

ASSOCIATED VARIETIES AND HIGGS BRANCHES (A SURVEY)

TOMOYUKI ARAKAWA

ABSTRACT. Associated varieties of vertex algebras are analogue of the associated varieties of primitive ideals of the universal enveloping algebras of semisimple Lie algebras. They not only capture some of the important properties of vertex algebras but also have interesting relationship with the Higgs branches of four-dimensional $N = 2$ superconformal field theories (SCFTs). As a consequence, one can deduce the modular invariance of Schur indices of 4d $N = 2$ SCFTs from the theory of vertex algebras.

1. ASSOCIATED VARIETIES OF VERTEX ALGEBRAS

A *vertex algebra* consists of a vector space V with a distinguished vacuum vector $|0\rangle \in V$ and a vertex operation, which is a linear map $V \otimes V \rightarrow V((z))$, written $u \otimes v \mapsto Y(u, z)v = (\sum_{n \in \mathbb{Z}} u_{(n)} z^{-n-1})v$, such that the following are satisfied:

- (Unit axioms) $(|0\rangle)(z) = 1_V$ and $Y(u, z)|0\rangle \in u + zV[[z]]$ for all $u \in V$.
- (Locality) $(z - w)^n[Y(u, z), Y(v, w)] = 0$ for a sufficiently large n for all $u, v \in V$.

The operator $\partial : u \mapsto u_{(-2)}|0\rangle$ is called the translation operator and it satisfies $Y(Tu, z) = \partial_z Y(u, z)$. The operators $u_{(n)}$ are called *modes*.

To each vertex algebra V one associates a Poisson algebra R_V , called the *Zhu's C_2 -algebra*, as follows ([Zhu]). Let $C_2(V)$ be the subspace of V spanned by the elements $a_{(-2)}b$, $a, b \in V$, and set $R_V = V/C_2(V)$. Then R_V is a Poisson algebra by

$$\bar{a} \cdot \bar{b} = \overline{a_{(-1)}b}, \quad \{\bar{a}, \bar{b}\} = \overline{a_{(0)}b},$$

where \bar{a} denote the image of $a \in V$ in R_V .

A vertex algebra is called *strongly finitely generated* if R_V is finitely generated. In this note we assume that all the vertex algebras are finitely strongly generated.

The *associated variety* X_V of a vertex algebra V is the affine Poisson variety X_V defined by

$$X_V = \text{Specm}(R_V)$$

([A1]).

Let \mathfrak{g} be a simple Lie algebra over \mathbb{C} , $\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K$ be the affine Kac-Moody algebra associated with \mathfrak{g} and the normalized invariant inner product $(\cdot | \cdot)$. Set

$$V^k(\mathfrak{g}) := U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t] \oplus \mathbb{C}K)} \mathbb{C}_k,$$

where $k \in \mathbb{C}$ and \mathbb{C}_k is one-dimensional representation of $\mathfrak{g}[t] \oplus \mathbb{C}K$ on which $\mathfrak{g}[t]$ acts trivially and K acts as the multiplication by k . There is a unique vertex algebra structure on $V^k(\mathfrak{g})$ such that $|0\rangle = 1 \otimes 1$ is the vacuum vector and

$$Y(x, z) = x(z) := \sum_{n \in \mathbb{Z}} (xt^n) z^{-n-1} \quad (x \in \mathfrak{g}),$$

where we consider \mathfrak{g} as a subspace of $V^k(\mathfrak{g})$ by the embedding $\mathfrak{g} \hookrightarrow V^k(\mathfrak{g})$, $x \mapsto xt^{-1}|0\rangle$. $V^k(\mathfrak{g})$ is called the *universal affine vertex algebra associated with \mathfrak{g} at level k* .

One can regard $V^k(\mathfrak{g})$ as an analogue of the universal enveloping algebra in the sense that a $V^k(\mathfrak{g})$ -module is the same as a smooth $\widehat{\mathfrak{g}}$ -module of level k , where a $\widehat{\mathfrak{g}}$ -module M is called *smooth* if $x(z)m \in M((z))$ for all $m \in M$, $x \in \mathfrak{g}$, and called of level k if K acts as the multiplication by k .

