Representation theory of W-algebras and Higgs branch conjecture

ICM 2018 Session "Lie Theory and Generalizations"

Tomoyuki Arakawa

August 2, 2018

RIMS, Kyoto University

What are W-algebras?

- W-algebras are certain generalizations of infinite-dimensional Lie algebras such as affine Kac-Moody algebras and the Virasoro algebra.
- W-algebras can be also considered as affinizations of finite W-algebras ([Premet '02]) which are quantizations of Slodowy slices ([De-Sole-Kac '06]).
- W-algebras appeared in '80s in physics in the study of the two-dimensional conformal field theories.
- W-algebras are closely connected with integrable systems, (quantum) geometric Langlands program (e.g. [T.A.-Frenkel '18]), four-dimensional gauge theory ([Alday-Gaiotto-Tachikawa '10]), etc.

An example

The Zamolodchikov W_3-algebra

generators: L_n $(n \in \mathbb{Z})$, W_n $(n \in \mathbb{Z})$, $\mathbf{c}$,

relations: $[\mathbf{c}, W_3] = 0$, $[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m+n,0}\mathbf{c}$,

$[L_m, W_n] = (2m-n)W_{m+n}$,

$$
\begin{aligned}
&[W_m, W_n] \\
&= (m-n)\left(\tfrac{1}{15}(m+n+3)(m+n+2) - \tfrac{1}{6}(m+2)(n+2)\right) L_{m+n} \\
&+ \tfrac{16}{22+5\mathbf{c}}(m-n)\Lambda_{m+n} + \tfrac{1}{360}m(m^2-1)(m^2-4)\delta_{m+n,0}\mathbf{c},
\end{aligned}
$$

where $\Lambda_n = \sum_{k\geq 0} L_{n-k}L_k + \sum_{k<0} L_k L_{n-k} - \frac{3}{10}(n+2)(n+3)L_n$.

W-algebras are not Lie algebras in general but vertex algebras.

A representation of W_3 on a ($\mathbb{C}$-)vector space M makes sense by imposing the conditions

$$L_n m = W_n m = 0 \ (n \gg 0, \forall m \in M).$$

A highest weight representation of W_3 is a representation M that is generated by a vector v satisfying

$$\begin{aligned} &L_n v = W_n v = 0 \ (n > 0), \\ &L_0 v = a_1 v,\ W_0 v = a_2 v,\ \mathbf{c} v = c v, \quad \exists (a_1, a_2, c) \in \mathbb{C}^3. \end{aligned}$$

For a highest weight representation M of W_3 the (normalized) character

$$\chi_M(q) = \mathrm{tr}_M(q^{L_0 - \frac{\mathbf{c}}{24}})$$

makes sense.

Quantized Drinfeld-Sokolov reduction

In general, a W-algebra is defined by means of the (quantized) Drinfeld-Sokolov reduction ([Feigin-Frenkel '90,. . . , Kac-Roan-Wakimoto '03]).

$\mathfrak{g}$: a simple Lie algebra, $f \in \mathfrak{g}$: a nilpotent element,

$\rightsquigarrow \mathcal{W}^k(\mathfrak{g}, f) = H^0_{DS,f}(V^k(\mathfrak{g}))$: the W-algebra associated with $(\mathfrak{g}, f)$ at level $k \in \mathbb{C}$.

Here,

$H^\bullet_{DS,f}(M)$: the BRST cohomology of the Drinfeld-Sokolov reduction associated with $(\mathfrak{g}, f)$ with coefficient in M;

$V^k(\mathfrak{g})$: the universal affine vertex algebra associated with $\mathfrak{g}$ at level k (vertex algebra associated with the affine Kac-Moody algebra $\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K$).

Examples of $\mathcal{W}^k(\mathfrak{g}, f)$

1). $\mathcal{W}^k(\mathfrak{g}, 0) = V^k(\mathfrak{g}) = U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t]+\mathbb{C}K)} \mathbb{C}_k$
(a $V^k(\mathfrak{g})$-module = a smooth $\widehat{\mathfrak{g}}$-module of level k).

2). $\mathcal{W}^k(\mathfrak{sl}_2, f_{prin})$ = the Virasoro vertex algebra of central charge $1 - 6(k+1)^2/(k+2)$ (if k is not critical, i.e., $k \neq -2$).

3). $\mathcal{W}^k(\mathfrak{sl}_3, f_{prin}) = W_3$ with $\mathbf{c} = 2 - 24(k+2)^2/(k+3)$ (for a non-critical k).

4). $\mathcal{W}^k(\mathfrak{sl}_n, f_{prin})$ is the Fateev-Lukyanov W_n-algebra.

