Exact WKB Analysis and Cluster Algebras

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(joint work with Tomoki Nakanishi)

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Exact WKB analysis

Schrödinger equation:

$$\left(\frac{d^2}{dz^2} - \eta^2 Q(z)\right)\psi(z,\eta) = 0$$

where z is an complex variable, $\eta = \hbar^{-1} > 0$ is a **large parameter**.

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- **WKB (Wentzel-Kramers-Brillouin) solutions**:

$$\psi_\pm(z,\eta) = e^{\pm\eta\int^z \sqrt{Q(z')}dz'} \sum_{n=0}^{\infty} \eta^{-n-\frac{1}{2}} \psi_{\pm,n}(z)$$

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- **Exact WKB analysis** = **WKB method** + **Borel resummation**.

$$\mathcal{S}[\psi_\pm](z,\eta) \sim \psi_\pm(z,\eta) \text{ as } \eta \to +\infty$$

  Monodromy/connection matrices of (Borel resummed) WKB solutions are described by "**Voros symbols**".

[Voros 83], [Sato-Aoki-Kawai-Takei 91], [Delabaere-Dillinger-Pham 93], ...

Cluster algebras (of rank $n \geq 1$)

- A **cluster algebra** [Fomin-Zelevinsky 02] is defined in terms of **seeds**.
- A seed is a triplet $(B, \mathbf{x}, \mathbf{y})$ where
  - ∗ skew-symmetric integer matrix $B = (b_{ij})_{i,j=1}^n$
  - ∗ **cluster x-variables** $\mathbf{x} = (x_i)_{i=1}^n$
  - ∗ **cluster y-variables** $\mathbf{y} = (y_i)_{i=1}^n$

  These two variables satisfy $y_i = r_i \prod_{j=1}^n (x_j)^{b_{ji}}$ (r_i : "coefficient").

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- A **"signed" mutation** at $k \in \{1, \ldots, n\}$ with sign $\varepsilon \in \{\pm\}$:
  $\mu_k^{(\varepsilon)} : (B, \mathbf{x}, \mathbf{y}) \mapsto (B', \mathbf{x}', \mathbf{y}')$ defined by

$$b'_{ij} = \begin{cases} -b_{ij} & i = k \text{ or } j = k \\ b_{ij} + [b_{ik}]_+ b_{kj} + b_{ik}[b_{kj}]_+ & \text{otherwise.} \end{cases}$$

$$x'_i = \begin{cases} {x_k}^{-1} \left(\prod_{j=1}^{n} {x_j}^{[-\varepsilon b_{jk}]_+} \right) (1 + {y_k}^{\varepsilon}) & i = k \\ x_i & i \neq k. \end{cases} \qquad y'_i = \begin{cases} {y_k}^{-1} & i = k \\ y_i {y_k}^{[\varepsilon b_{ki}]_+} (1 + {y_k}^{\varepsilon})^{-b_{ki}} & i \neq k. \end{cases}$$

Here $[a]_+ = \max(a, 0)$. (The coefficients r_i also mutate.)

Results and Application [I-Nakanishi 14]

- Cluster algebraic structure appears in many contexts:
  - representation of quivers
  - Teichmüller theory
  - hyperbolic geometry
  - discrete integrable systems
  - Donaldson-Thomas invariants and their wall-crossing
  - supersymmetric gauge theory
  - ...

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- **Main result**: We add **Exact WKB analysis** in the above list:

skew-symmetric matrix B	$\leftrightarrow$	Stokes graph
cluster variables	$\leftrightarrow$	Voros symbols
cluster mutation	$\leftrightarrow$	Stokes phenomenon (for $\eta \to \infty$)

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- **Application**: **Identities of Stokes automorphsims** in the exact WKB analysis (c.f., [Delabaere-Dillinger-Pham 93]) follow from **periodicity** of corresponding cluster algebras.

For example: $\mathfrak{S}_{\gamma_1} \mathfrak{S}_{\gamma_2} = \mathfrak{S}_{\gamma_2} \mathfrak{S}_{\gamma_2+\gamma_1} \mathfrak{S}_{\gamma_1}$

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- **Generalized cluster algebras** ([Chekhov-Shapiro 11]) also appear when Schrödinger equation has a certain type of regular singularity.

