Research Institute for Mathematical Sciences - Kyoto University

Homotopical Algebraic and Arithmetic Geometry

Seven Topics of Homotopic Arithmetic Geometry – 七福話

An exceptional virtual seminar

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First Semester 2020-2021

Introduction

The goal of this seminar is for PhD and postdoctoral researchers to gather on a regular basis for sharing actively their interest in some seminal and recent advances of arithmetic geometry. The *Homotopical, Algebraic, and Arithmetic Geometry* seminar provides the opportunity for speakers and attendants (1) to strengthen their expertise on a specific point of their research field, or (2) to broaden their cultures by acquiring new techniques of arithmetic geometry.

The programme of this semester, in order to reflect the various interests of the participants, presents 7 topics around *(1) the existence of rational points and Grothendieck's section conjecture* (see T1 & T6), and *(2) some potential obstructions to Grothendieck's anabelian conjecture* (see T4, T5, also the previous Homotopical Anabelian Geometry Seminar.) These questions are presented from the points of view of the étale homotopy type and $\mathbb{A}^1$-geometry, which via *simplicial and homotopical techniques* provides some complementary anabelian and abelian approaches via higher homotopy groups and cohomology theories.
Additional grounding is given by three seminal arithmetic constructions that are the compactification of moduli problems (T2), some (an)abelian good reduction criteria for (poly)curves (T3), and the arithmetic of G-covers (T7). A common aspect of these topics is the consideration of *anabelian constructions in higher dimensions or for higher symmetries.*

The non-rigidity of the programme should provide a free space for the speakers where to develop their own understanding of the theories.

Contents

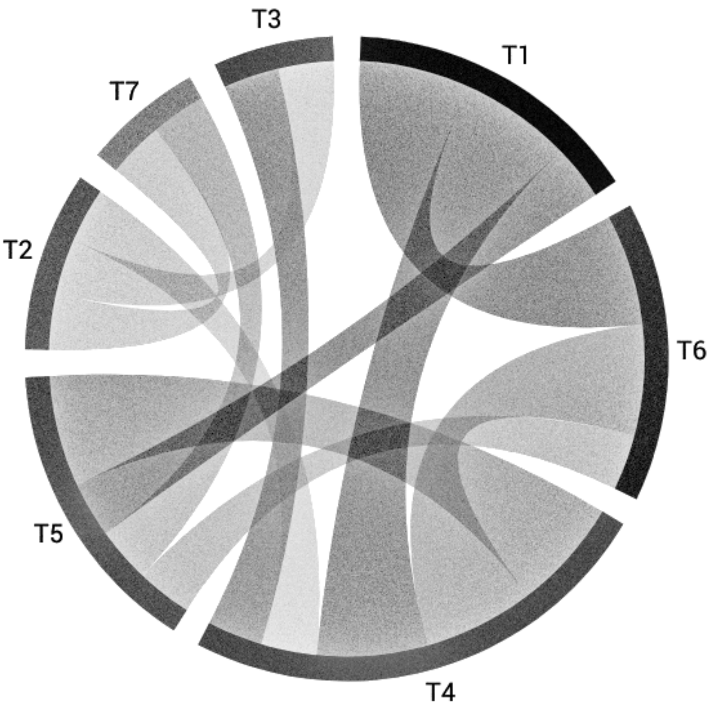

Fig 1. *Interconnections of topics in terms of Grothendieck's anabelian and section conjectures, $\mathbb{A}^1$-homotopy and étale topological type, higher symmetries and dimensions, abelian and non-abelian methods.*

Programme

The following programme proposes various topics of general interest, in-between classical topics and recent progress of arithmetic geometry, that the participants must feel free to adapt and to complete. Depending of the audience interest, some topic can support one or two talks, that will be given in English or in Japanese. Interested speaker should contact the organizers for coordination on the topic of their choice.

Each topic should be introduced by a 30 minutes introductory part for non-experts (definition, motivation, examples.)

Topic 1 - Rational Obstruction: Brauer Group and Étale Homotopy Type. The question of finding an adelic obstruction to the existence of rational points in a scheme X – i.e. *the local-global principle* – is a central one in arithmetic geometry, for example with respect to Grothendieck's section conjecture – see also Topic 6, the Noether programme for torsors or Regular Inverse Theory via Hurwitz spaces – see also Topic 7. Étale cohomology leads to the classical Brauer-Manin obstruction $X(\mathbb{A})^{Br}$, descent techniques for G-torsors to a certain $X(\mathbb{A})^{desc}$, and the étale homotopy type to Harpaz-Schlank's $X(\mathbb{A})^h$.

