Positivity of canonical bases of quantum coideal algebras and geometry of flag varieties

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Kyoto, 1/23/2015

References

[Bao-W] *A new approach to Kazhdan-Lusztig theory of type B via quantum symmetric pairs*, arXiv:1310.0103.

[Bao-Kujawa-Yiqiang Li-W] *Geometric Schur duality of classical type*, arXiv:1404.4000v2.

[Li-W] *Positivity vs negativity of canonical bases*, arXiv:1501.00688.

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(2) iCanonical basis (iCB) of quantum coideal algebras

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Schur duality

- Let $\mathbf{U} = \langle E_i, F_i, K_i^{\pm 1} \rangle$ be the quantum group of type $\mathfrak{gl}_N$.
- $\exists$ a bar-involution on $\mathbf{U}$ such that $\overline{q} = q^{-1}$, $\overline{E}_i = E_i$, $\overline{F}_i = F_i$ and $\overline{K}_i = K_i^{-1}$.
- Let $\mathbb{V}$ be the natural representation of $\mathbf{U}$.
Then $\mathbb{V}^{\otimes d}$ is a $\mathbf{U}$-module, via the coproduct $\Delta : \mathbf{U} \to \mathbf{U} \otimes \mathbf{U}$.
- Let $H_{S_d} = \langle H_i, 1 \leq i \leq d-1 \rangle$ be Hecke algebra of type A.
There is a bar-involution on H_{S_d} such that $\overline{H}_i = H_i^{-1}$.

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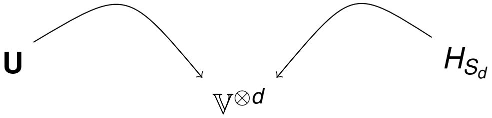

- [Jimbo] Action of **U** via coproduct and action of H_{S_d} via R-matrix on $\mathbb{V}^{\otimes d}$ form double centralizers.
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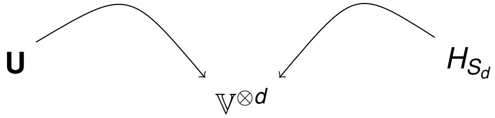

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Motivation

Motivating problem: Develop a Kazhdan-Lusztig theory in Category $\mathcal{O}$ of Lie superalgebras ($\mathfrak{osp}$), say of type B.

• The Hecke algebra of type B_d, $H_{B_d} = \langle , H_{S_d}, H_0 \rangle$, acts naturally on $\mathbb{V}^{\otimes d}$:
if we choose the standard basis $\{v_i\}$ of $\mathbb{V}$ to run over indices of the form $[-a, a]$, then H_0 acts on the first tensor factor by

$$H_0 : v_i \mapsto \begin{cases} v_{-i}, & \text{if } i > 0, \\ v_{-i} + (q - q^{-1})v_i, & \text{if } i < 0. \end{cases}$$

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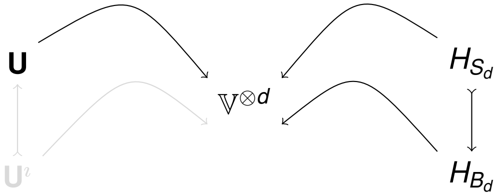
A double centralizer question

- Question: What quantum algebra centralizes H_{B_d}?



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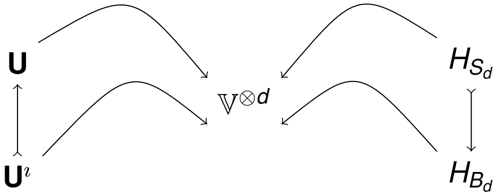

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Quantum coideal subalgebras

- $\mathbf{U}^\imath$ comes in 2 forms, depending on the parity of N (ignore!)
- $\exists$ a presentation $\mathbf{U}^\imath = \{e_i, f_i, ...\}_{i>0}$, but "bad" Serre relations;

  $\exists$ an imbedding $\imath : \mathbf{U}^\imath \hookrightarrow \mathbf{U}$, e.g., $e_i \to E_i + K_i^{-1} F_{-i}$.
- $\mathbf{U}^\imath$ is a coideal subalgebra of $\mathbf{U}$, i.e., $\Delta : \mathbf{U}^\imath \to \mathbf{U}^\imath \otimes \mathbf{U}$.
- The algebra $\mathbf{U}^\imath$ admits a bar-involution $\psi_\imath$, $\psi_\imath(e_i) = e_i$, ...