Contents

Refferences

- A. Voros, "The return of the quartic oscillator. The complex WKB method", Ann. Inst. Henri Poincaré **39** (1983), 211–338.
- T. Kawai and Y. Takei, "*Algebraic Analysis of Singular Perturbations*", AMS translation, 2005.

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Schrödinger equation and WKB solutions

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$$\boxed{\left(\frac{d^2}{dz^2} - \eta^2 Q(z)\right)\psi(z,\eta) = 0}$$

  - $\eta = \hbar^{-1}$: large parameter
  - $Q(z)$: rational function ("potential")
  - Assume that all zeros of $Q(z)$ are of order 1, and all poles of $Q(z)$ are of order ≥ 2.

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- WKB solutions are divergent in general: ($|\psi_{\pm,n}(z)| \sim C A^n n!$).

Borel resummation method

- Expansion of WKB solution:

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- The **Borel sum** of $\psi_\pm$ (as a formal series of η^{-1}):

$$\mathcal{S}[\psi_\pm] = \int_{\mp a(z)}^{\infty} e^{-y\eta}\psi_{\pm,B}(z,y)dy.$$

Here $a(z) = \int_{z_0}^{z}\sqrt{Q(z')}dz'$ and

$$\psi_{\pm,B}(z,y) = \sum_{n=0}^{\infty}\frac{\psi_{\pm,n}(z)}{\Gamma(n+\frac{1}{2})}(y \pm a(z))^{n-\frac{1}{2}}$$: **Borel transform** of $\psi_\pm$

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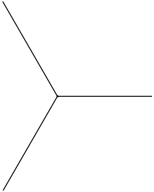
- If the Borel sums $\mathcal{S}[\psi_\pm]$ are well-defined, they give analytic solutions of the Schödinger equation and $\mathcal{S}[\psi_\pm] \sim \psi_\pm$ when $\eta \to +\infty$.

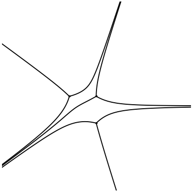
Stokes graph and Stokes segent

- **Stokes graph**:
  - Vertices: **turning points** (i.e., zeros of $Q(z)$) and singular points.
  - Edges: **Stokes curves** emanating from turning points.
    (real one-dimensional curves defined by $\mathrm{Im} \int^z \sqrt{Q(z')}dz' = \mathrm{const.}$)

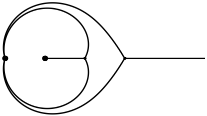
Stokes curves are **trajectories** of the quadratic differential $Q(z)dz^{\otimes 2}$.



$Q(z) = z.$



$Q(z) = z(z+1)(z+i).$



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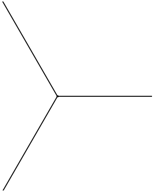

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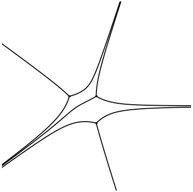

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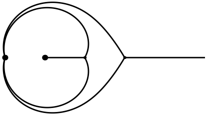

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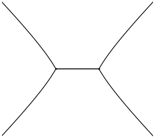

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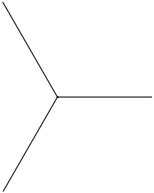

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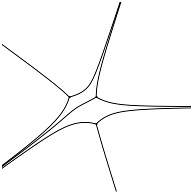

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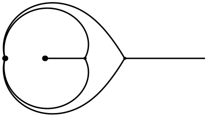

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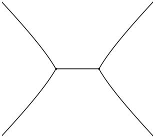
- **Stokes segment** is a Stokes curve connecting turning points (= saddle trajectory of $Q(z)dz^{\otimes 2}$).
- Stokes graph is said to be **saddle-free** if it doesn't contain Stokes segments.



