Character sheaves and modular generalized Springer correspondence
Part 2: The generalized Springer correspondence

Anthony Henderson
(joint with Pramod Achar, Daniel Juteau, Simon Riche)

University of Sydney

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Simplifying the context

To get a ‘toy model’ of character sheaves on $G$:

1. Instead of $G$-equivariant perverse sheaves on $G$, consider $G$-equivariant perverse sheaves on the **unipotent variety** $U_G$. This is simpler because there are only finitely many $G$-orbits, but still highly relevant e.g. for cuspidal character sheaves.

2. Assume $p$ is large enough so that there is a $G$-equivariant isomorphism $U_G \sim \to N_G$ where $N_G$ is the **nilpotent cone** in the Lie algebra $\mathfrak{g}$; then we can use Fourier transform on $\mathfrak{g}$.

3. The behaviour for large $p$ is no different from considering $G$ over $\mathbb{C}$ with the usual topology rather than étale topology.

This setting (first with simplification 1 only, later with 2 also) was studied by Lusztig in the case of $\mathbb{Q}_\ell$-sheaves: one of his main results here was the ‘generalized Springer correspondence’.

Aim: to prove an analogue in the **modular** case where $\text{char}(k) = \ell$, as a first step towards understanding modular character sheaves.
The new set-up

New notation:

- $G$ is a connected reductive algebraic group over $\mathbb{C}$,
- $\mathfrak{g}$ is its Lie algebra, on which $G$ has the adjoint action,
- $\mathcal{N}_G = \{ x \in \mathfrak{g} | x \text{ nilpotent} \}$ is the nilpotent cone, on which $G$ has finitely many orbits,
- $k$ is a sufficiently large field of characteristic $\ell \geq 0$,
- $\mathcal{D}_G(\mathcal{N}_G, k)$ is the constructible equivariant derived category.

For any $A \in \mathcal{D}_G(\mathcal{N}_G, k)$ and $G$-orbit $\mathcal{O}$ in $\mathcal{N}_G$, the restrictions $\mathcal{H}^i A|_{\mathcal{O}}$ are $G$-equivariant local systems (i.e. $G$-equivariant sheaves of finite-dimensional $k$-vector spaces) on $\mathcal{O}$, so they correspond to finite-dimensional representations over $k$ of the finite group

$$A_G(x) = G_x / G^o_x,$$

where $G_x$ is the stabilizer in $G$ of $x \in \mathcal{O}$.

Let $\mathfrak{M}_{G,k}$ denote the set of pairs $(\mathcal{O}, \mathcal{E})$ where $\mathcal{O}$ is a $G$-orbit in $\mathcal{N}_G$ and $\mathcal{E}$ is an irreducible $G$-equivariant local system on $\mathcal{O}$. 
Example ($G = GL_n$)

When $G = GL_n$, $g = \text{Mat}_n$ and $\mathcal{N}_G = \{x \in \text{Mat}_n \mid x^n = 0\}$. By the Jordan form theorem, we have a bijection

\[ G \backslash \mathcal{N}_G \longleftrightarrow \mathcal{P}_n = \{\text{partitions } \lambda \text{ of } n\}, \]

where $x \in \mathcal{O}_\lambda$ means that $x$ has Jordan blocks of sizes $\lambda_1, \lambda_2, \cdots$. In this case $A_G(x) = 1$ for all $x$, so $\mathcal{N}_{G,k} \longleftrightarrow \mathcal{P}_n$ for all fields $k$.

Example ($G$ of type $G_2$)

The five nilpotent orbits, in order of decreasing dimension, are:

\[ G_2 \text{ (regular), } G_2(a_1) \text{ (subregular), } \widetilde{A}_1, \ A_1, \ 0. \]

These Bala–Carter labels record the type of the smallest Levi subalgebra meeting the orbit (where $\widetilde{A}_1$ means the short-root $A_1$).