  (This bar map was **independently** noted by Ehrig-Stroppel)
- $(\mathbf{U}, \mathbf{U}^\imath)$ is an example of quantum symmetric pairs [Noumi, Letzter, Kolb] in different presentations

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iSchur duality

Theorem 1 (Bao-W2012)

The actions of $\mathbf{U}^\imath$ and H_{B_m} on $\mathbb{V}^{\otimes m}$ form double centralizers.

Canonical basis

- Let $L(\lambda)$ be the simple **U**-module with h.wt. $\lambda \in X^+$.
- $L(\lambda)$ admits a bar-involution, which is compatible with the bar-involution on **U**.
- [Lusztig, Kashiwara] $L(\lambda)$ admit a *canonical/crystal basis* (CB).
- Via $\imath : \mathbf{U}^\imath \to \mathbf{U}$, $L(\lambda)$ becomes a $\mathbf{U}^\imath$-module.

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Quasi-R-matrix

- Lusztig's quasi-R-matrix Θ is a variant of Drinfeld's R-matrix.
- The coproduct $\Delta : \mathbf{U} \to \mathbf{U} \otimes \mathbf{U}$ does not commute with the bar maps, i.e., $\Delta \neq \overline{\Delta}$.
- Θ intertwines the coproduct Δ and $\overline{\Delta}$, i.e., $\Theta\Delta = \overline{\Delta}\Theta$. iIt leads to a bar-involution and CBs in tensor product $\mathbf{U}$-module $L(\lambda_1) \otimes \cdots \otimes L(\lambda_\ell)$ [Lusztig].
- The bar map on $\mathbf{U}^\imath$ is not compatible with the bar map on $\mathbf{U}$, i.e., $\imath(\psi_\imath(u)) \neq \overline{\imath(u)}, \forall u$ (recall $e_i \to E_i + K_i^{-1} F_{-i}$).
- One key theorem of [Bao-W] is the existence of Υ which intertwines the imbedding $\imath : \mathbf{U}^\imath \to \mathbf{U}$ and its bar-conjugate $\bar{\imath}$.
- This leads to a quasi-R-matrix $\Theta^\imath$ which intertwines $\Delta : \mathbf{U}^\imath \to \mathbf{U}^\imath \otimes \mathbf{U}$ and its bar-conjugate; $\Theta^\imath \neq \Theta|_{\mathbf{U}^\imath}$.

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1. *The $\Theta^\imath$ induces a bar involution $\psi_\imath$ on the tensor product* $\mathbf{U}^\imath$*-module* $L(\lambda_1) \otimes \cdots \otimes L(\lambda_\ell)$, *and in particular on* $L(\lambda)$.
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The iCB on $\mathbb{V}^{\otimes m} \otimes (\mathbb{V}^*)^{\otimes n}$ can be expanded with respect to the simple tensor basis, with coefficients $t^{\imath}_{\lambda\mu}(q) \in \mathbb{Z}[q]$.

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$\imath$Schur duality recalled

Recall iSchur duality

$$\mathbf{U}^\imath \twoheadrightarrow \ S_d^\imath \quad \circlearrowright \mathbb{V}^{\otimes d} \circlearrowleft \quad H_{B_d}.$$

Here $S_d^\imath := \mathrm{End}_{H_{B_d}}(\mathbb{V}^{\otimes d})$ is a q-Schur-type algebra.

Type B flag varieties

Set $N = 2n+1,\ D = 2d+1$. Let $\mathbb{q}$ be an odd prime power.

