Workshop on Motives--the 1-st

Japanese Page
Workshop on Motives--2-nd

Date: 19(Mon)-22(Thu)/Dec/2005
Place: Room 052, Graduate School of Mathematics, University of Tokyo, Komaba, Meguro-ku, Tokyo, Japan,

With financial supports of JSPS (B)#17340008 "数論的多様体のp進的手法による研究 "(representative:Nobuo Tsuzuki),
we will hold a workshop as follows.

Speakers:
H. Furusho (Nagoya University), T. Geisser (USC), K. Hagihara (University of Tokyo), M. Hanamura (Tohoku University),
L. Hesselholt (MIT), S. Kimura (Hiroshima University), S. Mochizuki (University of Tokyo), S. Saito (University of Tokyo),
K. Sato (Nagoya University), A. Shiho (University of Tokyo), N. Takahashi (Hiroshima University),
G. Yamashita (University of Tokyo), T.Yamazaki (Tsukuba University), T. Yasuda (RIMS)

Schedule:
19(Mon)/Dec
9:30-10:30: S. Kimura (Hiroshima Univ.) Mumfordの反例, Bloch予想, Bloch-Beilinson予想.
11:00-12:00: G. Yamashita (Univ. of Tokyo) Introduction to Voevodsky's category of mixed motives 1.
13:30-14:30: N. Takahashi (Hiroshima Univ.) Motivic Zeta の紹介.
15:00-16:00: K. Sato (Nagoya Univ.) Bloch-Kato conjecture and Beilinson-Lichtenbaum conjecture.
16:30-17:30: S. Saito (Univ. of Tokyo) Overview on finiteness results for motivic cohomology.

20(Tue)/Dec
9:30-10:30: T. Geisser (USC) Motivic cohomology and special values of zeta-functions.
11:00-12:00: K. Hagihara (Univ. of Tokyo) Introduction to Voevodsky's category of mixed motives 2.
13:30-14:30: T. Yasuda (RIMS) モティヴィック積分概論.
15:00-16:00: A. Shiho (Univ. of Tokyo) On (Hodge realization of) polylogarithm.
16:30-17:30: S. Mochizuki (Univ. of Tokyo) The category of mixed Tate motives.
Reception

21(Wed)/Dec
9:30-10:30: S. Kimura (Hiroshima Univ.) Chow homology, Chow cohomology.
11:00-12:00: K. Hagihara (Univ. of Tokyo) Introduction to Voevodsky's category of mixed motives 3.
13:30-14:30: L. Hesselholt (MIT/Nagoya Univ.) An introduction to model categories.
15:00-16:00: G. Yamashita (Univ. of Tokyo) The category of mixed Tate motives over the ring of

integers.
16:30-17:30: T. Yamazaki (Tsukuba Univ.) Chow motive of a product of curves and Milnor K-groups.

22(Thu)/Dec
9:30-10:30: T. Geisser (USC) TBA.
11:00-12:00: K. Hagihara (Univ. of Tokyo) Introduction to Voevodsky's category of mixed motives 4.
13:30-14:30: H. Furusho (Nagoya Univ.) Grothendieck-Teichmuller group.
15:00-16:00: M. Hanamura (Tohoku Univ.) Comparison of motivic theories.

The main theme of this workshop is to understand the theory of mixed motives due to Voevodsky. This workshop is the first time of series of workshops, so lectures are introductory. Researchers on other areas and undergraduate students are heartily welcomed.

Organizer:
T. Geisser (USC), S. Kimura (Hiroshima University) kimura@math.sci.hiroshima-u.ac.jp ,
G. Yamashita (University of Tokyo) gokun@ms.u-tokyo.ac.jp ,

Back

なぜモチーフを勉強したいのか？ 12/19 Kimura. S

・異なるコホモロジー理論が、似たふるまいをする。

同じXに対して、ほぼ同値な情報を与える。

理由：Motiveという親玉がいる。

例：Abel-Jacobiの定理　X：リーマン面，$J(X)=H^0(X,\Omega_X)^*/H_1(X,\mathbb{Z})$

$$X \xrightarrow{AJ} J(X) \simeq \mathbb{C}^g/\mathbb{Z}^{2g}, \quad Q \mapsto \left(\int_P^Q : \eta \mapsto \int_P^Q \eta\right)$$

$$S^nX = \underbrace{X\times\cdots\times X}_{n}/\mathfrak{S}_n \xrightarrow{AJ} J(X), \quad [Q_1]+\cdots+[Q_n] \mapsto \sum_{i=1}^n AJ(Q_i)$$

$\{P_1,\cdots,P_n\}$ というXのnコの重複を許す点。

$\{P_1,\cdots,P_n\}\cap\{Q_1,\cdots,Q_n\}=\phi$　X上の有理型関数 f が存在して、

$P_1,\cdots,P_n$ が0点で、$Q_1,\cdots,Q_n$ が極。

$\iff$ $AJ_n\left(\sum[P_i]\right) = AJ_n\left(\sum[Q_i]\right)$

f があれば、$X \xrightarrow{f} \mathbb{P}^1$　$\mathbb{P}^1 \to S^nX$, $t \mapsto f^{-1}(t)$　$\left(\begin{array}{l} f^{-1}(0)=\sum[P_i] \\ f^{-1}(\infty)=\sum[Q_i] \end{array}\right) \Rightarrow$

S^nX の中の2点が、$\mathbb{P}^1$ で結べる為の必要充分条件が、$J(X)$ で一致。（同じ点）

Xのホモロジー

$$H_*(X;\mathbb{Z}) = \{\gamma \mid \partial\gamma = 0\}/\sim$$

位相的

位相的元

位相的変型 パラメータ $I=[0,1]$

代数幾何的ホモロジー

$$CH_*X = \left\{\sum n_i[V_i]\right\}/\sim$$

*次元 subvar.

代数幾何的変型 パラメーターは $\mathbb{P}^1$.

$cl : CH_*X \to H_{2*}(X;\mathbb{Z})$: cycle map が出来る。

$$CH_1X \xrightarrow{\sim} H_2(X;\mathbb{Z})$$

$$H_1(X;\mathbb{Z}) \simeq \mathbb{Z}^{2g} \hookrightarrow H^0(X,\Omega_X)^*$$

$$0 \to J(X) \to CH_0X \to H_0(X;\mathbb{Z}).$$

$h^1(X)$

Abel-Jacobiの一般化

X: smooth proj. $\quad X \to \mathrm{Alb}(X) := H^0(X,\Omega_X)^* \big/ H_1(X,\mathbb{Z})$

$/\mathbb{C}$ $\quad CH_0(X) \twoheadrightarrow \mathbb{Z}\oplus \mathrm{Alb}(X) \qquad \dim X = d$

一般には，全射にならない。

$$0\to 2\pi i\mathbb{Z}\to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\times \to 0 \qquad \mathrm{Pic}\,X = H^1(X,\mathcal{O}_X^\times)$$

$$\underbrace{H^1(X,\mathbb{Z})\to H^1(X,\mathcal{O}_X)}_{\text{coker}}\to H^1(X,\mathcal{O}_X^\times)\xrightarrow{cl} H^2(X,\mathbb{Z})$$

$H^1(X,\mathcal{O}_X^\times) = CH_{d-1}(X)$ $\qquad$ image は，$H^{1,1}(X,\mathbb{Z})$と交わり

定理(Mumford 1968) $\quad 0\to \mathrm{Pic}^0X\to CH_{d-1}(X)\xrightarrow{cl} H^2(X,\mathbb{Z})$

Xが，surface. base field は，非可算 $\quad p_g(X)>0$

$(\iff cl: CH_1X_{\mathbb{Q}} \to H^2(X,\mathbb{Q})$ が全射でない。$)$

ならば，$\quad CH_0X\to\mathbb{Z}\oplus \mathrm{Alb}(X)$ の kernel (Albanese kernel が) が，巨大.

$$\begin{array}{l}
CH_2X \xrightarrow{\sim} H_4^{全} \\
\qquad\qquad\quad H_3 \\
\mathrm{Pic}^0X \to CH_1X \to H_2 \to \mathrm{Coker}(CH_1\to H_2) \\
\qquad\qquad\quad H_1 \\
\qquad \to CH_0X \to H_0 \to \mathbb{Z} \\
\text{Albanese kernel} \oplus \mathrm{Alb}X
\end{array}$$

予想(Bloch 1975)

逆に，Xが，surface. $p_g(X)=0$

$(\iff cl: CH_1X_{\mathbb{Q}} \to H^2(X,\mathbb{Q})$ が全射$)$

$\implies CH_0X \xrightarrow{\sim} \mathbb{Z}\oplus \mathrm{Alb}(X)$

$\kappa(X)\leq 1$の時，O.K.

↑

小平次元. $\qquad \kappa(X)=2$の時 $p_g(X)=0$ なら，$q=0$ より

仮定 $\iff$ cl 全体が全射. $\quad \bigoplus_{i=0}^{2\dim X} CH_i(X)_{\mathbb{Q}} \to \bigoplus_{i=0}^{2\dim X} H_i(X,\mathbb{Q})$

Bloch予想の一般化：cycle map が全射 $\Rightarrow$ 単射. (X：任意次元)

定理：(Jannsen 1995)

AMS Motives cycle map が、単射 ⇒ 全射.