5). Almost all superconformal algebras are realized as the W-algebra $\mathcal{W}^k(\mathfrak{g}, f_{min})$ associated with some Lie superalgebra $\mathfrak{g}$ and a minimal nilpotent element f_{min} ([Kac-Roan-Wakimoto '03]).

Presentation of $\mathcal{W}^k(\mathfrak{g}, f)$ by generators and relations are not known in general.

Drinfeld-Sokolov reduction functor

The definition of $\mathcal{W}^k(\mathfrak{g}, f)$ by the quantized Drinfeld-Sokolov reduction gives rise to a functor

$$\begin{gathered} V^k(\mathfrak{g})\text{-Mod} \to \mathcal{W}^k(\mathfrak{g}, f)\text{-Mod}, \\ M \mapsto H^0_{DS,f}(M). \end{gathered}$$

$\mathcal{O}_k$: the category $\mathcal{O}$ of $\widehat{\mathfrak{g}}$ at level k.

$L(\lambda) \in \mathcal{O}_k$: the irreducible highest weight representation of $\widehat{\mathfrak{g}}$ with highest weight λ of level k.

Theorem (T.A. '05, $f = f_{min}$ = minimal nilpotent element)

1). $H^{i\neq 0}_{DS,f_{min}}(M) = 0$ *for any* $M \in \mathcal{O}_k$. *Therefore, the functor* $\mathcal{O}_k \to \mathcal{W}^k(\mathfrak{g}, f_{min})$-Mod, $M \mapsto H^0_{DS,f_{min}}(M)$, *is exact.*

2). $H^0_{DS,f_{min}}(L(\lambda))$ *is zero or simple. Moreover, any irreducible highest weight representation of* $\mathcal{W}^k(\mathfrak{g}, f_{min})$ *arises in this way.*

By the Euler-Poincaré principle, the character ch $H^0_{DS,f_{min}}(L(\lambda))$ is expressed in terms of the character of $L(\lambda) \Rightarrow$ get the character of irreducible highest weight representations of $\mathcal{W}^k(\mathfrak{g}, f_{min})$.

Remark

The above theorem holds for Lie superalgebras as well. This in particular proves the Kac-Roan-Wakimoto conjecture '03.

Principal W-algebras and W-algebras of type A

One can extend the previous results for more general nilpotent elements by modifying the DS functor following Frenkel-Kac-Wakimoto '92.

As a result, we obtain

- characters of all irreducible highest weight representations of principal W-algebras $\mathcal{W}^k(\mathfrak{g}, f_{prin})$ ([T.A. '07]), which in particular proves the conjecture of Frenkel-Kac-Wakimoto '92 on the existence and construction of modular invariant representations of principal W-algebras;
- characters of all (ordinary) representations of W-algebras $\mathcal{W}^k(\mathfrak{sl}_n, f)$ of type A ([T.A.'12]), which in particular proves the similar conjecture of Kac-Wakimoto '08.

Rationality and the lisse condition

Theorem (Zhu '96)

Let V be a "nice" vertex (operator) algebra. Then the character $\chi_M(e^{2\pi i\tau})$ converges to a holomorphic function on the upper half plane for any $M \in \mathrm{Irrep}(V)$. Moreover, the space spanned by the characters $\chi_M(e^{2\pi i\tau})$, $M \in \mathrm{Irrep}(V)$, is invariant under the natural action of $SL_2(\mathbb{Z})$.

Here a vertex operator algebra V is calle "nice" if

- V is lisse (or C_2-cofinite), that is, $\mathrm{Specm}(\mathrm{gr}\, V) = \{0\}$.
- V is rational, that is, any positively graded V-modules are completely reducible.

Example of a "nice" vertex algebra

The universal affine vertex algebra $V^k(\mathfrak{g})$ is not lisse.

Indeed, $V^k(\mathfrak{g}) \cong U(t^{-1}\mathfrak{g}[t^{-1}])$, and we have

$$\operatorname{gr} V_k(\mathfrak{g}) = S(t^{-1}\mathfrak{g}[t^{-1}]) = \mathbb{C}[J_\infty \mathfrak{g}^*].$$

Here $J_\infty X$ is the arc space of X:
$\mathrm{Hom}(\operatorname{Spec} R, J_\infty X) = \mathrm{Hom}(\operatorname{Spec} R[[t]], X)$, $R : \mathbb{C}-$algebra.

Let $L_k(\mathfrak{g})$ be the simple (graded) quotient $L(k\Lambda_0)$ of $V^k(\mathfrak{g})$ (simple affine vertex algebra).