After the presentation of the local-global principle, the speaker will choose which of the above obstructions to construct, will then illustrate some of their properties on examples, and finally present their comparison. The construction of $X(\mathbb{A})^h$ can rely on the previous semester seminar. We also refer to [HS13] for overview and references.

Note that via a conjecture of Colliot-Thélène, Sansuc, Kato and Saito, the obstruction translates in terms of a 0-cycles of degree 1 property. A similar question has been developed in terms of $\mathbb{A}^1$-algebraic geometry by Asok, Häsemayer and Morel – see also Topics 4 & 5.

References.

[HS13] Y. Harpaz and T. M. Schlank. "Homotopy obstructions to rational points". In: *Torsors, étale homotopy and applications to rational points.* Vol. 405. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 2013, pp. 280–413. eprint: `https://arxiv.org/pdf/1110.0164.pdf`.

Topic 2 - Compactifications of Spaces: Moduli Stack of Curves. The construction of compactified spaces that classifies families of given type of objects relies on finding a "canonical" species of degenerated geometric objects. In the case of curves, this problem is solved with the addition of stable curves in the one hand by Deligne and Mumford [DM69], and by the use of log-structure by F. Kato [Kat00] on the other hand. The latter provides a functorial and canonical compactification for general schemes and stacks.

The speaker(s) will choose to present either the Deligne-Mumford construction [DM69] (including or not the notion of algebraic stacks), or to follow Kato's log-smooth geometry approach [Kat00]. We refer to [Abr+13] for a general introduction to log-structures.

References.

[Abr+13] D. Abramovich, Q. Chen, D. Gillam, Y. Huang, et al. "Logarithmic geometry and moduli". In: *Handbook of moduli. Vol. I.* Vol. 24. Adv. Lect. Math. (ALM). Int. Press, Somerville, MA, 2013, pp. 1–61. eprint: `https://arxiv.org/pdf/1006.5870v1.pdf`.

[DM69] P. Deligne and D. Mumford. "The irreducibility of the space of curves of given genus". In: *Inst. Hautes Études Sci. Publ. Math.* 36 (1969), pp. 75–109.

[Kat00] F. Kato. "Log smooth deformation and moduli of log smooth curves". In: *Internat. J. Math.* 11.2 (2000), pp. 215–232. eprint: http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.37.1907&rep=rep1&type=pdf.

Topic 3 - Good Reduction Criteria in Anabelian Geometry. Starting with Serre-Tate's result for abelian varieties, one obtains a criterion for good reduction in terms of the action of Galois inertia group on ℓ-adic étale cohomology (Néron-Ogg-Shafarevich: the ℓ-adic Tate module is unramified, see [ST68].) A "non-abelian" criterion was later given in terms of central series by Oda [Oda90; Oda95] and Tamagawa [Tam97, §5] respectively for proper and non-proper hyperbolic curves. This approach was recently investigated for polycurves in terms of Tannaka category or fibration methods in anabelian geometry by Nagamachi [Nag20].

The speaker(s) will present the abelian criterion (T3a), the first classical Oda-Tamagawa non-abelian approach (T3b), or the most recent work for polycurves (T3c). A special care will be given to the definition of the good reduction that is considered. We refer to [Tam97, §5] for a presentation of the topic.

References.

[Nag20] I. Nagamachi. "Criteria for good reduction of proper polycurves". In: *Arxiv* (2020). eprint: https://arxiv.org/abs/1801.08728.

[Oda90] T. Oda. "A note on ramification of the Galois representation on the fundamental group of an algebraic curve". In: *J. Number Theory* 34.2 (1990), pp. 225–228.

[Oda95] T. Oda. "A note on ramification of the Galois representation on the fundamental group of an algebraic curve. II". In: *J. Number Theory* 53.2 (1995), pp. 342–355.

[ST68] J.-P. Serre and J. Tate. "Good reduction of abelian varieties". In: *Ann. of Math. (2)* 88 (1968), pp. 492–517.

[Tam97] A. Tamagawa. "The Grothendieck conjecture for affine curves". In: *Compositio Math.* 109.2 (1997), pp. 135–194.

Topic 4 - $\mathbb{A}^1$-Homotopy Theory and Anabelian Geometry. The $\mathbb{A}^1$-homotopy theory is a theory of smooth schemes over a field k developed by Morel and Voevodsky where the affine line is rendered contractible [MV99]. By relying on Quillen model category theory for simplicial sheaves such as developed by Jardine, it stands in between algebraic geometry and algebraic topology, suports the higher symmetries of stacks, and reunite via the unstable motivic homotopy category $\mathcal{H}(k)$ classical algebraic geometry and motivic theory.