Albanese kernel の構造は？ /$\mathbb{C}$

例： X : Abel 多様体 $\underset{\text{位相}}{\simeq} (S^1)^{2g}$

$$H_*(X;\mathbb{Q}) \simeq \bigotimes^{2g} (H_1(S^1;\mathbb{Q}) \oplus H_0(S^1;\mathbb{Q}))$$

$N : X \to X$: N 倍写像 $\qquad H_1(S^1;\mathbb{Q}) \to H_1(S^1;\mathbb{Q})$: N 倍

$H_0(S^1;\mathbb{Q}) \to H_0(S^1;\mathbb{Q})$: 1 倍

⇒ $H_i(X;\mathbb{Q})$ に於ては、N^i 倍 $\qquad \mu : X \times X \to X$: 群の演算

Pontragin Product for CH_0X $\qquad [V]*[W] := \mu_*[V\times W]$

CH_0X 上では、$[P]*[Q] = [P+Q]$ これで、CH_0X : 可換環.

$\underset{\uparrow}{I} \subseteq CH_0X \xrightarrow{\deg} \mathbb{Z}$ が、環の準同型.

ideal に成る。

$\ker(\deg)$ $\qquad CH_0X \supsetneq I \supset I^{*2} \supset \cdots \supset I^{*n} \supset \cdots$

Claim： $I/(I^*)^2 \xrightarrow{\sim} X$ $\qquad I$ は、$[P]-[0]$ で、生成される。

$\overline{[P]-[0]} \longmapsto [P]$ $\qquad I^*$ は $([P]-[0])*([Q]-[0])$

$= [P+Q]-[P]-[Q]+[0]$

N_* による固有値は、N のみ. $\qquad$ で生成.

$I^{*k}/(I^*)^{k+1} \twoheadleftarrow (I/I^{*2})^{\otimes k}$ より、$I^{*k}/I^{*(k+1)}$ では、N_* は、N^k 倍.

Bloch-Beilinson 予想 $\qquad d = \dim X$ $\qquad I^{*(g+1)} = 0$

$CH^iX := CH_{d-i}X$

$CH^iX_{\mathbb{Q}} = F^0CH^iX_{\mathbb{Q}} \supset F^1CH^iX_{\mathbb{Q}} \supset \cdots$ と filt. が入り.

①：$F^1CH^iX_{\mathbb{Q}}$ は、cycle map の kernel.

②：Intersection Product について、

$$F^rCH^iX_{\mathbb{Q}} \otimes F^sCH^jX_{\mathbb{Q}} \subseteq F^{r+s}CH^{i+j}X_{\mathbb{Q}}.$$

③：f_*, f^* は、filt. を保つ

④：$F^nCH^iX_{\mathbb{Q}} / F^{n+1}CH^iX_{\mathbb{Q}}$ は、$H^{2i-n}(X, \mathbb{Q})$ で control される.

⑤：$F^{i+1}CH^iX_{\mathbb{Q}} = 0$

④については、$Gr^n CH^i(X) \simeq \mathrm{Ext}^n_{\mathcal{MM}}(\mathbb{1}, h^{2i-n}(X)(i))$

$\mathbb{1} = \mathbb{Q}$

と予想される。

goal: to construct Voevodsky's tensor triangulated category of mixed motive

properties $(n \geq 0)$

Tate object. $\mathbb{Z}(n) \leftarrow$ complex of Zariski sheaves. degree $\leq n$

$(Sm/k)_{Zar}$

Beilinson:

①: $\mathbb{Z}(0) = \mathbb{Z}$

②: $\mathbb{Z}(1) = \mathcal{O}^{\times}[-1]$

③: F: field$/k$

$\mathbb{H}^{n}_{Zar}(\mathrm{Spec} F, \mathbb{Z}(n)) \simeq K^{M}_{n}(F) \leftarrow$ Milnor K-group

④: $\mathbb{H}^{2n}_{Zar}(X, \mathbb{Z}(n)) \simeq CH^{n}(X)$

$\left(\mathbb{H}^{p}_{Zar}(X, \mathbb{Z}(q)) \simeq CH^{q}(X, 2q-p) \right)$.

$\uparrow$ higher Chow 群.

⑤: $X \in Sm/k$

$\Rightarrow$ spectral sequence. $E_2^{p,q} = \mathbb{H}^{p-q}_{Zar}(X, \mathbb{Z}(-q)) \Rightarrow K_{-p-q}(X)$

: Bloch-Litchtenbaum, Friedlander-Suslin.

: Voevodsky, Levine

: Grayson-Suslin.

$\mathbb{H}^{p}_{Zar}(X, \mathbb{Z}(q)) \otimes \mathbb{Q} \simeq gr^{q}_{\gamma} K_{2q-p}(X) \otimes \mathbb{Q}$

Beilinson-Litchtenbaum Conj:

F: field$/k$, l: prime $\neq$ char k

$$\mathbb{H}^{p}_{Zar}(F, \mathbb{Z}(q) \otimes \mathbb{Z}/l) \simeq \begin{cases} H^{p}_{et}(F, \mu_l^{\otimes q}) & p \neq q \\ 0 & p > q. \end{cases}$$

$\Longleftrightarrow$ (Suslin-Voevodsky) $\mathbb{Z}(q) \otimes \mathbb{Z}/l \xrightarrow{qis} \tau_{\leq q} R\alpha_{*} \mu_l^{\otimes q}$

(Bloch-Kato Conj. generalized Hilbert 90) $\alpha: (Sm/k)_{et} \longrightarrow (Sm/k)_{Zar}$

$\left((\mathbb{Z}(q) \otimes \mathbb{Z}/l)_{et} \simeq \mu_l^{\otimes q} \right)$

<u>Beilinson–Soulé</u>: <u>Conj.</u> $\mathbb{H}^p_{Zar}(X, \mathbb{Z}(q)) \overset{df}{=}: H^p_{\mathcal{M}}(X, \mathbb{Z}(q))$ <u>No. 2</u>

<u>Vanishing</u> : motivic coh.

$X \in Sm/k \Rightarrow \mathbb{H}^i_{Zar}(X, \mathbb{Z}(n)) = 0$ for $i < 0$.

$DM_{gm}(k)$ $\quad M : Sm/k \longrightarrow DM_{gm}(k)$

$$X \longmapsto M(X)$$ motive of X

$$H^p_{\mathcal{M}}(X, \mathbb{Z}(q)) \cong \hom_{DM_{gm}(k)}(M(X), \mathbb{Z}(q)[p])$$

$$DM_{gm}(k) \rightsquigarrow DM^{eff}_{gm}(k) \subseteq DM^{eff}_{-}(k)$$

"invert $\mathbb{Z}(1)$"

bounded above complexes of Nisnevich shf with transfers with homotopy invariant cohomologies.

<u>three key words</u>:

- homotopy invariant
- Nisnevich shf.
- with transfers.

$$Sm/k \rightsquigarrow SmCor/k \rightsquigarrow D^-(Shv_{Nis}, (SmCor(k)))$$

$$\rightsquigarrow DM^{eff}_{-}(k)$$

<u>Def</u>: $SmCor(k)$

Object: X: smooth$/k$ $\quad$ <u>Mor</u>: $\hom_{SmCor(k)}(X, Y)$

$$Sm/k \longrightarrow SmCor/k$$

$$X \longmapsto X$$

$$f \longmapsto \Gamma_f : \text{graph}.$$

$= \langle Z \mid Z \subseteq X \times Y$: closed subscheme. integral, $Z \to X$ finite surj over irred. comp. of $X \rangle_{\mathbb{Z}}$

Def: $F : SmCor/k \to Ab$: additive contravariant functor. presheaf with transfer.

(= pretheory)

Def: F : presheaf homotopy invariant

$\overset{}{\underset{\text{def}}{\iff}}$ $\forall X \in Sm/k$ $\quad F(X) \xrightarrow[\sim]{pr_1^*} F(X \times A^1)$: isom.

Def: X : scheme. $\{U_i \to X\}$ が Nisnevich covering

$\underset{\text{def}}{\iff}$ $\underbrace{\forall x \in X \text{ について}, \exists i, \exists u_i \in U_i}_{\text{étale covering であって,}}$ $\quad k(x) \xrightarrow{\sim} k(u_i)$: isom.

Zar Nis. Et

coarse $\longleftrightarrow$ fine $\quad \rightsquigarrow (Sm/k)_{Nis}$

Zar loc. ~~hensel~~ strict hensel

Def:

F : Nis. sheaf with transfer

F : presheaf with transfer

Nis sheaf on $(Sm/k)_{Nis}$. $\nearrow Sm/k$ $\quad SmCor/k \to Ab$.

$Shv_{Nis}(SmCor(k))$

$D^-(Shv_{Nis}(SmCor(k)))$: derived category of Nisnevich sheaf with transfers.

$\cup|$

$DM^{eff}_-(k)$: bounded above cpxes of. cohomology が homotopy invariant.

Def: $X \in Sm/k$

$\mathbb{Z}_{tr}(X)$: representable presheaf by X on $SmCor(k)$

$$Y \mapsto \hom_{SmCor(k)}(Y, X) \overset{\text{df}}{=}: \mathbb{Z}_{tr}(X)(Y)$$

Def: (Suslin complex)

F: presheaf. $C_*(F)$: Suslin complex

$$\Delta^n := \mathrm{Spec}\, k[T_0, \dots, T_n] \Big/ \left(\sum_{i=0}^{n} T_i - 1\right) \qquad C_n(F) := F(\Delta^n \times -)$$

associated cpx.

$$RC_* : D^-(\mathrm{Shv}_{Nis}(\mathrm{SmCor}(k))) \supset DM_-^{eff}(k)$$

$\to C_*(F)$: has homotopy invariant cohomologies

각 F, Nisnevich shf.

Def: $C_*(\mathbb{Z}_{tr}(X)) \in DM_-^{eff}(k)$ eff.