Any graded quotient V of $V^k(\mathfrak{g})$ as a $\widehat{\mathfrak{g}}$ -module has the structure of the quotient vertex algebra. In particular the unique simple graded quotient $L_k(\mathfrak{g})$ is a vertex algebra and is called the *simple affine vertex algebra associated with \mathfrak{g} at level k* .

For any quotient vertex algebra V of $V^k(\mathfrak{g})$, we have $R_V = V/\mathfrak{g}[t^{-1}]t^{-2}V$, and the surjective linear map

$$(1) \quad \mathbb{C}[\mathfrak{g}^*] = S(\mathfrak{g}) \rightarrow R_V, \quad x_1 \dots x_r \mapsto \overline{(x_1 t^{-1} \dots x_r t^{-1} | 0\rangle)} \quad (x_i \in \mathfrak{g})$$

is a homomorphism of Poisson algebras. In particular $X_{L_k(\mathfrak{g})}$ is a subvariety of \mathfrak{g}^* , which is G -invariant and conic. We note that on the contrary to the associated variety of a primitive ideal of $U(\mathfrak{g})$, $X_{L_k(\mathfrak{g})}$ is not necessarily contained in the nilpotent cone \mathcal{N} of \mathfrak{g} . Indeed, $L_k(\mathfrak{g}) = V^k(\mathfrak{g})$ for a generic k and $X_{V^k(\mathfrak{g})} = \mathfrak{g}^*$ as (1) is an isomorphism for $V = V^k(\mathfrak{g})$ by the PBW theorem.

For a nilpotent element f of \mathfrak{g} , let $\mathcal{W}^k(\mathfrak{g}, f)$ be the universal W -algebra associated with (\mathfrak{g}, f) at level k :

$$\mathcal{W}^k(\mathfrak{g}, f) = H_{DS, f}^0(V^k(\mathfrak{g})),$$

where $H_{DS, f}^\bullet(?)$ is the BRST cohomology functor of the quantized Drinfeld-Sokolov reduction associated with (\mathfrak{g}, f) ([FF, KRW]). The associated variety $X_{\mathcal{W}^k(\mathfrak{g}, f)}$ is isomorphic to the *Slodowy slice* $\mathcal{S}_f = f + \mathfrak{g}^e$, where $\{e, f, h\}$ is an \mathfrak{sl}_2 -triple and \mathfrak{g}^e is the centralizer of e in \mathfrak{g} ([DSK]). For any quotient V of $V^k(\mathfrak{g})$, $H_{DS, f}^0(V)$ is a quotient vertex algebra of $\mathcal{W}^k(\mathfrak{g}, f)$ provided that it is nonzero, and we have

$$(2) \quad X_{H_{DS, f}^0(V)} = X_V \cap \mathcal{S}_f,$$

which is a \mathbb{C}^* -invariant subvariety of \mathcal{S}_f ([A2]).

2. LISSE AND QUASI-LISSE VERTEX ALGEBRAS

A vertex algebra V is called *lisse* (or C_2 -cofinite) if $\dim X_V = 0$, or equivalently, R_V is finite-dimensional. For instance, $L_k(\mathfrak{g})$ is lisse if and only if $L_k(\mathfrak{g})$ is integrable as a $\widehat{\mathfrak{g}}$ -module, or equivalently, $k \in \mathbb{Z}_{\geq 0}$ ([DM]). Therefore, the lisse condition generalizes the integrability to an arbitrary vertex algebra. Indeed, lisse vertex algebras are analogue of finite-dimensional algebras in the following sense.

Lemma 2.1 ([A1]). *A vertex algebra V is lisse if and only if $\dim \operatorname{Spec}(\operatorname{gr} V) = 0$, where $\operatorname{gr} V$ is the associated graded Poisson vertex algebra with respect to the canonical filtration on V ([Li]).*

It is known that lisse vertex algebras have various nice properties such as modular invariance of characters of V -modules under some mild assumptions ([Zhu, Miy]). However, there are significant vertex algebras that do not satisfy the lisse condition. For instance, an *admissible affine vertex algebra* $L_k(\mathfrak{g})$ (see below) has a complete reducibility property ([A4]) and the modular invariance property ([KW1], see also [AvE]) in the category \mathcal{O} although it is not lisse unless it is integrable. So it is natural to try to relax the lisse condition.

