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Theory of Principal Partitions Revisited
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Summary. The theory of principal partitions of discrete systems such as graphs, matrices, matroids, and submodular systems has been developed since 1968. In the early stage of the developments during 1968–75 the principal partition was considered as a decomposition of a discrete system into its components together with a partially ordered structure of the set of the components. It then turned out that such a decomposition with a partial order on it arises from the submodularity structure pertinent to the system and it has been realized that the principal partitions are closely related to resource allocation problems with submodular structures, which are kind of dual problems.

The aim of this paper is to give an overview of the developments in the theory of principal partitions and some recent extensions with special emphasis on its relation to associated resource allocation problems in order to make it better known to researchers in combinatorial optimization.

1 Introduction

The concept of principal partition originated from Kishi and Kajitani's pioneering work [46] in 1968, which is concerned with the tri-partition of a graph determined by a maximally distant pair of spanning trees of the graph. Since then the theory of principal partitions has been extended from graphs [71] to matrices [31, 32], matroids [5, 67, 80], and submodular systems [18, 19, 33, 35, 65, 66, 81].

In the early stage of the developments around 1968–75 the principal partition was considered as a decomposition of a discrete system into its components together with a partially ordered structure of the set of the components. It then turned out that such a decomposition and the associated partial order (poset) come from the submodularity structure pertinent to the system and that the principal partition is closely related to resource allocation problems with submodular constraints.

The decomposition and its associated poset structure arise from minimization of a submodular function underlying the discrete system under consideration. We have a min-max theorem that characterizes the submodular function minimization, and we can relate optimal solutions of the dual maximization problem to a resource allocation problem with submodular constraints.

It should be noted that research developments closely related to principal partitions have been independently made for parametric optimization problems with special emphasis put on monotonicity of optimal solutions, by Topkis et al. (see, e.g., [4, 27, 52, 82, 83]).

The aim of this paper is to give an overview of the developments in the theory of principal partitions and some recent extensions to make it better known to and fully understood by researchers in combinatorial optimization.

The present paper is organized as follows. Section 2 gives basics of submodular functions that lay the foundations of principal partitions. We make a historical overview of principal partitions in Section 3 and some recent extensions in Section 4. Section 5 describes some applications of principal partitions and related topics.

2 Fundamentals of Submodular Functions and Associated Polyhedra

In this section we describe basic properties and facts in the theory of submodular functions, which will play a fundamental rôle in the developments of the theory of principal partitions (also see [20]).

2.1 Posets, distributive lattices, and submodular functions

Let E be a finite nonempty set and $\mathcal{D}$ be a collection of subsets of E such that for every $X, Y \in \mathcal{D}$ we have $X \cup Y, X \cap Y \in \mathcal{D}$. Then $\mathcal{D}$ is a *distributive lattice* (or a *ring family*) with set union and intersection as the lattice operations, join and meet.

Let $\preceq$ be a *partial order* on set E, i.e., $\preceq$ is a binary relation on E such that (i) (reflexive) $e \preceq e$ for all $e \in E$, (ii) (antisymmetric) $e \preceq e'$ and $e' \preceq e$ imply $e = e'$ for all $e, e' \in E$, and (iii) (transitive) $e \preceq e'$ and $e' \preceq e''$ imply $e \preceq e''$ for all $e, e', e'' \in E$. The pair $(E, \preceq)$ is called a *partially ordered set* (or a *poset* for short). A subset I of E is called an *order-ideal* (or an *ideal*) of poset $(E, \preceq)$ if $e \preceq e' \in I$ implies $e \in I$ for all $e, e' \in E$.

Theorem 1 (Birkhoff). *Let $\mathcal{D}$ be a set of subsets of a finite set E with $\emptyset, E \in \mathcal{D}$. Then $\mathcal{D}$ is a distributive lattice with set union and intersection as the lattice operations if and only if there exists a poset $(\Pi(E), \preceq)$ on a partition $\Pi(E)$ of E such that $\mathcal{D}$ is expressed as follows:*

For any $X \in \mathcal{D}$ there exists an ideal $\mathcal{J}$ of $(\Pi(E), \preceq)$ such that $X = \bigcup_{F \in \mathcal{J}} F.$

□

We denote the poset $(\Pi(E), \preceq)$ appearing in Theorem 1 by $\mathcal{P}(\mathcal{D})$. Conversely, for any poset $\mathcal{P}$ on a partition of E there uniquely exists a distributive lattice $\mathcal{D} \subseteq 2^E$ with set union and intersection as the lattice operations such that $\emptyset, E \in \mathcal{D}$, and $\mathcal{P} = \mathcal{P}(\mathcal{D})$. We denote such a distributive lattice by $\mathcal{D}(\mathcal{P})$.

Remark 1. The original Birkhoff theorem [3] says that a finite lattice (not necessarily given as a set lattice) is a distributive lattice if and only if it is isomorphic to the set lattice of ideals of a finite poset. It is a crucial observation in principal partitions that a finite distributive lattice given as a set lattice induces a partition of the underlying set and a partial order on it, which conversely gives the distributive lattice as a set of ideals of the poset. This was explicitly mentioned by Iri in [33, 35]. □

Let $\mathcal{D} \subseteq 2^E$ be a finite distributive lattice with set union and intersection as the lattice operations. Also suppose that $f : \mathcal{D} \to \mathbf{R}$ satisfies

$$f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y) \qquad (X, Y \in \mathcal{D}). \tag{1}$$

Then, f is called a *submodular function* on $\mathcal{D}$. When f is a submodular function, $-f$ is called a *supermodular function.* A function that is simultaneously submodular and supermodular is called a *modular function.*

Lemma 1. *For any submodular function $f : \mathcal{D} \to \mathbf{R}$ define*

$$\mathcal{D}_{\min}(f) = \{X \in \mathcal{D} \mid f(X) = \min\{f(Z) \mid Z \in \mathcal{D}\}\}. \tag{2}$$

Then, the collection $\mathcal{D}_{\min}(f)$ of minimizers of f forms a distributive lattice with set union and intersection as the lattice operations.

(Proof) For any $X, Y \in \mathcal{D}_{\min}(f)$ we have

$$f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y) \geq f(X) + f(Y) \tag{3}$$

where note that $f(X) = f(Y) \leq \min\{f(X \cup Y), f(X \cap Y)\}$. Hence $X \cup Y, X \cap Y \in \mathcal{D}_{\min}(f)$. □

Combining Theorem 1 and Lemma 1, we observe that the collection of minimizers of submodular function f gives a partition of the underlying set E and a partial order on it. More precisely,

Theorem 2. *Let $\mathcal{D}_{\min}(f)$ be the collection of minimizers of a submodular function f as in Lemma* 1*. Let $E_{\min}$ be the minimum element of $\mathcal{D}_{\min}(f)$ and $E_{\max}$ the maximum element of $\mathcal{D}_{\min}(f)$. Then, E is partitioned into*

$$E_{\min}, \qquad F_i \quad (i \in I), \qquad E \setminus E_{\max} \tag{4}$$

in such a way that every $X \in \mathcal{D}_{\min}(f)$ is expressed as

$$X = \left(\bigcup\{F_i \mid i \in I,\ F_i \subseteq X\}\right) \cup E_{\min}. \tag{5}$$

Here, $\{F_i \mid i \in I\}$ is a partition of $E_{\max} \setminus E_{\min}$. □

Remark 2. Note that expressions (5) correspond to ideals of a poset. We have a partial order on $\{F_i \mid i \in I\}$ as in the Birkhoff theorem. Hence, adding to it $E_{\min}$ and $E \setminus E_{\max}$ as the minimum element and the maximum element of the poset, respectively, we get a partial order on the partition of the whole set E. □

Remark 3. There is a large class of combinatorial optimization problems that have min-max relations expressed by submodular functions. For such problems we often encounter the problem of submodular function minimization that characterizes the minimization side of the min-max relation. Then, this naturally leads us to the decomposition of the discrete system under consideration, due to Theorem 2, i.e., we obtain a partition and a partial order on it derived from the submodular function minimization. This is the essence of principal partitions in the early stage of its developments. Related arguments for cut functions of networks were made in [73]. □

We call $\mathcal{D}$ a *simple* distributive lattice if the length of a maximal chain of $\mathcal{D}$ is equal to $|E|$, where note that all the maximal chains of $\mathcal{D}$ have the same length. For a simple distributive lattice $\mathcal{D}$ and its corresponding poset $\mathcal{P}(\mathcal{D}) = (\Pi(E), \preceq)$ the partition $\Pi(E)$ consists of singletons only. Hence we regard poset $\mathcal{P}(\mathcal{D})$ as a poset $\mathcal{P}(\mathcal{D}) = (E, \preceq)$ on E.

For a poset $\mathcal{P} = (E, \preceq)$ with $m = |E|$ a sequence or ordering $(e_1, e_2, \cdots, e_m)$ of elements of E is called a *linear extension* of $\mathcal{P} = (E, \preceq)$ if for all $i, j = 1, \cdots, m,\ \ e_i \prec e_j$ implies $i < j$. Every linear extension $(e_1, e_2, \cdots, e_m)$ of $(E, \preceq)$ determines a maximal chain $S_0 = \emptyset \subset S_1 \subset \cdots \subset S_m = E$ of $\mathcal{D}(\mathcal{P})$ by defining S_i as the set of the first i elements of the linear extension for each $i = 0, 1, \cdots, m$. Conversely, every maximal chain $S_0 = \emptyset \subset S_1 \subset \cdots \subset S_m = E$ of simple $\mathcal{D}$ determines a linear extension $(e_1, e_2, \cdots, e_m)$ of $\mathcal{P}(\mathcal{D}) = (E, \preceq)$ by defining $\{e_i\} = S_i \setminus S_{i-1}$ for each $i = 1, \cdots, m$.

2.2 Submodular functions and associated polyhedra

Let $f : \mathcal{D} \to \mathbf{R}$ be a submodular function on a distributive lattice $\mathcal{D} \subseteq 2^E$. Assume that $\emptyset, E \in \mathcal{D}$ and $f(\emptyset) = 0$. Then we call the pair $(\mathcal{D}, f)$ a *submodular system* on E. If $\mathcal{D}$ is simple, we call $(\mathcal{D}, f)$ a *simple* submodular system. Similarly we define a (simple) *supermodular system*.

We define two polyhedra associated with submodular system $(\mathcal{D}, f)$ as follows.

$$\mathrm{P}(f) = \{x \mid x \in \mathbf{R}^E,\ \forall X \in \mathcal{D} : x(X) \le f(X)\}, \tag{6}$$

$$\mathrm{B}(f) = \{x \mid x \in \mathrm{P}(f),\ x(E) = f(E)\}, \tag{7}$$

where for any $X \subseteq E$ and $x \in \mathbf{R}^E$ we define $x(X) = \sum_{e \in X} x(e)$. We call $\mathrm{P}(f)$ and $\mathrm{B}(f)$, respectively, the *submodular polyhedron* and the *base polyhedron* associated with submodular system $(\mathcal{D}, f)$. Informally, submodular polyhedron

$\mathrm{P}(f)$ is the set of vectors $x \in \mathbf{R}^E$ 'smaller' than or equal to f. Base polyhedron $\mathrm{B}(f)$ is the face of $\mathrm{P}(f)$ determined by the hyperplane $x(E) = f(E)$ and is the set of all maximal vectors in submodular polyhedron $\mathrm{P}(f)$, which is always nonempty. Define $f^{\#}(E \setminus X) = f(E) - f(X)$ for all $X \in \mathcal{D}$. We call $f^{\#}$ the *dual supermodular function* of f, and $(\overline{\mathcal{D}}, f^{\#})$ the *dual supermodular system* of $(\mathcal{D}, f)$, where $\overline{\mathcal{D}} = \{E \setminus X \mid X \in \mathcal{D}\}$. Similarly, for any supermodular system $(\mathcal{D}, g)$ on E we define the *dual submodular function* $g^{\#}$ of g by $g^{\#}(E \setminus X) = g(E) - g(X)$ for all $X \in \mathcal{D}$.

For a supermodular system $(\mathcal{D}, g)$ we define the *supermodular polyhedron* $\mathrm{P}(g)$ and the *base polyhedron* $\mathrm{B}(g)$ by $\mathrm{P}(g) = \{x \mid x \in \mathbf{R}^E,\ \forall X \in \mathcal{D} : x(X) \geq g(X)\}$ and $\mathrm{B}(g) = \{x \mid x \in \mathrm{P}(g),\ x(E) = g(E)\}$, respectively. Note that $\mathrm{B}(g) = \mathrm{B}(g^{\#})$.