$\overset{\text{df}}{=} M(X)$ motive of X

tensor str: $\mathbb{Z}_{tr}(X) \otimes \mathbb{Z}_{tr}(Y) := \mathbb{Z}_{tr}(X \times Y)$

F: presheaf with transfer.

take res. of F, G by "$\mathbb{Z}_{tr}(X)$" $\rightsquigarrow F \otimes G$

$\rightsquigarrow$

$DM_-^{eff}(k) \ni M, M'$

$$M \otimes M' \overset{\text{dfn}}{:=} RC_*(M \overset{L}{\otimes} M')$$

$M(X) \otimes M(Y) \simeq M(X \times Y)$

internal hom: F, G: presheaf with transfer.

$$\underline{\mathrm{Hom}}(F,G)(X) \underset{\text{df}}{:=} \mathrm{Hom}(F \otimes \mathbb{Z}_{tr}(X), G)$$

$$\Rightarrow \mathrm{Hom}(F, \underline{\mathrm{Hom}}(G,H)) \simeq \mathrm{Hom}(F \otimes G, H)$$

$$\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n}) := \mathrm{coker}\left(\bigoplus_{i=0}^{n-1} \mathbb{Z}_{tr}(\mathbb{G}_m^{\times n-1}) \to \mathbb{Z}_{tr}(\mathbb{G}_m^{\times n})\right)$$

$$\mathbb{Z}(n) \overset{\mathrm{dfn}}{:=} C_*\left(\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n})\right)[-n]$$

$$\begin{array}{ccc} Sm/k & \longrightarrow & \underline{DM}^{\mathrm{eff}}_{-}(k) \\ \Psi & & \Psi \\ X & \longmapsto & M(X) \end{array}$$

an algebraic ~~corr~~

char $k = 0$

$$M(X)(\mathrm{Spec}\,k) = \left\langle \begin{array}{c} \text{a} \\ Z \subseteq \Delta^{\cdot} \times X \\ \text{fini. surj.} \searrow \quad \downarrow \\ \Delta^{\cdot} \end{array} \right\rangle_{\mathbb{Z}}.$$

$$\simeq \hom\left(\Delta^{\cdot}, \coprod_{d=0}^{\infty} \mathrm{Sym}^d(X)\right)^{+} \nwarrow \text{gr. completion.}$$

$$\mathrm{Hom}_{\substack{\text{conti.}\\\text{map.}}}\left(\Delta^{\cdot}_{top}, \coprod_{d=0}^{\infty} \mathrm{Sym}^d(X(\mathbb{C}))\right)^{+}$$

$$\xleftarrow[\text{q-isom}]{} \mathbb{Z}\left(\mathrm{Hom}_{\substack{\text{conti.}\\\text{map}}}\left(\Delta^{\cdot}_{top}, X(\mathbb{C})\right)\right)$$

12/19 Sato. K

B-K conj. & B-L conj. (cycle class maps)

important notation

§.1. Cycle Class. (complement)

§.2. higher Chow Groups. & B-L-conj.

§.3. B-K conj.

§.4. examples in arithmetic situation

we like consider under the case where $\exists X_0/\mathbb{R}$ s.t. $X_0 \otimes_{\mathbb{R}} \mathbb{C} \cong X$

§.1. Cycle Class.

X : smooth alg. var $/\mathbb{C}$

$$CH^c(X) \longrightarrow H^{2c}_{an}(X(\mathbb{C})^{an}, \mathbb{Z})$$

not compatible with complex conj.

Case $n=1$

$$0 \to 2\pi i \mathbb{Z} \xrightarrow{\text{標準的}} \mathcal{O}_{X^{an}} \to \mathcal{O}^{\times}_{X^{an}} \to 0 \quad \text{(exact)} \text{ on } X(\mathbb{C})^{an}$$

$$H^1(X(\mathbb{C})^{an}, \mathcal{O}^{\times}_{X^{an}}) \longrightarrow H^2_{an}(X(\mathbb{C})^{an}, 2\pi i \mathbb{Z})$$

complex conj.

$$\mathrm{cl}_X(Z)$$

Isom class gp of complex line b'dle

$H^1_{Zar}(X, \mathcal{O}^*_X)$

isom class gr of algebraic line b'dle on X

$\mathrm{Pic}\, X$

$CH^1(X)$

$$CH^1(X) \longrightarrow H^2_{an}(X(\mathbb{C})^{an}, 2\pi i \mathbb{Z})$$

compatible with complex conj.

Case. $n \geq 2$.

For $Z \subseteq X$: integral closed alg. subvar. of codim $= c$ on X

$$\mathrm{cl}_X(Z) \in H^{2c}_{an}(X(\mathbb{C})^{an}, \underbrace{(2\pi i)^c \mathbb{Z}})$$

complex conj. acts by $(-1)^c$

Ex: $c=2$ $Z = D_1 \cap D_2$: smooth divisors intersecting transversally.

$$\mathrm{cl}_X(Z) := \mathrm{cl}_X(D_1) \cap \mathrm{cl}_X(D_2)$$

$$H^2(X, (2\pi i)\mathbb{Z}) \qquad H^2(X, (2\pi i)\mathbb{Z})$$

cycle class gives

$$\begin{pmatrix}\text{rational}\\ \text{equivalence}\\ \text{to } 0\end{pmatrix} \subseteq Z^c(X) \longrightarrow H^{2c}_{an}(X(\mathbb{C})^{an}, (2\pi i)^c\mathbb{Z})$$

$Z^c(X)$ = gp of alg. cycles on X of codim $= c$

$\{0\}$? $\subseteq H^{2c}_{an}(X(\mathbb{C})^{an}, (2\pi i)^c\mathbb{Z})$

これは、

① : homotopy invariance of $H^{2c}_{an}(X(\mathbb{C})^{an}, (2\pi i)^c\mathbb{Z})$

② : intersection product と cup 積 の compatibility

$Z \subseteq X \times \mathbb{A}^1$

Algebraic Version

k : field X : smooth algebraic variety $/ k$

$n \in \mathbb{N}_{\geq 2}$ $c \in \mathbb{Z}_{\geq 0}$

$\frac{1}{n} \in k$; assumption.

μ_n : étale sheaf of n-th roots of unity on X.

$$(U \xrightarrow{\text{étale}} X) \longmapsto \{x \in \Gamma(U, \mathcal{O}_U) \mid x^n = 1\}$$ という sheaf.

Rem : $k = \mathbb{C}$ の場合,

$$\begin{array}{ccc} ((2\pi i)^c\mathbb{Z}) \otimes \mathbb{Z}/n\mathbb{Z} & \longrightarrow & \mu_n^{\otimes c} \\ \Psi & & \Psi \\ (2\pi i)^c \otimes \bar{1} & \longmapsto & \underbrace{\exp\left(\frac{2\pi i}{n}\right) \otimes \cdots \otimes \exp\left(\frac{2\pi i}{n}\right)}_{c\text{ コ}} \end{array}$$

$\mathbb{Z}/n\mathbb{Z}$ 一加群

Returning to the algebraic situation we have.

$$\begin{array}{ccc} Z^c(X) & \longrightarrow & H^{2c}_{\acute{e}t}(X, \mu_n^{\otimes c}) \\ \downarrow & & \\ CH^c(X)/n & & \end{array}$$

in a similar way.

Rem : $k = \mathbb{C}$ の場合.

$$\begin{array}{ccc} CH^c(X) & \longrightarrow & H^{2c}_{an}(X(\mathbb{C})^{an}, (2\pi i)^c\mathbb{Z}) \\ \downarrow & & \downarrow \\ H^{2c}_{\acute{e}t}(X, \mu_n^{\otimes c}) & \xrightarrow{\sim} & H^{2c}(X(\mathbb{C})^{an}, \mu_n^{\otimes c}) \end{array}$$

exponential map

§. 2. higher Chow Groups.

X : alg. var. smooth.

$$\bigoplus_{i=0}^{\dim X} CH^i(X)_{\mathbb{Q}} \xrightarrow{\cong} K_0(X)_{\mathbb{Q}}$$

$$[Z] \longmapsto [\text{proj. res. of } \mathcal{O}_Z]$$

Def (Bloch):

k, X, c algebraic situation $\downarrow$ higher Chow Group. cycle.

$\downarrow$ $K_q(X)$: higher K 理論. homotopy theoretic.

$Z^c(X,*)$: $\quad \to Z^c(X,q) \xrightarrow{d} \cdots \to Z^c(X,1) \xrightarrow{d} Z^c(X,0) \to 0 \to$

cpx of abelian gps. determined by c

$Z^c(X,q)$ closed integral subvariety codim $= c$.