The speaker(s) will first introduce $\mathcal{H}(k)$ in terms of model structures and Bousfield localisation – [Sev06] Chap.1-6, then establish the factorisation of the $\mathbb{A}^1$-type by the étale topological type by following either Schmidt [Sch12] or Isaksen [Isa04] – which illustrate how non $\mathbb{A}^1$-rigid schemes provide some anabelian obstruction.

References.

[Isa04] D. C. Isaksen. "Etale realization on the $\mathbb{A}^1$-homotopy theory of schemes". In: *Adv. Math.* 184.1 (2004), pp. 37–63. eprint: https://arxiv.org/abs/math/0106158.

[MV99] F. Morel and V. Voevodsky. "$\mathbf{A}^1$-homotopy theory of schemes". In: *Inst. Hautes Études Sci. Publ. Math.* 90 (1999), 45–143 (2001). eprint: https://faculty.math.illinois.edu/K-theory/305/nowmovo.pdf.

[Sch12] A. Schmidt. "Motivic aspects of anabelian geometry". In: *Galois-Teichmüller theory and arithmetic geometry*. Vol. 63. Adv. Stud. Pure Math. Math. Soc. Japan, Tokyo, 2012, pp. 503–517. eprint: https://www.mathi.uni-heidelberg.de/~schmidt/papers/63117.pdf.

[Sev06] M. Severitt. "Motivic Homotopy Types of Projective Curves (SE)". Diplomarbeit. Universität Bielfeld, 2006. eprint: https://www.math.uni-bielefeld.de/~mseverit/mseverittse.pdf.

TOPIC 5 - HOMOTOPICAL AND SPECTRAL METHODS IN MOTIVIC THEORY. In homotopy theory, it follows in the one hand Brown's representability theorem that the notion of Spectra is the proper one for the representation of cohomology functors, while Quillen model category theory gives access to functorial homotopical constructions and to contractions in various topologies. A key aspect of Morel-Voevodsky motivic homotopy theory is to develop these principles in algebraic geometry with the construction of a triangulated (stable) motivic homotopy category $\mathcal{SH}(k)$ for smooth schemes [Voe98].

The speaker(s) will present the construction of the model category of T-spectra and its homotopy category $\mathcal{SH}(k)$ via Bousfield localisation, recall the definition of Voevodsky's derived category of motives $DM\_(k)$ of Nisnevich sheaves with transfer in terms of Verdier localisation, and finally present the functor $(\mathcal{S})\mathcal{H}(k) \to DM\_(k)$ via the normalised cochain complex – see [Sev06] Chap. 7-8 and also [VRØ07].

References.

[Sev06] M. Severitt. "Motivic Homotopy Types of Projective Curves (SE)". Diplomarbeit. Universität Bielfeld, 2006. eprint: https://www.math.uni-bielefeld.de/~mseverit/mseverittse.pdf.

[Voe98] V. Voevodsky. "$\mathbf{A}^1$-homotopy theory". In: *Proceedings of the International Congress of Mathematicians, Vol. I (Berlin, 1998).* Extra Vol. I. 1998, pp. 579–604. eprint: https://www.math.uni-bielefeld.de/documenta/xvol-icm/00/Voevodsky.MAN.html.

[VRØ07] V. Voevodsky, O. Röndigs, and P. A. Østvær. "Voevodsky's Nordfjordeid lectures: motivic homotopy theory". In: *Motivic homotopy theory.* Universitext. Springer, Berlin, 2007, pp. 147–221.

TOPIC 6 - ÉTALE HOMOTOPY TYPE AND GROTHENDIECK'S SECTION CONJECTURE. Considering the étale topological type $(X)_{et}$ of a quasi-projective smooth variety over k as an object over BG, $G = \mathrm{Gal}(\bar{k}/k)$ leads first to consider the set X_{et}^{hG} of homotopy fixed points of X as a potential obstruction to the existence of k-points in X, then to a homotopy limit reformulation of Grothendieck's Section Conjecture [Qui13].

The speaker will follow the construction of [Qui13] in terms of Friedlander (rigid) étale topological type and a certain fibrant profinite model in terms of Postnikov tower, then of the set of continuous étale homotopy fixed points X_{pf}^{hG}, before stating the homotopy limit criterion.

References.

[Qui13] G. Quick. "Existence of rational points as a homotopy limit problem". In: *Arxiv* (2013). eprint: https://arxiv.org/pdf/1309.0463.pdf.