An element of $\mathrm{B}(f)$ is called a *base* of submodular system $(\mathcal{D}, f)$. An extreme point of $\mathrm{B}(f)$ is called an *extreme base*. An element of submodular polyhedron $\mathrm{P}(f)$ is called a *subbase* of $(\mathcal{D}, f)$. The following theorem in the case when $\mathcal{D} = 2^E$ is due to Edmonds [11] and Shapley [78].

Theorem 3 (Edmonds, Shapley). *For a simple submodular system $(\mathcal{D}, f)$ let $(e_1, \cdots, e_m)$ be a linear extension of poset $\mathcal{P}(\mathcal{D}) = (E, \preceq)$, and let $S_i = \{e_1, \cdots, e_i\}$ for $i = 1, \cdots, m$ and $S_0 = \emptyset$. Define a vector $x \in \mathbf{R}^E$ by*

$$x(e_i) = f(S_i) - f(S_{i-1}) \qquad (i = 1, \cdots, m). \tag{8}$$

Then x is an extreme base of submodular system $(\mathcal{D}, f)$.

Conversely, every extreme base of submodular system $(\mathcal{D}, f)$ is generated in this way. □

We say that the base x defined by (8) is the *extreme base corresponding to the linear ordering* $(e_1, \cdots, e_m)$. Similarly, for a simple supermodular system $(\mathcal{D}, g)$ on E the extreme base y corresponding to a linear ordering $(e_1, \cdots, e_m)$ (a linear extension of $\mathcal{P}(\mathcal{D})$) is given by $y(e_i) = g(S_i) - g(S_{i-1})$ $(i = 1, \cdots, m)$, where S_i is the set of the first i elements of the linear ordering.

Remark 4. Note that $S_i = \{e_1, \cdots, e_i\}$ for $i = 0, 1, \cdots, m$ in Theorem 3 is a maximal chain of $\mathcal{D}$. Any (not necessarily maximal) chain

$$\mathcal{C} : C_0 = \emptyset \subset C_1 \subset \cdots \subset C_k = E \tag{9}$$

of $\mathcal{D}$ determines a face $\mathbf{F}(\mathcal{C})$ of $\mathrm{B}(f)$ by

$$\mathbf{F}(\mathcal{C}) = \{x \mid x \in \mathrm{B}(f),\ \forall i = 1, \cdots, k : x(C_i) = f(C_i)\}, \tag{10}$$

which is nonempty. Every maximal chain containing $\mathcal{C}$ determines an extreme point of the face $\mathbf{F}(\mathcal{C})$. It should also be noted that the face $\mathbf{F}(\mathcal{C})$ is again a base polyhedron, which is a direct sum of bases of minors of submodular system $(\mathcal{D}, f)$ defined in the sequel. □

Let $G(\mathcal{P}) = (E, A)$ be the graph with vertex set E and arc set A representing the Hasse diagram of $\mathcal{P} = (E, \preceq)$, where $(e, e') \in A$ if and only if $e' \prec e$ and there exists no element e'' such that $e' \prec e'' \prec e$ in $\mathcal{P}$. Any function $\varphi : A \to \mathbf{R}$ is a *flow* in $G(\mathcal{P})$. The *boundary* $\partial\varphi : E \to \mathbf{R}$ of flow φ is defined by

$$\partial\varphi(e) = \sum_{(e,e')\in A} \varphi(e, e') - \sum_{(e'',e)\in A} \varphi(e'', e) \qquad (e \in E). \tag{11}$$

Theorem 4. *The characteristic cone* Cone(B(f)) *of base polyhedron* B(f) *associated with a simple submodular system* $(\mathcal{D}, f)$ *on* E *is given by*

$$\mathrm{Cone}(\mathrm{B}(f)) = \{\partial\varphi \mid \varphi : \text{a nonnegative flow in } G(\mathcal{P}(\mathcal{D}))\}. \tag{12}$$

□

Consider a submodular system $(\mathcal{D}, f)$ on E. For any $X \in \mathcal{D}$ the *reduction* or *restriction* of submodular system $(\mathcal{D}, f)$ by X is a submodular system $(\mathcal{D}^X, f^X)$ on X defined by

$$\mathcal{D}^X = \{Z \mid Z \in \mathcal{D},\ Z \subseteq X\}, \tag{13}$$

$$f^X(Z) = f(Z) \qquad (Z \in \mathcal{D}^X). \tag{14}$$

Also the *contraction* of $(\mathcal{D}, f)$ by $X \in \mathcal{D}$ is a submodular system $(\mathcal{D}_X, f_X)$ on $E \setminus X$ defined by

$$\mathcal{D}_X = \{Z \setminus X \mid Z \in \mathcal{D},\ Z \supseteq X\}, \tag{15}$$

$$f_X(Z) = f(Z \cup X) - f(X) \qquad (Z \in \mathcal{D}_X). \tag{16}$$

Note that $\mathcal{D}_\emptyset = \mathcal{D}$ and $\mathcal{D}^E = \mathcal{D}$.

For any $X, Y \in \mathcal{D}$ such that $X \subset Y$ define

$$\mathcal{D}^Y_X = (\mathcal{D}^Y)_X, \tag{17}$$

$$f^Y_X = (f^Y)_X. \tag{18}$$

Here note that $(\mathcal{D}^Y)_X = (\mathcal{D}_X)^{Y\setminus X}$ and $(f^Y)_X = (f_X)^{Y\setminus X}$. We call the submodular system $(\mathcal{D}^Y_X, f^Y_X)$ on $Y \setminus X$ a *minor* of $(\mathcal{D}, f)$.

Theorem 5. *Let* $\mathcal{C}$ *be a chain of* $\mathcal{D}$ *given by* (9) *and* $\mathbf{F}(\mathcal{C})$ *be the face of the base polyhedron* B(f) *determined by* (10)*. Then,* $\mathbf{F}(\mathcal{C})$ *is expressed as*

$$\mathbf{F}(\mathcal{C}) = \bigoplus_{i=1}^{k} \mathrm{B}(f^{C_i}_{C_{i-1}}), \tag{19}$$

which is the direct sum of the base polyhedra associated with minors $(\mathcal{D}^{C_i}_{C_{i-1}}, f^{C_i}_{C_{i-1}})$ $(i{=}1, \cdots, k)$ *of* $(\mathcal{D}, f)$*.* □

We need some other definitions. Consider a submodular system $(\mathcal{D}, f)$ on E. For any $x \in \mathrm{P}(f)$ we call $X \in \mathcal{D}$ a *tight set* for x if $x(X) = f(X)$, and let $\mathcal{D}_f(x)$ denote the collection of all tight sets for x. Note that $\mathcal{D}_f(x) = \mathcal{D}_{\min}(f - x)$, so that it is closed with respect to set union and intersection.

For any $x \in \mathrm{P}(f)$ define

$$\mathrm{sat}(x) = \bigcup \{X \mid X \in \mathcal{D}_f(x)\}, \tag{20}$$

which is defined to be the empty set if $\mathcal{D}_f(x) = \emptyset$. When $\mathcal{D}_f(x)$ is nonempty, $\mathrm{sat}(x)$ is the unique maximal element of the distributive lattice $\mathcal{D}_f(x)$ and can be expressed as

$$\mathrm{sat}(x) = \{e \in E \mid \forall \alpha > 0 : x + \alpha\chi_e \notin \mathrm{P}(f)\}. \tag{21}$$

Here χ_e is the unit vector in $\mathbf{R}^E$ with $\chi_e(e') = 1$ if $e' = e$ and $\chi_e(e') = 0$ if $e' \in E \setminus \{e\}$. We call $\mathrm{sat} : \mathrm{P}(f) \to 2^E$ the *saturation function*. Note that $\mathrm{sat}(x)$ is empty if and only if x lies in the interior of $\mathrm{P}(f)$.

Also define for any $x \in \mathrm{P}(f)$ and any element $e \in \mathrm{sat}(x)$

$$\mathrm{dep}(x, e) = \bigcap \{X \mid e \in X \in \mathcal{D}_f(x)\}, \tag{22}$$

which can be rewritten as

$$\mathrm{dep}(x, e) = \{e' \in E \mid \exists \alpha > 0 : x + \alpha(\chi_e - \chi_{e'}) \in \mathrm{P}(f)\}, \tag{23}$$

and we also define $\mathrm{dep}(x, e) = \emptyset$ if $e \notin \mathrm{sat}(x)$. We call $\mathrm{dep} : \mathrm{P}(f) \times E \to 2^E$ the *dependence function*. Note that when $e \in \mathrm{sat}(x)$, $\mathrm{dep}(x, e)$ is the unique minimal element of the distributive lattice $\{X \mid e \in X \in \mathcal{D}_f(x)\}$.

3 An Overview of Principal Partitions

We make an overview of the developments in the theory of principal partitions.

3.1 Kishi and Kajitani's tri-partition for graphs

Suppose that we are given a connected graph $G = (V, E)$ with a vertex set V and an edge set E. We identify a spanning tree with its edge set. Let $\mathcal{T} \subseteq 2^E$ be the set of all the spanning trees of G. For any two spanning trees T_1 and T_2 in $\mathcal{T}$ we denote by $\mathrm{dist}(T_1, T_2)$ the *distance* $|T_1 \setminus T_2|$ of T_1 and T_2. A pair of spanning trees T_1 and T_2 is called a *maximally distant pair of spanning trees* if it attains the maximum of the distance.

Kishi and Kajitani's principal partition [46] of graph $G = (V, E)$ is the ordered tri-partition of the edge set into (E^-, E^0, E^+) such that the following three hold.

(−) For any $e \in E^-$ there exists a maximally distant pair of spanning trees T_1 and T_2 such that $e \notin T_1 \cup T_2$.
(0) For any maximally distant pair of spanning trees T_1 and T_2 we have a bi-partition of E^0 into $E^0 \cap T_1$ and $E^0 \cap T_2$, i.e., for any $e \in E^0$ we have either $e \in T_1$ or $e \in T_2$.
(+) For any $e \in E^+$ there exists a maximally distant pair of spanning trees T_1 and T_2 such that $e \in T_1 \cap T_2$.

Graph $G = (V, E)$ is decomposed into $G \cdot E^-$, $G \cdot (E^0 \cup E^-)/E^-$, and $G/(E^0 \cup E^-)$, where for any edge set $F \subseteq E$, $G \cdot F$ is the restriction of G on F and G/F is the graph obtained by contraction of all the edges in F.

It can be shown that Kishi and Kajitani's tri-partition is characterized by the following theorem, which is a matroidal min-max theorem known earlier in matroid theory [11, 12, 13].

Theorem 6. *For a connected graph $G = (V, E)$ with rank function $r_G : 2^E \to \mathbf{Z}_+$,*

$$\max\{|T_1 \cup T_2| \mid T_1, T_2 : \text{spanning trees of } G\} = \min\{2r_G(X) + |E \setminus X| \mid X \subseteq E\}. \tag{24}$$

□

Theorem 7. *The set $\mathcal{D}_G$ of all the minimizers of the submodular function $f(X) = 2r_G(X) + |E \setminus X|$ in $X \in 2^E$ is closed with respect to set union and intersection and forms a distributive lattice. The unique minimal element of $\mathcal{D}_G$ is given by E^- and the unique maximal element of $\mathcal{D}_G$ by $E^- \cup E^0 (= E \setminus E^+)$, where (E^-, E^0, E^+) is the Kishi-Kajitani tri-partition of E for $G = (V, E)$.* □

Ozawa [71] generalized Kishi and Kajitani's principal partition of a graph to a pair of graphs, which is a special case of the principal partition of a pair of (poly-)matroids to be discussed in Section 3.5.

Remark 5. For an electrical network the *topological degrees of freedom* is the minimum number of current and voltage variables whose values uniquely determine all current and voltage values of arcs through Kirchhoff's current and voltage laws. It was noticed that Kishi and Kajitani's principal tri-partition could be used to resolve the problem of determining the topological degrees of freedom (see [31, 34, 46, 69] and also [68, Chapter 14]).

It should also be noted that Kishi and Kajitani's principal tri-partition gives a solution of *Shannon's switching game* (see [5, 10]). □

3.2 Iri's maximum-rank minimum-term-rank theorem for pivotal transforms of a matrix

Iri [31, 32] considered a generalization of Kishi and Kajitani's framework for graphs to that for matrices and related the matroidal min-max theorem to what is called the maximum-rank minimum-term-rank theorem for pivotal transforms of a matrix. Moreover, he derived a finer poset structure on E^0 part, based on the Dulmage-Mendelsohn decomposition of bipartite graphs.

Suppose that we are given an $m \times n$ real matrix $M = [I_m|A]$, where I_m is the identity matrix of order m and A an $m \times (n-m)$ matrix. Let E be the index set of the columns of M. Then consider the matroid $\mathbf{M}$ on E represented by the matrix M defined by the linear independence among the column vectors of M.

