MODULAR AFFINE VERTEX ALGEBRAS AND BABY WAKIMOTO MODULES

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Abstract. We develop some basic properties such as p-centers of affine vertex algebras and free field vertex algebras in prime characteristic. We show that the Wakimoto-Feigin-Frenkel homomorphism preserves the p-centers by providing explicit formulas. This allows us to formulate the notion of baby Wakimoto modules, which in particular provides an interpretation in the context of modular vertex algebras for Mathieu's irreducible character formula of modular affine Lie algebras at the critical level.

1. Introduction

Let $\mathbb{K}$ be an algebraically closed field of prime characteristic p. Denote by $U = \mathbb{K} \otimes_{\mathbb{Z}} U_{\mathbb{Z}}$, where $U_{\mathbb{Z}}$ is the Kostant-Garland $\mathbb{Z}$-form (including divided powers) of the universal enveloping algebra of $\mathfrak{g}$. Mathieu [Ma] established a character formula for the irreducible highest weight U-module $L(-\rho)$ at the critical level (see (5.3)), which can be rephrased as that the Wakimoto module of highest weight $-\rho$ over the complex field $\mathbb{C}$ remains irreducible over U after reduction modulo p. Mathieu also gave a character formula for $\mathfrak{l}(-\rho)$ (and also for $L((p-1)\rho)$); see (5.1)-(5.2). Here $\mathfrak{l}(-\rho)$ denotes the irreducible quotient $\mathfrak{g}$-module of the Verma $\mathfrak{g}$-module of high weight $-\rho$, which can be regarded as an irreducible module over the restricted enveloping algebra $\mathfrak{u}_0(\mathfrak{g})$ (and $\mathfrak{u}_0(\mathfrak{g}) \subset U$). These two irreducible character formulas are equivalent by the Steinberg tensor product theorem and noting that $(p-1)\rho$ is a restricted weight.

Modular vertex algebras (i.e., vertex algebras in prime characteristic) were first considered in [BR] by Borcherds and Ryba in their study of modular moonshine. This paper is motivated by putting Mathieu's result in a proper context of modular Lie algebras and modular vertex algebras (where the algebra U plays no role). We formulate the notion of p-centers for vertex algebras associated to Heisenberg algebras, affine algebras, and some other free fields, and this gives rise to corresponding p-restricted vertex algebras. We show that the p-centers and the state-field correspondence for these vertex algebras are compatible in a simple manner; cf. Proposition 2.6.

Wakimoto modules (over $\mathbb{C}$) were introduced by Wakimoto [Wak] for $\mathfrak{sl}_2$ and then by Feigin and E. Frenkel for general semisimple Lie algebras [FF]. Wakimoto modules have played a fundamental role in the affine vertex algebra setting and applications to the geometric Langlands program, cf. [Fr1, Fr2]. The construction of Wakimoto modules relies on the Wakimoto-Feigin-Frenkel homomorphism $\mathtt{w}$ from an affine vertex algebra to a bosonic free field vertex algebra. As a main result of this note we show that $\mathtt{w}$ (over the field $\mathbb{K}$) preserves the p-centers, and indeed we provide explicit formulas for the restriction of $\mathtt{w}$ on the p-center. This allows us to formulate a notion of baby Wakimoto modules, which is analogous to the more familiar notion of baby Verma modules for modular Lie algebras. Now Mathieu's result can be restated that the baby Wakimoto module of highest weight $-\rho$ is irreducible as module over $\mathfrak{g}$ or over $\mathfrak{u}_0(\mathfrak{g})$ (that is, it coincides with $\mathfrak{l}(-\rho)$ in the above notation).

This paper is organized as follows. In Section 2, we prove some basic properties of the modular affine vertex algebras including the p-centers. In Section 3, we describe the p-centers of the Heisenberg vertex algebra and of a symplectic bosonic vertex algebra. We formulate the main construction of the baby Wakimoto modules. In Section 4, we establish the formulas for the WFF homomorphism on the p-center of the affine vertex algebra. In Section 5, we give a reformulation of Mathieu's main result in terms of the irreducibility of the baby Wakimoto module of highest weight $-\rho$. We end with some conjectures and open problems on further development of modular representation theory of affine Lie algebras.

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2. Modular affine algebras and modular vertex algebras

2.1. **Affine Lie algebra in prime characteristic.** Let $\bar{\mathfrak{g}}$ be a finite-dimensional semisimple Lie algebra, which is a Lie algebra of a simply connected algebraic group $\bar{G}$ over an algebraically closed field $\mathbb{K}$ of characteristic $p > 0$. Then $\bar{\mathfrak{g}}$ is a restricted Lie algebra (also called a p-Lie algebra) with p-power map denoted by $-^{[p]}$; cf. [Jan] for a review of modular Lie algebras. Moreover, $\bar{\mathfrak{g}}$ affords a non-degenerate bilinear form $\langle \cdot, \cdot \rangle$, which induces a linear isomorphism $\bar{\mathfrak{g}} \to \bar{\mathfrak{g}}^*$. We fix a Chevalley basis $h_i (1 \le i \le \ell), e_\alpha, f_\alpha (\alpha \in \bar{\Delta}^+)$ of $\bar{\mathfrak{g}}$, where $\bar{\Delta}^+$ is a set of positive roots for $\bar{\mathfrak{g}}$ corresponding to a set of simple roots $\bar{\Pi} = \{\alpha_1, \ldots, \alpha_\ell\}$. We further write $e_i = e_{\alpha_i}, f_i = f_{\alpha_i}$. We denote by $\bar{B}$ (respectively, $\bar{B}_-$) the Borel subgroup of $\bar{G}$ whose Lie algebra $\bar{\mathfrak{b}}$ (respectively, $\bar{\mathfrak{b}}_-$) is spanned by root vectors from $\bar{\Delta}^+$ (respectively, $\bar{\Delta}^- = -\bar{\Delta}^+$).

We consider the affine Lie algebra

$$\mathfrak{g} \cong L\bar{\mathfrak{g}} \oplus \mathbb{K}c$$

where $L\bar{\mathfrak{g}} \cong \mathbb{K}[t, t^{-1}] \otimes \bar{\mathfrak{g}}$. We shall write $x_n = t^n \otimes x$ for $x \in \bar{\mathfrak{g}}$ and $n \in \mathbb{Z}$. Then $\bar{\mathfrak{g}}$ is naturally a Lie subalgebra of $\mathfrak{g}$ by the identification $1 \otimes \bar{\mathfrak{g}} \cong \bar{\mathfrak{g}}$. We denote by $h^\vee$ the dual Coxeter number for the affine Lie algebra $\mathfrak{g}$.

A Cartan subalgebra $\mathfrak{h}$ of the affine Lie algebra $\mathfrak{g}$ is

$$\mathfrak{h} = \bar{\mathfrak{h}} + \mathbb{K}c$$

and a Borel subalgebra of $\mathfrak{g}$ is $\mathfrak{b} = \mathbb{K}c + t\mathbb{K}[t]\otimes\bar{\mathfrak{g}} + \bar{\mathfrak{b}}$ with nilradical $\mathfrak{n} = t\mathbb{K}[t]\otimes\bar{\mathfrak{g}} + \bar{\mathfrak{n}}$, so that $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$. Denote by Δ_+ the set of positive roots associated to $\mathfrak{n}$, and by Δ_+^{re} the subset of real roots in Δ_+. Let $\mathfrak{g}^*$ denote the restricted dual of $\mathfrak{g}$ associated to the root space decomposition of $\mathfrak{g}$.

Denote by $\bar{T} \subset \bar{G}$ the maximal torus with Lie algebra $\bar{\mathfrak{h}}$. Let $\mathbb{K}^* = \mathbb{K} - \{0\}$ be the torus corresponding to the derivation d on $\mathfrak{g}$, where $[d, c] = 0$ and $[d, t^n \otimes x] = -nt^n \otimes x$ for $x \in \bar{\mathfrak{g}}$ and $n \in \mathbb{Z}$. Set $T = \bar{T} \times \mathbb{K}^*$.