Fact (Frenkel-Zhu '92, Zhu '96, Dong-Mason '06)

$L_k(\mathfrak{g})$ is lisse $\iff$ $L_k(\mathfrak{g})$ is integrable ($\iff$ $k \in \mathbb{Z}_{\geq 0}$).

If this is the case,
$L_k(\mathfrak{g})$-Mod $= \{$integrable $\widehat{\mathfrak{g}}$-modules of level $k\}$. Thus, $L_k(\mathfrak{g})$ is rational as well.

Lisse condition and associated varieties

V: vertex algebra

$\rightsquigarrow R_V = V/C_2(V)$: Zhu's C_2-algebra (a Poisson algebra)

$\rightsquigarrow X_V := \mathrm{Specm}(R_V)$: the associated variety of V ([T.A. '12])

Lemma (T.A. '12)

V is lisse iff $X_V = \{0\}$.

Examples

1). $X_{V^k(\mathfrak{g})} = \mathfrak{g}^*$, and so $X_{L_k(\mathfrak{g})} \subset \mathfrak{g}^*$, G-invariant and conic.

2). $X_{\mathcal{W}^k(\mathfrak{g},f)} \cong \mathcal{S}_f := f + \mathfrak{g}^e \subset \mathfrak{g} = \mathfrak{g}^*$, the Slodowy slice at f ([De-Sole-Kac '06]), where $\{e, f, h\}$ is an $\mathfrak{sl}_2$-triple.

Associated varieties of W-algebras

Let $\mathcal{W}_k(\mathfrak{g}, f)$ be the simple quotient of $\mathcal{W}^k(\mathfrak{g}, f)$.

$\rightsquigarrow X_{\mathcal{W}_k(\mathfrak{g},f)} \subset X_{\mathcal{W}^k(\mathfrak{g},f)} = \mathcal{S}_f$, invariant under the natural $\mathbb{C}^*$-action which contracts to f. So $\mathcal{W}^k(\mathfrak{g}, f)$ is lisse iff $X_{\mathcal{W}_k(\mathfrak{g},f)} = \{f\}$.

One can show that $\mathcal{W}_k(\mathfrak{g}, f)$ is a quotient of the vertex algebra $H^0_{DS,f}(L_k(\mathfrak{g}))$, provided that it is nonzero ([T.A. '16]).

Theorem (T.A. '16)

We have

$$X_{H^0_{DS,f}(L_k(\mathfrak{g}))} = X_{L_k(\mathfrak{g})} \cap \mathcal{S}_f$$

(holds as schemes). Hence,

(i). $H^0_{DS,f}(L_k(\mathfrak{g})) \neq 0$ *iff* $X_{L_k(\mathfrak{g})} \supset \overline{G.f}$;

(ii). *If* $X_{L_k(\mathfrak{g})} = \overline{G.f}$, $X_{H^0_{DS,f}(L_k(\mathfrak{g}))} = \{f\}$. *Hence* $H^0_{DS,f}(L_k(\mathfrak{g}))$ *is lisse, and so is its quotient* $\mathcal{W}_k(\mathfrak{g}, f)$.

Admissible representations of affine Kac-Moody algebras

Note that $H^0_{DS,f}(L_k(\mathfrak{g})) = 0$ if $L_k(\mathfrak{g})$ is integrable. Therefore we need to study more general representations of $\widehat{\mathfrak{g}}$ to obtain lisse W-algebras using the previous result.

There is a nice class of representations of $\widehat{\mathfrak{g}}$ which are called admissible representations (Kac-Wakimoto '88):

$$\{\text{integrable rep.}\} \subsetneq \{\text{admissible rep.}\} \subsetneq \{\text{highest weight rep.}\}$$

The simple affine vertex algebra $L_k(\mathfrak{g})$ is admissible as a $\widehat{\mathfrak{g}}$-module iff

$$k + h^\vee = \frac{p}{q}, \quad p, q \in \mathbb{N},\ (p,q) = 1,\ p \geq \begin{cases} h^\vee & \text{if } (q, r^\vee) = 1, \\ h & \text{if } (q, r^\vee) = r^\vee. \end{cases}$$

Here h is the Coxeter number of $\mathfrak{g}$ and $r^\vee$ is the lacity of $\mathfrak{g}$.

Theorem (T.A. '16)

Let $L_k(\mathfrak{g})$ be an admissible affine vertex algebra.