For any base B of matroid $\mathbf{M}$ we can transform the original matrix $M = [I_m|A]$ so that the submatrix corresponding to the columns B becomes the identity matrix I_m by fundamental row operations. After an appropriate column permutation we obtain a new matrix $M(B) = [I_m|A(B)]$. We call $A(B)$ a *pivotal transform* of A. Define

$$\mathcal{A}(M) = \{A(B) \mid B : \text{a base of } \mathbf{M}\}. \tag{25}$$

For any matrix $C \in \mathcal{A}(M)$ consider the bipartite graph $G(C)$ corresponding to the nonzero elements of matrix C. The size of a maximum matching in the bipartite graph $G(C)$ is the *term rank* of C, which we denote by t-rank C.

Now we have

Theorem 8 (Iri).

$$\max\{\text{rank } C \mid C \in \mathcal{A}(M)\} = \min\{\text{t-rank } C \mid C \in \mathcal{A}(M)\}, \tag{26}$$

where the maximum and the minimum can be attained simultaneously by a matrix $C \in \mathcal{A}(M)$. □

This theorem can be considered as a matrix variant, in terms of term rank, of the following matroidal min-max theorem about the union of matroids [13, 75]. We denote by $r_{\mathbf{M}}$ the rank function of matroid $\mathbf{M}$.

Theorem 9. *For any matroid* $\mathbf{M}$ *with rank function* $r_{\mathbf{M}} : 2^E \to \mathbf{Z}_+$,

$$\max\{|B_1 \cup B_2| \mid B_1, B_2 : \text{bases of } \mathbf{M}\} = \min\{2r_{\mathbf{M}}(X) + |E \setminus X| \mid X \subseteq E\}, \tag{27}$$

or equivalently,

$$\max\{|B_1 \setminus B_2| \mid B_1, B_2 : \text{bases of } \mathbf{M}\} = \min\{r_{\mathbf{M}}(X) + r^*_{\mathbf{M}}(E \setminus X) \mid X \subseteq E\}, \tag{28}$$

where $r^*_{\mathbf{M}}$ *is the corank function of matroid* $\mathbf{M}$. □

The left-hand side of (28) is equal to that of (26) when matroid $\mathbf{M}$ is represented by matrix M, but it is nontrivial to directly show the equality of the right-hand sides of (28) and (26). It is mentioned in [32] that D. R. Fulkerson noticed the matroidal structure of the result of Iri, which can be derived from [13].

3.3 The principal partition of matroids by Bruno and Weinberg, Tomizawa, and Narayanan

Bruno and Weinberg [5] also noticed the matroidal structure of the result of Kishi and Kajitani. With any positive integer $k \geq 2$ as a parameter they considered the union of k copies of a given matroid. This leads us to the following min-max relation with integer parameter $k \geq 2$, known for unions of matroids (see [13, 75]).

Theorem 10. *For a matroid* $\mathbf{M}$ *on* E *with the base family* $\mathcal{B}$ *and the rank function* ρ*,*

$$\max\{\Big|\bigcup_{i=1}^{k} B_i\Big| \mid B_i \in \mathcal{B}\} = \min\{k\rho(X) + |E \setminus X| \mid X \subseteq E\}. \tag{29}$$

□

For each positive integer k we have the distributive lattice $\mathcal{D}_k$ of the minimizers of a submodular function $f_k(X) = k\rho(X) + |E \setminus X|$ appearing in the right-hand side of (29). Denote by E_k^- and E_k^+ the minimum and the maximum element of $\mathcal{D}_k$, respectively. It follows from Theorem 2 that we have a partition of the underlying set E as in (4) and a poset structure on the partition of $E_k^+ \setminus E_k^-$ for each integer $k \geq 2$. Suppose that the collection of distinct $\mathcal{D}_k$ is given by $\mathcal{D}_{k_i}$ $(i = 1, \cdots, l)$ with $k_1 < \cdots < k_l$.

Then we have the following theorem, which will be shown for more general setting later.

Theorem 11.

$$E_{k_1}^+ \supseteq E_{k_1}^- \supseteq E_{k_2}^+ \supseteq E_{k_2}^- \supseteq \cdots \supseteq E_{k_l}^+ \supseteq E_{k_l}^-. \tag{30}$$

□

Remark 6. For each $i = 1, \cdots, l$ we have a partition of the difference set $E_{k_i}^+ \setminus E_{k_i}^-$ and a poset on it determined by the distributive lattice $\mathcal{D}_{k_i}$. □

Remark 7. If the difference set $E_{k_i}^+ \setminus E_{k_i}^-$ is nonempty, the minor of matroid $\mathbf{M}$ on $E_{k_i}^+ \setminus E_{k_i}^-$ with rank function $\rho_{E_{k_i}^-}^{E_{k_i}^+}$ has disjoint k_i bases that partition $E_{k_i}^+ \setminus E_{k_i}^-$. □

Tomizawa [80] and Narayanan [67] independently generalized the decomposition scheme of Bruno and Weinberg by considering rational numbers instead of integers k. For a positive rational $\frac{l}{k}$ for positive integers l and k they find a minor that has k bases that uniformly cover each element of the underlying set l times. The Bruno-Weinberg decomposition corresponds to the case when $l = 1$.

The min-max theorem associated with the Tomizawa-Narayanan decomposition is given parametrically as follows. This will also be proved in a more general setting later.

Theorem 12. *For any positive integers k and l,*

$$\max\{\sum_{i=1}^{k}|I_i| \mid I_i \in \mathcal{I}\ (i=1,\cdots,k),\ \forall e \in E : |\{i \mid i \in \{1,\cdots,k\},\ e \in I_i\}| \leq l\}$$
$$= \min\{k\rho(X) + l|E \setminus X| \mid X \subseteq E\}, \tag{31}$$

where $\mathcal{I}$ is the family of the independent sets of matroid $\mathbf{M}$. □

Note that when $l = 1$, Theorem 12 is reduced to Theorem 10.

For a nonnegative rational number $\lambda = \frac{l}{k}$ let $\mathcal{D}_\lambda$ be the distributive lattice formed by the minimizers of the submodular function $f_\lambda(X) = k\rho(X) + l|E \setminus X|$.

We call the value λ *critical* if $\mathcal{D}_\lambda$ contains more than one element. Because of the finiteness character we have a finite set of critical vales, which are supposed to be given by $0 \leq \lambda_1 < \cdots < \lambda_p$. For each $i = 1, \cdots, p$ let $E^-_{\lambda_i}$ and $E^+_{\lambda_i}$ be the minimum and the maximum element of $\mathcal{D}_{\lambda_i}$, respectively.

Theorem 13.

$$E^-_{\lambda_1} \subset E^+_{\lambda_1} = E^-_{\lambda_2} \subset E^+_{\lambda_2} = E^-_{\lambda_3} \subset \cdots \subset E^+_{\lambda_{l-1}} = E^-_{\lambda_l} \subset E^+_{\lambda_l}. \tag{32}$$

□

For each nonempty difference set $E^+_i \setminus E^-_i$ we have a partition of it with a partial order associated with the distributive lattice $\mathcal{D}_{\lambda_i}$. Also note that the union of $\mathcal{D}_{\lambda_i}$ $(i = 1, \cdots, p)$ as a whole is again a distributive lattice, which determines the decomposition of matroid $\mathbf{M}$ and a poset structure on it. Each minor $\mathbf{M}^{E^+_i}_{E^-_i}$ of $\mathbf{M}$ on $E^+_i \setminus E^-_i$, the restriction of $\mathbf{M}$ to E^+_i followed by the contraction by E^-_i, with critical value $\lambda = l/k$ has k bases of the minor that cover uniformly l times every element of $E^+_i \setminus E^-_i$. The decomposition given above is the finest one that has such a property. This is the principal partition of matroid $\mathbf{M}$ in the sense of Tomizawa and Narayanan.

3.4 A polymatroidal approach to the principal partition of Tomizawa and Narayanan: a lexicographically optimal base

The author [18, 19] noticed that Tomizawa and Narayanan's principal partition was polymatroidal. Readers will see that a polymatroidal approach to the principal partition is quite natural and easy to understand. Also this can easily be extended to general submodular systems.

Let $\mathbf{P} = (E, \rho)$ be a polymatroid with a rank function $\rho : 2^E \to \mathbf{R}_+$ and let $w : E \to \mathbf{R}$ be a positive weight vector on E. Then we have the following min-max relation for polymatroids [11].

Theorem 14 (Edmonds). *For any real parameter* λ,

$$\max\{x(E) \mid x \in \mathrm{P}(\rho),\ x \leq \lambda w\} = \min\{\rho(X) + \lambda w(E \setminus X) \mid X \subseteq E\}, \quad (33)$$

where $\mathrm{P}(\rho)$ *is the submodular polyhedron associated with the rank function* ρ. □

It should be noted that when $\lambda \geq 0$, $\mathrm{P}(\rho)$ in (33) can be replaced by the polymatroid polyhedron $\mathrm{P}(\rho) \cap \mathbf{R}_+^E$ and that when $\lambda < 0$, the right-hand side of (33) has the unique minimizer $X = \emptyset$. Relation (33) in the form given above can more naturally be extended to submodular systems.

Lemma 2. *For any reals* λ_1 *and* λ_2 *with* $\lambda_1 < \lambda_2$ *there exist a maximizer* $x{=}b_1$ *of the left-hand side of* (33) *for* $\lambda{=}\lambda_1$ *and a maximizer* $x{=}b_2$ *for* $\lambda{=}\lambda_2$ *such that* $b_1 \leq b_2$.

(Proof) Let b_1 be any maximizer for $\lambda = \lambda_1$. Since $\{x \mid x \in \mathrm{P}(\rho),\ x \leq \lambda_2 w\}$ is a submodular polyhedron (the vector reduction of $\mathrm{P}(\rho)$ by $\lambda_2 w$) and b_1 belongs to it, there exists a base b_2 of the reduction such that $b_1 \leq b_2$. Here, b_2 is a maximizer of the left-hand side of (33) for $\lambda = \lambda_2$. □

Because of this fact the following was observed in [18].

Theorem 15. *For any given positive weight vector* w *there uniquely exists a base* b^* *of polymatroid* (E, ρ) *such that* $b^* \wedge \lambda w$ *is a maximizer of the left-hand side of* (33) *for each* λ, *where* $b^* \wedge \lambda w = (\min\{b^*(e), \lambda w(e)\} \mid e \in E)$. □

The base b^* appearing in Theorem 15 is called the *universal base* for polymatroid (E, ρ) with weight vector w.

Remark 8. The universal base b^* can be defined geometrically as follows. We start with $b = \lambda w$ for a sufficiently small λ such that b lies in the interior of $\mathrm{P}(\rho)$ (we can take any negative λ in the present case of polymatroid rank function ρ). Then increase λ until we reach the boundary of $\mathrm{P}(\rho)$. Let $b_1 = \lambda_1 w$ be the boundary point of $\mathrm{P}(\rho)$. Put S_1 as the maximum minimizer of the submodular function $\rho(X) - b_1(X)$ (note that $S_1 = \mathrm{sat}(b_1)$). Now fix the components $b(e)$ as $b_1(e)$ for $e \in S_1$ and increase the other components $b(e)$ $(e \in E \setminus S_1)$ in proportion to $w(e)$ until we cannot increase them without leaving $\mathrm{P}(\rho)$. Let b_2 be the new boundary point of $\mathrm{P}(\rho)$, find the maximum minimizer $S_2 (= \mathrm{sat}(b_2))$ of the submodular function $\rho(X) - b_2(X)$, and fix the components $b(e)$ as $b_2(e)$ for $e \in S_2$, where note that we have $S_1 \subset S_2$. Repeat this process until all the components of b are fixed. The finally obtained base b is the universal base b^*. □

In the same way as in the principal partition of Tomizawa and Narayanan we call the value λ *critical* if $\mathcal{D}_\lambda$ contains more than one element. We have a finite set of critical values $0 \leq \lambda_1 < \cdots < \lambda_p$. For each $i = 1, \cdots, p$ let E_i^- and E_i^+ be the minimum and the maximum element of $\mathcal{D}_{\lambda_i}$, respectively.

Theorem 16.

$$E^-_{\lambda_1} \subset E^+_{\lambda_1} = E^-_{\lambda_2} \subset E^+_{\lambda_2} = E^-_{\lambda_3} \subset \cdots \subset E^+_{\lambda_{p-1}} = E^-_{\lambda_p} \subset E^+_{\lambda_p}. \qquad (34)$$

(Proof) For the universal base b^* let the distinct values of $b^*(e)/w(e)$ $(e \in E)$ be given by $\beta_1 < \cdots < \beta_q$ and define

$$S_i = \{e \mid e \in E,\ b(e)/w(e) \le \beta_i\} \qquad (i = 1, \cdots, q). \qquad (35)$$

Then we can show that $q = p$, $S_i = E^+_{\lambda_i}$ $(i = 1, \cdots, p)$, and $S_i = E^-_{\lambda_{i+1}}$ $(i = 0, \cdots, p-1)$ where $S_0 = \emptyset$. □

For any base $b \in \mathrm{B}(\rho)$ let the distinct values of $b(e)/w(e)$ $(e \in E)$ be given by

$$\lambda_1 < \cdots < \lambda_p, \qquad (36)$$

and define

$$S_i = \{e \mid e \in E,\ b(e)/w(e) \le \lambda_i\} \qquad (37)$$

for each $i = 1, \cdots, p$.

