra, in: R. Guy, H. Hanani, N. Sauer and J. Schönheim, eds., *Combinatorial Structures and Their Applications* (Gordon and Breach, New York, 1970) 69–87.

[5] P. Favati and F. Tardella, Convexity in nonlinear integer programming, *Ricerca Operativa* 53 (1990) 3–44.

[6] A. Frank, An algorithm for submodular functions on graphs, *Annals of Discrete Mathematics* 16 (1982) 97–120.

[7] A. Frank and É. Tardos, Generalized polymatroids and submodular flows, *Mathematical Programming* 42 (1988) 489–563.

[8] S. Fujishige, Theory of submodular programs: A Fenchel-type min-max theorem and subgradients of submodular functions, *Mathematical Programming* 29 (1984) 142-155.

[9] S. Fujishige, On the subdifferential of a submodular function, *Mathematical Programming* 29 (1984) 348-360.

[10] S. Fujishige, *Submodular Functions and Optimization* (Annals of Discrete Mathematics 47, North-Holland, Amsterdam, 1991).

[11] S. Fujishige and K. Murota, On the relationship between L-convex functions and submodular integrally convex functions, RIMS Preprint 1152, Kyoto University (August 1997).

[12] S. Fujishige and K. Murota, Short proofs of the separation theorems for L-convex/concave and M-convex/concave functions, RIMS Preprint 1167, Kyoto University (October 1997).

[13] S. Iwata and M. Shigeno, Conjugate scaling technique for Fenchel-type duality in discrete optimization, IPSJ SIG Notes 98-AL-65 (1998).

[14] N. Katoh and T. Ibaraki, Resource allocation problems, *Handbook of Combinatorial Optimization*, Vol.2 (1998), (D. -Z. Du and P. M. Pardalos, eds.), Kluwer, 159–260.

[15] B. L. Miller, On minimizing nonseparable functions defined on the integers with an inventory application, *SIAM Journal on Applied Mathematics* 21 (1971) 166–185.

[16] L. Lovász, Submodular functions and convexity, in: A. Bachem, M. Grötschel and B. Korte, eds., *Mathematical Programming – The State of the Art* (Springer-Verlag, Berlin, 1983) 235–257.

[17] K. Murota, Valuated matroid intersection, I: optimality criteria, II: algorithms, *SIAM Journal on Discrete Mathematics* 9 (1996) 545–561, 562–576.

[18] K. Murota, On exchange axioms for valuated matroids and valuated delta-matroids. *Combinatorica* 6 (1996) 591–596.

[19] K. Murota, Structural approach in systems analysis by mixed matrices – An exposition for index of DAE. ICIAM 95 (Proceedings of the Third International Congress on Industrial and Applied Mathematics, Hamburg, July 3–7, 1995) K. Kirchgässner, O. Mahrenholtz, and R. Mennicken, eds., Mathematical Research, Vol. 87 (1996), Akademie Verlag, pp. 257–279.

[20] K. Murota, Fenchel-type duality for matroid valuations, *Mathematical Programming* 82 (1998) 357–375.

[21] K. Murota, Matroid valuation on independent sets, *Journal of Combinatorial Theory* B69 (1997) 59–78.

[22] K. Murota, Submodular flow problem with a nonseparable cost function, *Combinatorica*, to appear.

[23] K. Murota, Convexity and Steinitz's exchange property, *Advances in Mathematics* 124 (1996) 272–311.

[24] K. Murota, Discrete convex analysis, *Mathematical Programming* 83 (1998) 313–371.

[25] K. Murota, "Discrete convex analysis," in: S. Fujishige (ed.), *Discrete Structures and Algorithms V*, Kindai-Kagaku-sha, Tokyo, 51–100 (1998). [In Japanese.]

[26] K. Murota, On the degree of mixed polynomial matrices, *SIAM Journal on Matrix Analysis and Applications* 20 (1999) 196–227.

[27] K. Murota, L-convex functions and M-convex functions, *Encyclopaedia of Optimization*, (P. M. Pardalos and C. A. Floudas, eds.) Kluwer Academic Publishers, to appear.

[28] K. Murota and A. Shioura, M-convex function on generalized polymatroid, Mathematics of Operations Research, to appear.

[29] K. Murota and A. Shioura, M-convex and L-convex functions over the real space — Two classes of combinatorial convex functions, RIMS preprint, 1999.

[30] R. T. Rockafellar, *Convex Analysis* (Princeton University Press, Princeton, NJ, 1970).

[31] R. T. Rockafellar, *Conjugate Duality and Optimization* (SIAM Regional Conference Series in Applied Mathematics 16, SIAM, Philadelphia, 1974).

[32] M. Shigeno, A Dual Approximation Approach to Matroid Optimization Problems. Doctor's dissertation, Tokyo Institute of Technology, 1996.

[33] A. Shioura, A constructive proof for the induction of M-convex functions through networks, *Discrete Applied Mathematics* 82 (1998), 271–278.

[34] A. Shioura, Minimization of an M-convex function, *Discrete Applied Mathematics* 84 (1998), 215–220.

[35] A. Shioura, Level set characterization of M-convex functions, Research Report No. 21, Department of Mechanical Engineering, Sophia University (February 1998).