Since X_V is a Poisson variety we have a finite partition

$$X_V = \bigsqcup_{k=0}^r X_k,$$

where each X_k is a smooth analytic Poisson variety. Thus for any point $x \in X_k$ there is a well defined symplectic leaf through it. A vertex algebra V is called *quasi-lisse* ([AK]) if X_V has only finitely many symplectic leaves. Clearly, lisse vertex algebras are quasi-lisse.

For example, consider the simple affine vertex algebra $L_k(\mathfrak{g})$. Since symplectic leaves in $X_{L_k(\mathfrak{g})}$ are the coadjoint G -orbits contained in $X_{L_k(\mathfrak{g})}$, where G is the adjoint group of \mathfrak{g} , it follows that $L_k(\mathfrak{g})$ is quasi-lisse if and only if $X_{L_k(\mathfrak{g})} \subset \mathcal{N}$. Hence [FM, A2], admissible affine vertex algebras are quasi-lisse.

A theorem of Etingof and Schelder [ES] says that if a Poisson variety $\operatorname{Specm}(R)$ has finitely many symplectic leaves then the zeroth Poisson homology $R/\{R, R\}$ is finite-dimensional. It follows [AK] that a quasi-lisse conformal vertex algebra has only finitely many simple ordinary representations. Here a V -module M is called ordinary if it is a positive energy representation on which L_0 acts semisimply and each L_0 -eigenspace is finite-dimensional, so that the normalized character

$$\chi_M(\tau) = \operatorname{tr}_M(q^{L_0 - \frac{c}{24}})$$

is well-defined.

By extending Zhu's argument [Zhu] using the theorem of Etingof and Schelder, we get the following assertion.

Theorem 2.2 ([AK]). *Let V be a quasi-lisse vertex algebra and M a ordinary V -module. Then χ_M satisfies a modular linear differential equation.*

Since the space of solutions of a modular linear differential equation (MLDE) is invariant under the action of $SL_2(\mathbb{Z})$, this implies that a quasi-lisse vertex algebra possesses a certain modular invariance property, although we do not claim that the normalized characters of V -modules span the space of the solutions.

3. IRREDUCIBILITY CONJECTURE AND EXAMPLES OF QUASSI-LISSE VERTEX ALGEBRAS

Let $\hat{\Delta}^{re}$ be the set of real roots of $\hat{\mathfrak{g}}$, $\hat{\Delta}_+^{re}$ the set of real positive roots. For a weight λ of $\hat{\mathfrak{g}}$, let $\hat{\Delta}(\lambda) = \{\alpha \in \hat{\Delta}^{re} \mid \langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z}\}$, the integral roots system of λ . An irreducible highest weight representation $L(\lambda)$ of $\hat{\mathfrak{g}}$ with highest weight λ is called admissible if λ is regular dominant, that is, $\langle \lambda + \rho, \alpha^\vee \rangle \notin \{0, -1, -2, \dots\}$ for all positive $\alpha \in \Delta_+$, and $\mathbb{Q}\hat{\Delta}(\lambda) = \mathbb{Q}\hat{\Delta}^{re}$ ([KW1]). The simple affine vertex algebra $L_k(\mathfrak{g})$ is called admissible if it is admissible as a $\hat{\mathfrak{g}}$ -module. This condition is equivalent to that

$$k + h^\vee = \frac{p}{q}, \quad p, q \in \mathbb{N}, \quad (p, q) = 1, \quad p \geq \begin{cases} h^\vee & \text{if } (r^\vee, q) = 1, \\ h & \text{if } (r^\vee, q) \neq 1, \end{cases}$$

where h , h^\vee , and r^\vee is the Coxeter number, the dual Coxeter number, and the lacing number of \mathfrak{g} , respectively ([KW2]).