1). *(Feigin-Frenkel conjecture) $X_{L_k(\mathfrak{g})} \subset \mathcal{N}$, the nilpotent cone of $\mathfrak{g}$.*
2). *$X_{L_k(\mathfrak{g})}$ is irreducible, that is, $\exists$ a nilpotent orbit $\mathbb{O}_k$ of $\mathfrak{g}$ such that $X_{L_k(\mathfrak{g})} = \overline{\mathbb{O}_k}$.*

By previous theorems we immediately obtain the following assertion, which was (essentially) conjectured by Kac-Wakimoto '08.

Theorem (T.A. '16)

Let $L_k(\mathfrak{g})$ be an admissible affine vertex algebra, and let $f \in \mathbb{O}_k$. Then the simple affine W-algebra $\mathcal{W}_k(\mathfrak{g}, f)$ is lisse.

Frenkel-Kac-Wakimoto conjecture

An admissible affine vertex algebra $L_k(\mathfrak{g})$ is called *non-degenerate* if

$$X_{L_k(\mathfrak{g})} = \mathcal{N} = \overline{G.f_{prin}}.$$

If this is the case k is called a *non-degenerate admissible number* for $\widehat{\mathfrak{g}}$. For a non-degenerate admissible number k, the simple principal W-algebra $\mathcal{W}_k(\mathfrak{g}, f_{prin})$ is lisse by the previous theorem.

Theorem (T.A. '15, Frenkel-Kac-Wakimoto conjecture '92)

Let k be a non-degenerate admissible number. Then the simple principal W-algebra $\mathcal{W}_k(\mathfrak{g}, f_{prin})$ is rational.

For $\mathfrak{g} = \mathfrak{sl}_2$, the corresponding rational W-algebras are exactly the minimal series of the Virasoro (vertex) algebra.

Adamović-Milas conjecture

The proof of the previous theorem is based on the following assertion on admissible affine vertex algebras.

Theorem (T.A. '16, Adamović-Milas conjecture '95)

Let $L_k(\mathfrak{g})$ be an admissible affine vertex algebra. Then $L_k(\mathfrak{g})$ is rational in the category $\mathcal{O}$, that is, any $L_k(\mathfrak{g})$-module that belongs to $\mathcal{O}$ is completely reducible.

$4d$-$2d$ duality

Recently, Beem, Lemos, Liendo, Peelaers, Rastelli, and van Rees '15 have constructed a remarkable map

$$\Phi : \{\text{4d } N = 2 \text{ SCFTs}\} \to \{\text{vertex algebras}\}$$

such that, among other things, the character of the vertex algebra $\Phi(\mathcal{T})$ coincides with the Schur index of the corresponding 4d $N = 2$ SCFT $\mathcal{T}$, which is an important invariant of the theory $\mathcal{T}$.

VOAs coming from $4d$ theory

How do vertex algebras coming from 4d $N = 2$ SCFTs look like?

We have

$$c_{2d} = -12 c_{4d}.$$

So the vertex algebras obtained by Φ are never unitary. In particular integrable affine vertex algebras never appear by this correspondence.

The main examples of vertex algebras considered by Rastelli *et al.* '15. are the simple affine vertex algebras $L_k(\mathfrak{g})$ of types D_4, E_6, E_7, E_8 at level $k = -h^\vee/6 - 1$, which are non-rational, non-admissible affine vertex algebras at negative integer levels.

Higgs branch conjecture

There is another important invariant of a 4d $N = 2$ SCFT $\mathcal{T}$, called the Higgs branch. The Higgs branch $Higgs_{\mathcal{T}}$ is an affine algebraic variety that has a hyperKähler structure in its smooth part. In particular, $Higgs_{\mathcal{T}}$ is a (possibly singular) symplectic variety.

Let $\mathcal{T}$ be one of the 4d $N = 2$ SCFTs such that $\Phi(\mathcal{T}) = L_k(\mathfrak{g})$ with $k = h^\vee/6 - 1$ for types D_4, E_6, E_7, E_8 appeared previously. It is known that $Higgs_{\mathcal{T}} = \overline{\mathbb{O}_{min}}$, and it turned out that this equals to the associated variety $X_{\Phi(\mathcal{T})}$ ([T.A.-Moreau '18]).

Conjecture (Beem and Rastelli '17)

For *any* 4d $N = 2$ SCFT $\mathcal{T}$, we have

$$Higgs_{\mathcal{T}} = X_{\Phi(\mathcal{T})}.$$

Higgs branch conjecture

So we are expected to recover the Higgs branch of a 4d $N = 2$ SCFT from the corresponding vertex algebra, which is purely an algebraic object!