• Fix a non-degenerate symmetric bilinear form Q on $\mathbb{F}_{\mathbb{q}}^D$.

• X: the variety of N-step isotropic flags (i.e. $V_{-i} = V_i^{\perp}$):
$V = (0 = V_{-n-1} \subseteq \cdots \subseteq V_{-1} \subseteq V_1 \subseteq \cdots \subseteq V_{n+1} = \mathbb{F}_{\mathbb{q}}^D)$

• $\mathcal{B}$: the complete flag variety of type B_d.

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Type B flag varieties

Set $N = 2n + 1, \ D = 2d + 1$. Let q be an odd prime power.

- Fix a non-degenerate symmetric bilinear form Q on $\mathbb{F}_{\mathrm{q}}^{D}$.
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Convolution algebras

- $\mathscr{A}_{O(D)}(X \times X)$: the space of $O(D)$-invariant $\mathscr{A}$-valued functions on $X \times X$. ($\mathscr{A} = \mathbb{Z}[q, q^{-1}]$)
- Similarly define $\mathscr{A}_{O(D)}(X \times \mathcal{B})$ and $\mathscr{A}_{O(D)}(\mathcal{B} \times \mathcal{B})$.
- $\mathscr{A}_{O(D)}(X \times X)$ and $\mathscr{A}_{O(D)}(\mathcal{B} \times \mathcal{B})$ are $\mathscr{A}$-algebras by convolution products.
- The convolution products also produce the following duality (i.e. double centralizing action):

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The algebra $\mathbf{K}^\imath$

Similar to [BLM90], we show

- Structure constants in $\mathscr{A}_{O(D)}(X \times X)$ stabilize as $D \to \infty$, which gives rise to a "limit" algebra $\mathbf{K}^\imath$.
- $\exists$ a natural surjective algebra homomorphism ϕ_d:

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Geometric iSchur duality

Theorem 4 (Bao-Kujawa-Li-W2014)

$$\begin{array}{ccccccc} \dot{\mathbf{U}}^\imath \twoheadrightarrow & S^\imath_d & \circlearrowleft & \mathbb{V}^{\otimes d} & \circlearrowright & H_{B_d} \\ & & & & & \downarrow \cong \text{[Iwahori]} \\ \mathbf{K}^\imath \twoheadrightarrow & \mathscr{A}_{O(D)}(X \times X) & \circlearrowleft & \mathscr{A}_{O(D)}(X \times \mathcal{B}) & \circlearrowright & \mathscr{A}_{O(D)}(\mathcal{B} \times \mathcal{B}) \end{array}$$

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iCanonical Basis of $\dot{\mathbf{U}}^\imath$

Facts:

- $\mathscr{A}_{O(D)}(X \times X)$ admits a [geometric] bar involution and a canonical basis with positivity $\{A\}_d$, where A runs over

$$\Xi_d := \{A \in \mathrm{Mat}_{N\times N}(\mathbb{Z}_{\geq 0}) \mid |A| = D = 2d+1, A^\imath = A\}$$

(Here $\imath$ denotes the involution of rotation by 180 degree.)

- The bar involutions stabilize as $D \to \infty$, and lifts to the algebra $\mathbf{K}^\imath$.

Recall $\mathbb{Q}(q) \otimes \mathbf{K}^\imath \cong \dot{\mathbf{U}}^\imath$.

Theorem 5 (BKLW2014)

The algebra $\mathbf{K}^\imath$, and hence $\dot{\mathbf{U}}^\imath$, admits a stably CB $\{A\}$, where $A \in \widetilde{\Xi} := \{(a_{ij}) \in Mat_{N\times N}(Z) \mid a_{ij} \geq 0\ (i \neq j), |A| \text{ is odd}\}$.