We have $A_G(x) = 1$ for all $x$ except $A_G(x) = S_3$ for $x \in G_2(a_1)$, so $|\mathcal{N}_{G,k}| = 7$ usually, $|\mathcal{N}_{G,k}| = 6$ if $\text{char}(k) \in \{2, 3\}$. 
There is an anti-autoequivalence $D$ of $\mathcal{D}_G(\mathcal{N}_G, k)$, Verdier duality. We study the abelian subcategory $\text{Perv}_G(\mathcal{N}_G, k)$ of $G$-equivariant perverse $k$-sheaves on $\mathcal{N}_G$, where $A \in \mathcal{D}_G(\mathcal{N}_G, k)$ is perverse if

$$\mathcal{H}^i A|_{\mathcal{O}} = \mathcal{H}^i(DA)|_{\mathcal{O}} = 0 \text{ whenever } i > -\dim \mathcal{O}. $$

The simple objects in $\text{Perv}_G(\mathcal{N}_G, k)$ are in bijection with $\mathfrak{N}_{G,k}$:

$$\text{IC}(\mathcal{O}, \mathcal{E}) = \text{‘intermediate extension’ of } \mathcal{E}[\dim \mathcal{O}] \text{ to } \overline{\mathcal{O}}, \text{ extended by zero to the whole of } \mathcal{N}_G.$$

**Example ($G = GL_2$, cf. Juteau–Mautner–Williamson)**

The two orbits are $\mathcal{O}_{(1,1)} = \{0\}$ and $\mathcal{O}_{(2)} = \mathcal{N}_G \setminus \{0\}$. We have

$$\text{IC}(\mathcal{O}_{(1,1)}, k) = k_0 \text{ (skyscraper sheaf)},$$

$$\text{IC}(\mathcal{O}_{(2)}, k) = k_{\mathcal{N}_G}[2] \text{ if } \ell \neq 2.$$

The $\ell = 2$ case is different, because then $H^1(\mathcal{O}_{(2)}, k) \neq 0$. 
Cuspidal pairs and induction series

Let $P$ be a parabolic subgroup of $G$ and $L$ a Levi factor of $P$. We have a geometric parabolic induction functor

$$I_{L \subset P}^G = \text{Ind}_P^G \circ \text{Res}_P^L : \mathcal{D}_L(N_L, k) \to \mathcal{D}_G(N_G, k),$$

defined in the same way as for character sheaves:

$$\text{Res}_P^L : \mathcal{D}_L(N_L, k) \xrightarrow{\sim} \mathcal{D}_P(N_L, k) \xrightarrow{(\cdot)^*} \mathcal{D}_P(N_P, k),$$

$$\text{Ind}_P^G : \mathcal{D}_P(N_P, k) \xrightarrow{\sim} \mathcal{D}_G(G \times_P N_P, k) \xrightarrow{(\cdot)_!} \mathcal{D}_G(N_G, k).$$

Lemma (Lusztig when $\ell = 0$, [AHR] when $\ell > 0$)

$I_{L \subset P}^G$ commutes with $D$ and maps $\text{Perv}_L(N_L, k)$ to $\text{Perv}_G(N_G, k)$. It has left adjoint $R_{L \subset P}^G = \text{Ind}_P^L \circ \text{Res}_P^G$ and right adjoint $R_{L \subset P}^G$ where $P^-$ denotes the opposite parabolic with the same Levi $L$. 
We say that a pair \((\mathcal{O}, \mathcal{E}) \in \mathcal{N}_{G,k}\), or the corresponding \(\text{IC}(\mathcal{O}, \mathcal{E})\), is \textit{cuspidal} if the following equivalent conditions hold:

1. \(R_{L \subseteq P}^G(\text{IC}(\mathcal{O}, \mathcal{E})) = 0\) for all \(L \subset P \subsetneq G\);
2. \(\text{IC}(\mathcal{O}, \mathcal{E})\) is not a quotient of \(I_{L \subseteq P}^G(A)\) for any \(L \subset P \subsetneq G\) and any \(A \in \text{Perv}_L(\mathcal{N}_L, k)\);
3. \(\text{IC}(\mathcal{O}, \mathcal{E})\) is not a subobject of \(I_{L \subseteq P}^G(A)\) for any \(L \subset P \subsetneq G\) and any \(A \in \text{Perv}_L(\mathcal{N}_L, k)\).

**Remark**

When \(\ell = 0\), the Decomposition Theorem of [BBD] implies that if \(A \in \text{Perv}_L(\mathcal{N}_L, k)\) is simple, then \(I_{L \subseteq P}^G(A)\) is semisimple, so one can replace ‘quotient’/‘subobject’ with ‘summand’. Semisimplicity can fail if \(\ell > 0\), and cuspidals \textbf{can} occur as constituents of \(I_{L \subseteq P}^G(A)\). This is analogous to modular representations of \(G(\mathbb{F}_q)\) when \(\ell \neq p\).