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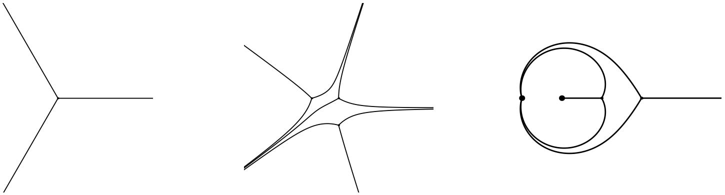
Stokes graph and Borel summability



Theorem (Koike-Schäfke)

Suppose that the Stokes graph is saddle-free. Then,

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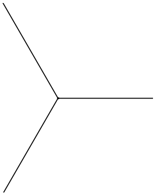

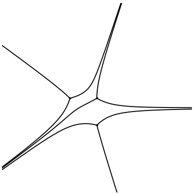

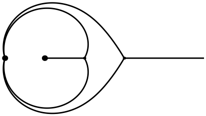

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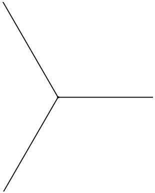

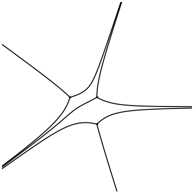

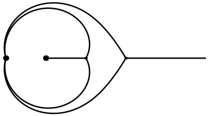

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- $\psi_{\pm}(z,\eta)$ are **Borel summable** (as a formal series of η^{-1}) on each **Stokes region** (= a face of the Stokes graph).
- The Borel sums $\mathcal{S}[\psi_{\pm}](z,\eta)$ give **analytic** (in both z and η) solutions of the Schrödinger equation on each Stokes region satisfying

$$\mathcal{S}[\psi_{\pm}](z,\eta) \sim \psi_{\pm}(z,\eta) \quad \text{as} \quad \eta \to +\infty.$$

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- Connection formulas and monodromy matrices of WKB solutions are written by (the Borel sum of) **Voros symbols** $e^{W_\beta(\eta)}$ and $e^{V_\gamma(\eta)}$, where

$$W_\beta(\eta) = \int_\beta \left(S_{\mathrm{odd}}(z,\eta) - \eta \sqrt{Q(z)} \right) dz, \quad V_\gamma(\eta) = \oint_\gamma S_{\mathrm{odd}}(z,\eta) dz.$$

(c.f., [Kawai-Takei 05, §3]). Here

- $S_\pm(z,\eta) = \dfrac{d}{dz} \log \psi_\pm(z,\eta) = \pm\eta\sqrt{Q(z)} + \cdots$, and

$$S_{\mathrm{odd}}(z,\eta) = \frac{1}{2}\left(S_+(z,\eta) - S_-(z,\eta)\right) = \eta\sqrt{Q(z)} + \cdots$$

- $\beta \in H_1(\mathcal{R}, P; \mathbb{Z})$ ("path"), $\gamma \in H_1(\mathcal{R}; \mathbb{Z})$ ("cycle").

  $\mathcal{R}$ = Riemann surface of $\sqrt{Q(z)}$, P = the set of poles of $Q(z)$.

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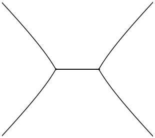
- Voros symbols $e^{W_\beta(\eta)}$ and $e^{V_\gamma(\eta)}$ (for any path β and any cycle γ) are **Borel summable** if the Stokes graph is saddle-free.

Mutation of Stokes graphs



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(The figure describes a part of Stokes graph.)

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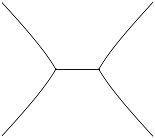

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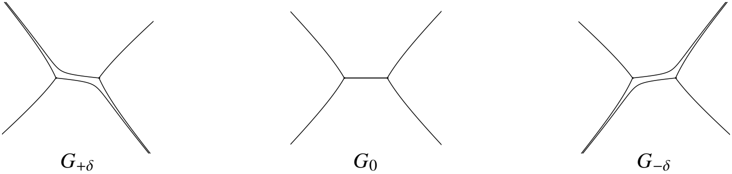
- Suppose that the Stokes graph G_0 has a Stokes segment.
- Consider the S^1-family of the potential: $Q^{(\theta)}(z) = e^{2i\theta}Q(z) \quad (\theta \in \mathbb{R})$.
  G_θ : Stokes graph for $Q^{(\theta)}(z)$.