Lemma 2.1 (cf. [Ma], (1.4))**.** *There is a restricted Lie algebra structure on the affine Lie algebra $\mathfrak{g}$ as an extension of the one on $\bar{\mathfrak{g}}$, whose p-power map is given by*

$$c^{[p]} = c, \quad (t^n \otimes x)^{[p]} = t^{np} \otimes x^{[p]}, \quad \textit{for } n \in \mathbb{Z}, x \in \bar{\mathfrak{g}}.$$

Then as usual one has the p-center $\mathcal{Z}_0(\mathfrak{g})$ in the enveloping algebra $U(\mathfrak{g})$ which is generated by $x^p - x^{[p]}$ for all $x \in \mathfrak{g}$. The subalgebra of $\mathcal{Z}_0(\mathfrak{g})$ generated by $x^p - x^{[p]}$ for all $x \in L\bar{\mathfrak{g}}$ will be denoted by $\mathcal{Z}_0'(\mathfrak{g})$ and referred to as the *proper p-center.*

Each $\chi \in (L\bar{\mathfrak{g}})^*$ defines a p-character and gives rise to the reduced enveloping algebra by

$$\mathfrak{u}_\chi(\mathfrak{g}) = U(\mathfrak{g})/I_\chi$$

where I_χ is the ideal generated by $a^p - a^{[p]} - \chi(a)^p$ for all $a \in L\bar{\mathfrak{g}}$. In particular, $\mathfrak{u}_0(\mathfrak{g})$ is called the restricted enveloping algebra of $\mathfrak{g}$. Note that according to our definition $c^p - c$ is not in the ideal I_0.

A distinguished restricted Lie subalgebra of $\mathfrak{g}$ is the Heisenberg algebra

$$\mathfrak{hs} = L\bar{\mathfrak{h}} \oplus \mathbb{K}c = \mathfrak{hs}^- \oplus \mathfrak{h} \oplus \mathfrak{hs}^+,$$

where $\mathfrak{hs}^\pm = \oplus_{n\in\pm\mathbb{N}} t^n \otimes \bar{\mathfrak{h}}$. The Lie algebra $\mathfrak{hs}$ has a large center spanned by $c, t^{pn} \otimes \bar{\mathfrak{h}}$ for $n \in \mathbb{Z}$.

The p-center $\mathcal{Z}_0(\mathfrak{hs})$ of $U(\mathfrak{hs})$ is generated by $x^p - x^{[p]}$ for all $x \in \mathfrak{hs}$ and the proper p-center $\mathcal{Z}_0'(\mathfrak{hs})$ of $U(\mathfrak{hs})$ is by definition the subalgebra generated by $x^p - x^{[p]}$ for all $x \in L\bar{\mathfrak{h}}$. The whole center of $U(\mathfrak{hs})$ is generated by $\mathcal{Z}_0(\mathfrak{hs})$ and $c, t^{pn} \otimes \bar{\mathfrak{h}}$ for $n \in \mathbb{Z}$, though this fact will not be needed below.

2.2. **Vertex algebras in prime characteristic.** The usual notion of vertex algebras can be readily made sense over the field $\mathbb{K}$ of characteristic $p > 0$ (cf. Borcherds-Ryba [BR]). All one needs is to use the divided power of the translation operator $T^{(i)} = T^i/i!, i \geq 1$ and noting that

$$Y(T^{(i)}a, z) = \partial^{(i)} Y(a, z),$$

where $\partial^{(i)}$ denotes the ith divided power of the derivative with respect to z.

Denote $L_+\bar{\mathfrak{g}} = \sum_{n\in\mathbb{Z}_+} t^n \otimes \bar{\mathfrak{g}}$. It is well known that the vacuum $\mathfrak{g}$-module of level $\kappa \in \mathbb{K}$

$$V^\kappa(\mathfrak{g}) = U(\mathfrak{g}) \bigotimes_{U(L_+\bar{\mathfrak{g}}+\mathbb{K}c)} \mathbb{K}_\kappa$$

carries a canonical structure of a vertex algebra (cf. e.g. [Fr2]), where $L\bar{\mathfrak{g}}^+$ acts on $\mathbb{K}_\kappa = \mathbb{K}$ trivially and c as scalar κ. Denote by $|0\rangle = 1 \otimes 1$ the vacuum vector in $V^\kappa(\mathfrak{g})$.

2.3. **The p-centers of modular vertex algebras.** Let

$$x(z) = \sum_{n \in \mathbb{Z}} x_n z^{-n-1}, \qquad x \in \bar{\mathfrak{g}}.$$

The next lemma on vertex operators is standard (cf. [Fr2]), except the divided power notation.

Lemma 2.2. *The following formulas hold in the vertex algebra $V^\kappa(\mathfrak{g})$:*

$$\begin{aligned} Y(x_{-r}|0\rangle, z) = \partial^{(r-1)} x(z) &= \sum_{n \in \mathbb{Z}} \binom{-n-1}{r-1} x_n z^{-n-r}, \quad (2.1) \\ Y(x_{-r_1} y_{-r_2} \cdots |0\rangle, z) &= :\partial^{(r_1-1)} x(z)\, \partial^{(r_2-1)} y(z) \cdots : \end{aligned}$$

for $x, y, \ldots \in \bar{\mathfrak{g}}$, and $r, r_1, r_2, \ldots \in \mathbb{N}$.

We shall need some more formulas for vertex operators in characteristic p.

Lemma 2.3. *The following identities hold for the vertex algebra $V^\kappa(\mathfrak{g})$: for $x \in \bar{\mathfrak{g}}$ and $r \geq 1$, we have*

$$Y(x_{-rp}|0\rangle, z) = \partial^{(rp-1)} x(z) = \sum_{n \in \mathbb{Z}} \binom{-n-1}{r-1} x_{np}\, z^{-np-rp}, \quad (2.2)$$

$$Y(x_{-r}^p|0\rangle, z) = :(\partial^{(r-1)} x(z))^p: = \sum_{n \in \mathbb{Z}} \binom{-n-1}{r-1} x_n^p\, z^{-np-rp}. \quad (2.3)$$

The special case of Lemma 2.3 for $r = 1$ reads:

$$Y(x_{-p}|0\rangle, z) = \partial^{(p-1)} x(z) = \sum_{n \in \mathbb{Z}} x_{np}\, z^{-np-p}, \quad (2.4)$$

$$Y(x_{-1}^p|0\rangle, z) = :x(z)^p: = \sum_{n \in \mathbb{Z}} x_n^p\, z^{-np-p}. \quad (2.5)$$

To prove Lemma 2.3, we shall need the following classical formula.

Lemma 2.4. *For $a = a_0 + pa' \in \mathbb{Z}_{\geq 0}, b = b_0 + pb'$ with $0 \leq a_0, b_0 \leq p-1$ and $a' \geq 0$, we have*

$$\binom{b}{a} \equiv \binom{b'}{a'} \binom{b_0}{a_0} \mod p.$$

(All the a's and b's involved are integers.)

Proof of Lemma 2.3. By Lemma 2.4, we obtain that $\binom{-m-1}{rp-1} \equiv 0 \mod p$ if $p \nmid m$, and $\binom{-np-1}{rp-1} \equiv \binom{-n-1}{r-1} \mod p$ for $n \in \mathbb{Z}$. Now (2.2) follows from (2.1).

We write $A(z) \equiv \partial^{(r-1)}x(z) = \sum_{n\in\mathbb{Z}} \binom{-n-1}{r-1} x_n z^{-n-r} = A_+(z) + A_-(z)$, where $A_\pm(z) = \sum_{n \lesseqgtr -r} \binom{-n-1}{r-1} x_n z^{-n-r}$. By the definition of normal ordered product and induction on $m \geq 1$, we have

$$\begin{aligned} :A(z)^m: &= A_+(z):A(z)^{m-1}: + :A(z)^{m-1}:A_-(z) \\ &= \sum_{i=0}^{m} \binom{m}{i} A_+(z)^i A_-(z)^{m-i}. \end{aligned}$$

Note that $A_+(z)^p = \sum_{n\leq -r} \binom{-n-1}{r-1} x_n^p z^{-np-p}$ since x_n with $n < 0$ commute and $b^p = b$ for $b \in \mathbb{F}_p$. Similarly, $A_-(z)^p = \sum_{n\geq 0} \binom{-n-1}{r-1} x_n^p z^{-np-p}$. Hence, $:A(z)^p: = A_+(z)^p + A_-(z)^p$, whence (2.3). $\square$

Remark 2.5. Lemmas 2.2 and 2.3 are applicable to other modular vertex algebras, e.g. $\mathcal{F}$.