Then we have

Theorem 17. *A base $b \in \mathrm{B}(\rho)$ is the universal base of (E, ρ) for weight vector w if and only if the sets S_i $(i = 1, \cdots, p)$ defined by* (36) *and* (37) *are tight sets of b, i.e.,*

$$\rho(S_i) = b(S_i) \qquad (i = 1, \cdots, p). \qquad (38)$$

□

Note that $\lambda_i w(E^+_i \setminus E^-_i) = \rho(E^+_i) - \rho(E^-_i)$ $(i = 1, \cdots, q)$. Hence the critical values for the principal partition of Tomizawa and Narayanan are rational, where $w(X) = |X|$ $(X \subseteq E)$ and ρ is a matroid rank function.

The universal base b^* can be characterized as a lexicographically optimal base of polymatroid (E, ρ) with weight vector w and as a base that minimizes a separable convex function. Both were discussed in [18].

Given a positive weight vector $w \in \mathbf{R}^E$, for any vector $x \in \mathbf{R}^E$ define a sequence of ratios $x(e)/w(e)$ $(e \in E)$

$$T_w(x) = (x(e_1)/w(e_1), \cdots, x(e_m)/w(e_m)) \qquad (39)$$

such that

$$x(e_1)/w(e_1) \le \cdots \le x(e_m)/w(e_m), \qquad (40)$$

where $E = \{e_1, \cdots, e_m\}$. A base $b \in \mathrm{B}(\rho)$ is called a *lexicographically optimal base* with respect to the weight vector w if it lexicographically maximizes $T_w(x)$ among all the bases $x \in \mathrm{B}(\rho)$. We can easily see that a lexicographically optimal base with respect to the weight vector w uniquely exists.

Theorem 18. *The lexicographically optimal base with respect to the weight vector w coincides with the universal base b^* for the same w.*

(Proof) We can show that a base $\hat{b} \in \mathrm{B}(\rho)$ is the lexicographically optimal base with respect to the weight vector w if and only if for all $e, e' \in E$ such that $\hat{b}(e)/w(e) < \hat{b}(e')/w(e')$ we have $e' \notin \mathrm{dep}(\hat{b}, e)$. (Recall (22) and (23).) The latter condition is equivalent to (36) and (37). □

We also have

Theorem 19. *Let $x = \hat{b}$ be an optimal solution of the following problem.*

$$\text{Minimize} \sum_{e \in E} \frac{x^2(e)}{w(e)} \quad \text{subject to } x \in \mathrm{B}(\rho). \tag{41}$$

Then $\hat{b}$ is the universal base b^* for w.

(Proof) We can also show that a base $\hat{b}$ is an optimal solution of (41) if and only if for all $e, e' \in E$ such that $\hat{b}(e)/w(e) < \hat{b}(e')/w(e')$ we have $e' \notin \mathrm{dep}(\hat{b}, e)$. □

Fujishige [18] gave an $\mathrm{O}(|E|\mathrm{SFM})$ algorithm for finding a lexicographically optimal base with respect to weight w, where SFM denotes the complexity of submodular function minimization (see [39, 49] for submodular function minimization). When specialized to multi-terminal flows, this improved Megiddo's algorithms for lexicographically optimal multi-terminal flows [50, 51]. Also, Gallo, Grigoriadis, and Tarjan [26] devised a faster algorithm for finding a lexicographically optimal multi-terminal flow with weights, which requires running time of a single max-flow computation. More general separable convex function minimization problems over polymatroids and their incremental algorithms were considered by Federgruen and Groenevelt [15, 28]. An O(SFM) algorithm for finding a lexicographically optimal base with weights has been obtained by Fleischer and Iwata [16] (also see related recent algorithms by Nagano [63, 64]).

The results in this subsection do not depend on the monotonicity of the rank function ρ, so that we can easily extend the results to those for general submodular systems with positive weight vectors. (Just replace the polymatroid rank function ρ with the rank function f of any submodular system. For details see [20].)

Getting rid of the monotonicity assumption on the rank function is very important and extends the applicability of the theory of principal partitions.

Remark 9. The concept of a lexicographically optimal base of a polymatroid was rediscovered in convex games by Dutta and Ray [8, 9] (also see [29, 30]), where the lexicographically optimal base is called the *egalitarian solution* of a convex game. Note that the core of a convex game is the same as the base polyhedron of a polymatroid ([78]). □

Remark 10. Consider a submodular system $(\mathcal{D}, f)$ on E and a positive weight vector w. If we are given the universal base b^* (or the lexicographically optimal base) with respect to weight w, $b^* \wedge \mathbf{0} = (\min\{b^*(e), 0\} \mid e \in E)$ is a maximizer of

$$\max\{x(E) \mid x \in \mathrm{P}(f),\ x \le \mathbf{0}\} = \min\{f(X) \mid X \in \mathcal{D}\} \tag{42}$$

due to a generalized version of Theorem 14. Moreover, $A^- = \{e \mid e \in E,\ b^*(e) < 0\}$ and $A^0 = \{e \mid e \in E,\ b^*(e) \le 0\}$ are, respectively, the unique minimal minimizer and the unique maximal minimizer of f, which minimize the right-hand side of (42). Hence we can minimize a given submodular function by solving the minimum-norm-point problem (41). Here we may choose a uniform weight vector w with $w(X) = |X|$ for all $X \subseteq E$ to get the Euclidean norm. Polynomial algorithms for submodular function minimization have been developed so far [37, 38, 40, 70, 76] (also see [39, 49]), but it seems to be worth investigating to apply the minimum-norm-point algorithm of Wolfe [84] to submodular function minimization (see [21]). □

3.5 The principal partition of a pair of polymatroids of Iri and Nakamura

Let (E, ρ_i) $(i = 1, 2)$ be two polymatroids. Then we have the following min-max theorem parametrically.

Theorem 20 (Edmonds). *For any $\lambda \ge 0$ we have*

$$\max\{x(E) \mid x \in \mathrm{P}(\rho_1) \cap \mathrm{P}(\lambda\rho_2)\} = \min\{\rho_1(X) + \lambda\rho_2(E \setminus X) \mid X \subseteq E\}. \tag{43}$$

□

For the sake of simplicity we suppose that ρ_2 is strictly monotone increasing, i.e., all the extreme bases of (E, ρ_2) are positive vectors (or $\mathrm{B}(\rho_2)$ is included in the interior of the nonnegative orthant $\mathbf{R}^E_+$).

Iri and Nakamura [33, 35, 65, 66] developed the principal partition of a pair of polymatroids, based on Theorem 20. Define $\mathcal{D}(\rho_1, \lambda\rho_2)$ as the collection of minimizers of the submodular function $\rho_1(X) + \lambda\rho_2(E \setminus X)$ in X. Let E^-_λ and E^+_λ be, respectively, the minimum and the maximum element of the distributive lattice $\mathcal{D}(\rho_1, \lambda\rho_2)$ for all $\lambda \ge 0$. We call λ a *critical value* if $\mathcal{D}(\rho_1, \lambda\rho_2)$ contains more than one element. It should be noted that when ρ_2 is a modular function represented by a positive vector $w \in \mathbf{R}^E$, Theorem 20 reduces to Theorem 14.

Theorem 21 (Iri, Nakamura). *For two critical values λ and λ' with $\lambda < \lambda'$ we have*

$$E^-_\lambda \subset E^+_\lambda \subseteq E^-_{\lambda'} \subset E^+_{\lambda'}. \tag{44}$$

Moreover, for any $X \in \mathcal{D}(\rho_1, \lambda\rho_2)$ and $X' \in \mathcal{D}(\rho_1, \lambda'\rho_2)$ we have

$$X \cap X' \in \mathcal{D}(\rho_1, \lambda\rho_2), \qquad X \cup X' \in \mathcal{D}(\rho_1, \lambda'\rho_2). \tag{45}$$

(Proof) For any $\lambda < \lambda'$ (not necessarily critical values) and for any $X \in \mathcal{D}(\rho_1, \lambda\rho_2)$ and $X' \in \mathcal{D}(\rho_1, \lambda'\rho_2)$ we have

$$\begin{aligned}
&\rho_1(X') + \lambda'\rho_2(E \setminus X') + \rho_1(X) + \lambda\rho_2(E \setminus X) \\
&\geq \rho_1(X \cup X') + \lambda'\rho_2(E \setminus (X \cup X')) + \rho_1(X \cap X') + \lambda\rho_2(E \setminus (X \cap X')) \\
&\quad + (\lambda' - \lambda)(\rho_2(E \setminus (X \cap X')) - \rho_2(E \setminus X)) \\
&\geq \rho_1(X \cup X') + \lambda'\rho_2(E \setminus (X \cup X')) + \rho_1(X \cap X') + \lambda\rho_2(E \setminus (X \cap X')). \quad (46)
\end{aligned}$$

This implies (45) and hence

$$E_\lambda^- \subseteq E_{\lambda'}^-, \qquad E_\lambda^+ \subseteq E_{\lambda'}^+. \tag{47}$$

When λ is a critical value, for a sufficiently small $\epsilon > 0$ $\mathcal{D}(\rho_1, (\lambda + \epsilon)\rho_2)$ contains only one element E_λ^+ since $\rho_2(X) < \rho_2(Y)$ for all $X \subset Y \subseteq E$ by the assumption that $\mathrm{B}(\rho_2)$ lies in the interior of $\mathbf{R}_+^E$. This together with (47) implies (44). □

Note that (45) and (47) hold without the assumption that $\mathrm{B}(\rho_2)$ lies in the interior of the nonnegative orthant $\mathbf{R}_+^E$.

It follows from Theorem 21 that there exist a finite number of critical values $\lambda_1 < \cdots < \lambda_p$ and that

$$\bigcup_{i=1}^{p} \mathcal{D}(\rho_1, \lambda_i\rho_2) \tag{48}$$

forms a distributive lattice, which leads us to a decomposition of the pair of polymatroids (ρ_1, ρ_2) as follows ([33, 35, 65, 66, 81]).

The whole distributive lattice (48) yields a chain

$$E_{\lambda_1}^- \subset E_{\lambda_1}^+ = E_{\lambda_2}^- \subset \cdots \subset E_{\lambda_{p-1}}^+ = E_{\lambda_p}^- \subset E_{\lambda_p}^+. \tag{49}$$

Then polymatroids $\mathbf{P}_i = (E, \rho_i)$ $(i = 1, 2)$ are decomposed into

$$\begin{aligned}
&\mathbf{P}_1 \cdot E_{\lambda_1^-}, && \mathbf{P}_2/\overline{E}_{\lambda_1^-} && (50)\\
&\mathbf{P}_1 \cdot E_{\lambda_1^+}/E_{\lambda_1^-}, && \mathbf{P}_2 \cdot \overline{E}_{\lambda_1^-}/\overline{E}_{\lambda_1^+} && (51)\\
& && \vdots && \\
&\mathbf{P}_1 \cdot E_{\lambda_p^+}/E_{\lambda_p^-}, && \mathbf{P}_2 \cdot \overline{E}_{\lambda_p^-}/\overline{E}_{\lambda_p^+} && (52)\\
&\mathbf{P}_1/E_{\lambda_p^+}, && \mathbf{P}_2 \cdot \overline{E}_{\lambda_p^+}, && (53)
\end{aligned}$$

where for any $X \subseteq E$ we denote by $\overline{X}$ its complement $E \setminus X$, by $\mathbf{P}_i \cdot X$ the restriction of $\mathbf{P}_i$ to X, and by $\mathbf{P}_i/X$ the contraction of $\mathbf{P}_i$ by X. For any $\lambda > 0$ we denote $\lambda\mathbf{P}_i = (E, \lambda\rho_i)$.

Theorem 22 (Iri, Nakamura). *The minors of polymatroids* $\mathbf{P}_i = (E, \rho_i)$ $(i = 1, 2)$ *in* (50)–(53) *are uniquely determined, independently of the choice of a maximal chain* (49). *Moreover,* (*a*) *The pair of* $\mathbf{P}_1 \cdot E_{\lambda_1^-}$ *and* $\lambda_1 \mathbf{P}_2/\overline{E}_{\lambda_1^-}$ *has a maximum common subbase* $b^{(0)}$ *that is a base of* $\mathbf{P}_1 \cdot E_{\lambda_1^-}$.