Mutation of Stokes graphs



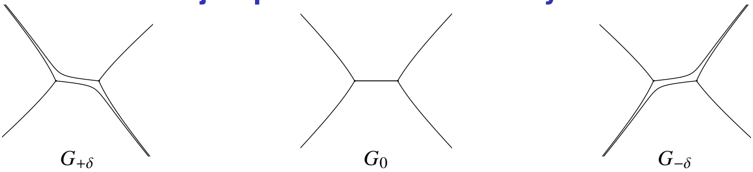
(The figure describes a part of Stokes graph.)

- Suppose that the Stokes graph G_0 has a Stokes segment.
- Consider the S^1-family of the potential: $Q^{(\theta)}(z) = e^{2i\theta}Q(z) \quad (\theta \in \mathbb{R})$.
  G_θ : Stokes graph for $Q^{(\theta)}(z)$.
- For any sufficiently small $\delta > 0$, $G_{\pm\delta}$ are saddle-free since the existence of the Stokes segment implies

$$\int_{\text{along Stokes segment}} \sqrt{Q(z)}dz \in \mathbb{R}_{\neq 0}$$

- S^1-action causes a "**mutation of Stokes graphs**" (= a discontinuous change of topology of Stokes graphs caused by a Stokes segment).

DDP's jump formula of Voros symbols



- Suppose that G_0 has a Stokes segment connecting two distinct turning points, and no other Stokes segments.

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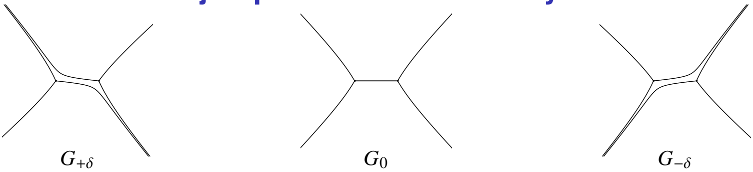

- Suppose that G_0 has a Stokes segment connecting two distinct turning points, and no other Stokes segments.
- Let $\mathcal{S}[e^{W_\beta^{(\theta)}}]$, $\mathcal{S}[e^{V_\gamma^{(\theta)}}]$ be the Borel sum of Voros symbols for $Q^{(\theta)}(z)$ and

$$\mathcal{S}_\pm[e^{W_\beta}] := \lim_{\theta \to \pm 0} \mathcal{S}[e^{W_\beta^{(\theta)}}], \quad \mathcal{S}_\pm[e^{V_\gamma}] := \lim_{\theta \to \pm 0} \mathcal{S}[e^{V_\gamma^{(\theta)}}].$$

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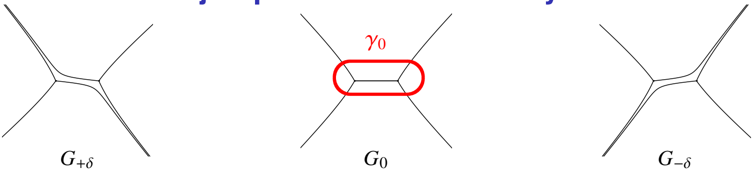

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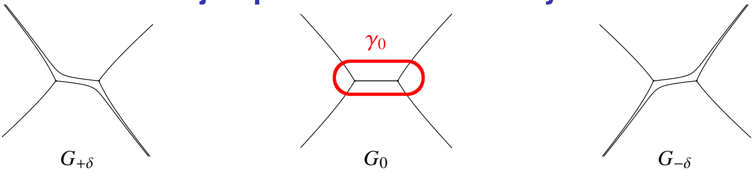
Theorem (Delabaere-Dillinger-Pham 93)

$$\mathcal{S}_-[e^{W_\beta}] = \mathcal{S}_+[e^{W_\beta}](1 + \mathcal{S}_+[e^{V_{\gamma_0}}])^{-\langle\gamma_0,\beta\rangle},$$

$$\mathcal{S}_-[e^{V_\gamma}] = \mathcal{S}_+[e^{V_\gamma}](1 + \mathcal{S}_+[e^{V_{\gamma_0}}])^{-\langle\gamma_0,\gamma\rangle}.$$

Here $\langle\ ,\ \rangle$ is the intersection form (normalized as $\langle x\text{-axis}, y\text{-axis}\rangle = +1$), and γ_0 is the cycle around the Stokes segment oriented as $\oint_{\gamma_0} \sqrt{Q(z)}dz \in \mathbb{R}_{<0}$.