Denote

$$\iota(x_n) = x_n^p - x_{np}^{[p]}, \quad x \in \bar{\mathfrak{g}}.$$

We also denote $\iota(z) = z^p$ (the Frobenius morphism). In the next proposition, which follows directly from Lemmas 2.2 and 2.3, we formulate a basic property of modular affine vertex algebras.

Proposition 2.6 (Commutativity of ι and Y)**.** *For $x \in \bar{\mathfrak{g}}$ and $r \geq 1$, we have*

$$\begin{aligned} Y(\iota(x_{-r})|0\rangle, z) &= Y\left((x_{-r}^p - (x^{[p]})_{-rp})|0\rangle, z\right) \\ &= \left(\partial^{(r-1)}x(z)\right)^p - \partial^{(rp-1)}x^{[p]}(z) = \iota\left(\partial^{(r-1)}x(z)\right). \end{aligned} \tag{2.6}$$

When $r = 1$, we have

$$Y(\iota(x_{-1})|0\rangle, z) = \iota Y(x_{-1}|0\rangle, z) = \sum_{n\in\mathbb{Z}} (x_n^p - (x^{[p]})_{np})\, z^{-np-p}.$$

By definition, the center of a vertex algebra V consists of all vectors $v \in V$ such that $Y(a, z)v \in V[[z]]$ for all $a \in V$. The center of a vertex algebra is a commutative vertex algebra (cf. [Fr2]).

Definition 2.7. The *p-center* (or the *Frobenius center*) $\mathfrak{z}_0(V^\kappa(\mathfrak{g}))$ of the vertex algebra $V^\kappa(\mathfrak{g})$ is defined to be the subspace $\mathcal{Z}_0'(\mathfrak{g})|0\rangle \subset V^\kappa(\mathfrak{g})$.

Clearly these p-centers (and other p-centers below) are vertex subalgebras of the centers of the corresponding vertex algebras.

Proposition 2.8. (1) *The p-center $\mathfrak{z}_0(V^\kappa(\mathfrak{g}))$ is a commutative vertex subalgebra of $V^\kappa(\mathfrak{g})$.*

(2) *$U(\mathfrak{g}) \cdot \mathfrak{z}_0(V^\kappa(\mathfrak{g}))$ is an ideal of the vertex algebra $V^\kappa(\mathfrak{g})$, and so the quotient*

$$V_0^\kappa(\mathfrak{g}) \stackrel{def}{=} V^\kappa(\mathfrak{g})/(U(\mathfrak{g}) \cdot \mathfrak{z}_0(V^\kappa(\mathfrak{g})))$$

carries an induced vertex algebra structure.

Proof. Part (2) follows from (1) easily, and we shall prove (1).

Observe that the Fourier components of (2.6) are of the form $x_n^p - (x^{[p]})_{np}$ (up to a scalar multiple), and hence belong to the p-center $\mathfrak{z}_0(V^\kappa(\mathfrak{g}))$. By definition, the p-center $\mathfrak{z}_0(V^\kappa(\mathfrak{g}))$ is spanned by elements of the form $x = \iota(a_{-r_1} b_{-r_2} \cdots |0\rangle)$ with $a, b \ldots \in \bar{\mathfrak{g}}$ and $r_1, r_2, \ldots > 0$. By Proposition 2.6, the vertex operator

$$Y(x, z) = :Y(\iota(a_{-r_1})|0\rangle, z) Y(\iota(b_{-r_2})|0\rangle, z) \cdots:$$

is a linear combination of operators composed from those of the form $x_n^p - (x^{[p]})_{np}$, and hence clearly preserves $\mathfrak{z}_0(V^\kappa(\mathfrak{g}))$. □

Following the standard terminology in the theory of modular Lie algebras, we shall refer to the vertex algebras $V_0^\kappa(\mathfrak{g})$ as the *restricted* (or more precisely *p-restricted*) vertex algebras associated to $\mathfrak{g}$.

Remark 2.9. Since $c^p - c \notin \mathcal{Z}_0'(\mathfrak{g})$ by definition, the central charges for the restricted vertex algebras $V_0^\kappa(\mathfrak{g})$ can be any scalar in $\mathbb{K}$.

A *baby Verma* $\mathfrak{g}$*-module* (associated to a weight λ on $\mathfrak{h}$ of level κ) is a $\mathfrak{g}$-module of the form

$$V(\lambda) \equiv V^\kappa(\lambda) = \mathfrak{u}_0(\mathfrak{g}) \bigotimes_{\mathfrak{u}_0(\mathfrak{n}+\mathfrak{h})} \mathbb{K}_\lambda$$

where $\mathfrak{n}$ acts trivially on the one-dimensional space $\mathbb{K}_\lambda \cong \mathbb{K}$ and $\mathfrak{h}$ acts by the weight $\lambda \in \mathfrak{h}^*$. These baby Verma modules are modules of the restricted vertex algebra $V_0^\kappa(\mathfrak{g})$.

3. The baby Wakimoto modules

3.1. **A vertex algebra** M**.** Let $\mathcal{A}^\mathfrak{g}$ be the Weyl algebra over $\mathbb{K}$ with generators $a_{\alpha,n}, a^*_{\alpha,n}$ with $\alpha \in \overline{\Delta}_+, n \in \mathbb{Z}$, and relations

$$[a_{\alpha,n}, a^*_{\beta,m}] = \delta_{\alpha,\beta}\delta_{n,-m}, \quad [a_{\alpha,n}, a_{\beta,m}] = [a^*_{\alpha,n}, a^*_{\beta,m}] = 0.$$

A restricted Lie algebra structure on $\mathcal{A}^\mathfrak{g}$ is given as follows:

$$a_{\alpha,n}^{[p]} = (a^*_{\alpha,n})^{[p]} = 0, \quad n \in \mathbb{Z}.$$

Introduce the fields

$$a_\alpha(z) = \sum_{n\in\mathbb{Z}} a_{\alpha,n} z^{-n-1}, \quad a^*_\alpha(z) = \sum_{n\in\mathbb{Z}} a^*_{\alpha,n} z^{-n}, \qquad \alpha \in \overline{\Delta}_+.$$

Let M be the Fock representation of $\mathcal{A}^\mathfrak{g}$ generated by $|0\rangle$ such that

$$a_{\alpha,n}|0\rangle = 0, \quad n \geq 0; \qquad a^*_{\alpha,n}|0\rangle = 0, \quad n > 0.$$

As a vector space, $M \cong \mathbb{K}[a_{\alpha,n-1}, a^*_{\alpha,n}]_{\alpha\in\bar{\Delta}_+, n\leq 0}$. It is well known that M carries a vertex algebra structure with state-field correspondence

$$Y(a_{\alpha_1,-r_1} \cdots a_{\alpha_k,-r_k} a^*_{\beta_1,-s_1} \cdots a^*_{\beta_m,-s_m}|0\rangle)$$
$$= :\partial^{(r_1-1)} a_{\alpha_1}(z) \cdots \partial^{(r_k-1)} a_{\alpha_k}(z) \partial^{(s_1)} a^*_{\beta_1}(z) \cdots \partial^{(s_m)} a^*_{\beta_m}(z):$$

and with the translation operator T such that

$$T|0\rangle = 0, \ [T, a_{\alpha,n}] = -n a_{\alpha,n-1}, \ [T, a^*_{\alpha,n}] = -(n-1) a^*_{\alpha,n-1}.$$

Proposition 3.1. (1) *The p-center* $\mathcal{Z}_0(\mathcal{A}^{\mathfrak{g}})$ *is equal to* $\mathbb{K}[a^p_{\alpha,n}, (a^*_{\alpha,n})^p]_{\alpha\in\bar{\Delta}_+, n\in\mathbb{Z}}$*; and moreover,* $\mathfrak{z}_0(M) \cong \mathbb{K}[a^p_{\alpha,n-1}, (a^*_{\alpha,n})^p]_{\alpha\in\bar{\Delta}_+, n\leq 0}$.

(2) *The space* $U(\mathcal{A}^{\mathfrak{g}})\cdot\mathfrak{z}_0(M)$ *is an ideal of the vertex algebra* M*, so the quotient*

$$M_0 := M/(U(\mathcal{A}^{\mathfrak{g}})\cdot\mathfrak{z}_0(M))$$

carries an induced vertex algebra structure.