(*b*) *For each* $i = 1, \cdots, p$ *the pair of* $\mathbf{P}_1 \cdot E_{\lambda_i^+}/E_{\lambda_i^-}$ *and* $\lambda_i \mathbf{P}_2 \cdot \overline{E}_{\lambda_i^-}/\overline{E}_{\lambda_i^+}$ *has a common base* $b^{(i)}$.

(*c*) *The pair of* $\mathbf{P}_1/E_{\lambda_p^+}$ *and* $\lambda_p \mathbf{P}_2 \cdot \overline{E}_{\lambda_p^+}$ *has a maximum common subbase* $b^{(p+1)}$ *that is a base of* $\lambda_p \mathbf{P}_2 \cdot \overline{E}_{\lambda_p^+}$.

□

Define $b_1 = b^{(0)} \oplus b^{(1)} \oplus \cdots \oplus b^{(p)} \oplus b^{(p+1)}$ and $b_2 = (1/\lambda_1) b^{(0)} \oplus (1/\lambda_1) b^{(1)} \oplus \cdots \oplus (1/\lambda_p) b^p \oplus (1/\lambda_p) b^{(p+1)}$. Choose any bases $\hat{b}_i \in \mathrm{B}(\rho_i)$ such that $\hat{b}_i \geq b_i$ $(i = 1, 2)$, $\hat{b}_1^{E^-_{\lambda_1}}$ is a base of $\mathbf{P}_1 \cdot E^-_{\lambda_1}$, and $\hat{b}_2^{\overline{E^+_{\lambda_p}}}$ is a base of $\mathbf{P}_2 \cdot \overline{E^+_{\lambda_p}}$, where for any vector $x \in \mathbf{R}^E$ and any set $A \subseteq E$ define a vector x^A in $\mathbf{R}^A$ as $x^A(e) = x(e)$ $(e \in A)$. Then for any $\lambda \geq 0$ $\hat{b}_1 \wedge \lambda \hat{b}_2$ is a maximum common base of $\mathbf{P}_1$ and $\lambda \mathbf{P}_2$. (Note that $\hat{b}_1(e) = b_1(e)$ for $(e \in E \setminus E^-_{\lambda_1})$ and $\hat{b}_2(e) = b_2(e)$ for $(e \in E^+_{\lambda_p})$.) Hence,

Theorem 23 (Nakamura). *There exist a base* b_1 *of* $\mathbf{P}_1$ *and a base* b_2 *of* $\mathbf{P}_2$ *such that for any* $\lambda \geq 0$ $b_1 \wedge \lambda b_2$ *is a maximum common base of* $\mathbf{P}_1$ *and* $\lambda \mathbf{P}_2$. □

This generalizes Theorem 15. The pair (b_1, b_2) is called a *universal pair of bases*, where note that such a pair is not necessarily unique (also see [54]).

It is not difficult to generalize the principal partition of a pair of polymatroids to that of a submodular system and a polymatroid. The range of parameter λ can also be extended to negative values by defining $\lambda\rho$ for $\lambda < 0$ by

$$\lambda\rho(X) = \lambda\rho^{\#}(X) \qquad (X \subseteq E) \tag{54}$$

(see [20, 81]).

We shall discuss a further generalization later in Section 4.

3.6 The principal structure of a submodular system

A related decomposition slightly different from principal partitions was considered in [19].

Let $(\mathcal{D}, f)$ be any submodular system on E. Then for any $e \in E$ define

$$\mathcal{D}_f(e) = \{X \mid e \in X \in \mathcal{D},\ f(X) = \min\{f(Y) \mid e \in Y \in \mathcal{D}\}\}. \tag{55}$$

Note that $\mathcal{D}_f(e)$ is a distributive lattice with set union and intersection as the lattice operations. Denote by $D_f(e)$ the minimum element of $\mathcal{D}_f(e)$.

Now we have the following.

Theorem 24. *For any* $e_1, e_2 \in E$ *such that* $e_2 \in D_f(e_1)$ *we have*

$$D_f(e_2) \subseteq D_f(e_1). \tag{56}$$

(Proof) Putting $F_i = D_f(e_i)$ for $i = 1, 2$, we have

$$f(F_1) \leq f(F_1 \cup F_2), \tag{57}$$

since $e_1 \in F_1 \cup F_2$. It follows from (57) and the submodularity of f that

$$\begin{aligned} f(F_2) &\geq f(F_1 \cap F_2) + f(F_1 \cup F_2) - f(F_1) \\ &\geq f(F_1 \cap F_2). \end{aligned} \tag{58}$$

This implies (56) since $e_2 \in F_1 \cap F_2$, and hence $F_2 \subseteq F_1 \cap F_2$. □

Let $\mathcal{F}$ be the collection of $D_f(e)$ $(e \in E)$. Then we see from this theorem that for any $F_1, F_2 \in \mathcal{F}$ we have $F_1 \cap F_2 = \bigcup_{e \in F_1 \cap F_2} D_f(e)$.

We can define a transitive binary relation $\rightarrow$ on E by

$$e_1 \rightarrow e_2 \quad \Longleftrightarrow \quad e_2 \in D_f(e_1). \tag{59}$$

The transitive binary relation $\rightarrow$ on E naturally defines a directed graph G_f with a vertex set E whose strongly connected components are complete directed graphs with selfloops at every vertex. Decomposing G_f into strongly connected components, we obtain a decomposition with a poset structure on it, which is called the *principal structure* of the submodular system $(\mathcal{D}, f)$.

Remark 11. For a submodular system $\mathbf{S} = (\mathcal{D}, f)$ on E the principal structure of submodular system $\mathbf{S}$ furnishes a further decomposition of $E \setminus D_f^{\max}$, where $D_f^{\max}$ is the maximum element of the set of minimizers of f. □

Remark 12. The concepts of principal structure and principal partition have been effectively applied to systems analysis and examined in details in matric and matroidal frameworks in [36, 41, 42, 53, 55, 61] (see Murota's book [57]). □

4 Extensions

In the principal partitions viewed in Section 3 we have considered submodular functions with a parameter that appears linearly as follows. Vector $\mathbf{1}$ denotes the vector of all ones.

- $\rho(X) + \lambda w(E \setminus X) \quad (X \subseteq E)$,
  $\rho = r_G$, $w = \mathbf{1}$, $\lambda = \frac{1}{2}$ (Kishi and Kajitani)
  ρ: a matroid rank function, $w = \mathbf{1}$, $\lambda \geq 0$ (Tomizawa and Narayanan)
  ρ: a polymatroid rank function, a positive weight w, $\lambda \geq 0$ (Fujishige)
  extension to submodular systems, a positive weight w, $\lambda \in \mathbf{R}$ (Fujishige)

- $\rho_1(X) + \lambda\rho_2(E \setminus X) \quad (X \subseteq E)$,
  ρ_1, ρ_2: polymatroid rank functions, $\lambda \geq 0$ (Iri and Nakamura)
  extension to submodular systems, $\lambda \in \mathbf{R}$ (Fujishige and Tomizawa)

We shall examine how the linear form in the parameter can be extended to a nonlinear form in Section 4.1. We also examine possible extension of the domain 2^E or $\mathcal{D}$ to the integer lattice $\mathbf{Z}^E$ in Section 4.2.

4.1 Parameters nonlinearly

The result of this section is based on joint work with Nagano [24] (also see [64]).

In the principal partition with a parameter λ described in Section 3 a kind of monotonicity of λw and $\lambda\rho_2$ plays a crucial rôle. The essence of the monotonicity is the strong map relation of submodular systems.

Consider two submodular systems $\mathbf{S}_i = (\mathcal{D}_i, f_i)$ $(i = 1, 2)$ on E. The ordered pair $(\mathbf{S}_1, \mathbf{S}_2)$ is called a *strong map* if for all $X \in \mathcal{D}_1$ and $Y \in \mathcal{D}_2$ such that $X \subseteq Y$ we have

$$f_1(Y) - f_1(X) \geq f_2(Y) - f_2(X), \tag{60}$$

where if $X \notin \mathcal{D}_2$ or $Y \notin \mathcal{D}_1$, we understand that (60) holds. Following the convention, we write $f_1 \to f_2$ if $(\mathbf{S}_1, \mathbf{S}_2)$ is a strong map. For two supermodular functions g_1 and g_2 we write $g_1 \to g_2$ if we have a strong map relation $g_2^{\#} \to g_1^{\#}$, where recall that $g_i^{\#}$ is the dual submodular function of g_i.

The strong map relation is the monotonicity that we need to extend the principal partition having a parameter linearly.

Consider parameterized submodular systems $(\mathcal{D}, f_\lambda)$ $(\lambda \in \mathbf{R})$ and supermodular systems $(\mathcal{D}, g_\lambda)$ $(\lambda \in \mathbf{R})$ such that for all λ and λ' with $\lambda < \lambda'$

$$f_\lambda \to f_{\lambda'}, \qquad g_\lambda \to g_{\lambda'}. \tag{61}$$

We assume that for each $X \in \mathcal{D}$ the values of $f_\lambda(X)$ and $g_\lambda(X)$ are continuous in $\lambda \in \mathbf{R}$.

Now we have the following min-max theorem due to Edmonds. For any $x \in \mathbf{R}^E$ define $x^- = (\min\{x(e), 0\} \mid e \in E)$.

Theorem 25.

$$\max\{(x - y)^-(E) \mid x \in \mathrm{B}(f_\lambda),\ y \in \mathrm{B}(g_\lambda)\} = \min\{f_\lambda(X) - g_\lambda(X) \mid X \in \mathcal{D}\}. \tag{62}$$

□

Define a parameterized submodular function $h_\lambda(X)$ in $X \in \mathcal{D}$ as

$$h_\lambda(X) = f_\lambda(X) - g_\lambda(X) \qquad (X \in \mathcal{D}). \tag{63}$$

It should be noted that for any λ and λ' such that $\lambda < \lambda'$ we have a strong map relation

$$h_\lambda \rightarrow h_{\lambda'}. \tag{64}$$

For any λ let $\mathcal{D}(h_\lambda)$ be the set of minimizers of h_λ.

Theorem 26. *For any λ and λ' such that $\lambda < \lambda'$ and for any $X \in \mathcal{D}(h_\lambda)$ and $Y \in \mathcal{D}(h_{\lambda'})$ we have*

$$X \cap Y \in \mathcal{D}(h_\lambda), \qquad X \cup Y \in \mathcal{D}(h_{\lambda'}). \tag{65}$$

(Proof) Under the assumption of the present theorem,

$$\begin{aligned} h_{\lambda'}(X) + h_\lambda(Y) &= h_{\lambda'}(X) + h_{\lambda'}(Y) - h_{\lambda'}(Y) + h_\lambda(Y) \\ &\geq h_{\lambda'}(X \cup Y) + h_{\lambda'}(X \cap Y) - h_{\lambda'}(Y) + h_\lambda(Y) \\ &= h_{\lambda'}(X \cup Y) + h_\lambda(X \cap Y) \\ &\quad + h_\lambda(Y) - h_\lambda(X \cap Y) - h_{\lambda'}(Y) + h_{\lambda'}(X \cap Y) \\ &\geq h_{\lambda'}(X \cup Y) + h_\lambda(X \cap Y). \end{aligned} \tag{66}$$

Hence we have (65). □

It follows that the union of distributive lattices $\mathcal{D}(h_\lambda)$ ($\lambda \in \mathbf{R}$) is again a distributive lattice, denoted by $\mathcal{D}(h)$. For each $\lambda \in \mathbf{R}$ denote the maximum and the minimum element of $\mathcal{D}(h_\lambda)$ by S_λ^+ and S_λ^-, respectively. From Theorem 26 we have

Theorem 27. *For any λ and λ' such that $\lambda < \lambda'$,*

$$S_\lambda^- \subseteq S_{\lambda'}^-, \qquad S_\lambda^+ \subseteq S_{\lambda'}^+. \tag{67}$$

□

Hence there exist finitely many distinct S_λ^+ ($\lambda \in \mathbf{R}$), which are given by

$$S_0 \subset S_1 \subset \cdots \subset S_p. \tag{68}$$

Because of the finiteness character and the continuity of $h_\lambda(X)$ in λ, for each λ we have $\mathcal{D}(h_\lambda) \supseteq \mathcal{D}(h_{\lambda+\epsilon})$ for a sufficiently small $\epsilon > 0$. Hence, from Theorem 27, $\mathbf{R}$ is divided into the intervals

$$\Lambda_0 = (-\infty, \lambda_1), \quad \Lambda_1 = [\lambda_1, \lambda_2), \quad \cdots, \quad \Lambda_p = [\lambda_p, +\infty) \tag{69}$$

such that for any $i = 0, 1, \cdots, p$ and any $\lambda \in \Lambda_i$ we have $S_\lambda^+ = S_i$. We call λ_i ($i = 1, \cdots, p$) *upper critical values.*

For simplicity we assume that

$$S_0 = \emptyset, \qquad S_p = E. \tag{70}$$

Lemma 3. *For any* $i = 2, \cdots, p$ *we have* $S^+_{\lambda_{i-1}} \in \mathcal{D}(h_{\lambda_i})$ *and* $\emptyset \in \mathcal{D}(h_{\lambda_1})$.