DDP's jump formula of Voros symbols



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- This formula describes the **Stokes phenomenon** for Voros symbols.

Contents

Refferences

- K. I and T. Nakanishi, "*Exact WKB analysis and cluster algebras*", J. Phys. A: Math. Theor. 47 (2014) 474009.
- K. I and T. Nakanishi, "*Exact WKB analysis and cluster algebras II: simple poles, orbifold points, and generalized cluster algebras*", arXiv:1401.7094.

Dictionary

Exact WKB analysis	Cluster algebras
saddle-free Stokes graph	skew-symmetric matirx B
mutation of Stokes graphs	mutation of B
(Borel sum of) Voros symbol $e^{W_{\beta_i}}$	cluster x-variable x_i
(Borel sum of) Voros symbol $e^{V_{\gamma_i}}$	cluster y-variable y_i
$e^{\eta \oint_{\gamma_i} \sqrt{Q(z)}dz}$	coefficient r_i
Stokes phenomenon for Voros symbols	mutation of cluster variables

$$W_\beta(\eta) = \int_\beta \left(S_{\rm odd}(z,\eta) - \eta\sqrt{Q(z)}\right) dz, \quad V_\gamma(\eta) = \oint_\gamma S_{\rm odd}(z,\eta)dz.$$

$$b'_{ij} = \begin{cases} -b_{ij} & i = k \text{ or } j = k \\ b_{ij} + [b_{ik}]_+ b_{kj} + b_{ik}[b_{kj}]_+ & \text{otherwise.} \end{cases}$$

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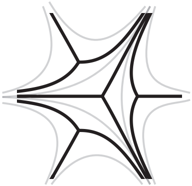
Stokes graph $\rightsquigarrow$ Skew-symmetric matrix

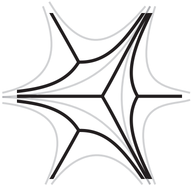
- A saddle-free Stokes graph



Stokes graph

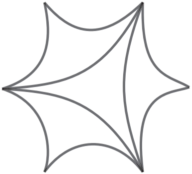
Stokes graph $\rightsquigarrow$ Skew-symmetric matrix

- A saddle-free Stokes graph $\rightsquigarrow$ A triangulated surface:
  (Three Stokes curve emanate from an order 1 turning point.)
  [Gaiotto-Moore-Neitzke 09]



Stokes graph

$\rightsquigarrow$



Triangulated surface

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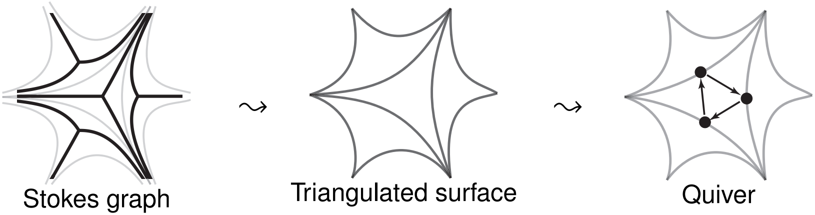
- A triangulated surface $\rightsquigarrow$ A quiver [Fomin-Shapiro-Thurston 08]:
  - ∗ Put vertices on edges of triangulation.
  - ∗ Draw arrows on each triangle in clockwise direction.
  - ∗ Remove vertices on "boundary edges" together with attached arrows.
    (boundary / internal edge ↔ digon-type / rectangular Stokes region)