3.2. **Realization of the contragredient Verma modules.** We first recall (cf. e.g. [Fr2, pp.135]) that the contragredient Verma module of $\bar{\mathfrak{g}}$ can be realized via its identification with the space of regular functions $\mathcal{O}_{\bar{N}_+}$ on the unipotent subgroup $\bar{N}_+$ of the algebraic group $\bar{G}$; equivalently, this is described as follows: let $\bar{U} = \bar{N}_+\bar{B}_-/\bar{B}_-$ be the open cell of the flag variety $\bar{\mathcal{B}} := \bar{G}/\bar{B}_-$. Let

$$\phi : U(\bar{\mathfrak{g}}) \longrightarrow \mathcal{D}_{\bar{\mathcal{B}}}(\bar{U}) \tag{3.1}$$

denote the composition of the restriction to the open cell $\Gamma(\bar{\mathcal{B}}, \mathcal{D}_{\bar{\mathcal{B}}}) \to \mathcal{D}_{\bar{\mathcal{B}}}(\bar{U})$ with an algebra homomorphism $U(\bar{\mathfrak{g}}) \to \Gamma(\bar{\mathcal{B}}, \mathcal{D}_{\bar{\mathcal{B}}})$ where $\mathcal{D}_{\bar{\mathcal{B}}}$ denotes the sheaf of crystalline differential operators on the flag variety $\bar{\mathcal{B}}$ (i.e. no divided powers of differential operators, cf. e.g., [BMR]). Then the contragredient Verma module of $\bar{\mathfrak{g}}$ is the pullback of the $\mathcal{D}_{\bar{\mathcal{B}}}(\bar{U})$-module $\mathcal{O}_{\bar{U}}$ via the algebra homomorphism ϕ. The restriction $\phi|_{\bar{\mathfrak{g}}} : \bar{\mathfrak{g}} \to \mathrm{Vect}_{\bar{U}}$ is a Lie algebra homomorphism, where $\mathrm{Vect}_{\bar{U}}$ the vector fields over $\bar{U}$.

Let us fix some coordinates y_γ (and $\partial_\gamma := \frac{\partial}{\partial y_\gamma}$) for the open cell $\bar{U}$. The following lemma is standard.

Lemma 3.2. *We have* $\mathcal{Z}_0(\mathcal{D}_{\bar{\mathcal{B}}}(U)) = \mathbb{K}[y^p_\gamma, \partial^p_\gamma]_{\gamma\in\bar{\Delta}_+}$.

The following is known (cf., e.g., [BMR, §1.3]).

Lemma 3.3. *The restriction* $\phi : \bar{\mathfrak{g}} \longrightarrow \mathrm{Vect}_{\bar{U}}$ *is a homomorphism of restricted Lie algebras, and the homomorphism* $\phi : U(\bar{\mathfrak{g}}) \longrightarrow \mathcal{D}_{\bar{\mathcal{B}}}(\bar{U})$ *maps the p-center of* $U(\bar{\mathfrak{g}})$ *to the p-center of* $\mathcal{D}_{\bar{\mathcal{B}}}(\bar{U})$.

In some cases, we can make this fairly explicit as follows. For $x \in \bar{\mathfrak{n}} \oplus \bar{\mathfrak{h}}$, one can write

$$\phi(x) = \sum_{\beta\in\bar{\Delta}_+} c_\beta m_\beta(y_\gamma)\partial_\beta \tag{3.2}$$

where $c_\beta \in \mathbb{K}$ and $m_\beta(y_\gamma)$ denotes some monomials in the variables y_γ, for $\gamma \in \bar{\Delta}_+$.

Lemma 3.4. *Let* $x \in \bar{\mathfrak{n}} \oplus \bar{\mathfrak{h}}$ *and retain the above notation* (3.2)*. Then we have*

$$\phi(\iota(x)) = \sum_{\beta\in\bar{\Delta}_+} c^p_\beta m_\beta(y^p_\gamma)\partial^p_\beta.$$

Proof. We know that

$$\phi(\iota(x)) = \phi(x)^p - \phi(x^{[p]}) = \big(\sum c_\beta m_\beta(y_\gamma)\partial_\beta\big)^p - \phi(x^{[p]})$$

lies in the p-center $\mathcal{Z}_0(\mathcal{D}_{\bar{\mathcal{B}}}(U))$ and $\phi(x^{[p]})$ is a sum of differential operators of order one (plus some possible constants). We expand this pth power and move the differential operator ∂_β to the right by using commutators. Lemma 3.2 ensures all the commutators will cancel out with each other as they would produce differential operators of order between 1 and $p-1$. $\square$

Remark 3.5. Let $\mu : T^*\bar{\mathcal{B}} \to \bar{\mathcal{N}}$ be the Springer resolution, $\mu^{(1)} : T^*\bar{\mathcal{B}}^{(1)} \to \bar{\mathcal{N}}^{(1)}$ be the induced map between the corresponding Frobenius twists. Then the same argument as in proof of Lemma 3.4 shows that $\phi(\iota(x)) = (\mu^{(1)})^*(\bar{x})|_U$ ([BMR, 1.3.3]), where $\bar{x}$ is the image of $x \in \bar{\mathfrak{g}}$ by the projection $\mathbb{K}[\bar{\mathfrak{g}}^*] \to \mathbb{K}[\bar{\mathcal{N}}]$.

3.3. **Heisenberg vertex algebra π^κ.** Let $\mathcal{B}^{\mathfrak{h}}_\kappa$ be a copy of Heisenberg algebra (of the affine Lie algebra $\mathfrak{g}$), with generators $\mathbf{1}$ and $b_{i,n}$ $(i = 1, \dots, \ell, n \in \mathbb{Z})$ and subject to the relations

$$[b_{i,n}, b_{j,m}] = n\kappa\langle h_i, h_j\rangle\delta_{n,-m}\mathbf{1}.$$

Denote by π^κ the vertex algebra $\mathbb{K}[b_{i,n}]_{1\le i\le \ell; n<0}$ (where $\mathbf{1}$ acts as the identity map), with

$$Y(b_{i,-1}, z) \equiv b_i(z) = \sum_{n<0} b_{i,n} z^{-n-1},$$

and the translation T given by

$$T \cdot b_{i_1,n_1} \cdots b_{i_m,n_m} = -\sum_{j=1}^{m} n_j b_{i_1,n_1} \cdots b_{i_j,n_j-1} \cdots b_{i_m,n_m}.$$

A restricted Lie algebra structure on $\mathcal{B}^{\mathfrak{h}}_\kappa$ is given by

$$b_{i,n}^{[p]} = b_{i,np}, \quad \mathbf{1}^{[p]} = \mathbf{1}, \qquad 1 \le i \le \ell, n \in \mathbb{Z}.$$

The proper p-center of the vertex algebra π^κ (which excludes $\mathbf{1}^p - \mathbf{1}$) is then equal to

$$\mathfrak{z}'_0(\pi^\kappa) = \mathbb{K}[b_{i,n}^p - b_{i,np}]_{1\le i\le\ell; n<0}.$$

Denote by π^κ_0 the quotient vertex algebra of π^κ by the ideal (in the sense of vertex algebras) generated by $\mathfrak{z}'_0(\pi^\kappa)$.

For $\kappa = 0$, π^0 is naturally a commutative vertex algebra.