(Proof) It follows from Theorems 26 and 27 that for any $\lambda \in \mathbf{R}$ we have

$$S^-_{\lambda-\epsilon} = S^-_\lambda, \qquad S^+_{\lambda+\epsilon} = S^+_\lambda \tag{71}$$

for a sufficiently small $\epsilon > 0$. That is to say, S^-_λ is left-continuous in λ and S^+_λ is right-continuous in λ.

For any $i = 2, \cdots, p$ and a sufficiently small $\epsilon > 0$ we have $S^-_{\lambda_i} \in \mathcal{D}(h_{\lambda_i - \epsilon})$ and $S^+_{\lambda_i-\epsilon} = S^+_{\lambda_{i-1}}$. Hence,

$$S^-_{\lambda_i} \subseteq S^+_{\lambda_{i-1}}. \tag{72}$$

It follows from Theorem 26 and (72) that $S^+_{\lambda_{i-1}} = S^+_{\lambda_{i-1}} \cup S^-_{\lambda_i} \in \mathcal{D}(h_{\lambda_i})$.

Similarly we can show $(S_0 =)\ \emptyset \in \mathcal{D}(h_{\lambda_1})$. □

For each λ let $\mathbf{S}_\lambda$ be the submodular system $(\mathcal{D}, h_\lambda)$ on E and for each $i = 1, \cdots, p$ consider minors $\mathbf{S}_{\lambda_i} \cdot S_i / S_{i-1}$. Note that for each $i = 1, \cdots, p$

$$\mathbf{S}_{\lambda_i} \cdot S_i/S_{i-1} = (\mathcal{D}^{S_i}_{S_{i-1}}, h_{\lambda_i}{}^{S_i}_{S_{i-1}}) \tag{73}$$

is a submodular system on $S_i \setminus S_{i-1}$ with rank function $h_{\lambda_i}{}^{S_i}_{S_{i-1}}$.

We use $\mathbf{0}$ to denote a zero vector of appropriate dimension. Its dimension is determined by the context.

Lemma 4. *For each* $i = 1, \cdots, p$ *we have* $\mathbf{0} \in \mathrm{B}(h_{\lambda_i}{}^{S_i}_{S_{i-1}})$.

(Proof) We see from Lemma 3 that $S_{i-1} \subset S_i$ is a chain of $\mathcal{D}(h_{\lambda_i})$ for each $i = 1, \cdots, p$. Hence $h_{\lambda_i}{}^{S_i}_{S_{i-1}}$ is nonnegative and $h_{\lambda_i}{}^{S_i}_{S_{i-1}}(S_i \setminus S_{i-1}) = 0$, which shows the present lemma. □

Now we assume that $\mathcal{D}$ is simple, i.e., $\mathcal{D}$ is the collection of (lower) order-ideals of a poset $\mathcal{P} = (E, \preceq)$ on E. Let $G(\mathcal{P})$ be the graph representing the Hasse diagram of poset $\mathcal{P}$. Recall that for any $x \in \mathbf{R}^E$ and $F \subseteq E$ we denote $x^F = (x(e) \mid e \in F)$.

Then,

Theorem 28. *There exist at most* $|E|$ *linear extensions of poset* $\mathcal{P}$ *identified with linear orderings* σ_i $(i \in I)$ *of* E*, a nonnegative flow* φ *in* $G(\mathcal{P})$*, and coefficients* $\mu_i > 0$ $(i \in I)$ *with* $\sum_{i \in I} \mu_i = 1$ *such that for all* $\lambda \in \mathbf{R}$*, defining a base* b_λ *of submodular system* $\mathbf{S}_\lambda$ *by*

$$b_\lambda = \sum_{i \in I} \mu_i b^{\sigma_i}_\lambda + \partial\varphi, \tag{74}$$

the base b_λ *satisfies*

$$(b_{\lambda_i})^{S_i \setminus S_{i-1}} = \mathbf{0} \quad (i \in I), \tag{75}$$

where for each $i \in I$ $b^{\sigma_i}_\lambda$ *appearing in* (74) *is the extreme base of* $\mathrm{B}(h_\lambda)$ *corresponding to the linear ordering* σ_i *and* $\partial\varphi$ *is the boundary of flow* φ *in*

$G(\mathcal{P})$.

(Proof) For each $i = 1, \cdots, p$ base $b_i \equiv \mathbf{0} \in \mathrm{B}(h_{\lambda_i}{}^{S_i}_{S_{i-1}})$ is expressed by a convex combination of at most $|S_i \setminus S_{i-1}|$ extreme bases $b^{\sigma_{ij}}_{\lambda_i}$ $(j \in I_i)$ of $\mathrm{B}(h_{\lambda_i}{}^{S_i}_{S_{i-1}})$ and a nonnegative flow φ_i in $G(\mathcal{P}) \cdot (S_i \setminus S_{i-1})$ as follows.

$$b_i = \sum_{j \in I_i} \mu_{ij} b^{\sigma_{ij}}_{\lambda_i} + \partial \varphi_i. \tag{76}$$

Hence we can have an expression (74) satisfying (75), where we need at most $|E|$ extreme bases of $\mathrm{B}(h_\lambda)$ since

$$|S_1| + |S_2 \setminus S_1| + \cdots + |S_p \setminus S_{p-1}| = |E|. \tag{77}$$

For, the expression (74) can be constructed by the following procedure. Put $I = \emptyset$.

1. For each $i = 1, \cdots, p$ choose an index $k_i \in I_i$.
2. Find $i^* \in \{1, \cdots, p\}$ such that $\mu_{i^* k_{i^*}} = \min\{\mu_{i k_i} \mid i = 1, \cdots, p\}$.
3. Put $I \leftarrow I \cup \{i^*\}$.
   Let σ_{i^*} be the concatenation of $\sigma_{k_1}, \cdots, \sigma_{k_p}$ and define $\bar{\mu}_{i^*} = \mu_{i^* k_{i^*}}$.
4. For each $i = 1, \cdots, p$
   put $\mu_{i k_i} \leftarrow \mu_{i k_i} - \bar{\mu}_{i^*}$ and
   if $\mu_{i k_i} = 0$, then $I_i \leftarrow I_i \setminus \{k_i\}$ and
   if $I_i \neq \emptyset$, then choose an index $k_i \in I_i$,
   else go to Step 5.
   Go to Step 2.
5. Return σ_i, $\bar{\mu}_i$ $(i \in I)$, and I.

(Here we assume that I_i $(i = 1, \cdots, p)$ are disjoint.)

It should be noted that the linear ordering defined by the concatenation of $\sigma_{k_1}, \cdots, \sigma_{k_p}$ in Step 2 is a linear extension of $\mathcal{P}$, so that it gives an extreme base of $\mathrm{B}(h_\lambda)$. We can see that $|I| \leq |E|$, because of (77). Then we have

$$b_\lambda = \sum_{i \in I} \bar{\mu}_i b^{\sigma_i}_\lambda + \partial \varphi, \tag{78}$$

where $\varphi = \oplus_{i=1}^p \varphi_i$. We can also show that b_λ defined by (78) satisfies

$$(b_{\lambda_i})^{S_i \setminus S_{i-1}} = \mathbf{0} \tag{79}$$

for all $i = 1, \cdots, p$. □

Moreover, we have

Theorem 29. *For any $\lambda \in \mathbf{R}$ the base $b_\lambda \in \mathrm{B}(h_\lambda)$ in Theorem 28 satisfies*

$$b^-_\lambda(E) (= \sum \{b_\lambda(e) \mid e \in E,\ b_\lambda(e) < 0\}) = \max\{x^-(E) \mid x \in \mathrm{B}(h_\lambda)\}. \tag{80}$$

(Proof) Consider any $i \in \{0, 1, \cdots, p\}$ and $\lambda \in \Lambda_i$. Then, since S_i is a minimizer of h_λ, it suffices to show that

$$b_\lambda(e) \leq 0 \qquad (e \in S_i), \tag{81}$$

$$b_\lambda(e) \geq 0 \qquad (e \in E \setminus S_i), \tag{82}$$

and S_i is a tight set for b_λ in $\mathrm{B}(h_\lambda)$.

Because of Theorem 28 and the strong map relation we have (81) and (82) for $\lambda \in \Lambda_i$, where note that for any λ' and λ'' with $\lambda' < \lambda''$ we have $b^{\sigma_i}_{\lambda'} \geq b^{\sigma_i}_{\lambda''}$. Moreover, by the definitions of σ_k $(k \in I)$ and φ we have

$$b^{\sigma_k}_\lambda(S_i) = h_\lambda(S_i) \qquad (k \in I), \tag{83}$$

$$\partial\varphi(S_i) = 0. \tag{84}$$

It follows that S_i is a tight set. □

From Theorems 28 and 29 we have

Theorem 30. *There exist at most $|E|$ linear orderings σ_i $(i \in I)$ of E, coefficients μ_i $(i \in I)$ of convex combination, and nonnegative flows $\bar{\varphi}$ and $\underline{\varphi}$ in $G(\mathcal{P})$ such that for all $\lambda \in \mathbf{R}$, defining*

$$\bar{b}_\lambda = \sum_{i \in I} \mu_i \bar{b}^{\sigma_i}_\lambda + \partial\bar{\varphi}, \qquad \underline{b}_\lambda = \sum_{i \in I} \mu_i \underline{b}^{\sigma_i}_\lambda - \partial\underline{\varphi} \tag{85}$$

by extreme bases $\bar{b}^{\sigma_i}_\lambda$ of $\mathrm{B}(f_\lambda)$ and $\underline{b}^{\sigma_i}_\lambda$ of $\mathrm{B}(g_\lambda)$ corresponding to linear orderings σ_i $(i \in I)$, we have

$$(\bar{b}_\lambda - \underline{b}_\lambda)^-(E) = \max\{(x-y)^-(E) \mid x \in \mathrm{B}(f_\lambda),\ y \in \mathrm{B}(g_\lambda)\} \tag{86}$$

for all $\lambda \in \mathbf{R}$.

Moreover, we have

$$(\bar{b}_\lambda)^{S_i \setminus S_{i-1}} \in \mathrm{B}(f_\lambda{}^{S_i}_{S_{i-1}}), \qquad (\underline{b}_\lambda)^{S_i \setminus S_{i-1}} \in \mathrm{B}(g_\lambda{}^{S_i}_{S_{i-1}}) \tag{87}$$

for all $\lambda \in \mathbf{R}$ and $i = 1, \cdots, p$, and

$$(\bar{b}_{\lambda_i})^{S_i \setminus S_{i-1}} = (\underline{b}_{\lambda_i})^{S_i \setminus S_{i-1}} \tag{88}$$

for all $i = 1, \cdots, p$. □

It should be noted that Theorem 30 generalizes Theorems 22 and 23.

Remark 13. Besides upper critical values we can also define *lower critical values* as follows. Recall that S^-_λ is the minimum element of $\mathcal{D}(h_\lambda)$. Since we have $\mathcal{D}(h_{\lambda-\epsilon}) \subseteq \mathcal{D}(h_\lambda)$ for each λ and a sufficient small $\epsilon > 0$, let $S'_1 \subset S'_2 \subset \cdots \subset S'_q$ be the distinct elements of S^-_λ $(\lambda \in \mathbf{R})$. Then, $\mathbf{R}$ is divided into the intervals

$$\Lambda'_0 = (-\infty, \lambda'_1],\ \Lambda'_1 = (\lambda'_1, \lambda'_2],\ \cdots,\ \Lambda'_q = (\lambda'_q, +\infty) \tag{89}$$

such that for any $j = 0, 1, \cdots, q$ and any $\lambda \in \Lambda'_j$ we have $S^-_\lambda = S'_j$. We call each λ'_j a *lower critical value*. By means of lower critical values and the chain $S'_1 \subset \cdots \subset S'_q$ we can develop the similar arguments as the above-mentioned principal partitions. □

For any $\lambda, \lambda' \in \mathbf{R}$ with $\lambda < \lambda'$, if h_λ and $h_{\lambda'}$ satisfy

$$h_\lambda(Y) - h_\lambda(X) > h_{\lambda'}(Y) - h_{\lambda'}(X) \tag{90}$$

for all $X, Y \in \mathcal{D}$ with $X \subset Y$, we call $(h_\lambda, h_{\lambda'})$ a *strict strong map* and write $h_\lambda \twoheadrightarrow h_{\lambda'}$.