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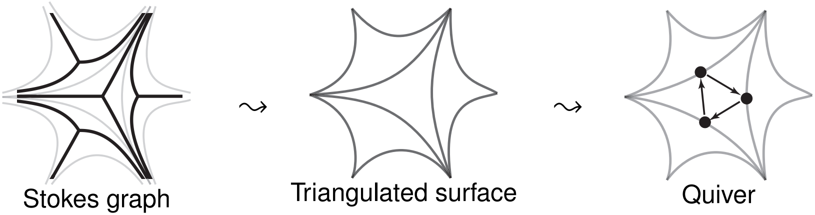

- A quiver $\rightsquigarrow$ A skew-symmetric matrix $B = (b_{ij})_{i,j=1}^{n}$ by

$$b_{ij} = (\sharp \text{ of arrows } \circ_i \to \circ_j) - (\sharp \text{ of arrows } \circ_j \to \circ_i)$$

(Assign labels $i \in \{1, \ldots, n\}$ to rectangular Stokes regions.)

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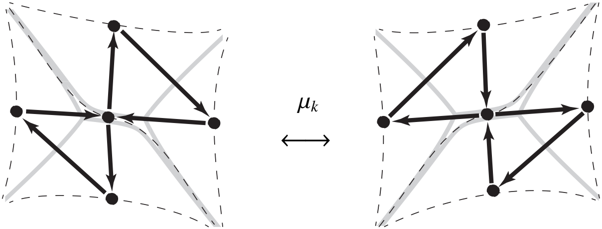

$G_{+\delta}$ $G_{-\delta}$

(Figures describes a part of Stokes graphs.)

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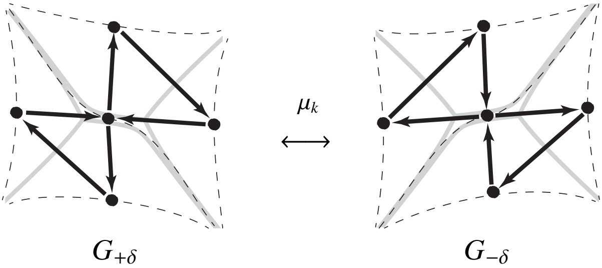

(Figures describes a part of Stokes graphs.)

- Quiver muation is compatible with **mutation** of B-matix:

$$b'_{ij} = \begin{cases} -b_{ij} & i = k \text{ or } j = k \\ b_{ij} + [b_{ik}]_+ b_{kj} + b_{ik}[b_{kj}]_+ & \text{otherwise.} \end{cases} \qquad ([a]_+ = \max(a, 0))$$

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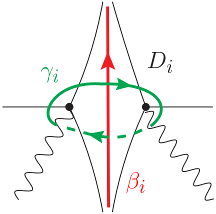
($[a]_+ = \max(a, 0)$ and $y_i = r_i \prod_{j=1}^{n} (x_j)^{b_{ji}}$.)

Simple paths and simple cycles

- For a *saddle-free* Stokes graph, label **horizontal strips** (= rectangular Stokes regions) as $D_1, \ldots, D_n$.
- n = the number of horizontal strips.
- For each D_i we associate a path β_i (called "**simple path**") and a cycle γ_i (called "**simple cycle**") on the Riemann surface of $\sqrt{Q(z)}$.



  * The simple path β_i is oriented so that the function $\mathrm{Re}\left(\int^z \sqrt{Q(z)}dz\right)$ increases along the positive direction of β_i.
  * The orientation of the simple cycle γ_i is given so that $\langle \gamma_i, \beta_i \rangle = +1$.