3.4. **The Wakimoto-Feigin-Frenkel (WFF) homomorphism.** Let κ_c denote the critical level for $\mathfrak{g}$. There exists a homomorphism of vertex algebras

$$\mathtt{w} = \mathtt{w}_\kappa : V^\kappa(\mathfrak{g}) \to M \otimes \pi^{\kappa-\kappa_c},$$

which is roughly speaking an affinization of ϕ defined in (3.1); see [Fr2, Theorem 6.1.6]. We shall call $\mathtt{w}$ the *WFF homomorphism*, since this was introduced by Wakimoto [Wak] in the $\mathfrak{sl}_2$ case and by Feigin-Frenkel [FF] for general semisimple

Lie algebras $\bar{\mathfrak{g}}$. On generating fields, the formulas for $\mathtt{w}$ read as follows:

$$e_i(z) \mapsto a_{\alpha_i}(z) + \sum_{\alpha_i \neq \beta \in \bar{\Delta}_+} {:}P^i_\beta\big(a^*_\alpha(z)\big)a_\beta(z){:}, \tag{3.3}$$

$$h_i(z) \mapsto \sum_{\beta \in \bar{\Delta}_+} \beta(h_i){:}a^*_\beta(z)a_\beta(z){:} + b_i(z), \tag{3.4}$$

$$f_i(z) \mapsto \sum_{\beta \in \bar{\Delta}_+} {:}Q^i_\beta\big(a^*_\alpha(z)\big)a_\beta(z){:} + \big(c_i + (\kappa - \kappa_c)\langle e_i, f_i\rangle\big)\partial_z a^*_{\alpha_i}(z) + a^*_{\alpha_i}(z)b_i(z). \tag{3.5}$$

We shall take for granted that c_i are integers. The polynomial $Q^i_\beta\big(a^*_\alpha(z)\big)$ for $\beta = \alpha_i$ can be determined explicitly as follows; cf. [Fr2].

Lemma 3.6. *We have* $Q^i_{\alpha_i}\big(a^*_\alpha(z)\big) = -{:}a^*_{\alpha_i}(z)^2{:}$.

Example 3.7. For $\bar{\mathfrak{g}} = \mathfrak{sl}_2$, the formulas for $\mathtt{w}$ are greatly simplified (where we drop the indices of the Chevalley generators and of the free fields) as follows:

$$\begin{aligned} e(z) &\mapsto a(z), \qquad h(z) \mapsto -2{:}a^*(z)a(z){:} + b(z), \\ f(z) &\mapsto -{:}a^*(z)^2a(z){:} + \kappa\partial_z a^*(z) + a^*(z)b(z). \end{aligned} \tag{3.6}$$

The following is a main result of this paper, which will be proved in Section 4.

Theorem 3.8. *The homomorphism* $\mathtt{w} : V^\kappa(\mathfrak{g}) \to M \otimes \pi^{\kappa-\kappa_c}$ *sends the* p*-center* $\mathfrak{z}_0(V^\kappa(\mathfrak{g}))$ *to the* p*-center* $\mathfrak{z}_0(M) \otimes \mathfrak{z}_0(\pi^{\kappa-\kappa_c})$.

We have the following immediate consequence.

Corollary 3.9. *The homomorphism* $\mathtt{w}$ *induces naturally a homomorphism of vertex algebras* $\mathtt{w}_0 : V^\kappa_0(\mathfrak{g}) \to M_0 \otimes \pi_0^{\kappa-\kappa_c}$.

3.5. **The baby Wakimoto modules.** We define baby Wakimoto modules (associated to p-characters) at the critical level κ_c as follows.

Let ξ be a p-character on the Lie algebra $\mathcal{A}^{\mathfrak{g}}$ such that

$$\xi((a^*_{\alpha,n})^p) = 0 = \xi(a^p_{\alpha,n-1}), \qquad \text{for } n > 0, \alpha \in \bar{\Delta}_+. \tag{3.7}$$

Let ξ^π be a p-character on the Lie algebra $\mathcal{B}^{\mathfrak{h}}$. Take a weight $\lambda(t) = \sum_{n\in\mathbb{Z}} \lambda_n t^{-n-1} \in \bar{\mathfrak{h}}^*((t))$ which is compatible with the p-character ξ^π in the sense that

$$\lambda^p_{i,n} - \lambda_{i,np} = \xi^\pi(b_{i,n})^p, \quad \text{for all } 1 \le i \le \ell, n \in \mathbb{Z},$$

where $\lambda_{i,n} = \lambda_n(h_i)$. Such a weight gives rise to the one-dimensional π^0–module $\mathbb{K}_{\lambda(t)}$ on which $b_{i,n}$ acts by multiplication by $\lambda_{i,n}$. This defines a $\mathfrak{g}$-module at the critical level on M (which is identified with $M \otimes \mathbb{K}_{\lambda(t)}$). Identifying M as the polynomial algebra $\mathbb{K}[a_{\alpha,n-1}, a^*_{\alpha,n}]_{\alpha\in\bar{\Delta}_+, n\le 0}$, we let I_ξ be the subalgebra of M spanned by $(a^*_{\alpha,n})^p - \xi((a^*_{\alpha,n})^p), a^p_{\alpha,n-1} - \xi(a^p_{\alpha,n-1})$, where $n \le 0$ and $\alpha \in \bar{\Delta}_+$. Since all the generators of I_ξ are central in $U(\mathfrak{g})$, I_ξ is clearly a $\mathfrak{g}$-submodule of M, and this gives rise to a quotient $\mathfrak{g}$-module

$$\mathfrak{w}_\xi(\lambda) := M/I_\xi.$$

We refer to the $\mathfrak{g}$-module $\mathfrak{w}_\xi(\lambda)$ as the *baby Wakimoto module* of high weight λ (and p-character ξ).

Now assume $\kappa \neq \kappa_c$. Given $\lambda \in \mathfrak{h}^*$, let $\pi_0^{\kappa-\kappa_c}(\lambda)$ be the Fock representation of $\mathcal{A}^{\mathfrak{g}}$ generated by a vector $|\lambda\rangle$ such that

$$b_{i,n}|\lambda\rangle = 0 \quad (n > 0), \qquad b_{i,0}|\lambda\rangle = \lambda(h_i)|\lambda\rangle, \qquad \mathbf{1}|\lambda\rangle = |\lambda\rangle.$$

Any module over the vertex algebra $M \otimes \pi^{\kappa-\kappa_c}$ becomes a module over $V^\kappa(\mathfrak{g})$ via the pullback of homomorphism $\mathtt{w} : V^\kappa(\mathfrak{g}) \to M \otimes \pi^{\kappa-\kappa_c}$. In particular,

$$W(\lambda) := M \otimes \pi^{\kappa-\kappa_c}(\lambda)$$

becomes a module over $\mathfrak{g}$ of level κ. This is the *Wakimoto module* of high weight λ and level κ defined in [FF].

Let ξ be a p-character on the Lie algebra $\mathcal{A}^{\mathfrak{g}}$ satisfying (3.7). Let ξ^π be a p-character on the Lie algebra $\mathcal{B}^{\mathfrak{h}}$ such that

$$\xi^\pi(b_{i,n}^p - b_{i,np}) = 0, \qquad \text{for } n \geq 0.$$

Assume that $\lambda \in \mathfrak{h}^*$ is compatible with the p-character ξ^π in the sense that

$$\lambda(h_i)^p - \lambda(h_i) = \xi^\pi(h_i)^p, \quad \text{for all } 1 \leq i \leq \ell.$$

Identifying $\pi^{\kappa-\kappa_c}(\lambda)$ as the polynomial algebra $\mathbb{K}[b_{i,n}]_{n<0,1\leq i\leq \ell}$, we let I_{ξ^π} be the subalgebra of $\pi^{\kappa-\kappa_c}(\lambda)$ spanned by $b_{i,n}^p - b_{i,np} - \xi^\pi(b_{i,n})^p$, where $n < 0$ and $1 \leq i \leq \ell$. Since all the generators of I_{ξ^π} are central in $U(\mathfrak{g})$, I_{ξ^π} is clearly a $\mathfrak{g}$-submodule of $\pi^{\kappa-\kappa_c}(\lambda)$, and this gives rise to a quotient $\mathfrak{g}$-module

$$\mathfrak{w}^\kappa_{\xi,\xi^\pi}(\lambda) := (M \otimes \pi^{\kappa-\kappa_c}(\lambda))/(I_\xi \otimes I_{\xi^\pi}) \cong (M/I_\xi) \otimes (\pi^{\kappa-\kappa_c}(\lambda)/I_{\xi^\pi}).$$

We refer to the $\mathfrak{g}$-module $\mathfrak{w}^\kappa_{\xi,\xi^\pi}(\lambda)$ as the *baby Wakimoto module* of high weight λ (and p-character (ξ, ξ^π)).