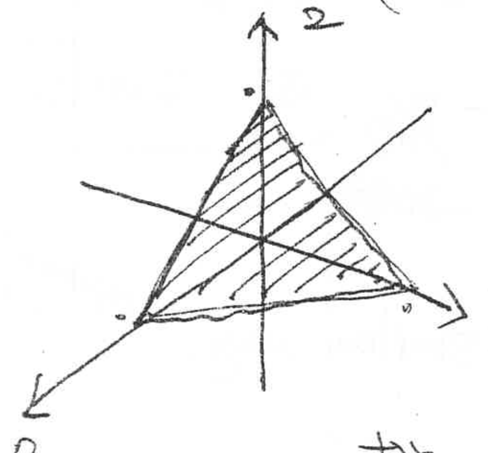
$$:\underset{\text{dfn}}{=} \left\{ I \overset{c}{\subset} (X \times \Delta^q) \;\middle|\; \begin{array}{l} I \text{ meets properly} \\ \text{with all faces on } X \times \Delta^q \end{array} \right\}$$

$$\Delta^q = \operatorname{Spec} k[T_0, \cdots, T_q]/(T_0 + \cdots + T_q - 1) \xrightarrow{\sim} \mathbb{A}^q_k$$

$$\Delta^q \text{の faces} :\overset{\text{dfn}}{=} \left\{ \begin{array}{l} T_{i_1} = \cdots = T_{i_r} = 0 \\ \text{で, def される } \Delta^q \text{ の} \\ \text{closed subvariety.} \end{array} \right\}$$

$q=2$

$$\Delta^2 = \operatorname{Spec}\left(k[T_0, T_1, T_2] / (T_0 + T_1 + T_2 - 1) \right)$$



face of $X \times \Delta^q = X \times (\text{face of } \Delta^q)$

$d :=$ alternate sum of pull-backs

$$CH^c(X,q) \overset{\text{dfn}}{=} H_q(Z^c(X,*)) \text{ とおく。}$$

(Bloch)

すると,

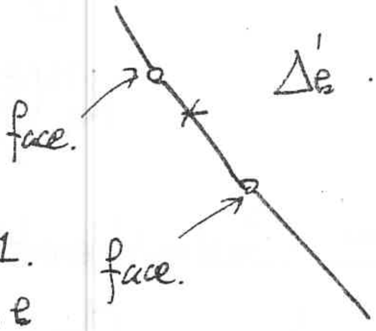
$$\bigoplus_{q=0} CH^{2m-q}(X,q)_{\mathbb{Q}} \simeq K_m(X)_{\mathbb{Q}}$$

<u>Ex.1</u>: $CH^1(\mathrm{Spec}\, k, q) \xrightarrow{\sim} \begin{cases} k^\times & (q=1) \\ 0 & \text{otherwise}. \end{cases}$

(Bloch's Rersults)

Nesterenko - Suslin - Totaro

$$\begin{array}{ccc} k^\times & \longrightarrow & CH^1(\mathrm{Spec}\, k, 1) \\ \Psi & & \Psi \\ a & \longmapsto & \left(\frac{1}{1-a}, \frac{-a}{1-a}\right) \in \Delta^1_k \\ 1 & \longmapsto & 0. \end{array}$$



<u>Ex 2</u>: $(q \geq 2)$

$$\underbrace{k^\times \otimes \cdots \otimes k^\times}_{q} \xrightarrow{\cong} CH^1(k,1) \otimes \cdots \otimes CH^1(k,1) \xrightarrow{\text{product}} CH^q(k,q)$$

is surjective and the kernel is generated by Steinberg Symbol.

Steinberg's Symbol: Symbol of the form $\{a_1, \ldots, a_q\}$ $\exists i\ \exists j$ $a_i + a_j = 1$ $a_j, a_i \in k \setminus \{0,1\}$

X: smooth$/k$ $\Longrightarrow$ we have canonical map.

$$CH^c(X,q) \longrightarrow H_{et}^{2c-q}(X, \mu_n^{\otimes c})$$

$$CH^c(X, q; \mathbb{Z}/n\mathbb{Z}) = H_q\left(Z^c(X,*) \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}\right) \xrightarrow{\alpha^{c,q}} H_{et}^{2c-q}(X, \mu_n^{\otimes c})$$

<u>Conj.(B-L)</u>:

$$CH^c(X, q; \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\alpha^{c,q}} H_{et}^{2c-q}(X, \mu_n^{\otimes c})$$

$\alpha^{c,q}$: bijective for any c, q with $q \geq c$.

Case: $c = q$ $\quad CH^q(k, q) \xrightarrow{\alpha^{q,q}} H^q_{et}(X, \mu_n^{\otimes q})$

$\quad\quad\quad\quad\quad\quad \wr\!| \; ; \mathbb{Z}/n\mathbb{Z}$

$\quad\quad\quad\quad\quad\quad K_q^M(k)/n$

: Bloch-Kato Conjecture

Thm (Suslin-Voevodsky / Geisser-Levine)

Assume. Conj. (B-K) holds for any finitely generated fields $/k$ $\Rightarrow$ (B.L)-conj. holds for any smooth var. $/k$.

$X/\mathbb{Z}$: of finite type.

12/20 Geisser :

$$\zeta(X,s) = \prod_{x \in |X|} \frac{1}{1 - Nx^{-s}}$$

want to understand this.
Next time, special values

$X/\mathbb{F}_q$: finite field X ; smooth, projective

Grothendieck ; Use $\boxed{\text{pure motive}}$ to prove / understand Weil Conjecture

arbitrary $X \rightsquigarrow$ mixed motive.

k : field $\mathcal{V}_k$: smooth projective ~~variety~~ k-schemes

$Z^d(X)$: free Abelian group on irred. subvar. of codim $= d$.

those equivalence relations $\sim$ s.t. pull-back, push-forward.

$$\underset{\substack{\uparrow \\ \text{Chow Motives.}}}{\text{rat. eq.}} \subseteq \cdots \subseteq \text{alg. eq.} \overset{\text{conjectually equal.}}{\subseteq} \underset{\substack{\uparrow \\ \text{very good eq. (Jannsen)}}}{\text{arith. eq.}}$$

$A^d(X) = Z^d(X)/\sim$ Remark : Don't read $\sim$ in Voevodsky's situation

$\phi : X \to Y$ induces $\phi^* : A^*(Y) \to A^*(X)$

$\phi_* : A^*(X) \to A^{*+\dim Y - \dim X}(Y)$

product : $A^d(X) \otimes A^e(X) \to A^{d+e}(X)$

if, $\dim X = d$ define $\mathrm{Corr}^r(X \times Y) = A^{d+r}(X \times Y)$

extend linearly to $X = \coprod_{i \in I} X_i$. get

$$\mathrm{Corr}^r(X,Y) \times \mathrm{Cor}^s(Y,S) \longrightarrow \mathrm{Corr}^{r+s}(X,S)$$

$$f \otimes g \longmapsto g \circ f = \underbrace{(p_{12}^* f . p_{23}^* g)}_{(p_{13})_*}$$

$$\begin{tikzcd}
 & X \times Y \times S \arrow[dl, "p_{12}"'] \arrow[d, "p_{13}"] \arrow[dr, "p_{23}"] & \\
X \times Y & X \times S & Y \times S
\end{tikzcd}$$

$\mathcal{M}_k$: cat. objects (X,p,m) $X \in Var_k$, $m \in \mathbb{Z}$. $p = p^2 \in Cor^0(X,X)$

$Hom((X,p,m),(Y,q,n)) = q \cdot Cor^{n-m}(X,Y) \circ p$

Thm: $\mathcal{M}_k$ is additive $\mathbb{Q}$-linear category, which is pseudo-Abelian

Pf. $M = (X,p,m)$ and $f = pfp \in End(M)$ then

$M = (X, p-pfp, m) \oplus (X, pfp, m)$ $\quad (X,p,m) \oplus (Y,q,m) = (X \sqcup Y, p \oplus q, m)$

In general,

$\mathcal{M}_k$ is not Abelian, but if $F : M \to N$ has a left inverse $g \circ f = id_M$, then $F : M \xrightarrow{\sim} fg N \subseteq N$

right $f \circ g = id_N$ then $M \supseteq gfN \xleftarrow{\sim} N : g$

There is a functor $h : V_k^{opp} \longrightarrow \mathcal{M}_k$ $\quad \phi : Y \to X$

$X \longmapsto (X, id, 0) = h(X)$ $\quad h(\phi) = \phi^* = [{}^t\Gamma_\phi] \in Corr^0(X,Y) = Hom_{\mathcal{M}_k}(h(X), h(Y))$

Tensor Product

$(X,p,n) \otimes (Y,q,m) = (X \times Y, p \times q, m+n)$

on morph $q_1 f_1 p_1 \otimes q_2 f_2 p_2 = (q_1 \otimes q_2)(f_1 \otimes f_2)(p_1 \otimes p_2) \in Corr(X_1 \times X_2, Y_1 \times Y_2)$

$1 = (Spec\, k, id, 0)$ identity for $\otimes$

$\mathcal{L} = (Spec\, k, id, -1)$ $\quad ((X,p,m) = ph(X) \otimes \mathcal{L}^{-m} \subseteq h(X) \otimes \mathcal{L}^{-m})$

$\mathcal{L}^n \overset{df}{:=} \mathcal{L}^{\otimes n} = (Spec\, k, id, -n)$

$\phi : Y \to X$ $\dim X = d$, $\dim Y = e$. $\quad ({}^t\Gamma_\phi) \in A^d(Y \times X) = Corr^{d-e}(Y, X)$

$\rightsquigarrow \quad \phi_*: h(Y) \longrightarrow h(X) \otimes \mathbb{L}^{e-d}$ X : irred. $\dim X = d$, $x \in X(k)$

$\alpha: X \to \mathrm{Spec}\, k$ $\quad x^*\alpha^* = \mathrm{id}$ so $\alpha^*: \mathbb{1} \hookrightarrow h(X)$: subobject $h^0(X)$

$\alpha_* x_* = \mathrm{id}$ so $\alpha_*: h(X) \longrightarrow \mathbb{L}^d$ quotient $h^{2d}(X)$

in fact $\quad h^0(X) = (X, \{x\} \times X, 0) = \mathbb{1}$

$h^{2d}(X) = (X, X \times \{x\}, 0) = \mathbb{L}^d$ $\quad$ diagonal $\Delta \sim X \times \{x\} + \{x\} \times X$

e.g. $\quad h(\mathbb{P}^1) = h^0(\mathbb{P}^1) \oplus h^2(\mathbb{P}^1) = \mathbb{1} \oplus \underline{\mathbb{L}}$

<u>direct sum</u>: $\quad M = (X, p, m) \quad N = (Y, q, n)$ if $m < n$

then $M \doteq (X, p, n) \otimes \mathbb{L}^{n-m} = (X, p, n) \otimes h^2(\mathbb{P}^1)^{n-m}$

$= (X \times (\mathbb{P}^1)^{n-m}, p', n)$

$M \oplus N = (X \times (\mathbb{P}^1)^{n-m}, p' \oplus q, n)$

$\vee : \mathcal{M}_k^{opp} \longrightarrow \mathcal{M}_k \quad (X, p, m)^\vee = (X, {}^t p, d-m) \quad d = \dim X$

transpose on morph

$h(X)^\vee = h(X) \otimes \mathbb{L}^{-d}$ $\quad$ "Poincaré Duality" $\quad M^{\vee\vee} = M$.