Lemma

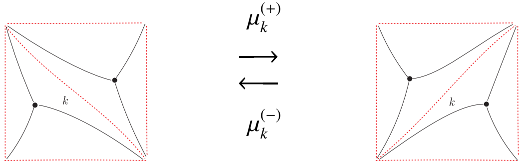
$$\gamma_i = \sum_{j=1}^{n} b_{ji}\, \beta_j \quad (i = 1, \ldots, n).$$

Voros symbols for simple path and simple cycles

- Fix a sign $\varepsilon \in \pm$. Suppose that the saddle-free Stokes graphs $G = G_{\varepsilon\delta}$ and $G' = G_{-\varepsilon\delta}$ are related by the "signed mutation" $\mu_k^{(\varepsilon)}$:

G if $\varepsilon = +$

G' if $\varepsilon = -$



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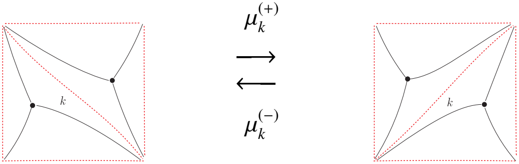

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- Define the skew-symmetric matrix B (resp., B'), simple paths/cycles $(\beta_i)_{i=1}^n, (\gamma_i)_{i=1}^n$ (resp., $(\beta_i')_{i=1}^n, (\gamma_i')_{i=1}^n$) for G (resp., G'). We also set

$$x_i = \mathcal{S}_\varepsilon\left[e^{W_{\beta_i}}\right], \quad y_i = \mathcal{S}_\varepsilon\left[e^{V_{\gamma_i}}\right], \quad r_i = \exp\left(\eta \oint_{\gamma_i} \sqrt{Q(z)}dz\right).$$

$$x_i' = \mathcal{S}_{-\varepsilon}\left[e^{W_{\beta_i'}}\right], \quad y_i' = \mathcal{S}_{-\varepsilon}\left[e^{V_{\gamma_i'}}\right], \quad r_i' = \exp\left(\eta \oint_{\gamma_i'} \sqrt{Q(z)}dz\right).$$

(Recall: $\mathcal{S}_\pm[e^{W_\beta}] = \lim_{\theta \to \pm 0} \mathcal{S}[e^{W_\beta^{(\theta)}}]$ etc, where

$e^{W_\beta^{(\theta)}}$ is the Voros symbol for $Q^{(\theta)}(z) = e^{2i\theta}Q(z)$.)

Voros symbols as cluster variables

- Decomposition formula imples the following:

Proposition

$$y_i = r_i \prod_{j=1}^{n} (x_j)^{b_{ji}}, \quad y'_i = r'_i \prod_{j=1}^{n} (x'_j)^{b'_{ji}} \quad (i = 1, \ldots, n).$$

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- Under the mutation of Stokes graphs relevant to a Stokes segment connecting two distinct simple turning points, the Borel sum of Voros symbols mutate as cluster variables:

Main Theorem ([I-Nakanishi 14])

In the signed muation $\mu_k^{(\varepsilon)}$ of Stokes graphs, we have

$$x'_i = \begin{cases} {x_k}^{-1} \left(\prod_{j=1}^{n} {x_j}^{[-\varepsilon b_{jk}]_+} \right) (1 + {y_k}^{\varepsilon}) & i = k \\ x_i & i \neq k. \end{cases} \qquad y'_i = \begin{cases} {y_k}^{-1} & i = k \\ y_i {y_k}^{[\varepsilon b_{ki}]_+} (1 + {y_k}^{\varepsilon})^{-b_{ki}} & i \neq k. \end{cases}$$

Proof of the main formula

The main theorem follows from the DDP formula and the following:

Proposition

$$\beta_i{}' = \begin{cases} -\beta_k + \sum_{j=1}^{n}[-\varepsilon b_{jk}]_+\beta_j & i = k \\ \beta_i & i \neq k. \end{cases} \qquad \gamma_i{}' = \begin{cases} -\gamma_k & i = k \\ \gamma_i + [\varepsilon b_{ki}]_+\gamma_k & i \neq k. \end{cases}$$

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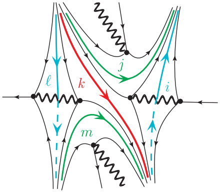
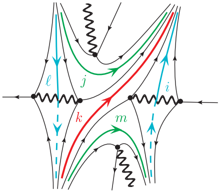


