Character sheaves and modular generalized Springer correspondence
Part 1: Introduction to character sheaves

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January 2015

The idea of character sheaves

Fix some notation:

- $\mathbb{F}$ is an algebraic closure of $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ for some prime p,
- G is a linear algebraic group over $\mathbb{F}$,
- q is a power of p and $\mathbb{F}_q = \{\alpha \in \mathbb{F} \,|\, \alpha^q = \alpha\}$ is the finite subfield of $\mathbb{F}$ with q elements,
- $F : G \to G$ is the Frobenius endomorphism resulting from an $\mathbb{F}_q$-structure of G [$\sim$ raising matrix entries to the qth power],
- $G^F = G(\mathbb{F}_q)$ is the associated finite group,
- $\mathrm{Rep}(G^F, k)$ is the category of representations of G^F on finite-dimensional vector spaces over a field k.

Idea: to define a geometric category $\mathrm{Ch}(G, k)$ that in some sense 'unifies' all the algebraic categories $\mathrm{Rep}(G^F, k)$ as q, F vary.

Algebra: every $V \in \mathrm{Rep}(G^F, k)$ has a *character*

$$\chi_V : G^F \to k : g \mapsto \mathrm{tr}(g, V),$$

which lies in the vector space $\mathcal{C}(G^F, k)$ of class functions.
When k is algebraically closed of characteristic 0, $\mathrm{Rep}(G^F, k)$ is a 'categorification' of $\mathcal{C}(G^F, k)$ in the sense that

$$K_0(\mathrm{Rep}(G^F, k)) \otimes_{\mathbb{Z}} k \to \mathcal{C}(G^F, k) : [V] \mapsto \chi_V \text{ is an isomorphism.}$$

Geometry: if $\mathcal{L}$ is a G-equivariant (for conjugation) sheaf of finite-dimensional k-vector spaces on G and $\varphi : F^*\mathcal{L} \xrightarrow{\sim} \mathcal{L}$, we define the *characteristic function*

$$\chi_{\mathcal{L},\varphi} : G^F \to k : g \mapsto \mathrm{tr}(\varphi_g, \mathcal{L}_g).$$

This lies in $\mathcal{C}(G^F, k)$ also. So we can imagine a category of sheaves on G giving another 'categorification' of $\mathcal{C}(G^F, k)$.

Lusztig's theory ('Character sheaves' I–V, mid-1980s)

Let G be **connected** and **reductive**, and take $k = \overline{\mathbb{Q}_\ell}$ where $\ell \neq p$. Lusztig defined a set $\widehat{G}$ of irreducible G-equivariant perverse $\overline{\mathbb{Q}_\ell}$-sheaves on G, called *character sheaves*. His definition is geometric and makes no reference to q or F.

Theorem (Lusztig)

[Exclude some small p if G has factors of exceptional type.]
For any F as above,

$$\{\chi_{A,\varphi} \,|\, A \in \widehat{G} \text{ such that } F^*A \cong A,\ \varphi : F^*A \xrightarrow{\sim} A \text{ normalized}\}$$

is an orthonormal basis of $\mathcal{C}(G^F, \overline{\mathbb{Q}_\ell})$.

Under various further assumptions on G, Lusztig and others showed that this basis is 'almost' the basis of irreducible characters. This resulted in an algorithm for computing the character table of G^F.

Remarks:

1. $\overline{\mathbb{Q}_\ell}$-sheaves on $G/\mathbb{F}$ are defined using the étale topology. But Lusztig's definition of $\widehat{G}$ can be adapted for $G/\mathbb{C}$ with the usual topology; there one can take $k = \mathbb{C}$ rather than $\overline{\mathbb{Q}_\ell}$.
2. Perverse sheaves are not sheaves; the elements of $\widehat{G}$ actually belong to the equivariant derived category $\mathcal{D}_G(G, \overline{\mathbb{Q}_\ell})$.
3. Boyarchenko (2013) proved an analogous theorem for G unipotent, using a definition of $\widehat{G}$ due to him and Drinfeld.
4. We can define $\mathrm{Ch}(G, \overline{\mathbb{Q}_\ell})$ to be the (semisimple abelian) full subcategory of $\mathcal{D}_G(G, \overline{\mathbb{Q}_\ell})$ consisting of all direct sums of $\widehat{G}$. Then Lusztig's theorem says that $\mathrm{Ch}(G, \overline{\mathbb{Q}_\ell})$ 'categorifies all $\mathcal{C}(G^F, \overline{\mathbb{Q}_\ell})$ at once', in that sense unifying all $\mathrm{Rep}(G^F, \overline{\mathbb{Q}_\ell})$.
5. One could hope to define a category $\mathrm{Ch}(G, k) \subset \mathcal{D}_G(G, k)$ for general k, perhaps abelian, but not semisimple in general.