As we have already mentioned above an admissible affine vertex algebra $L_k(\mathfrak{g})$ is quasi-lisse, that is, $X_{L_k(\mathfrak{g})} \subset \mathcal{N}$. In fact, the following assertion holds.

Theorem 3.1 ([A2]). *For an admissible affine vertex algebra $L_k(\mathfrak{g})$, $X_{L_k(\mathfrak{g})}$ is an irreducible variety contained in \mathcal{N} , that is, there exists a nilpotent orbit \mathbb{O} such that $X_{L_k(\mathfrak{g})} = \overline{\mathbb{O}}$.*

See [A2] for a concrete description of the orbit \mathbb{O} that appears in the above theorem.

For $\mathfrak{g} = \mathfrak{sl}_2$, it is not difficult to check that $L_k(\mathfrak{g})$ is quasi-lisse if and only if $L_k(\mathfrak{g})$ is admissible for a non-critical¹ k , see [Mal, GK]. However, there are non-admissible affine vertex algebras that are quasi-lisse for higher rank \mathfrak{g} .

Recall that the *Deligne exceptional series* [Del] is the sequence of simple Lie algebras

$$A_1 \subset A_2 \subset G_2 \subset D_4 \subset F_4 \subset E_6 \subset E_7 \subset E_8.$$

Let \mathbb{O}_{min} be the unique non-trivial nilpotent orbit of \mathfrak{g} .

Theorem 3.2 ([AM1]). *Let \mathfrak{g} be a simple Lie algebra that belongs to the Deligne exceptional series, and let k be a rational number of the form $k = -h^\vee/6 - 1 + n$, $n \in \mathbb{Z}_{\geq 0}$, such that $k \notin \mathbb{Z}_{\geq 0}$. Then*

$$X_{L_k(\mathfrak{g})} = \overline{\mathbb{O}_{min}}.$$

For types A_1, A_2, G_2, D_4, F_4 , the simple affine vertex algebra $L_k(\mathfrak{g})$ appearing Theorem 3.2 is admissible, and hence, the statement is the special case of [A2]. However, this is not the case for types D_4, F_4, E_6, E_7, E_8 and Theorem 3.2 gives examples of non-admissible quasi-lisse affine vertex algebras

Except for $\mathfrak{g} = \mathfrak{sl}_2$, the classification problem of quasi-lisse affine vertex algebras is wide open. (See [AM2, AM3] for more examples of lisse affine vertex algebras.)

¹If k is critical, that is, if $k = -h^\vee$, then $X_{L_k(\mathfrak{g})} = \mathcal{N}$ by [FF, EF, FG] for all simple Lie algebra \mathfrak{g} .

All the associated varieties are irreducible in the above examples of quasi-lisse affine vertex algebras. We conjecture that this is true in general:

Conjecture 1 ([AM2]). The associated variety of an quasi-lisse conical vertex algebra is irreducible.

Recall the description of associated variety of W -algebras given by (2). This implies that if $L_k(\mathfrak{g})$ is quasi-lisse and $f \in X_{L_k(\mathfrak{g})}$, then the W -algebra $H_{DS,f}^0(L_k(\mathfrak{g}))$ is quasi-lisse as well, and so is its simple quotient $\mathcal{W}_k(\mathfrak{g}, f)$. In this way we obtain a huge number of quasi-lisse W -algebras. (See [AM3] for the irreducibility of the corresponding associated varieties.) Moreover, if $X_{L_k(\mathfrak{g})} = \overline{G} \cdot f$, then $X_{H_{DS,f}^0(L_k(\mathfrak{g}))} = \{f\}$ by the transversality of \mathcal{S}_f to G -orbits, so that $\mathcal{W}_k(\mathfrak{g}, f)$ is in fact lisse. Thus, Conjecture 1 in particular says that a quasi-lisse affine vertex algebra produces exactly one lisse simple W -algebra.

Lisse W -algebras thus obtained from admissible affine vertex algebras contain all the *exceptional W -algebras* discovered by Kac and Wakimoto [KW2] ([A2]), in particular, the *minimal series principal W -algebras* [FKW], which are natural generalization of minimal series Virasoro vertex algebras [BPZ]. The rationality of the minimal series principal W -algebras has been recently recently proved by the author ([A3]).