Assume for now that $\pi^{\kappa-\kappa_c}(\lambda)/I_{\xi^\pi}$ is a module over the restricted vertex algebra $\pi_0^{\kappa-\kappa_c}$. Since $\iota(b_{i,-1})|0\rangle = 0 \in \pi_0^{\kappa-\kappa_c}$, we have

$$0 = Y(\iota(b_{i,-1})|0\rangle, z) = \sum_{n\in\mathbb{Z}} (b_{i,n}^p - b_{i,np}) z^{-np-p}$$

when acting on $\pi^{\kappa-\kappa_c}(\lambda)/I_{\xi^\pi}$. Hence $\xi^\pi = 0$, and $\lambda(h_i) \in \mathbb{F}_p$ for $1 \leq i \leq \ell$. The converse is also true: if $\xi^\pi = 0$ and $\lambda(h_i) \in \mathbb{F}_p$ for $1 \leq i \leq \ell$, then $\pi^{\kappa-\kappa_c}(\lambda)/I_{\xi^\pi}$ is a module over the restricted vertex algebra $\pi_0^{\kappa-\kappa_c}$. Similarly, M/I_ξ is a module over the restricted vertex algebra M_0 if and only if $\xi = 0$. Summarizing, we have proved the following.

Proposition 3.10. *Let $\kappa \neq \kappa_c$. If $\lambda \in \mathfrak{h}^*$ satisfies $\lambda(h_i) \in \mathbb{F}_p$ for each $1 \leq i \leq \ell$ and $\lambda(c) = \kappa$, then $\mathfrak{w}^\kappa_{0,0}(\lambda)$ is a module over the vertex algebra $M_0 \otimes \pi_0^{\kappa-\kappa_c}$, and hence a module over the vertex algebra $V^\kappa(\mathfrak{g})$.*

4. Proof of Theorem 3.8

4.1. **Restriction of $\mathtt{w}$ to the p-center.** Theorem 3.8 follows readily from the explicit description of the restriction map of $\mathtt{w}$ to $\mathfrak{z}_0(V^\kappa(\mathfrak{g}))$ given in the following theorem. Recall that

$$\iota(e_i(z)) = e_i(z)^p, \qquad \iota(h_i(z)) = h_i(z)^p - h_i(z), \qquad \iota(f_i(z)) = f_i(z)^p. \tag{4.1}$$

Theorem 4.1. *The restriction of w to $\mathfrak{z}_0(V^\kappa(\mathfrak{g}))$ is given in terms of fields as follows:*

$$\iota(e_i(z))^w = a_{\alpha_i}(z)^p + \sum_{\alpha_i \neq \beta \in \bar{\Delta}_+} P^i_\beta\big(a^*_\alpha(z)\big)^p a_\beta(z)^p, \tag{4.2}$$

$$\iota(h_i(z))^w = -\sum_{\beta \in \bar{\Delta}_+} \beta(h_i) a^*_\beta(z)^p a_\beta(z)^p + b_i(z)^p - \partial^{(p-1)} b_i(z), \tag{4.3}$$

$$\iota(f_i(z))^w = \sum_{\beta \in \bar{\Delta}_+} Q^i_\beta\big(a^*_\alpha(z)\big)^p a_\beta(z)^p + (\kappa^p - \kappa)\langle e_i, f_i\rangle \big(\partial_z a^*_{\alpha_i}(z)\big)^p + a^*_{\alpha_i}(z)^p\big(b_i(z)^p - \partial^{(p-1)} b_i(z)\big). \tag{4.4}$$

Actually $P^i_\beta\big(a^*_\alpha(z)\big)^p$ and $Q^i_\beta\big(a^*_\alpha(z)\big)^p$ are simply polynomials in the commuting vertex operators $a^*_\alpha(z)^p$, for $\alpha \in \bar{\Delta}_+$. Note that the normal orderings are no longer needed in the above formulas.

Example 4.2. The formulas above read in the case of $\widehat{\mathfrak{sl}}_2$ as follows:

$$\begin{aligned}
&\iota(e(z))^{\mathtt{w}} = a(z)^p,\\
&\iota(h(z))^{\mathtt{w}} = -2a^*(z)^p a(z)^p + b(z)^p - \partial^{(p-1)} b(z),\\
&\iota(f(z))^{\mathtt{w}} = -a^*(z)^{2p} a(z)^p + (\kappa^p - \kappa)\partial_z a^*(z) + a^*(z)^p \left(b(z)^p - \partial^{(p-1)} b(z)\right).
\end{aligned}$$

The remainder of this section is devoted to the proof of Theorem 4.1.

4.2. **Proof of Theorem 4.1.** Let

$$A = \mathbb{K}[a^*_{\alpha,0}]_{\alpha \in \bar{\Delta}_+},$$

which can be identified with $\mathbb{K}[\bar{N}_+] = \mathbb{K}[\bar{U}]$. The space $V := \sum_{\alpha \in \bar{\Delta}_+} A a_{\alpha,-1}$ is identified with $\mathcal{T}_{\bar{\mathcal{B}}}(\bar{U})$, where $\mathcal{T}_{\bar{\mathcal{B}}}$ is the tangent sheaf of the flag variety $\bar{\mathcal{B}}$. Set $\Omega := TA$, where T is the translation operator in the vertex algebra M. Note that $(M_0)_0 = A$ and $(M_0)_1 = V \oplus \Omega$. An element $D \in V$ acts on A as a derivation by the correspondence $D \mapsto D_{(0)}$, where we denote $a_{(n)} = Y(a,z)_{(n)} = \mathrm{Res}_z z^n Y(a,z)$. One has

$$[D_{(m)}, f_{(n)}] = (Df)_{(m+n)} \quad \text{for } D \in V \text{ and } f \in A. \tag{4.5}$$

Recall $\phi : U(\bar{\mathfrak{g}}) \to \mathcal{D}_{\bar{\mathcal{B}}}(\bar{U})$, $x \mapsto D^x$. The algebra $\mathcal{D}_{\bar{\mathcal{B}}}(\bar{U})$ can be identified with the A-algebra generated by V such that $D \cdot f = f \cdot D + D_{(0)} f$, and thus we have $D^x \in V$ for each $x \in \bar{\mathfrak{g}}$. It follows by (3.3)-(3.5) that the image of $x_{-1}|0\rangle$ under the vertex algebra homomorphism $\mathtt{w} : V^\kappa(\mathfrak{g}) \to M \otimes \pi^{\kappa-\kappa_c}$, denoted by $x^{\mathtt{w}}_{-1}|0\rangle$, is of the form

$$x^{\mathtt{w}}_{-1}|0\rangle = D^x_{(-1)} + f^x_{(-2)} + b^x_{(-1)} \tag{4.6}$$

for $x \in \bar{\mathfrak{g}}$, some $f^x \in A$ and some conformal weight one vector $b^x \in \pi^{\kappa-\kappa_c}$ depending on x.

Lemma 4.3. *Let $x \in \bar{\mathfrak{g}}$ and $m \in \mathbb{Z}$. Then $\big(\iota(x_{-1})^{w}|0\rangle\big)_{(m)}$ commutes with $a^*_{\alpha,n}$ for all $\alpha \in \bar{\Delta}_+$ and $n \in \mathbb{Z}$.*

Proof. Since the algebra homomorphism $\phi : U(\bar{\mathfrak{g}}) \to \mathcal{D}_{\bar{\mathcal{B}}}(U)$ is compatible with the restricted Lie algebra structures (see Lemma 3.3), we have, for $f \in A$, that

$$[D^x, D^y]f = D^{[x,y]}f, \quad (D^x)^p f = D^{x^{[p]}} f. \tag{4.7}$$

Recall from Proposition 2.6 that

$$Y(\iota(x_{-1})^{\mathtt{w}}|0\rangle, z) = \sum_{m\in\mathbb{Z}} \left((x^{\mathtt{w}}_m)^p - (x^{[p]})^{\mathtt{w}}_{mp}\right) z^{-mp-p}.$$

By (4.5), (4.6) and (4.7), one has, for $f \in A$,

$$\begin{aligned}[(x^{\mathtt{w}}_m)^p, f_{(n)}] &= (\operatorname{ad} x^{\mathtt{w}}_m)^p(f_{(n)}) = (\operatorname{ad} D^x_{(m)})^p(f_{(n)}) \\ &= ((D^x)^p f)_{(n+mp)} = (D^{x^{[p]}} f)_{(n+mp)} \\ &= [D^{x^{[p]}}_{(mp)}, f_{(n)}] = [(x^{[p]})^{\mathtt{w}}_{mp}, f_{(n)}].\end{aligned}$$