Example ($G = GL_1 = \mathbb{F}^\times$)

The Frobenius $F : GL_1 \to GL_1$ is either $\alpha \mapsto \alpha^q$ (**split** case) or $\alpha \mapsto \alpha^{-q}$ (**non-split** case), so

$$GL_1^F = \begin{cases} \{\alpha \in \mathbb{F} \,|\, \alpha^{q-1} = 1\} = \mathbb{F}_q^\times & \text{(split)}, \\ \{\alpha \in \mathbb{F} \,|\, \alpha^{q+1} = 1\} = \mu_{q+1} & \text{(non-split)}. \end{cases}$$

A trivial observation: every irreducible $V \in \mathrm{Rep}(GL_1^F, \overline{\mathbb{Q}_\ell})$ is one-dimensional and satisfies $V^{\otimes n} \cong \overline{\mathbb{Q}_\ell}$ for some n prime to p. Accordingly, $\widehat{GL_1}$ consists of all rank-one local systems $\mathcal{L}$ on GL_1 (i.e. locally constant sheaves of 1-dimensional $\overline{\mathbb{Q}_\ell}$-vector spaces) satisfying $\mathcal{L}^{\otimes n} \cong \underline{\overline{\mathbb{Q}_\ell}}$ (the constant sheaf) for some n prime to p.

$$\text{We have} \quad F^*\mathcal{L} \cong \mathcal{L} \iff \begin{cases} \mathcal{L}^{\otimes(q-1)} \cong \underline{\overline{\mathbb{Q}_\ell}} & \text{(split)}, \\ \mathcal{L}^{\otimes(q+1)} \cong \underline{\overline{\mathbb{Q}_\ell}} & \text{(non-split)}. \end{cases}$$

The characteristic functions of such $\mathcal{L}$ are exactly the irreducible characters of $\mathbb{F}_q^\times$ (split) or μ_{q+1} (non-split).

Example ($G = SL_2$)

Suppose $F : G \to G$ raises entries to the qth power. There are two types of F-stable maximal tori (i.e. subgroups isomorphic to GL_1):

split T contained in an F-stable Borel B,

e.g. $T = \{\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{smallmatrix}\right)\}$, $B = \{\left(\begin{smallmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{smallmatrix}\right)\}$,

non-split T not contained in an F-stable Borel.

Most irreducibles in $\mathrm{Rep}(G^F, \overline{\mathbb{Q}_\ell})$ have dimension $q \pm 1$. Those of dimension $q + 1$ are obtained by *parabolic induction*:

$$\mathbf{I}_{T^F \subset B^F}^{G^F}(V) = \mathrm{Ind}_{B^F}^{G^F} \mathrm{Res}_{B^F}^{T^F}(V)$$

where $V \in \mathrm{Rep}(T^F, \overline{\mathbb{Q}_\ell})$ is one-dimensional and $\mathrm{Res}_{B^F}^{T^F}$ is the pullback functor of the projection $B^F \to T^F$. So we need geometric analogues of these induction and pullback functors.