$\mathrm{Hom}(M \otimes N, P) = \mathrm{Hom}(M, N^\vee \otimes P)$ so $\underline{\mathrm{Hom}}(M, N) = M^\vee \otimes N$

$\rightsquigarrow$ "rigid additive tensor category"

<u>Manin's identity principal</u>:

$A^n(X) = \hom(\mathbb{L}^n, h(X)) \quad \xi^*: h(X) \longrightarrow \mathbb{L}^{\dim X - n}$; transpose.

$\xi \longmapsto \xi_* \quad \bar{\xi}: h(X) \otimes \mathbb{L}^d \longrightarrow h(X) \otimes h(X) = h(X \times X) \xrightarrow{\Delta^*} h(X)$

so defined $\quad A^n(M) = \hom(\mathbb{L}^n, M)$

$$\mathcal{M}_k \to \mathrm{Fct}(\mathcal{M}_k, \mathrm{Vect}_{\mathbb{Q}}) \qquad A^0(M \otimes N^\vee)$$

$$M \longmapsto A^0(M \otimes -^\vee) \qquad = \mathrm{Hom}(1, M \otimes N^\vee)$$

$$= \mathrm{Hom}(N, M)$$

: fully faithful

every $N \in \mathcal{M}_k$ is a summand of $h(Y) \otimes \mathcal{L}^n$

$A^0(M \otimes h(Y) \otimes \mathcal{L}^n) = A^{-n}(M \otimes h(Y))$ so $\mathcal{M}_k \to \mathrm{Fct}(\mathcal{V}_k^{opp}, \mathrm{Vect}_{\mathbb{Q}})$, $M \mapsto \omega_M$ is fully faithful

$\omega_M(Y) = A^*(M \otimes h(Y))$

MIP i): $f: M \to N$ is an isom $\iff$ $\omega(Y): A^*(M \otimes h(Y)) \longrightarrow A^*(N \otimes h(Y))$ is an isom for all $Y \in \mathcal{V}_k$

ii): $f, g: M \to N$ are equal $\iff$ $\omega_f(Y) = \omega_g(Y)$ $\forall Y \in \mathcal{V}_k$.

iii): $0 \to M' \to M \to M'' \to 0$

$\iff$ $0 \to A^0(M' \otimes h(Y)) \to A^0(M \otimes h(Y)) \to A^0(M'' \otimes h(Y)) \to 0$ (exact)

Thm: $S \in \mathcal{V}_k$ $\mathcal{E}$: locally free shf. $X = \mathbb{P}(\mathcal{E}) \xrightarrow[\pi]{} S$

$\xi = c_1(\mathcal{O}_X) \in A^1(X)$; then

$\sum \xi^i \pi^*$; $\bigoplus_{i=0}^{r} h(S) \otimes \mathcal{L}^i \xrightarrow{\sim} h(X)$ is isom.

Thm: $Y \hookrightarrow X$ codim > 1 $\qquad X'$: X の Y における blow-up.

$$\begin{array}{ccc} Y & \hookrightarrow & X \\ \uparrow & & \uparrow \\ Y' & \hookrightarrow & X' \end{array}$$

$Y' \to Y$: projective budle.

$0 \to h(Y) \otimes \mathcal{L}^{r+1} \xrightarrow{(i_*,\,)} h(X) \oplus h(Y') \otimes \mathcal{L} \longrightarrow h(X') \to 0$ (exact)

Curves:

$$h(X) = h^0(X) \oplus h^1(X) \oplus h^2(X) \quad h^1(X) = (X, id - p_1 - p_2, 0)$$

non-canonically. X, X': curves.

$$\mathrm{Hom}(h^1(X), h^1(X')) = \mathrm{Hom}(\mathcal{J}, \mathcal{J}')_{\mathbb{Q}}$$

if, $\mathcal{J}, \mathcal{J}'$ ~~and~~ are Jacobians of X, X'.

$$\mathrm{Hom}(\mathcal{L}, h^1(X)) = \begin{cases} 0 & \text{if } \sim \text{ is num.} \\ \mathcal{J}(k)_{\mathbb{Q}} & \text{if } \sim \text{ is rat.} \end{cases}$$

$$\mathrm{Hom}(\mathcal{L}, h(X)) = A^1(X)$$

$$\mathrm{hom}(\mathcal{L}, h^0(X)) = 0, \quad \mathrm{Hom}(\mathcal{L}, h^2(X)) = \mathbb{Q}$$

so $\mathrm{Hom}(\mathcal{L}, h^1(X)) = \cancel{\mathrm{hom}(\mathcal{L}, h^1(X))}$

$= \ker(\deg : A^1(X) \to \mathbb{Q}.)$

<u>Prop</u>: If k is not contained in $\overline{\mathbb{F}_q}$ then $\mathcal{M}_k^{rat}$ is not Abelian cat

<u>Prop</u>: (Jannsen) $\mathcal{M}_k^{num}$ is Abelian.

(<u>Proof</u>): There is an elliptic curve E/k. $P \in E(k)$

$\xi = (P) - (O) \in A^1(E)$ $\quad \xi_* : \mathcal{L} \to A^1(E)$ is non-zero.

by $\otimes$ $\quad \underset{\underset{0}{\shortparallel}}{\xi_* \circ \xi^*} : h^1(E) \otimes \mathcal{L} \to h^1(E) \rightsquigarrow \xi_*$ is not.

If $\mathcal{M}_k^{rat}$ were abelian, then $\ker \xi_*$ is a proper subobject of $\mathcal{L}$ ~~have~~ hence 1. $\quad \mathrm{Hom}(1,1) = \mathbb{Q}.$

~~and~~ has proper subobject

k : field

$Sch/k = \{\text{scheme sep. of.t}\}/k \supseteq Sm/k$

$Sm/k \longrightarrow SmCor(k)$ Obj : Same

$\uparrow$ additive cat.

closed. int.

$Hom(X, Y) = \left\{ \begin{array}{c} Z \subset X \times Y \\ \text{finite surj.} \searrow \downarrow \\ X \end{array} \right\} \mathbb{Z}$

$PSWT/k \supseteq NSWT/k$

$\left\{ \begin{array}{l} \text{add. contr.} \\ \text{fct. } SmCor(k) \to Ab \end{array} \right\} = PSWT/k$

$NSWT/k = \{ F \mid Sm/k \text{ is a Nisneich shf} \}$

Abelian cat. w enough proj. and inj.

$\mathbb{Z}_{tr}(X) \in PSWT/k$

rep. by $X \in SmCor(k)$

In fact, a Nisnevich sf $\quad M(X) \overset{dfn}{:=} C_*(\mathbb{Z}_{tr}(X))$

$DM_-^{eff} \subseteq D^-$: cpx. with $\quad \in D^- := D^-(NSWT/k)$

cohomology $\to$ homotopy inv. shf

$\mathbb{Z}(q) := C_*^*(\underbrace{\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q})}_{\uparrow})[+q] \in D^-$

a direct summand of $\mathbb{Z}_{tr}(\mathbb{G}_m^{\times q})$

$= [\to \underset{q-1}{C_*^{-1}(\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q}))} \to \underset{q}{C_*^{0}(\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q}))} \to]$

ex : $\mathbb{Z}(1) = C_*^*(Coker(\mathbb{Z} \to \mathbb{Z}_{tr}(\mathbb{G}_m)))[-1]$

Thm : 0) : $M(X), \mathbb{Z}(q) \in DM_-^{eff}(k)$

1) : $Hom_{DM_-^{eff}(k)}(M(X), \mathbb{Z}(q)[p])$

$\simeq H^p_{Nis}(X, \mathbb{Z}(q)) \simeq H^p_{Zar}(X, \mathbb{Z}(q))$

2): $\mathbb{Z}(1) \simeq \mathcal{O}^\times[-1]$

(Rem: $\mathcal{O}^\times \in \mathrm{NSW}$)

Similar Prop holds for étale NOT for Zariski

$\downarrow$

The category NSWT

Notation: C: site T: topology

$C^\wedge$: the category of Abel. presheaves on C

$C_T^\sim$: the category of Abel. T-sheaves on C

Prop: $F \in \mathrm{PSWT}/k$ F_{Nis}: the Nisnevich shf associated to F regarded as an object in $(Sm/k)^\wedge$.

1): F_{Nis} has a unique str. of PSWT s.t.

$F \longrightarrow F_{Nis}$ is a morph in PSWT!