Theorem 31. *If $h_\lambda \twoheadrightarrow h_{\lambda'}$ for all λ and λ' with $\lambda < \lambda'$, then the upper critical values coincide with the lower critical values and we have*

$$S^+_i = S^-_{i+1} \qquad (i = 0, \cdots, p-1). \tag{91}$$

(Proof) For any $\lambda < \lambda'$ and for any $X \in \mathcal{D}(h_\lambda)$ and $Y \in \mathcal{D}(h_{\lambda'})$,

$$\begin{aligned} h_\lambda(X) + h_{\lambda'}(Y) &\geq h_\lambda(X \cup Y) + h_\lambda(X \cap Y) - h_\lambda(Y) + h_{\lambda'}(Y) \\ &= h_\lambda(X \cap Y) + h_{\lambda'}(X \cup Y) \\ &\qquad + h_\lambda(X \cup Y) - h_\lambda(Y) - h_{\lambda'}(X \cup Y) + h_{\lambda'}(Y) \\ &\geq h_\lambda(X \cap Y) + h_{\lambda'}(X \cup Y). \end{aligned} \tag{92}$$

If $Y \subset X \cup Y$, i.e., $X \setminus Y \neq \emptyset$, then the second inequality is strict since $h_\lambda \twoheadrightarrow h_{\lambda'}$, which is a contradiction. Hence $X \setminus Y = \emptyset$, i.e., $X \subseteq Y$. The present theorem follows from this fact. □

For a related parametric submodular intersection problem see [43].

4.2 Extension to discrete convex functions

The result of this section is based on joint work with Hayashi and Nagano [22].

Let $f : \mathbf{Z}^E \to \mathbf{R} \cup \{+\infty\}$ be a function on the integer lattice $\mathbf{Z}^E$ such that its effective domain $\mathrm{dom} f \equiv \{x \in \mathbf{Z}^E \mid f(x) < +\infty\}$ is nonempty. We suppose the following.

(S) f is submodular on $\mathrm{dom} f$, i.e.,

$$f(x) + f(y) \geq f(x \vee y) + f(x \wedge y) \qquad (x, y \in \mathrm{dom} f), \tag{93}$$

where $(x \vee y)(e) = \max\{x(e), y(e)\}$ and $(x \wedge y)(e) = \min\{x(e), y(e)\}$ for $e \in E$.

Given a positive vector $w : E \to \mathbf{R}$, consider an optimization problem with a parameter $\lambda \in \mathbf{R}$ as follows.

$$(P_\lambda): \ \text{Minimize} \ \ f(x) - \lambda\langle w, x\rangle, \tag{94}$$

where $\langle w, x\rangle = \sum_{e\in E} w(e)x(e)$. It should be noted that Problem (P_λ) generalizes the minimization problem appearing in Theorem 14. (For any $z \in \mathbf{Z}^E$ define $\mathcal{D}_z = \{X \mid X \subseteq E,\ f(z + \chi_X) < +\infty\}$, where χ_X is the characteristic vector of X, and if $\mathcal{D}_z \neq \emptyset$, also define a set function $f_z(X) = f(z + \chi_X)$ $(X \in \mathcal{D}_z)$. Then f_z is a submodular set function on the distributive lattice $\mathcal{D}_z$.)

Define $\mathcal{Z}(\lambda)$ to be the set of minimizers of $f(x) - \lambda\langle w, x\rangle$ in $x \in \mathbf{Z}^E$. Then,

Theorem 32. *For any $\lambda, \lambda' \in \mathbf{R}$ such that $\lambda \le \lambda'$ and for any $x \in \mathcal{Z}(\lambda)$ and $x' \in \mathcal{Z}(\lambda')$ we have*

$$x \vee x' \in \mathcal{Z}(\lambda'), \quad x \wedge x' \in \mathcal{Z}(\lambda). \tag{95}$$

Moreover, if $\lambda < \lambda'$, then $x \le x'$.

(Proof) Under the assumption of the present theorem we have

$$\begin{aligned}
& f(x) - \lambda\langle w, x\rangle + f(x') - \lambda'\langle w, x'\rangle \\
& \ge f(x \vee x') - \lambda'\langle w, x \vee x'\rangle + f(x \wedge x') - \lambda\langle w, x \wedge x'\rangle \\
& \quad +\lambda'\langle w, x \vee x'\rangle + \lambda\langle w, x \wedge x'\rangle - \lambda'\langle w, x'\rangle - \lambda\langle w, x\rangle \\
& = f(x \vee x') - \lambda'\langle w, x \vee x'\rangle + f(x \wedge x') - \lambda\langle w, x \wedge x'\rangle \\
& \quad +(\lambda' - \lambda)\langle w, x \vee x' - x'\rangle \\
& \ge f(x \vee x') - \lambda'\langle w, x \vee x'\rangle + f(x \wedge x') - \lambda\langle w, x \wedge x'\rangle.
\end{aligned} \tag{96}$$

Hence (95) follows. Moreover, since the inequalities in (96) must be equalities, if $\lambda < \lambda'$, the last inequality (now equality) implies $x \vee x' = x'$, i.e., $x \le x'$, where note that $w > \mathbf{0}$. □

Remark 14. Theorem 32 is subsumed by a result of Topkis [82, 83] (also see [35]). Monotonicity of optimal solutions of parametric optimization problems has been investigated in the literature such as [4, 52, 82]. The theory of principal partitions has been developed independently of these results and deals primarily with the critical values and the decomposition of systems, while the monotonicity of primal and dual optimal solutions with respect to the parameter plays a crucial rôle in the principal partitions. □

Denote by z^+_λ and z^-_λ, respectively, the maximum and the minimum element of $\mathcal{Z}(\lambda)$. Define

$$\Lambda^* = \{\lambda \in \mathbf{R} \mid z^+_\lambda \neq z^-_\lambda\}. \tag{97}$$

Each $\lambda \in \Lambda^*$ is called a *critical value*.

Theorem 33. *Consider any critical values $\lambda, \lambda' \in \Lambda^*$ with $\lambda < \lambda'$. Then we have either $\mathcal{Z}(\lambda) \cap \mathcal{Z}(\lambda') = \emptyset$ or $z^+_\lambda = z^-_{\lambda'}$.*

(Proof) If $\mathcal{Z}(\lambda) \cap \mathcal{Z}(\lambda')$ contains two distinct elements x and x', then this contradicts the monotonicity in the last statement of Theorem 32. Hence we have $|\mathcal{Z}(\lambda) \cap \mathcal{Z}(\lambda')| = 0$ or 1. If $|\mathcal{Z}(\lambda) \cap \mathcal{Z}(\lambda')| = 1$, the element of $\mathcal{Z}(\lambda) \cap \mathcal{Z}(\lambda')$ must be z^+_λ that is equal to $z^-_{\lambda'}$, due to Theorem 32. □

For any two critical values $\lambda, \lambda' \in \Lambda^*$ with $\lambda < \lambda'$ we say that λ' *covers* λ if there is no critical value λ'' satisfying $\lambda < \lambda'' < \lambda'$.

Theorem 34. *For any critical values $\lambda, \lambda' \in \Lambda^*$ such that λ' covers λ we have*

$$z^+_\lambda = z^-_{\lambda'}. \tag{98}$$

Moreover,

$$z^+_{\lambda''} = z^-_{\lambda''} = z^+_\lambda (= z^-_{\lambda'}) \quad (\lambda < \lambda'' < \lambda'). \tag{99}$$

(Proof) Because of the continuity in the parameter, for any λ'' and sufficiently small $\epsilon > 0$ we have

$$\mathcal{Z}(\lambda'' \pm \epsilon) \subseteq \mathcal{Z}(\lambda''). \tag{100}$$

It follows from (100), Theorem 33, and the definition of a critical value that we have (98) and (99). □

Remark 15. We can consider more general parametric submodular functions corresponding to those treated in Section 4.1. For each $\lambda \in \mathbf{R}$ let h_λ be a submodular function on $\mathbf{Z}^E$ that satisfies the following.

- For any λ and λ' with $\lambda < \lambda'$ and for any $x, y \in \mathbf{Z}^E$ with $x \leq y$ we have

$$h_\lambda(y) - h_\lambda(x) \geq h_{\lambda'}(y) - h_{\lambda'}(x). \tag{101}$$

Then we say that $(h_\lambda, h_{\lambda'})$ is a *strong map* and write $h_\lambda \to h_{\lambda'}$. The arguments in Section 4.1 can be adapted to such parametric submodular functions on $\mathbf{Z}^E$ (cf. [82, 83]). If (101) holds with strict inequality for all $x, y \in \mathbf{Z}^E$ with $x \leq y$ and $x \neq y$, we say that $(h_\lambda, h_{\lambda'})$ is a *strict strong map* and write $h_\lambda \twoheadrightarrow h_{\lambda'}$. Theorems 33 and 34 hold for parametric submodular functions satisfying the strict strong map condition. □

Remark 16. It should be noted that Theorems 32–34 hold for f satisfying the submodularity condition (S). However, the submodularity on $\mathbf{Z}^E$ alone is not enough to treat the structure of $\mathcal{Z}(\lambda)$ $(\lambda \in \Lambda^*)$ algorithmically. In order to resolve this situation we consider discrete convex functions called *$L^\natural$-convex functions* by Murota [58]. □

Denote by Conv the convex hull operator in $\mathbf{R}^E$. For any $z \in \mathbf{Z}^E$ and any linear ordering σ of E define a simplex

$$\Delta^\sigma_z = \mathrm{Conv}(\{z + \chi_{S_i} \mid i = 1, \cdots, m,\ S_i \text{ is the set of the first } i \text{ elements of } \sigma\}). \tag{102}$$

The collection of all such simplices Δ_z^σ for all points $z \in \mathbf{Z}^E$ and linear orderings σ of E forms a simplicial division of $\mathbf{R}^E$, which is called the *Freudentahl simplicial division*. We also call each Δ_z^σ a *Freudentahl cell*.

In addition to the submodularity condition (S) suppose

(A1) $\mathrm{Conv}(\mathrm{dom} f) \cap \mathbf{Z}^E = \mathrm{dom} f$.

Informally, (A1) means that there is no hole in $\mathrm{dom} f$.

We further assume

(A2) The convex hull $\mathrm{Conv}(\mathrm{dom} f)$ of the effective domain of f is full-dimensional and is the union of some Freudentahl cells.

The assumption of the full dimensionality is not essential but we assume it here for simplicity. Under Assumptions (A1) and (A2) we can uniquely construct a piecewise linear extension $\hat{f}$ of f by means of the Freudentahl simplicial division as follows. For any $x \in \Delta_z^\sigma$ we have a unique expression of x as a convex combination of extreme points of the cell Δ_z^σ as

$$x = \sum_{i=1}^{m} \alpha_i (z + \chi_{S_i}), \tag{103}$$

where S_i is the set of the first i elements of σ. According to the expression (103) we define

$$\hat{f}(x) = \sum_{i=1}^{m} \alpha_i f(z + \chi_{S_i}). \tag{104}$$

For all x outside $\mathrm{Conv}(\mathrm{dom} f)$ we put $\hat{f}(x) = +\infty$. Note that $\hat{f}$ is well defined. It should also be noted that when $\mathrm{dom} f = \{\chi_X \mid X \subseteq E\}$, $\hat{f}$ is called the *Lovász extension* ([20, 48]).

We add one more, crucial assumption as follows.

(A3) The piecewise linear extension $\hat{f} : \mathbf{R}^E \to \mathbf{R} \cup \{+\infty\}$ of f by (104) is a convex function on $\mathbf{R}^E$.

Remark 17. A function $f : \mathbf{Z}^E \to \mathbf{R} \cup \{+\infty\}$ satisfying Conditions (A1), (A2), and (A3) is exactly an *$L^\natural$-convex function* on $\mathbf{Z}^E$ (with full-dimensional $\mathrm{dom} f$) of Murota [23, 56, 58, 60]. The original definition of an $\mathrm{L}^\natural$-convex function on $\mathbf{Z}^E$ is different, but see [20, Chapter VII] for the proof of their equivalence. Note that Conditions (A1), (A2), and (A3) imply submodularity (S). It should also be noted that a submodular function $f : \mathbf{Z}^E \to \mathbf{R} \cup \{+\infty\}$ satisfying Condition (A3) with its effective domain being a standard box $[z_1, z_2]$ between two integer vectors z_1 and z_2 was first considered by Favati and Tardella [14] and was called a *submodular integrally convex function*. □

Now, suppose that we are given a positive vector $w : E \to \mathbf{R}$, a real constant β, and an $\mathrm{L}^\natural$-convex function $f : \mathbf{Z}^E \to \mathbf{R} \cup \{+\infty\}$. Let us consider the following optimization problem with a linear inequality constraint.

$$
\begin{aligned}
(P^\circ): \quad & \text{Minimize} \quad \hat{f}(x) \\
& \text{subject to} \quad \langle w, x\rangle \le \beta,
\end{aligned} \tag{105}
$$

where $\hat{f}$ is the piecewise linear extension of f defined by (104).