Digression on functors

Suppose $\psi : H \to G$ is a homomorphism of algebraic groups commuting with their Frobenius endomorphisms. We then get a homomorphism $\psi : H^F \to G^F$ of finite groups, and functors

$$\mathrm{Res}_{H^F}^{G^F} : \mathrm{Rep}(G^F, k) \to \mathrm{Rep}(H^F, k) \text{ (pullback through } \psi\text{)},$$
$$\mathrm{Ind}_{H^F}^{G^F} : \mathrm{Rep}(H^F, k) \to \mathrm{Rep}(G^F, k) : V \mapsto k[G^F] \otimes_{k[H^F]} V,$$

where $\mathrm{Ind}_{H^F}^{G^F}$ is left adjoint to $\mathrm{Res}_{H^F}^{G^F}$.

- If $\psi : H^F \to G^F$ is the inclusion of a subgroup, these are the usual induction and restriction functors, which are biadjoint.
- If $\psi : H^F \to G^F$ is surjective with kernel Γ, then $\mathrm{Ind}_{H^F}^{G^F}$ is the functor of Γ-coinvariants. The functor of Γ-invariants is right adjoint to $\mathrm{Res}_{H^F}^{G^F}$.

We want geometric analogues with the right 'decategorifications'. On characters, $\mathrm{Res}_{H^F}^{G^F}$ is just pullback through ψ, so we define

$$\mathrm{Res}_H^G : \mathcal{D}_G(G,k) \xrightarrow{\mathrm{For}_H^G} \mathcal{D}_H(G,k) \xrightarrow{\psi^*} \mathcal{D}_H(H,k).$$

(This works for equivariant sheaves, without derived categories.) If ψ is injective, for $V \in \mathrm{Rep}(H^F,k)$ and $g \in G^F$ we have

$$\chi_{\mathrm{Ind}_{H^F}^{G^F}(V)}(g) = \sum_{\substack{[g',h] \in G^F \times_{H^F} H^F \\ g'\psi(h)(g')^{-1}=g}} \chi_V(h),$$

so we define

$$\mathrm{Ind}_H^G : \mathcal{D}_H(H,k) \xrightarrow{\sim} \mathcal{D}_G(G \times_H H, k) \xrightarrow{\pi_!} \mathcal{D}_G(G,k),$$

where $\pi : G \times_H H \to G : [g',h] \mapsto g'\psi(h)(g')^{-1}$. (Here $\pi_!$ must be the derived functor.) As expected, Ind_H^G is left adjoint to Res_H^G.

Example ($G = SL_2$, continued)

We now have a geometric version of parabolic induction:

$$\mathbf{I}_{T\subset B}^{G} = \mathrm{Ind}_B^G \circ \mathrm{Res}_B^T : \mathcal{D}_T(T, \overline{\mathbb{Q}_\ell}) \to \mathcal{D}_G(G, \overline{\mathbb{Q}_\ell}).$$

For $\mathcal{L} \in \widehat{T}$ we want $\mathbf{I}_{T\subset B}^{G}(\mathcal{L})$ to be in $\mathrm{Ch}(G, \overline{\mathbb{Q}_\ell})$.

- If $\mathcal{L}^{\otimes 2} \ncong \underline{\overline{\mathbb{Q}_\ell}}$, then $\mathbf{I}_{T\subset B}^{G}(\mathcal{L})$ is irreducible, so include it in $\widehat{G}$.
- By definition, $\mathbf{I}_{T\subset B}^{G}(\underline{\overline{\mathbb{Q}_\ell}}) = \pi_! \underline{\overline{\mathbb{Q}_\ell}}$ where $\pi : G \times_B B \to G$ is the Grothendieck–Springer map. This decomposes as $\underline{\overline{\mathbb{Q}_\ell}} \oplus \mathrm{St}$, so include both $\underline{\overline{\mathbb{Q}_\ell}}$, St in $\widehat{G}$.
- If $p \neq 2$, there is a unique $\mathcal{S} \in \widehat{T}$ with $\mathcal{S} \ncong \underline{\overline{\mathbb{Q}_\ell}}$, $\mathcal{S}^{\otimes 2} \cong \underline{\overline{\mathbb{Q}_\ell}}$. We have $\mathbf{I}_{T\subset B}^{G}(\mathcal{S}) = X \oplus Y$, so include both X, Y in $\widehat{G}$.