2): $G \in \mathrm{NSWT}/k$ $F \to G$: morph in PSWT

$\underset{\text{induce}}{\Longrightarrow}$ $F_{Nis} \to G$: induced map in $(Sm/k)^\sim_{Nis}$

$$\begin{tikzcd} F_{Nis} \arrow[r] & G \\ F \arrow[u] \arrow[ur] & \end{tikzcd}$$

Results

$$\begin{tikzcd} \mathrm{NSWT}/k \arrow[d, "\tilde{\varphi}"'] \arrow[r, hook, shift right, "\tilde{i}"'] & \mathrm{PSWT}/k \arrow[l, shift right, "\tilde{a}"'] \arrow[d, "\hat{\varphi}"] \\ (Sm/k)^\sim_{Nis} \arrow[r, hook, shift right, "i"'] & (Sm/k)^\wedge \arrow[l, shift right, "a"'] \end{tikzcd}$$

enough inj.

($\hat{\varphi}, \tilde{\varphi}$: forget.)

1): $\tilde{\varphi}\tilde{a} = a\hat{\varphi}$

2): $\tilde{a} \leftrightarrow \tilde{i}$: adjoint

$\tilde{a} \circ \tilde{i} \cong \mathrm{id}$

3): NSWT: Abel. $\tilde{a}$: exact.

4): $E = [0 \to F_1 \to F_2 \to F_3 \to 0]$ in NSWT/k.

E: exact $\Longleftrightarrow$ $\tilde{\varphi}E$: exact

Thm: $F \in NSWT$, $X \in Sm/k$

precisely

$$\mathrm{Ext}^{i}_{NSWT/k}(\mathbb{Z}_{tr}(X), F) \simeq H^{i}_{Nis}(X, \tilde{\varphi} F).$$

In particular $F \in DM^{eff}_{-}$, $X \in Sm/k$.

$$\Rightarrow \hom_{D^-}(\mathbb{Z}_{tr}(X), F[i]) \simeq H^i_{Nis}(X, F)$$

∵ Suffice to Prove 「I : inj obj. in NSWT $\Rightarrow$ $H^i_{Nis}(X, I) = 0$ $(i > 0)$」

This follows from.

Key lemma: $Y \xrightarrow{f} X$: Nisnevich cov.

$$\Rightarrow 0 \leftarrow \mathbb{Z}_{tr}(X) \leftarrow \mathbb{Z}_{tr}(Y) \leftarrow \mathbb{Z}_{tr}(Y \times_X Y) \leftarrow \cdots$$

(exact) in $(Sm/k)^{\sim}_{Nis}$ //

ideal of pf of key lemma.

$$0 \leftarrow \hom(S, X) \leftarrow \hom(S, Y) \leftarrow \hom(S, Y \times_X Y) \leftarrow \cdots$$

(S : loc. hen.)

(Hom = Hom in SmCor)

$$\begin{tikzcd} Z_Y \arrow[d] \arrow[r, phantom, "\subset"] & S \times Y \arrow[d] \\ Z \arrow[r, phantom, "\subset"] \arrow[dr, "\text{finite, surj.}"'] & S \times X \arrow[d] \\ & S \end{tikzcd}$$

generator

S : hensel

$\Rightarrow$ Z : hensel $\Rightarrow$ $Z_Y \to Z$: split

: Nisnevich covering

homotopy inv pre.shf w.T.:

Thm: k : perfect $F \in PSWT$; homotopy inv.

(1): F_{Nis} is also h.i. &

$$X \mapsto H^i_{Nis}(X, F_{Nis}) \text{ has a str. of}$$

ρ : pr. w. tr.

(2): $H^i_{Zar}(X, F_{Zar}) \simeq H^i_{Nis}(X, F_{Nis})$ $(i \geq 0)$

In particular $F_{Zar} (\simeq F_{Nis})$ has a str. of pres. w. tr.

これから、以下全て k: perfect とする

Cor: $HI(k) (\subseteq NSwT)$ is closed under taking ker, Coker, extension

↑ the full subcategory of h.i. NSwT

In particular $\begin{cases} \cdot\ HI(k) : \text{Abel \& the inclusion is exact.} \\ \cdot\ DM^{eff}_-(k) : \text{subtri. cat.} \end{cases}$

Cor: $F \in NSwT \Rightarrow C_*(F) \in DM^{eff}_-(k)$

($H_i(C_*(F))$ is h.i. so is $H_i(C_*(F))_{Nis}$.)

Ex: $M(X), \mathbb{Z}(q) \in DM^{eff}_-(k)$ #

Cor: $F \in DM^{eff}_-(k)$, $X \in Sm/k$

$$H^i_{Nis}(X, F) \simeq H^i_{Zar}(X, F)$$

$$RC^* ;\ D^- \longrightarrow DM^{eff}_- ;\ F^\cdot \longmapsto Tot(C^*(F^\cdot))$$

$$DM^{eff}_- \overset{i}{\hookrightarrow} D^-$$

Thm: $F \in D^-$, $G \in DM^{eff}_-$

$$\Rightarrow \mathrm{Hom}_{DM^{eff}_-}(RC^*(F), G) \simeq \mathrm{Hom}_{D^-}(F, G)$$

(i.e. RC^* は、i の left adj.)

In particular,

$$\mathrm{Hom}_{DM^{eff}_-}(M(X), F) \simeq \mathrm{Hom}_{D^-}(\mathbb{Z}_{tr}(X), F) \quad (F \in DM^{eff}_-)$$

Key lem's for pf:

Lem: $F, G \in NST/k$ F: contractible G: h.i.

$$\Rightarrow \mathrm{Ext}^i_{NST}(F, G) = 0 \ (\forall i)$$

Lem: $\mathrm{Coker}(F \to C_1(F))$: contractible.

F : contractible $\overset{\text{def.}}{\Leftrightarrow}$

$$\begin{tikzcd} & & F \\ F \arrow[r, dashed, "\exists"] \arrow[urr, bend left, "0"] \arrow[drr, bend right, "\mathrm{id}"'] & C_1F \arrow[ur, "\partial_0"] \arrow[dr, "\partial_1"'] & \\ & & F \end{tikzcd}$$

$C_1F : X \mapsto F(X \times \Delta^1)$

Sketch of prf(2): $C_*(\mathbb{Z}_{tr}(\mathbb{G}_m)) \simeq \mathbb{Z} \oplus \mathcal{O}^\times$

$$M^*(\mathbb{P}^1; 0, \infty) ; U \longmapsto \left\{ f \in k(U \times \mathbb{P}^1) \,\middle|\, \begin{array}{l} f \text{ equals to } 1 \\ \text{on } U \times \{0, \infty\} \end{array} \right\}$$

$$\mathrm{Pic}(\mathbb{P}^1; 0, \infty) : U \mapsto \mathrm{Pic}(U \times \mathbb{P}^1, U \times \{0, \infty\})$$

$$\overset{\text{df}}{:=} \left\{ (L, \phi) \,\middle|\, \begin{array}{l} L : \text{line bdle on } U \times \mathbb{P}^1 \\ L|_{U \times \{0,\infty\}} \underset{\phi}{\simeq} \mathcal{O} \end{array} \right\} \Big/ \mathrm{Isom}.$$

Then for $U \in Sm/k$.

$$\mathbb{Z} \longrightarrow \mathcal{O}(\mathbb{Z})$$

$$0 \to M^*(\mathbb{P}^1, 0, \infty)(U) \to \mathbb{Z}_{tr}(\mathbb{G}_m)(U) \to \mathrm{Pic}(\mathbb{P}^1, 0, \infty)(U) \to 0$$

$\uparrow$ U: affine

Ts).

$$f \mapsto \mathrm{div}(f) \qquad \mathbb{Z}_{tr}(\mathbb{G}_m)(U) = \left\{ \begin{array}{l} \text{divisors on } U \times \mathbb{P}^1. \\ \mathrm{supp} \cap (U \times \{0, \infty\}) \\ = \phi \end{array} \right\}$$

$\rightsquigarrow \quad (C_i = C^{-i})$

$$0 \to C_*\left(\mathcal{M}^*(\mathbb{P}^1, 0, \infty)\right) \longrightarrow C_*\left(\mathbb{Z}_{tr}(\mathbb{G}_m)\right) \to C_*\left(\mathrm{Pic}(\mathbb{P}^1, 0, \infty)\right) \longrightarrow 0$$

q.i.s. $\Big\|$ h.i. of Pic

$\uparrow$

In fact, acyclic for each U.

(can construct a homotopy explicitly)

$\mathrm{Pic}(\mathbb{P}^1, 0, \infty)[0]$

$\cong$

$\mathbb{Z} \oplus \mathcal{O}^\times$

[Batyrev] Birational CY mfds have same betti number.

12/20 Yasuda. T

(pf): p-adic integer + Weil Conj.

Kontsevich introduces the motivic integration as a complex analogue of p-adic units.
developped by Denef & Loeser & others

§. 1. Grothendieck ring of varieties $Y \underset{\text{closed}}{\subset} X$

/ k : field

$e_{top}(X) = e_{top}(X \setminus Y) + e_{top}(Y)$

- $e_{top}(X \times Y) = e_{top}(X)\, e_{top}(Y)$

- $\# X(\mathbb{F}_q) = \#(X \setminus Y)(\mathbb{F}_q) + \# Y(\mathbb{F}_q)$
- $\#(X \times Y)(\mathbb{F}_q) = \# X(\mathbb{F}_q) \cdot \# Y(\mathbb{F}_q)$

Some properties for other invariants

$\downarrow$

the universal one

$[\cdot] \in K_0(Var)$

$X = \coprod X_i$: stratification by locally closed subvarieties

$\Rightarrow [X] = \sum [X_i] \in K_0(Var)$

$X \supseteq C = \coprod_i C_i$, $C_i \subseteq X$: loc. closed

C : constructible subset

$\rightsquigarrow [C] \in K_0(Var)$

$[C] = \sum_i [C_i]$: well-defined.

§. 2. Motivic Measure.