**Lemma (Lusztig – same proof works for \(\ell > 0\))**

\textit{If \((\mathcal{O}, \mathcal{E})\) is cuspidal, \(\mathcal{O}\) is distinguished, i.e. meets no proper Levi.}
Let $\mathcal{M}_{G,k}$ be the set of cuspidal data $(L, O_L, E_L)$ where $L$ is a Levi subgroup of $G$ (take only one representative of each $G$-conjugacy class, allowing $L = G$) and $(O_L, E_L)$ is a cuspidal pair for $L$.

**Proposition (Lusztig when $\ell = 0$, [AHJR] when $\ell > 0$)**

For any $(L, O_L, E_L) \in \mathcal{M}_{G,k}$, $I^G_{L \subset P}(IC(O_L, E_L))$ is independent of the parabolic $P$, and its head and socle are isomorphic.

**Remark**

The analogue for modular representations is by Geck–Hiss–Malle.

The induction series associated to $(L, O_L, E_L) \in \mathcal{M}_{G,k}$ is the set of simple quotients (equivalently, subobjects) of $I^G_{L \subset P}(IC(O_L, E_L))$.

**Lemma (Lusztig – same proof works for $\ell > 0$)**

Any simple object $IC(O, E)$ in $\text{Perv}_G(N_G, k)$ belongs to the induction series associated to some $(L, O_L, E_L) \in \mathcal{M}_{G,k}$ as above. (If $IC(O, E)$ is cuspidal, then $(L, O_L, E_L) = (G, O, E)$.)
The (modular) generalized Springer correspondence is:

**Theorem (Lusztig when \( \ell = 0 \), [AHJR] when \( \ell > 0 \))**

1. **Induction series associated to different cuspidal data are disjoint:** in other words, a given \( \text{IC}(\mathcal{O}, \mathcal{E}) \) belongs to the induction series associated to a unique \( (L, \mathcal{O}_L, \mathcal{E}_L) \in \mathcal{M}_{G,k} \).

2. **The induction series associated to** \( (L, \mathcal{O}_L, \mathcal{E}_L) \) **is canonically in bijection with the set of irreducible** \( k \)-**reps of** \( N_G(L)/L \).

3. **Hence we have a bijection**

\[
\mathcal{M}_{G,k} \leftrightarrow \biguplus_{(L, \mathcal{O}_L, \mathcal{E}_L) \in \mathcal{M}_{G,k}} \text{Irr}(N_G(L)/L, k).
\]

The proof will be discussed in the next lecture.

**Remark**

The analogue of 1 holds for modular representations of \( G(\mathbb{F}_q) \) also; for the analogue of 2 one needs a \( q \)-deformed group algebra.
In the mid-1970s, Springer gave a geometric construction of the irreducible $\mathbb{Q}_\ell$-reps of the Weyl group $W = N_G(T)/T$.

As reformulated by Lusztig and Borho–Macpherson, this comes from an action of $W$ on the semisimple perverse sheaf

$$\text{Spr} = \mathbf{1}_T^G(\overline{\mathbb{Q}_\ell}) = \mu_! \overline{\mathbb{Q}_\ell}[\dim N_G] \in \text{Perv}_G(N_G, \overline{\mathbb{Q}_\ell}),$$

where $\mu : G \times_B N_B \to N_G$ is the Springer resolution of $N_G$. The Springer correspondence is the resulting bijection

$$\{\text{simple summands of Spr}\} \longleftrightarrow \text{Irr}(W, \overline{\mathbb{Q}_\ell})$$

$$\text{IC}(\mathcal{O}, \mathcal{E}) \mapsto \text{Hom}_{\text{Perv}_G(N_G, \overline{\mathbb{Q}_\ell})}(\text{Spr}, \text{IC}(\mathcal{O}, \mathcal{E})).$$

Lusztig then found that this was the $(L, \mathcal{O}_L, \mathcal{E}_L) = (T, 0, \overline{\mathbb{Q}_\ell})$ case of the generalized Springer correspondence, thus accounting for the IC$(\mathcal{O}, \mathcal{E})$’s that are not summands of Spr.