Which of these $A \in \widehat{G}$ satisfy $F^*A \cong A$, and are their characteristic functions equal to the irreducible characters of $G^F = SL_2(\mathbb{F}_q)$?

Example ($G = SL_2$, continued)

- $F^*(\mathbf{I}^G_{T\subset B}(\mathcal{L})) \cong \mathbf{I}^G_{T\subset B}(\mathcal{L})$ can happen in two ways:
  - T split F-stable, $F^*\mathcal{L} \cong \mathcal{L}$. These give the characters of the irreducibles $\mathbf{I}^{G^F}_{T^F\subset B^F}(V)$ of dimension $q+1$.
  - T non-split F-stable, $F^*\mathcal{L} \cong \mathcal{L}$. These give the characters of the irreducibles of dimension $q-1$; note that these are not obtained by parabolic induction in the finite group G^F.
- $F^*\overline{\mathbb{Q}_\ell} \cong \overline{\mathbb{Q}_\ell}$ and $F^*\mathrm{St} \cong \mathrm{St}$, giving the trivial and Steinberg characters of G^F (the constituents of $\chi_{\mathrm{Ind}^{G^F}_{B^F}(\overline{\mathbb{Q}_\ell})}$).
- If $p = 2$, this completes the list of irreducible characters, so we need no more character sheaves. Henceforth take $p \neq 2$.
- $F^*X \cong X$ and $F^*Y \cong Y$, but χ_X and χ_Y are **not** irreducible characters. If the remaining irreducible characters are χ_1 and χ_2 (dimension $\frac{q+1}{2}$) and χ_3 and χ_4 (dimension $\frac{q-1}{2}$), we have

$$\chi_X = \frac{\chi_1+\chi_2+\chi_3+\chi_4}{2},\ \chi_Y = \frac{\chi_1+\chi_2-\chi_3-\chi_4}{2}.$$

Example ($G = SL_2$, $p \neq 2$, continued)

Two more character sheaves are needed. By an easy calculation, any class function orthogonal to the characteristic functions found so far must be supported on $\mathcal{O}^F \sqcup (\mathcal{O}')^F$, where

$$\mathcal{O} = \{u \in G \mid u \text{ unipotent}, u \neq 1\} \quad \text{(the regular unipotent class)},$$
$$\mathcal{O}' = \{u \in G \mid -u \in \mathcal{O}\}.$$

Since the centralizer of $u \in \mathcal{O}$ has two connected components, there is a unique nontrivial rank-one G-equivariant local system $\mathcal{E}$ on $\mathcal{O}$, and a corresponding $\mathcal{E}'$ on $\mathcal{O}'$. It turns out that

$$\chi_{\mathcal{E}} = \frac{\chi_1 - \chi_2 + \chi_3 - \chi_4}{2}, \ \chi_{\mathcal{E}'} = \frac{\chi_1 - \chi_2 - \chi_3 + \chi_4}{2}.$$

So we regard $\mathcal{E}$ and $\mathcal{E}'$ as sheaves on G (extending by zero) and include them in $\widehat{G}$. This completes the definition of $\widehat{G}$ for $G = SL_2$.

The idea of cuspidal character sheaves

As in the $G = SL_2$ example, many character sheaves on G arise as summands of parabolic inductions $\mathbf{I}_{L\subset P}^{G}(A)$ where:

- P is a proper parabolic subgroup of G (e.g. a Borel),
- L is a Levi factor of P (a smaller connected reductive group),
- $A \in \widehat{L}$.

A character sheaf not arising in this way is called *cuspidal*: e.g., all character sheaves on GL_1 (or any torus), and $\mathcal{E}, \mathcal{E}' \in \widehat{SL_2}$ as above.

One way to define and classify character sheaves on G is:

1. define and classify cuspidal character sheaves on all Levi subgroups L of G (including G itself);
2. for each cuspidal $A \in \widehat{L}$, classify the summands of $\mathbf{I}_{L\subset P}^{G}(A)$, known as the *induction series* associated to A.

As in the $G = SL_2$ example, cuspidal character sheaves are closely related to sheaves on unipotent (more generally, isolated) classes.