X : variety / k

$\mathcal{C}(X) := \{ \text{constructible subsets of } X \}$

$C = \coprod C_i$

$\nu_X(C) = \sum \nu_X(C_i)$

ν_X is a "measure" in a broad sense.

$$\nu_X : \mathcal{C}(X) \longrightarrow K_0(Var)$$
$$C \longmapsto [C]$$

$F : X \longrightarrow K_0(\mathrm{Var})$: constructible function.

$\left(\underset{\text{def}}{\Longleftrightarrow} \right.$ every fibre $F^{-1}(a)$ が constructible. $\left. \right)$

$\# F(X) < \infty$ 有限和.

$$\rightsquigarrow \int F d\nu_X = \sum_{a \in K_0(\mathrm{Var})} \nu_X(F^{-1}(a)) \cdot a \in K_0(\mathrm{Var}_{k})$$

<u>Ex</u>: X : smooth var.

$\cup$

Y : smooth closed subvariety. of $\mathrm{codim} = r$.

$$F(x) := \begin{cases} 1 = [1\,pt] & (x \in X \setminus Y) \\ [\mathbb{P}^{r-1}] & (x \in Y) \end{cases} \qquad \int F d\nu_X$$

$$\int F d\nu_X = [X \setminus Y] + [\mathbb{P}^{r-1}][Y] = [\mathrm{Bl}_X Y]$$

§3. Jet & Arcs

X^d : smooth var / $k = \bar{k}$

Zariski Tangent vector.

$\mathrm{Spec}\left(k[t]/t^2\right) \longrightarrow X$

$TX = \{\text{tangent vectors}\}$ $\qquad TX \to X$: projection.

tangent bundle. $\qquad$ v.b. of rank $= d$.

$n \in \mathbb{Z}_{\geq 0}$ $\quad \boxed{n\text{-jet on } X}$: $\mathrm{Spec}\,\frac{k[t]}{t^{n+1}} \longrightarrow X$

$J_n X := \{n\text{-jets on } X\}$

smooth scheme.

$m \geq n$

$$\begin{array}{ccc} \mathrm{Spec}\,\frac{k[t]}{t^{n+1}} & \hookrightarrow & \mathrm{Spec}\,\frac{k[t]}{t^{m+1}} \\ & {\scriptstyle n\text{-jet}} \searrow & \downarrow {\scriptstyle m\text{-jet}} \\ & & X \end{array}$$

$\underset{\pi_n^m}{J_m X \longrightarrow J_n X}$ が存在する.

Fact:

π_n^{n+1} is a Zariski Local-trivial $\mathbb{A}^d$-fibration

arc on X : $\mathrm{Spec}\, k[[t]] \to X$

$J_\infty X \underset{\mathrm{dfn}}{:=} \{\text{arcs on } X\}$

$$\begin{tikzcd} \mathrm{Spec}\, \frac{k[[t]]}{t^{n+1}} \arrow[r] \arrow[dr, "n\text{-jet}"'] & \mathrm{Spec}\, k[[t]] \arrow[d, "\text{arc}"] \\ & X \end{tikzcd}$$

$J_0X \leftarrow J_1X \leftarrow J_2X \leftarrow \cdots \leftarrow J_\infty X$

$J_\infty X = \varprojlim_n J_nX$ ← この空間の上で積分したい。

$$\begin{tikzcd} \mathcal{C}(J_0X) \arrow[r, "(\pi_0^1)^{-1}"] \arrow[dr, "\nu_{J_0X}"'] & \mathcal{C}(J_1X) \arrow[r, "(\pi_1^2)^{-1}"] \arrow[d, "\mathbb{L}^{-d}\nu_{J_1X}"] & \mathcal{C}(J_2X) \arrow[dl, "\mathbb{L}^{-2d}\nu_{J_2X}"] \cdots \\ & K_0(\mathrm{Var}) & \end{tikzcd}$$

$\mathbb{L} := [\mathbb{A}^1]$

Motivic Measure on $J_\infty X$

$\underset{\text{roughly}}{:=} \lim_{n\to\infty} \mathbb{L}^{-nd}\nu_{J_nX}$

$\mathcal{M} := K_0(\mathrm{Var})[\mathbb{L}^{-1}]$

$\hat{\mathcal{M}}$: dimensional completion

$F_m \subseteq \mathcal{M}$: subgroup generated by $[V]\mathbb{L}^i$ with $\dim V + i \leq -m$.
$i \in \mathbb{Z}$

not ideal

$\{F_m\}_{m\in\mathbb{N}}$: descending filtration of $\mathcal{M}$

ring $\hat{\mathcal{M}} = \varprojlim \mathcal{M}/F_m$

Ex: $\sum_{i=0}^{\infty} [V_i]\mathbb{L}^{a_i}$ $\dim V_i + a_i \to -\infty$ $(i\to\infty)$

$\hat{\mathcal{M}}$ well-defined

motivic measure μ_X on $J_\infty X$

take values in $\hat{\mathcal{M}}$

$F : J_\infty X \to \hat{\mathcal{M}}$ "measurable" fcn.

Rem: $\hat{\mathcal{M}} \to \hat{K}_0(HS)$ ⇝ $\int F d\mu_X \in \hat{\mathcal{M}} \cup \{\infty\}$

weight completion

ex: $F \equiv 1 = [1pt]$

$$\int 1 \, d\mu_X = \mu_X(J_\infty X) = [X]$$

Birational Morphism & Transformation rule.

$$\begin{array}{ccc} f: Y & \to & X \\ \cup & & \cup \\ E & \to & D \\ \uparrow & & \| \\ \text{exc. sec.} & & f(E) \end{array}$$

: proper birational

一般に, $f: Y \to X$: morphism $\rightsquigarrow$ $0 \le n \le \infty$

$f_n : J_n Y \to J_n X$

: natural morphism

Fact ①

can neglect

$$\left\{ \begin{array}{l} J_\infty(E) \subseteq J_\infty(Y) \\ J_\infty(D) \subseteq J_\infty(X) \end{array} \right.$$

infinite codimensional of measure zero.

②: valuative criterion

$$f_\infty : J_\infty Y \setminus J_\infty E \overset{1:1}{\longleftrightarrow} J_\infty X \setminus J_\infty D$$

Th: (Transformation rule.) Change of variables formula)

$$\int F \, d\mu_X = \int (F \circ f_\infty) \, \underline{\mathbb{L}^{-\mathrm{ord} J_f}} \, d\mu_Y \in \hat{M} \cup \{\infty\}$$

$J_f \subseteq \mathcal{O}_X$: Jacobian ideal of f.

X & Y smooth, J_f is the ideal of $K_{Y/X}$

$= K_Y - f^* K_X$

relative can. div.

disopancy div.

Cor: X, Y, Z smooth projective $/\mathbb{C}$

$$\begin{array}{ccccc} & & Z & & \\ & \overset{f}{\swarrow} & & \overset{g}{\searrow} & \\ X & & & & Y \end{array}$$

f, g : birational

$K_{Z/X} = K_{Z/Y}$ ($\leftarrow$ OK. where X, Y : CY.)

$\Rightarrow$ $h^{p,q}(X) = h^{p,q}(Y)$

(prf): $[X] = \int 1 \, d\mu_X = \int \mathbb{L}^{-\mathrm{ord} J_f} \, d\mu_Z = \int 1 \, d\mu_Y = [Y]$ #.

No 5

Problem:
- $\exists$ natural isom. $H^i(X) \xrightarrow{\sim} H^i(\Sigma)$?
- Construct a "factorial" motivic integration!

The Category of mixed Tate Motives. 記 係. No. 1

Deligne – Goncharov, 12/20. Mochizuki. S

$\mathcal{MTM}(k)$:

$\otimes$

$\mathbb{Q}$-linear Bloch – Kriz

Properties : ① : Abelian category.

② : semi-simple obj. ; $\mathbb{Q}(n)$

③ : $\forall M \in \mathcal{MTM}(k)$

$\exists$ weight filtration

$gr^W_{2n+1} M = 0$.

$gr^W_{2n} M = \bigoplus \mathbb{Q}(-n)$

finite length.

④ : $\hom(\mathbb{Q}(p), \mathbb{Q}(r))$

$$= \begin{cases} \mathbb{Q} & p = r \\ 0 & p \neq r \end{cases}$$

⑤ $\mathrm{Ext}^i(\mathbb{Q}, \mathbb{Q}(p)) = 0$ ~~$i \geq p$~~. $i \leq 0$ $(i,p) \neq (0,0)$

$\phi : \mathcal{MTM}(k) \longrightarrow \{ \mathbb{Z}\text{-graded. vector sp}/\mathbb{Q} \}$

$\phi_{-n}(M) = \hom(\mathbb{Q}(n), gr^W_{2n} M) \Longrightarrow \mathrm{Rep}\, G \simeq \mathcal{MTM}(k)$.

Tannakian category. $G \simeq \mathbb{G}_m \ltimes U$

property $\mathrm{Ext}^i_{\mathcal{MTM}(k)}(\mathbb{Q}, \mathbb{Q}(p)) \simeq CH^p(k, 2p-i)_{\mathbb{Q}}$.

$\exists \mathcal{L}$: Graded Pro Lie-alg$/\mathbb{Q}$. grading

$\mathcal{MTM}(k) \xrightarrow{\sim} \{ \text{rep of } \mathcal{L} \text{ finite dim}/\mathbb{Q} \}$.

constructing $\mathcal{L}$. explicitly using alg. cycle.

"de Rham Complex" / $BG/B\mathbb{G}_m$. (using rational homotopy theory.)