Juteau (2007) showed that the Springer correspondence holds with $k$ instead of $\overline{\mathbb{Q}_\ell}$ and ‘quotients’ instead of ‘summands’.
Example \((G = GL_n, W = S_n)\)

- When \(\ell = 0\), \(|\text{Irr}(S_n, k)| = |\mathcal{P}_n|\), so every \(\text{IC}(\mathcal{O}_\lambda, k)\) is a summand of \(\text{Spr}\), i.e. the Springer correspondence for \(GL_n\) is already ‘generalized’. In particular, \(GL_n\) does not have a cuspidal pair unless \(n = 1\).

- When \(\ell > 0\), James constructed the irreps \(D^\lambda\) of \(S_n\) over \(k\), labelled by \(\lambda\) that are \(\ell\)-regular (no part occurs \(\geq \ell\) times). Under Juteau’s correspondence, \(D^\lambda\) maps to \(\text{IC}(\mathcal{O}_{\lambda^t}, k)\) where \(\lambda^t\) is the transpose partition; so these are the simple quotients of \(\text{Spr}\). (All simples occur as constituents of \(\text{Spr}\).) The only distinguished orbit in \(\mathcal{N}_G\) is \(\mathcal{O}_{(n)}\); we will see that

\[(\mathcal{O}_{(n)}, k)\text{ is cuspidal} \iff n \text{ is a power of } \ell.\]

So \(\mathcal{M}_{G,k}\) is essentially the set of Levis of the form \(\prod_{i \geq 0} GL_{\ell^i}^{m_i}\), where \(m_i\) are nonnegative integers such that \(\sum_{i \geq 0} m_i \ell^i = n\).
Example ($G = GL_n$, $\ell > 0$ continued)

For $L = \prod_{i \geq 0} GL_{\ell^i}^{m_i}$ such a Levi subgroup of $GL_n$, we have

$$N_G(L)/L \cong \prod_{i \geq 0} S_{m_i},$$

$$\text{Irr}(N_G(L)/L, k) \leftrightarrow \prod_{i \geq 0} \{\ell\text{-regular } \lambda^{(i)} \vdash m_i\}.$$

Under our correspondence, the collection $(\lambda^{(i)})$ maps to $\text{IC}(O_{\lambda}, k)$ where $\lambda = \sum_{i \geq 0} \ell^i (\lambda^{(i)})^t$. Note that $\text{IC}(O_{(n)}, \mathbb{k})$ occurs in the series of $L = \prod_{i \geq 0} GL_{\ell^i}^{b_i}$ where $\sum_{i \geq 0} b_i \ell^i = n$ and all $b_i < \ell$.

Remark

The above combinatorial correspondence is a simplified version of what appears in the analogous theory of induction series for modular representations of $GL_n(\mathbb{F}_q)$ (Dipper–Du).
Example ($G = G_2$, $W$ dihedral of order 12)

- When $\ell = 0$, $|\text{Irr}(W, k)| = 6 < 7 = |\mathfrak{N}_{G,k}|$. The non-Springer pair is $(G_2(a_1), \mathcal{E}_{\text{sign}})$, which must be cuspidal because the other proper Levi subgroups are both isomorphic to $GL_2$.

- When $\ell = 2$, $|\text{Irr}(W, k)| = 2$, and only $\text{IC}(0, k)$ and $\text{IC}(\widetilde{A}_1, k)$ belong to Juteau’s correspondence. The other series are:

  - $(L$ of type $A_1, \mathcal{O}_L, k), \ |N_G(L)/L| = 2 : \text{IC}(A_1, k),$
  - $(L$ of type $\widetilde{A}_1, \mathcal{O}_L, k), \ |N_G(L)/L| = 2 : \text{IC}(G_2(a_1), \mathcal{E}_{\text{refln}}),$

leaving 2 cuspidal pairs, $(G_2(a_1), k)$ and $(G_2, k)$.

The above $GL_n$ and $G_2$ examples illustrate:

**Theorem ([AHJR])**

*When $\ell > 0$, $\text{IC}(\mathcal{O}_{\text{reg}}, k)$ belongs to the induction series associated to $(L, \mathcal{O}_L, \text{reg}, k)$ where $L$ is minimal such that $\ell \nmid |W/W_L|$.***