TOPIC 7 - ARITHMETIC OF GERBES AND G-COVERS. Following an initial idea from Fried, the Regular Inverse Galois Problem translates the realization of a finite group G as finite quotient of $\mathrm{Gal}(\bar{K}/K)$ first into the existence of a K-point in some Hurwitz spaces of G-covers, then into going for a geometric G-cover from a certain field of moduli to a field of definition (fModDef). Dèbes and Douai developed some (fModDef) criteria first in terms of group cohomology then in terms of gerbes of models of the G-cover [DD99]. The latter provides a clear and motivated introduction to Giraud's more abstract theory of gerbes and bands in non-abelian cohomology [Gir71].

The speaker will present the self-content [DD99] by ignoring the case of mere-covers.

Mote that the non-functoriality with respect to the coefficients of Giraud's non-abelian cohomology can be solved by considering simplicial constructions and crossed-modules as established by Breen.

References.

[DD99] P. Dèbes and J.-C. Douai. "Gerbes and covers". In: *Comm. Algebra* 27.2 (1999), pp. 577–594.

[Gir71] J. Giraud. *Cohomologie non abélienne.* Die Grundlehren der mathematischen Wissenschaften, Band 179. Springer-Verlag, Berlin-New York, 1971, pp. ix+467.

Additional Topics. Participants should feel free to propose their own research interest, which can include and are not restricted to:

T+ **Anabelian Geometry by m-solvability for hyperbolic curves or number fields.** On the reconstruction problem of isomorphism classes of curves and field (Grothendieck Conjecture and Neukirch-Uchida Theorem) in terms of the maximal m-step solvable quotient of their fundamental group (resp. the absolute Galois group). Follows results from Nakamura, Mochizuki, Saïdi and Tamagawa – see *Saidi and Tamagawa's The m-step solvable anabelian geometry of number fields* available at arXiv:1909.08829 [math.NT].

T7+ **Hurwitz Spaces.** Prof. P. Dèbes, Lille University France.
Abstract: Hurwitz spaces are moduli spaces of covers of the line. We will present their construction and review their topological and geometrical properties. We will then turn to arithmetic issues: finding irreducible components defined over small fields, and rational points on them. The Regular Inverse Galois Problem will be one leading example.

※ *Note from the organizers: the homological stability for Hurwitz spaces results mentioned in this talk provides another connection with the "higher dimension" topic of the seminar, see [EVW16].*

References.

[EVW16] J. S. Ellenberg, A. Venkatesh, and C. Westerland. "Homological stability for Hurwitz spaces and the Cohen-Lenstra conjecture over function fields". In: *Ann. of Math.* 183.3 (2016), pp. 729–786. eprint: https://arxiv.org/abs/0912.0325.

Organization and Schedule

The seminar takes place *every two weeks* for 3 hours on Wednesday by Zoom meeting between 16:00-19:00, JP time (8:00-11:00, UK time; 9:00-12:00 FR time).

Speakers will present their talk either

(1) by video conferencing and black board,

(2) by a Zoom shared whiteboard via tablet, or

(3) by sharing a scanned/photographed version of their handwritten notes.

A permanent private Zoom group chat **"[RIMS] Homotopical Arithmetic Geometry"** has been setup for sharing resources, questions and informal discussions – add one of the organizers to your contacts list to be invited to the group.

Please feel free to contact the organizers for attending and for reserving a slot, or fill in the register form in the QR-clickable link.

TALKS AND SPEAKERS.

T+	T1	T7+	T7	T3a	T3b/T6	T3c	T2	T1'	T4	T5
Yamaguchi	Porowski	Dèbes	Ishii	Philip	Sawada/Colla	Nagamashi	*Available*	Porowski	*Available*	*Available*
10	24	08	22	05	19	02	16	23	30	14
June		July		August		September				October

This schedule can be updated following last minute availability of the group and speakers.

LIST OF PARTICIPANTS (12).

(i) Benjamin Collas, Lille University, France & RIMS - Kyoto University, Japan;
(ii) Pierre Dèbes, Lille University, France;
(iii) Shun Ishii, RIMS - Kyoto University, Japan;
(iv) Masaoki Mori, Osaka University, Japan;
(v) Ippei Nagamachi, Tokyo University, Japan
(vi) Séverin Philip, Université Grenoble Alpes, France;
(vii) Wojciech Porowski, Nottingham University, UK & RIMS - Kyoto University, Japan;
(viii) Kenji Sakugawa, Keio University, Japan (TBC);
(ix) Koichiro Sawada, Osaka University, Japan;
(x) Densuke Shiraishi, Osaka University, Japan;
(xi) Naganori Yamaguchi, RIMS - Kyoto University, Japan;
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