We can relate critical values for f to Problem (P°) as follows. Recall that $\mathcal{Z}(\lambda)$ is the collection of minimizers of $h_\lambda(x) = f(x) - \lambda\langle w, x\rangle$.

Theorem 35. *Suppose that for a parameter $\lambda^* < 0$ there exist $x, x' \in \mathcal{Z}(\lambda^*)$ such that*

$$
\langle w, x\rangle \le \beta, \qquad \langle w, x'\rangle \ge \beta. \tag{106}
$$

Then a vector x^* lying on the line segment between x and x' and satisfying $\langle w, x^*\rangle = \beta$ is an optimal solution of Problem (P°).

(Proof) For any feasible solution y of Problem (P°),

$$
\begin{aligned}
\hat{f}(y) &\ge \hat{f}(y) + \lambda^*(\beta - \langle w, y\rangle) \\
&\ge \min\{f(z) + \lambda^*(\beta - \langle w, z\rangle) \mid z \in \mathrm{dom} f\} \\
&= \hat{f}(x^*) + \lambda^*(\beta - \langle w, x^*\rangle) \\
&= \hat{f}(x^*),
\end{aligned} \tag{107}
$$

where note that $f(x) - \lambda^*\langle w, x\rangle = f(x') - \lambda^*\langle w, x'\rangle = \hat{f}(x^*) - \lambda^*\langle w, x^*\rangle$ because of (A1)–(A3). Hence x^* is an optimal solution of (P°). □

Remark 18. Since Problem (P°) is an ordinary convex program, if (P°) has an optimal solution x^*, then either it is a global minimizer of $\hat{f}$ or it is the one that satisfies the condition of Theorem 35. In the latter case it suffices to find a critical value λ^* such that for some $x^* \in \mathrm{Conv}(\mathcal{Z}(\lambda^*))$ we have $\langle w, x^*\rangle = \beta$. The last condition can be rephrased as $\langle w, z^+_{\lambda^*}\rangle \ge \beta$ and $\langle w, z^-_{\lambda^*}\rangle \le \beta$. □

When dom$f$ is bounded, we can apply Murota's weakly polynomial algorithm [59] for minimizing L$^\natural$-convex functions to find a vector in $\mathcal{Z}(\lambda)$ for each λ. We can perform a binary search to find an optimal critical value λ^* by making use of algorithms for the minimum ratio problem described in Section 5.1. This gives a weakly polynomial algorithm for Problem (P°) with rational data (see [22]).

We can also consider multiple inequality constraints as follows.

$$
\begin{aligned}
(P): \quad & \text{Minimize} \quad \hat{f}(x) \\
& \text{subject to} \quad \langle w_i, x\rangle \le \beta_i \quad (i = 1, \cdots, k),
\end{aligned} \tag{108}
$$

where w_i $(i = 1, \cdots, k)$ are positive vectors and β_i $(i = 1, \cdots, k)$ are real constants. This leads us to the following multiple-parameter submodular function.

$$
h_\lambda(x) = \hat{f}(x) - \sum_{i=1}^{k} \lambda_i \langle w_i, x\rangle, \tag{109}
$$

where $\lambda = (\lambda_i \mid i = 1, \cdots, k)$. The present problem can also be treated similarly (cf. [20, Section 7] and [35]).

Remark 19. We can consider a class $\mathcal{F}$ of discrete convex functions $f : \mathbf{Z}^E \to \mathbf{R} \cup \{+\infty\}$ as follows.

(i) f satisfies Conditions (A1), i.e., $\mathrm{Conv}(\mathrm{dom} f) \cap \mathbf{Z}^E = \mathrm{dom} f$.
(ii) For any $x \in \mathrm{dom} f$ there exists a vector $w : E \to \mathbf{R}$ such that x is a minimizer of $f(z) - \langle w, z \rangle$ $(z \in \mathbf{Z}^E)$.

Provided that we can perform the minimization of $f(z) - \langle w, z \rangle$ for $z \in \mathbf{Z}^E$, we can solve Problem (P°) in a similar way as described in this section. A typical example of such a class of discrete convex functions other than L$^\natural$-convex functions is that of M$^\natural$-convex functions on $\mathbf{Z}^E$ of Murota and Shioura [62]. □

5 Applications and Related Topics

We often encounter problems described by submodular functions with parameters, for which the theory of principal partitions furnishes a powerful tool.

5.1 The minimum ratio problem

Suppose that we are given a submodular system $(\mathcal{D}, f)$ and a supermodular system $(\mathcal{D}, g)$ on E, where $f(X) \geq 0$ $(X \in \mathcal{D})$, $g(X) \geq 0$ $(X \in \mathcal{D})$, and there exists an $X \in \mathcal{D}$ such that $g(X) > 0$.

Consider the minimum ratio problem described as follows.

$$\text{Minimize } \frac{f(X)}{g(X)} \quad \text{subject to } X \in \mathcal{D}, \quad g(X) > 0. \tag{110}$$

Define a submodular function h_λ on $\mathcal{D}$ with a real parameter λ by

$$h_\lambda(X) = f(X) - \lambda g(X) \qquad (X \in \mathcal{D}). \tag{111}$$

Then we have

Theorem 36. *Let $\hat{\lambda}$ be the minimum value of the objective function of Problem* (110)*. Then,*

$$\min\{h_\lambda(X) \mid X \in \mathcal{D}\} = 0 \qquad (0 \leq \lambda \leq \hat{\lambda}), \tag{112}$$

$$\min\{h_\lambda(X) \mid X \in \mathcal{D}\} < 0 \qquad (\hat{\lambda} < \lambda). \tag{113}$$

Moreover, the converse also holds. □

Remark 20. It should be noted that Theorem 36 does not depend on the submodularity (supermodularity) of f (g) and holds for any set functions. However, if f (g) is submodular (supermodular), then Problem (110) has a close relationship with the principal partition.

Theorem 36 means that $\hat{\lambda}$ is a critical value for h_λ such that

$$\mathcal{D}(h_\lambda) = \{\emptyset\} \quad (0 \le \lambda < \hat{\lambda}), \qquad \mathcal{D}(h_\lambda) \ne \{\emptyset\} \quad (\hat{\lambda} \le \lambda). \tag{114}$$

Hence the minimum ratio problem for submodular and supermodular functions f and g is reduced to finding such a critical value $\hat{\lambda}$ and a set $X \in \mathcal{D}(h_{\hat{\lambda}})$ for $h_{\hat{\lambda}} = f - \hat{\lambda} g$. □

The network attack problem of Cunningham

Cunningham [6] introduced a measure of network (anti-)vulnerability as follows. For a connected graph $G = (V, E)$ and a positive weight vector $w : E \to \mathbf{R}_+$ the *strength* of the weighted graph is defined by

$$\sigma(G, w) = \min\left\{ \frac{w(X)}{\kappa(X)} \;\middle|\; X \subseteq E,\ \kappa(X) > 0 \right\}, \tag{115}$$

where $\kappa(X)$ denotes the number of the connected components of the subgraph $G \cdot (E \setminus X)$ minus one. We can easily see that $\kappa : 2^E \to \mathbf{Z}_+$ is a supermodular function expressed in terms of the rank function r_G of G as

$$\kappa(X) = r_G(E) - r_G(E \setminus X) \ (= r_G^{\#}(X)) \qquad (X \subseteq E). \tag{116}$$

Hence the problem of computing the strength of G relative to weight w is a special case of the minimum ratio problem described above. Letting $\hat{\lambda}$ be the largest critical value for $r_G - \lambda w$, we obtain

$$\sigma(G, w) = 1/\hat{\lambda}. \tag{117}$$

Also see [1, 2] for related topics on *partition inequalities*, which is also closely related to the *principal lattice of partitions* of Narayanan [68] (also see [7, 72] for their applications). Note that for a given submodular function f the principal lattice of partitions for f is concerned with the Dilworth truncation of the submodular function $f - \lambda$ with a real parameter λ.

Maximum density subgraphs

For a graph $G = (V, E)$ define the density of G by

$$d(G) = \frac{|E|}{|V| - 1}. \tag{118}$$

A subgraph of G of maximum density is connected, so that the problem of finding a maximum-density subgraph $H = (W, F)$ of G is reduced to the following problem.

$$\text{Maximize} \quad \frac{|F|}{r_G(F)} \quad \text{subject to} \quad \emptyset \ne F \subseteq E, \tag{119}$$

which is equivalent to the minimum-ratio problem

$$\text{Minimize} \quad \frac{r_G(F)}{|F|} \quad \text{subject to} \quad \emptyset \neq F \subseteq E. \tag{120}$$

Hence the problem is reduced to finding the minimum critical value λ_1 for $r_G(X) - \lambda|X|$.

The concept of density of a graph is closely related to connectivity and reliability of networks, to which the principal partitions can be applied effectively.

5.2 Resource allocation problems

Since the canonical simplex

$$\Delta_\beta = \{x \mid x \in \mathbf{R}_+^E,\ x(E) = \beta\} \tag{121}$$

for $\beta > 0$ is a special case of a base polyhedron, base polyhedra naturally arise in resource allocation problems. Also the core of a convex game [78] is a base polyhedron, so that we often consider allocation problems over cores or base polyhedra.

Given a positive weight vector $w : E \to \mathbf{R}_+$, the weighted min-max resource allocation problem over the base polyhedron $\mathrm{B}(f)$ associated with a submodular system $(\mathcal{D}, f)$ on E is described as

$$\text{Minimize} \quad \max\{x(e)/w(e) \mid e \in E\} \quad \text{subject to} \quad x \in \mathrm{B}(f). \tag{122}$$

Also, the weighted max-min resource allocation problem is described as

$$\text{Maximize} \quad \min\{x(e)/w(e) \mid e \in E\} \quad \text{subject to} \quad x \in \mathrm{B}(f). \tag{123}$$

Then we can show the following (also see [18] and [20, Chapter V] for more general and detailed discussions).

Theorem 37. *Let b^* be the universal base (or the lexicographically optimal base) for submodular system $(\mathcal{D}, f)$ with weight w. Then $x = b^*$ is an optimal solution of both problems* (122) *and* (123).

Moreover, the minimum (resp. maximum) critical value for $f - \lambda w$ is equal to the optimal objective function value of the max-min (resp. min-max) resource allocation problem (123) *(resp.* (122)*).* □

The following *equitable resource allocation* problem was considered by Jain and Vazirani [44]. Let $w_\lambda : E \to \mathbf{R}$ be a vector with a parameter $\lambda \in \mathbf{R}$. We assume that for each $e \in E$ the component $w_\lambda(e)$ of w_λ is increasing in λ. Then, for a submodular system $(2^E, f)$ we have dual problems characterized by the following min-max relation for any λ.

$$\max\{x(E) \mid x \in \mathrm{P}(f),\ x \leq w_\lambda\} = \min\{f(X) + w_\lambda(E \setminus X) \mid X \subseteq E\}. \tag{124}$$

This can be seen as a special case of the min-max relation given in Theorem 25. Hence Theorem 30 implies that there exist a (unique) base $b^* \in \mathrm{B}(f)$ such that for all λ

$$(b^* - w_\lambda)^-(E) = \min\{f(X) - w_\lambda(X) \mid X \subseteq E\}. \tag{125}$$

This is also equivalent to

$$(b^* \wedge w_\lambda)(E) = \max\{x(E) \mid x \in \mathrm{P}(f),\ x \leq w_\lambda\} \tag{126}$$

for all λ. The universal base b^* is the desired equitable allocation.

More general convex minimization problems over base polyhedra have recently been examined by Nagano [63], which shows the equivalence between the lexicographic optimal base problem and the submodular utility allocation market problem [45]. Separable nonquadratic convex function minimization over base polyhedra is also considered in [20, Chapter V].

6 Concluding Remarks

Combinatorial optimization problems characterized by submodular functions arise in a lot of applications such as graph and network optimizations, scheduling problems, queueing network problems, information-theoretic data analysis and communication networks, games and economic equilibrium problems, etc. (see, e.g., [17, 20, 25, 45, 47, 57, 68, 74, 77, 79, 83, 85]). Such combinatorial optimization problems often lead us to submodular function minimization, where the theory of principal partitions can provide us with the powerful tool for extracting useful structural information about the problems under consideration.

The essence of the theory of principal partitions is given in the author's book [20] but it is rather scattered through the book (also see [81]). The author hope that the present article will help readers fully appreciate the usefulness of the theory of principal partitions.

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