Now the lemma follows by taking $f = a^*_{\alpha,0}$. □

Lemma 4.4. *Let $x \in \bar{\mathfrak{g}}$ and $m \in \mathbb{Z}$. Then $(\iota(x_{-1})^{w}|0\rangle)_{(m)}$ commutes with $a_{\alpha,n}$ for all $\alpha \in \bar{\Delta}_+$ and $n \in \mathbb{Z}$.*

Proof. Recall from [Fr1] that, for $\alpha \in \bar{\Delta}_+$, $e^{\mathtt{w}}_{\alpha,-1}$ is of the form

$$e^{\mathtt{w}}_{\alpha,-1} = a_{\alpha,-1} + \sum_{\substack{\beta\in\bar{\Delta}_+ \\ \beta>\alpha}} P^\alpha_\beta a_{\beta,-1}, \tag{4.8}$$

where P^α_β is a polynomial in $A = \mathbb{K}[a^*_{\alpha,0}]_{\alpha\in\bar{\Delta}_+}$ of weight $\alpha-\beta$. Indeed, the weight of the right-hand-side of (4.8) equals α (see for example the first line of [Fr1, §1.3]), and hence the coefficient $P^\alpha_\beta \in A$ must be zero unless $\beta > \alpha$ in the standard dominance order of the root lattice of $\bar{\mathfrak{g}}$. It follows that each $a_{\alpha,-1}$ can be written as

$$a_{\alpha,-1} = e^{\mathtt{w}}_{\alpha,-1} + \sum_{\substack{\beta\in\bar{\Delta}_+ \\ \beta>\alpha}} Q^\alpha_\beta e^{\mathtt{w}}_{\beta,-1} \tag{4.9}$$

for some polynomials $Q^\alpha_\beta \in A$.

Since $(\iota(x_{-1})|0\rangle)_{(m)} = x^p_m - (x^{[p]})_{mp}$ is central in $U(\mathfrak{g})$ and $\mathtt{w}$ is a $\mathfrak{g}$-homomorphism, $(\iota(x_{-1})^{\mathtt{w}}|0\rangle)_{(m)}$ commutes with $e^{\mathtt{w}}_{\gamma,n}$ for any n and any $\gamma \in \bar{\Delta}_+$ (in particular for $\gamma \geq \alpha$). By Lemma 4.3, $(\iota(x_{-1})^{\mathtt{w}}|0\rangle)_{(m)}$ also commutes with $(Q^\alpha_\beta)_{(n)}$ for any n. Now by applying $Y(-,z)_{(n)}$ to (4.9), $(\iota(x_{-1})^{\mathtt{w}}|0\rangle)_{(m)}$ commutes with $a_{\alpha,n}$. □

Remark 4.5. It is elementary to show by induction on k that for any k, n, m, i, β one has

$$[(h^{\mathtt{w}}_{i,n})^k, a_{\beta,m}] = \sum_{d=1}^{k} \binom{k}{d} \beta(h_i) \cdot a_{\beta,nd+m} (h^{\mathtt{w}}_{i,n})^{k-d}.$$

It follows that $[(h^{\mathtt{w}}_{i,n})^p - h^{\mathtt{w}}_{i,np}, a_{\beta,m}] = 0$. Similarly, $[(h^{\mathtt{w}}_{i,n})^p - h^{\mathtt{w}}_{i,np}, a^*_{\beta,m}] = 0$.

Proposition 4.6. *Formulas* (4.2)-(4.3) *hold.*

Proof. Let us fix i. The strategy is similar to the proof of Lemma 3.4. We extend the notation to write $P^i_{\alpha_i}(a^*_\alpha(z)) = a_{\alpha_i}(z)$. Then by (4.1) and (3.3), we can write

$$(e_i(z)^p)^{\mathtt{w}} = \sum_{\beta \in \bar{\Delta}_+} P^i_\beta\big(a^*_\alpha(z)\big)^p a_\beta(z)^p + \sum_{t=1}^{p-1} {:}Y_t(a^*_\alpha(z)) a_\beta(z)^t{:}, \tag{4.10}$$

for some differential polynomials Y_t, where the last summand arises from contractions in Wick's formula. Lemma 4.3 can be rephrased by saying that $(e_i(z)^p)^{\mathtt{w}}$ commutes with $a^*_\beta(z)$. The first summand on the right-hand side of (4.10) commute with $a^*_\beta(z)$, but $a_\beta(z)^t$, for $1 \le t \le p-1$, do not commute with $a^*_\beta(z)$. Hence we must have $Y_t = 0$ for all t by a downward induction on t. This proves (4.2).

Similarly, noting $\beta(h_i)$ is integral and using (4.1) and (3.4), we have

$$(h_i(z)^p - h_i(z))^{\mathtt{w}} = -\sum_{\beta \in \bar{\Delta}_+} \beta(h_i)\big[({:}a^*_\beta(z) a_\beta(z){:})^p - {:}a^*_\beta(z) a_\beta(z){:}\big] + b_i(z)^p - \partial^{(p-1)} b_i(z). \tag{4.11}$$

Now write $({:}a^*_\beta(z)a_\beta(z){:})^p - {:}a^*_\beta(z)a_\beta(z){:} = a^*_\beta(z)^p a_\beta(z)^p + \sum_{t=1}^{p-1} {:}X_t(a^*_\beta(z)) a_\beta(z)^t{:}$, for some differential polynomials X_t (Here β is fixed). But by considering the commutation of (4.11) with $a^*_\beta(z)$ and applying Lemma 4.3, we conclude that $X_t = 0$ for each t. This proves (4.3). □

To complete the proof of Theorem 4.1 it remains to prove (4.4). Denote by $\bar{\mathfrak{g}}_{\mathbb{Z}}$ the $\mathbb{Z}$-lattice generated by the Chevalley generators e_α, f_α and h_i, for $\alpha \in \bar{\Delta}^+$ and $i = 1, \ldots, \ell$. Denote by $V_{\mathbb{Z}}$ the $\mathbb{Z}$-lattice of $V^\kappa(\mathfrak{g})$ spanned by all possible $a_{-i_1} b_{-i_2} c_{-i_3} \ldots |0\rangle$, where $a, b, c \ldots \in \bar{\mathfrak{g}}_{\mathbb{Z}}$ and $i_1, i_2, i_3, \ldots \ge 1$. Writing a general vertex operator $Y(a,z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$, we recall a general formula from the theory of vertex algebras (cf. [Fr2]):

$$[x_{(m)}, y_{(n)}] = \sum_{i \ge 0} \binom{m}{i} \big(x_{(i)} y\big)_{(m+n-i)}, \qquad i \ge 0. \tag{4.12}$$

From a similar consideration as in the proof of Proposition 4.6 above, we conclude that $\iota(f_i(z))^{\mathtt{w}}$ is of the form

$$\iota(f_i(z))^{\mathtt{w}} = \sum_{\beta \in \bar{\Delta}_+} Q^i_\beta\big(a^*_\alpha(z)\big)^p a_\beta(z)^p + \eta\big(\partial_z a^*_{\alpha_i}(z)\big)^p + a^*_{\alpha_i}(z)^p R(b_i(z)), \tag{4.13}$$

where $\eta \in \mathbb{K}$ and

$$R(b_i(z)) = Y(r_i, z) = b_i(z)^p + \ldots \tag{4.14}$$

is a (normal ordered) polynomial in $b_i(z)$ and its derivatives and $r_i \in \pi^{\kappa-\kappa_c}$. Formula (4.4) now follows from the proposition below.

Proposition 4.7. *We have*

(1) $R(b_i(z)) = b_i(z)^p - \partial^{(p-1)} b_i(z)$;
(2) $\eta = (\kappa^p - \kappa)\langle e_i, f_i\rangle$.