$\mu_k^{(+)}$ $\longrightarrow$

$\longleftarrow$ $\mu_k^{(-)}$



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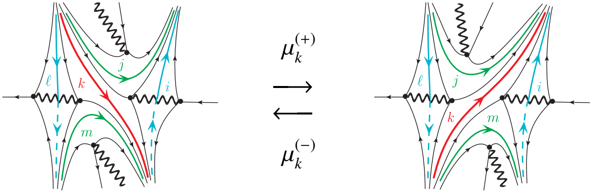

G' if $\varepsilon = +$

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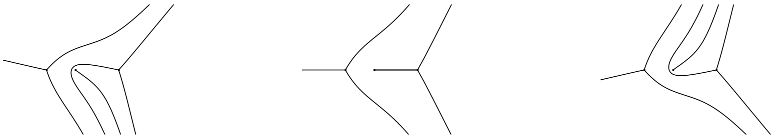
$$\begin{aligned} x_k' &= \mathcal{S}_{-\varepsilon}[e^{W_{\beta_k'}}] = \mathcal{S}_{-\varepsilon}\left[\left(e^{W_{\beta_k}}\right)^{-1}\left(\prod_{j=1}^{n}(e^{W_{\beta_j}})^{[-\varepsilon b_{jk}]_+}\right)\right] \\ &= \mathcal{S}_{+\varepsilon}\left[\left(e^{W_{\beta_k}}\right)^{-1}\left(\prod_{j=1}^{n}(e^{W_{\beta_j}})^{[-\varepsilon b_{jk}]_+}\right)\left(1 + e^{V_{\varepsilon\gamma_k}}\right)^{+\langle\gamma_k,\beta_k\rangle}\right] \quad \text{(DDP formula: } \gamma_0 = \varepsilon\gamma_k) \\ &= x_k{}^{-1}\left(\prod_{j=1}^{n} x_j{}^{[-\varepsilon b_{jk}]_+}\right)(1 + y_k^{\varepsilon}) \qquad (\langle\gamma_k,\beta_k\rangle = +1). \end{aligned}$$

Simple poles and generalized cluster algebras

We allow $Q(z)$ to have a simple pole, and consider the following mutation:



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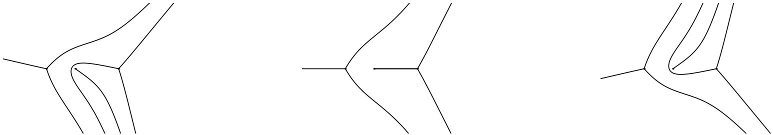

- Stokes graph defines a **triangulated orbifold**. We can associate a **skew-symmetrizable** matrix B: [Felikson-Shapiro-Tumarkin 12].

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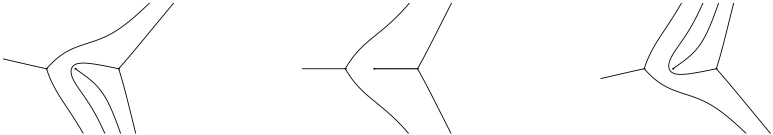

- Stokes graph defines a **triangulated orbifold**. We can associate a **skew-symmetrizable** matrix B: [Felikson-Shapiro-Tumarkin 12].
- The Stokes phenomenon for Voros symbols is an example of mutations in **generalized cluster algebra** [Chekhov-Shapiro 11]:

Theorem ([I-Nakanishi II 14])

$$x_i' = \begin{cases} x_k^{-1}\left(\prod_{j=1}^{n} x_j^{[-\varepsilon \tilde{b}_{jk}]_+}\right)^2 (1 + (t+t^{-1})y_k^{\varepsilon} + y_k^{2\varepsilon}) & i = k \\ x_i & i \neq k, \end{cases}$$

$$y_i' = \begin{cases} y_k^{-1} & i = k \\ y_i \left(y_k^{[\varepsilon \tilde{b}_{ki}]_+}\right)^2 (1 + (t+t^{-1})y_k^{\varepsilon} + y_k^{2\varepsilon})^{-\tilde{b}_{ki}} & i \neq k. \end{cases}$$

Here $\tilde{B} = DB$ is skew-symmetric, and t is defined from the characteristic exponents at the simple pole attached to the Stokes segment.