4. BL²PR² CORRESPONDENCE AND HIGGS BRANCH CONJECTURE

In [BLL⁺], Beem, Lemos, Liendo, Peelaers, Rastelli, and van Rees have constructed a remarkable map

$$\Phi : \{4d \ N = 2 \text{ SCFTs} \} \rightarrow \{ \text{vertex algebras} \},$$

such that, among other things, the character of the vertex algebra $\Phi(\mathcal{T})$ coincides with the *Schur index* of the corresponding 4d $N = 2$ SCFT \mathcal{T} , which is an important invariant.

How do vertex algebras coming from 4d $N = 2$ SCFTs look like? According to [BLL⁺], we have

$$c_{2d} = -12c_{4d},$$

where c_{4d} and c_{2d} are central charges of the 4d $N = 2$ SCFT and the corresponding vertex algebra, respectively. Since the central charge is positive for a unitary theory, this implies that the vertex algebras obtained in this way are never unitizable. In particular integrable affine vertex algebras never appear by this correspondence.

The main examples of vertex algebras considered in [BLL⁺] are affine vertex algebras $L_k(\mathfrak{g})$ of types D_4 , F_4 , E_6 , E_7 , E_8 at level $k = -h^\vee/6 - 1$, which are non-rational, non-admissible quasi-lisse affine vertex algebras appeared in Theorem 3.2. One can find more examples in the literature, see e.g. [BPRvR, XYY, CS, SXY].

Now, there is another important invariant of a 4d $N = 2$ SCFT \mathcal{T} , called the *Higgs branch*, which we denote by $Higgs_{\mathcal{T}}$. The Higgs branch $Higgs_{\mathcal{T}}$ is an affine algebraic variety that has the hyperKähler structure in its smooth part. In particular, $Higgs_{\mathcal{T}}$ is a (possibly singular) symplectic variety.

Let \mathcal{T} be one of the 4d $N = 2$ SCFTs studied in [BLL⁺] such that $\Phi(\mathcal{T}) = L_k(\mathfrak{g})$ with $k = h^\vee/6 - 1$ for types D_4, F_4, E_6, E_7, E_8 as above. It is known that $Higgs_{\mathcal{T}} = \mathbb{O}_{min}$, which equals to $X_{L_k(\mathfrak{g})}$ by Theorem 3.2. It is expected that this is not just a coincidence.

Conjecture 2 (Beem and Rastelli [BR]). For a 4d $N = 2$ SCFT \mathcal{T} , we have

$$Higgs_{\mathcal{T}} = X_{\Phi(\mathcal{T})}.$$

So we are expected to recover the Higgs branch of a 4d $N = 2$ SCFT from the corresponding vertex algebra, which is a purely algebraic object! Note that the associated variety of a vertex algebra is only a Poisson variety in general. Physical intuition expects that they are all quasi-lisse, and so vertex algebras that come from 4d $N = 2$ SCFTs via the map Φ form some special subclass of quasi-lisse vertex algebras.

We note that Conjecture 2 is a physical conjecture since the Higgs branch is not a mathematically defined object at the moment. The Schur index is not a mathematically defined object either. However, in view of [BLL⁺] and Conjecture 2, one can try to define both Higgs branches and Schur indices of 4d $N = 2$ SCFTs using vertex algebras. We note that there is a close relationship between Higgs branches of 4d $N = 2$ SCFTs and *Coulomb branches* of three-dimensional $N = 4$ gauge theories whose mathematical definition has been recently given by Braverman, Finkelberg and Nakajima [BFN1, BFN2] (see [A5, A6]).

In view of Conjecture 2, Theorem 2.2 implies that the Schur index of a 4d $N = 2$ SCFT satisfies a MLDE, which is something that has been conjectured in physics ([BR]).

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RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES, KYOTO UNIVERSITY, KYOTO 606-8502 JAPAN

DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, 77 MASSACHUSETTS AVENUE CAMBRIDGE, MA 02139-4307 USA

E-mail address: arakawa@kurims.kyoto-u.ac.jp