Sketch of a proof. It is possible to realize Wakimoto modules over $\mathbb{Z}$ as limits of twisting Verma modules, denoted by $W_{\mathbb{Z}}(\lambda)$, on which $e_{i,n}, h_{i,n}, f_{i,n}$ act. Then the formulas (3.3)-(3.5) are understood as a congruence equation modulo $pW_{\mathbb{Z}}(\lambda)$ when acting on any $v \in W_{\mathbb{Z}}(\lambda)$; moreover $(e^p_{-1}|0\rangle)^{\mathtt{w}}_{(n)} v \in pW_{\mathbb{Z}}(\lambda), (f^p_{-1}|0\rangle)^{\mathtt{w}}_{(n)} v \in pW_{\mathbb{Z}}(\lambda)$, thanks to Lemmas 4.3 and 4.4. From weight consideration, we have $(e^p_{-1}|0\rangle)_{(p-1)} f^p_{-1}|0\rangle \equiv -p(h^p_{-1} - h_{-p}) \mod p^2 V_{\mathbb{Z}}$.

On the other hand, for $n \geq 0$ and $v \in W_{\mathbb{Z}}(\lambda)$, we have

$$\Big((e^p_{-1}|0\rangle)_{(p-1)} f^p_{-1}|0\rangle\Big)^{\mathtt{w}}_{(n)} v = \sum_{i=0}^{p-1} (-1)^i \binom{p-1}{i} \Big((e^p_{-1}|0\rangle)^{\mathtt{w}}_{(p-1-i)} (f^p_{-1}|0\rangle)^{\mathtt{w}}_{(n+i)} v - (-1)^{p-1} (f^p_{-1}|0\rangle)^{\mathtt{w}}_{(n+p-1-i)} (e^p_{-1}|0\rangle)^{\mathtt{w}}_{(i)} v\Big). \quad (4.15)$$

But if we compute (4.15) by applying (3.3)–(3.5), the only term involving $b_{i,n}$ is given by $-r_i v$, which by (4.14) must be equal to $-(b^p_{i,-1} - b_{i,-p})v$ modulo $pW_{\mathbb{Z}}(\lambda)$. Part (1) now follows from this together with (4.14).

Part (2) reduces to the $\mathfrak{sl}_2$ case by Lemma 3.6. □

5. Irreducible baby Wakimoto modules $\mathfrak{w}(-\rho)$

5.1. **Mathieu's character formula reformulated.** For an integral weight $\lambda \in \mathfrak{h}^*$, denote by $\mathfrak{l}(\lambda)$ the irreducible quotient $\mathfrak{g}$-module of the Verma $\mathfrak{g}$-module of high weight λ. Recall the torus T from \$2.1. Then $\mathfrak{l}(\lambda)$ is naturally an $\mathfrak{g}$-T-module in the sense of Jantzen [Jan], and this allows one to makes sense its (formal) character $\mathrm{ch}\,\mathfrak{l}(\lambda)$ in the usual sense.

Mathieu [Ma] proved the following character formula

$$\mathrm{ch}\ \mathfrak{l}(-\rho) = e^{-\rho} \prod_{\alpha \in \Delta^{\mathrm{re}}_+} \frac{(1-e^{-p\alpha})}{(1-e^{-\alpha})}. \quad (5.1)$$

Note that $-\rho$ is a weight at the critical level κ_c. We have the following reformulation of a main result of Mathieu, which has the advantage that the irreducible $\mathfrak{g}$-module $\mathfrak{l}(-\rho)$ is realized explicitly as the baby Wakimoto module $\mathfrak{w}(-\rho)$ in terms of (restricted) free fields.

Theorem 5.1. *The baby Wakimoto module $\mathfrak{w}(-\rho)$ is the irreducible high weight $\mathfrak{g}$-module of high weight $-\rho$.*

Proof. By construction of the baby Wakimoto module, we have the following character formula:

$$\mathrm{ch}\ \mathfrak{w}(-\rho) = e^{-\rho} \prod_{\alpha \in \Delta^{\mathrm{re}}_+} \frac{(1-e^{-p\alpha})}{(1-e^{-\alpha})}.$$

The (obvious) surjective homomorphism $\mathfrak{w}(-\rho) \to \mathfrak{l}(-\rho)$ must be an isomorphism by a character comparison. □

As modules over $\mathfrak{g}$, we have $\mathfrak{l}(-\rho) = \mathfrak{l}((p-1)\rho)$. (More general $\mathfrak{l}(\lambda) = \mathfrak{l}(\mu)$ if $\lambda - \mu$ is a p-multiple of an integral weight of $\mathfrak{g}$.) Denote by $U = \mathbb{K} \otimes_{\mathbb{Z}} U_{\mathbb{Z}}$, where $U_{\mathbb{Z}}$ is the Kostant-Garland $\mathbb{Z}$-form of the universal enveloping algebra of $\mathfrak{g}$. Denote by $L(\lambda)$ (the notation $l(\lambda)$ was used in [Ma]) the irreducible highest weight U-module of highest weight λ (which is assumed to be integral). Note that the restricted enveloping algebra is a subalgebra of U, i.e., $\mathfrak{u}_0(\mathfrak{g}) \subseteq U$. Since $(p-1)\rho$ is a restricted weight, it follows by Mathieu [Ma, Lemma 1.7] that $L((p-1)\rho)$ when restricted to $\mathfrak{u}_0(\mathfrak{g})$ remains to be irreducible, and hence $L((p-1)\rho) \cong \mathfrak{l}((p-1)\rho)$ as $\mathfrak{g}$-modules. Therefore Theorem 5.1 and (5.1) have the following implication.

Corollary 5.2 ([Ma])**.** *We have the following character formulas:*

$$\mathrm{ch}\, l((p-1)\rho) = e^{(p-1)\rho} \prod_{\alpha \in \Delta_+^{\mathrm{re}}} \frac{(1-e^{-p\alpha})}{(1-e^{-\alpha})}, \tag{5.2}$$

$$\mathrm{ch}\, l(-\rho) = e^{-\rho} \prod_{\alpha \in \Delta_+^{\mathrm{re}}} \frac{1}{1-e^{-\alpha}}. \tag{5.3}$$

The above two formulas are equivalent by Steinberg tensor product theorem.

5.2. **Conjectures and further problems.** Recall the vertex algebra $V^{\kappa}(\mathfrak{g}_{\mathbb{C}})$ over $\mathbb{C}$ has trivial center at a non-critical level κ; at the critical level, $V^{\kappa_c}(\mathfrak{g}_{\mathbb{C}})$ has a large center, which is explicitly described in [Fr1, Fr2]. This center continues to make sense for $V^{\kappa_c}(\mathfrak{g})$ over $\mathbb{K}$ in characteristic p; we shall refer to this as the Harish-Chandra center of $V^{\kappa_c}(\mathfrak{g})$ and denote it by $\mathfrak{z}_{\mathrm{HC}}(V^{\kappa_c}(\mathfrak{g}))$.

Conjecture 5.3.
1. For $\kappa \neq \kappa_c$, the center of the vertex algebra $V^{\kappa}(\mathfrak{g})$ coincides with the p-center $\mathfrak{z}_0(V^{\kappa}(\mathfrak{g}))$.
2. The center of the vertex algebra $V^{\kappa_c}(\mathfrak{g})$ is generated by the Harish-Chandra center $\mathfrak{z}_{\mathrm{HC}}(V^{\kappa_c}(\mathfrak{g}))$ and the p-center $\mathfrak{z}_0(V^{\kappa_c}(\mathfrak{g}))$.

A p-character ξ^M of $\mathcal{A}^{\mathfrak{g}}$ is called graded if $\xi^M(a_{\alpha,n}) = 0 = \xi^M(a^*_{\alpha,n})$ for all $n \neq 0$. A p-character ξ^{π} of $\mathcal{B}^{\mathfrak{h}}_{\kappa}$ is graded if $\xi^{\pi}(b_{i,n}) = 0$ for all $n \neq 0$ and $1 \leq i \leq \ell$. Similarly, a graded p-character for $\mathfrak{g}$ can be defined.

The modular representation theory of (finite-dimensional) Lie algebras has been well developed; cf. the review of Jantzen [Jan]. It will be of great interest to develop modular representation theory for an affine Lie algebra $\mathfrak{g}$, say when the p-character is (graded) semisimple or nilpotent. In particular, one may ask if the baby Wakimoto modules are irreducible for *generic* (graded) semisimple p-characters. The modular representation theory of the algebra U (or the corresponding algebraic group of $\mathfrak{g}$) has been very challenging; we refer to [Lai] and the references therein for results in this direction. The modular representation theory of $\mathfrak{g}$ should be somewhat more accessible and flexible by imposing various conditions on p-characters.

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