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Negativity of stably CB

- Recall for type A one has

$$\dot{\mathbf{U}}(\mathfrak{sl}_N) \quad \text{vs} \quad \dot{\mathbf{U}}(\mathfrak{gl}_N)$$

- (Recall $N = 2n + 1$.) $\exists$ 2 versions of modified coideal algebras:

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- [Li-W15] The stably canonical basis of $\dot{\mathbf{U}}(\mathfrak{gl}_N)$ does not have positive structure constants;
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Transfer map

The transfer map $\phi^{\imath}_{d+N,d} : S^{\imath}_{d+N} \to S^{\imath}_d$ is the composition

$$S^{\imath}_{d+N} \xrightarrow{\cong} \mathrm{End}_{H_{B_{d+N}}}(\mathbb{V}^{\otimes(d+N)}) \longrightarrow \mathrm{End}_{H_{B_d} \times H_{S_N}}(\mathbb{V}^{\otimes(d+N)}) \xrightarrow{1\otimes\chi} \mathrm{End}_{H_{B_d}}(\mathbb{V}^{\otimes d}) \xrightarrow{\cong} S^{\imath}_d,$$

where χ is a “sign" homomorphism.

The algebra $\dot{\mathbf{U}}^{\imath}$ can be thought as an inverse limit of the family $\{S^{\imath}_d\}_{d\geq 1}$, i.e., $\exists$ homomorphism $\phi_d : \dot{\mathbf{U}}^{\imath}(\mathfrak{sl}_N) \to S^{\imath}_d$, compatible with the transfer map $\phi^{\imath}_{d+N,d}$.

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Positivity of iCB under a transfer map

Theorem 6 (Li-W15)

The transfer map sends each CB element to a sum of CB elements with coefficients in $\mathbb{Z}_{\geq 0}[q, q^{-1}]$.

The transfer map does preserve CB of $\imath$Schur algebras asymptotically; cf. [McGerty12] in type A.

For $A \in \Xi_{d_0}$ (recall $|A| = 2d + 1$), set ${}_{2p}A := A + 2pI \in \Xi_{d_0+pN}$.

Theorem 7 (Li-W15)

For each $A \in \Xi_{d_0}$, we have $\phi^{\imath}_{d,d-N}(\{{}_{2p}A\}_d) = \{{}_{2p-2}A\}_{d-N}$, for $p \gg 0$ (where $d = d_0 + pN$).

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Positivity again

The asymptotic property of transfer maps allows us to define the canonical basis $\mathbf{B}^{\imath}(\mathfrak{sl}_N)$ for $\dot{\mathbf{U}}^{\imath}(\mathfrak{sl}_N)$, parametrized by $\overline{A} \in \widetilde{\Xi}/\sim$, where the relation $\sim$ on $\widetilde{\Xi}$ is defined by $A \sim A + 2I$.

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The structure constants for the algebra $\dot{\mathbf{U}}^{\imath}(\mathfrak{sl}_N)$ with respect to the iCB $\mathbf{B}^{\imath}(\mathfrak{sl}_N)$ are in $\mathbb{Z}_{\geq 0}[q, q^{-1}]$.

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Positiivity again, and again

- The iCB $\mathbf{B}^\imath(\mathfrak{sl}_N)$ satisfies (and in turn is characterized by) the almost orthonormality with respect to a geometric bilinear form.
- The geometric bilinear form is positive with respect to the iCB.
- Recall $N = 2n + 1$. We have $\dot{\mathbf{U}}(\mathfrak{sl}_n) \subset \dot{\mathbf{U}}^\imath(\mathfrak{sl}_N)$, and that $\mathbf{B}^\imath(\mathfrak{sl}_N) \cap \dot{\mathbf{U}}(\mathfrak{sl}_n)$ is the CB of $\dot{\mathbf{U}}(\mathfrak{sl}_n)$.
- The action of $\dot{\mathbf{U}}(\mathfrak{sl}_n)$ (resp., the action of $S_d^\imath$) on $\mathbb{V}^{\otimes d}$ with respect to the corresponding iCB is positive.

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• Affinization of q-coideal algebras, iSchur duality, and iCB (via classical type affine flag variety, Steinberg variety, ...)

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