Remark

1. Higgs branch conjecture is a physical conjecture since the Higgs branch is not mathematically defined in general. The Schur index is not a mathematically defined object in general, either.
2. There is a close relationship between the Higgs branches of 4d $N = 2$ SCFTs and the Coulomb branches of three-dimensional $N = 4$ gauge theories whose mathematical definition has been given by Braverman-Finkelberg-Nakajima '16 (4d-3d duality).

Quasi-lisse vertex algebras

Note that the associated variety X_V of a vertex algebra V is only a Poisson variety in general.

Definition (T.A.-Kawasetsu '16)

A vertex algebra V is called *quasi-lisse* if X_V has only finitely many symplectic leaves.

- Lisse vertex algebras are quasi-lisse.
- The simple affine vertex algebra $L_k(\mathfrak{g})$ is quasi-lisse if and only if $X_{L_k(\mathfrak{g})} \subset \mathcal{N}$. In particular, admissible affine vertex algebras are quasi-lisse.
- Physical intuition expects that vertex algebras that come from 4d $N = 2$ SCFTs via the map Φ are quasi-lisse.

Modulaity of Schur indices

Theorem (T.A.-Kawasetsu'16)

Let V be a quasi-lisse vertex (operator) algebra (of CFT type). Then there are only finitely many simple ordinary V-modules. Moreover, for a finitely generated ordinary V-module M, the character $\chi_M(q)$ satisfies a modular linear differential equation *(MLDE).*

Since the space of solutions of a MLDE is invariant under the action of $SL_2(\mathbb{Z})$, the above theorem implies that a quasi-lisse vertex algebra possesses a certain modular invariance property, although we do not claim that the normalized characters of ordinary V-modules span the space of the solutions. In particular, this implies that the Schur indices of 4d $N = 2$ SCFTs have some modular invariance property. This is something that has been conjectured by physicists ([Beem-Rastelli '17]).

The theory of class $\mathcal{S}$

There is a distinct class of 4d $N = 2$ SCFTs called the theory of class $\mathcal{S}$ [Gaiotto '12], where $\mathcal{S}$ stands for 6. The vertex algebras obtained from the theory of class $\mathcal{S}$ are called the chiral algebras of class $\mathcal{S}$ [Rastelli *et al*. '15].

The Moore-Tachikawa conjecture '12, which was recently proved by Braverman-Finkelberg-Nakajima '17, describes the Higgs branches of the theory of class $\mathcal{S}$ in terms of 2d TQFT mathematically.

Rastelli *et al*. '15 conjectured that chiral algebras of class $\mathcal{S}$ can be also described in terms of 2d TQFT (see [Tachikawa] for a mathematical exposition of their conjecture and the background).

2d TQFT description of chiral algebras of class $\mathcal{S}$

Let $\mathbb{V}$ be the following category (the category of vertex algebras)

Objects: complex semisimple groups;

Morphisms:

$\mathrm{Hom}(G_1, G_2)$
$= \{\text{VOAs } V \text{ with a VA hom. } V^{-h_1^\vee}(\mathfrak{g}_1) \otimes V^{-h_2^\vee}(\mathfrak{g}_2) \to V\}/\sim.$

For $V_1 \in \mathrm{Hom}(G_1, G_2)$, $V_2 \in \mathrm{Hom}(G_2, G_3)$,

$$V_1 \circ V_2 = H^{\frac{\infty}{2}+\bullet}(\widehat{\mathfrak{g}}_2, \mathfrak{g}_2, V_1 \otimes V_2).$$

From a result of Arkhipov-Gaitsgory one finds that the identity morphism id_G is the algebra $\mathcal{D}_G^{ch}$ of *chiral differential operators* on G at the critical level, whose associated variety is canonically isomorphic to T^*G.

Higgs branch conjecture for class $\mathcal{S}$ theory

Theorem (T.A., to appear, conjectured by Rastelli et al.)

Let $\mathbb{B}_2$ be the category of 2-bordisms. For each semisimple group G, there exists a unique monoidal functor

$$\eta_G : \mathbb{B}_2 \to \mathbb{V}$$

which sends (1) the object S^1 to G, (2) the cylinder, which is the identity morphism id_{S^1}, to the identity morphism $\mathrm{id}_G = \mathcal{D}^{ch}_G$, and (3) the cap to $H^0_{DS,f_{prin}}(\mathcal{D}^{ch}_G)$. Moreover, we have

$$X_{\eta_G(B)} \cong \eta^{BFN}_G(B)$$

for any 2-bordism B, where η^{BFN}_G is the functor form $\mathbb{B}_2$ to the category of symplectic varieties constructed by Braverman-Finkelberg-Nakajima '17.

Thank you!