Cube version ← Bloch complex

alg cycle $\rightsquigarrow N^\cdot$. DGA

$\mathcal{L}$ $\rightsquigarrow$ $BN^\cdot \rightsquigarrow \chi_{mot} := H^0 BN^\cdot$

Bar construction.

$$\mathcal{L} = \left(\chi^+_{mot} / (\chi^+_{mot})^2 \right)^*$$

§. cubical obj. $\longleftrightarrow$ (simplicial. obj. みたいなもの.) No2

$\square \hookrightarrow$ (finite set の category)

sub cat. **object**: $\square^n$ $\square^0 = 0$

$\mathfrak{S}_n = \mathrm{Aut}_{\square} \square^n$ $\square^1 = \{0,1\}$

Morphism:

$\hom(\square^i, \square^j) = \{\text{injection}\} = \mathfrak{S}_n \ltimes \{\pm 1\}^n$ $\square^n = (\square^1)^{\times n}$

$c = 0, 1$

closed under finite product & their morphisms $\delta^n_{p,c}$

$: \square^n \to \square^{n+1}$

$\mathcal{C}$: permutative ~~&~~ cat.

$(i_1, \cdots, i_n) \mapsto (i_1, \cdots, c, \cdots i_n)$

p-番目

Strict tensor category. with unit.

$A \otimes (B \otimes C) = (A \otimes B) \otimes C$

$\times : \square \times \square \to \square$

$A : \square^{op} \to \mathcal{C}$. $m : A \times A \to A$ $(\square^n, \square^m) \mapsto \square^{n+m}$

: associative
commutative.

$A^p = A(\square^p)$

$A^p \to A^p$

$\mathrm{alt.} = \frac{1}{p!} \sum$, $: A^{p+1} \to A^p$.

$\mathrm{alt} = \frac{1}{2^p p!} \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma) \cdot \sigma$.

$\partial = \sum_k (-1)^k \{ A(\delta^p_{k,1}) - A(\delta^p_{k,0}) \}$.

$\mathrm{alt}\, A^*$: DGA.

$N(*) = \bigoplus_{r \geq 0} N(r)$ $N(r) = \mathrm{Alt}(\mathrm{Cycl.}^r \langle n \rangle)[2r]$

$\Delta^n = \mathrm{Spec}\, k[T_0, \cdots, T_n] / (\sum T_i - 1)$ $\square^n = (\mathbb{P}^1 - \{1\})^n$

(face. $(\cdots 0 \cdots)$
$(\cdots \infty \cdots)$)

$$\mathrm{cycl}^r\langle n\rangle = \left\langle V \subset \square^n \;\middle|\; \begin{array}{l} V : \mathrm{codim} = r \\ \text{integral closed subscheme} \\ \text{meet all faces properly} \end{array} \right\rangle_{\mathbb{Q}}$$

$$H^n(\mathcal{N}(r)) \cong CH^r(k, 2r-n)_{\mathbb{Q}} \qquad A^+ = \ker(A \to \mathbb{Q})$$

§ ~~Bar~~ Construction.

augmented $A^\bullet$: DGA $\qquad BA^\bullet = T(\mathbb{Q}, A, \mathbb{Q}) / D(\mathbb{Q}, A, \mathbb{Q}) \simeq T(\mathbb{Q}, A^+, \mathbb{Q})$

$$T(\mathbb{Q}, A, \mathbb{Q}) = \bigoplus_{n \geq 0} A^{\otimes n}. \qquad [a_1|a_2|\cdots|a_n] \overset{df}{:=} a_1 \otimes \cdots \otimes a_n$$

$$\deg[a_1|\cdots|a_n] = \sum \deg a_i - n.$$

$$\delta_p^{n+1} : A^{\otimes(n+1)} \longrightarrow A^{\otimes(n-1)} \qquad 1 \leq p \leq n$$

$$[a_1|\cdots|a_{n+1}] \longmapsto [a_1|\cdots|a_p a_{p+1}|\cdots|a_{n+1}]$$

$$s_p^n : A^{\otimes n} \longrightarrow A^{\otimes(n+1)} \qquad 1 \leq p \leq n+1$$

$$[a_1|\cdots|a_n] \longmapsto [a_1|\cdots|\underset{p\text{-th}}{1}|\cdots|a_n]$$

$$D(\mathbb{Q}, A, \mathbb{Q}) = \left\langle \bigcup s(T(\mathbb{Q}, A, \mathbb{Q})) \right\rangle$$

$$\partial = \sum_p (-1)^{n+1} \delta_p^{n+1}$$

$$\begin{tikzcd}
 & {} & A^+ \\
A^{+\otimes 3} \arrow[r, "\partial"] & A^{+\otimes 2} \arrow[r, "\partial"] \arrow[u] & A^+ \arrow[u] \\
A^{+\otimes 3} \arrow[r, "\partial"'] \arrow[u] & A^{+\otimes 2} \arrow[r, "\partial"'] \arrow[u] & A^{\oplus} \arrow[u]
\end{tikzcd}$$

$\psi : BA \to BA \otimes BA$: comulti.

$$[a_1 | \cdots | a_n] \mapsto \sum (-1)^p [a_1 | \cdots | a_p] \cdot [a_{p+1} | \cdots | a_n]$$

$$\underset{(\mathbb{k})}{\chi_{mot}} \overset{\text{dfn.}}{:=} H^0 BN(*) \qquad \underset{(\mathbb{k})}{\mathcal{L}} = \left(\chi^{+}_{mot} / (\chi^{+}_{mot})^2 \right)^*$$

k: perfect field $\quad D^- \overset{dfn}{:=} D^-(NSWT/k) \supseteq DM_-^{eff}(k)$: full subcategory consisting of cpx's whose cohomology sh's are h.i. No. 1

4. categories and 4 cohomologies

12/21. Hagihara. K

All are tensor triangulated functors are compatible

$$
\begin{array}{ccc}
DM_{gm}^{eff}(k) & \subseteq & DM_-^{eff}(k) \ (\subseteq D^-) \\
\downarrow & & \downarrow \\
DM_{gm}(k) & \subseteq & DM_-(k)
\end{array}
\quad \text{invert } \otimes \mathbb{Z}(1)
$$

$M(X) := C_*(\mathbb{Z}_{tr}(X)) \in DM_-^{eff}(k)$

DM_{gm}^{eff}: the minimal full subcategory

- containing $M(X)$ $(X \in Sm/k)$
- closed under direct summand.
- — — taking cones, shifts.

$DM(k)$: the category obtained from $DM_-^{eff}(k)$ by inverting $-\otimes\mathbb{Z}(1)$

$DM_{gm}(k)$: ——— ———, $DM_{gm}^{eff}(k)$ ———

More precisely,

$$\mathrm{Obj}(DM_-(k)) := \left\{ (M,m) \;\middle|\; \begin{array}{l} M \in DM_-^{eff}(k) \\ m \in \mathbb{Z} \end{array} \right\}$$

$$\mathrm{Hom}_{DM_-(k)}((M,m),(N,n)) := \operatorname*{colim}_{k \geq -m,-n} \mathrm{Hom}_{DM_-^{eff}(k)}(M(m+k), N(n+k))$$

we denote $(M,m) = M(m)$

Rem: (1): In fact, we have

$$\mathrm{Hom}_{DM_-^{eff}}(A,B) \xrightarrow{\simeq} \mathrm{Hom}_{DM_-^{eff}}(A(1), B(1))$$

(Cancellation Thm)

So the vertical fct's are in fact full faithful

(2): In order for DM_-, DM_{gm} to become symm. tensor cat, we have to check 「the cyclic permutation on $\mathbb{Z}(1)^{\otimes 3}$ $\mathfrak{S}_3 \curvearrowright$ is the identity」

(3): We easily see that DM_- also has $\underline{Hom}(M(X),-)$ but the two internal hom in DM_-, DM^{eff}_- does not coincide in gen. No.2

(ex): $\underline{Hom}_{DM_-}(\mathbb{Z}(1),\mathbb{Z}) =: \mathbb{Z}(-1) \notin DM^{eff}_-$

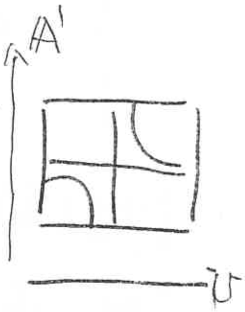
$\underline{Hom}_{DM^{eff}_-}(\mathbb{Z}(1),\mathbb{Z}) = 0.$

(4): Later we prove that DM_{gm} has $\underline{Hom}$ & "rigid"

For, $X \in Sch/k$ we construct $M(X)$, $M^c(X)$

$$\mathbb{Z}_{tr}(X) : U \longmapsto \mathbb{Z}\left\langle \begin{array}{c} Z \subseteq U \times X : \text{closed integral} \\ \downarrow \\ U \end{array} \right\rangle, \quad Z \to U \text{ finite surj. connected comp.}$$

$U \in Sm/k$



$$\mathbb{Z}^c_{tr}(X) : U \longmapsto \mathbb{Z}\left\langle \begin{array}{c} Z \subseteq U \times X : \text{closed int.} \\ \downarrow \\ U \end{array} \right\rangle, \quad Z \to U \text{ quasi-finite dominant a comp.com}$$

$\mathbb{Z}_{tr}(X)$, $\mathbb{Z}^c_{tr}(X) \in NSwT/k$

$M(X) \overset{dfn}{:=} C_*(\mathbb{Z}_{tr}(X)) \in DM^{eff}_-$

$M^c(X) := C_*(\mathbb{Z}^c_{tr}(X)) \in$

functoriality:

$f: X \to Y$

$\Rightarrow M(X) \to M(Y)$

$f: X \to Y$: proper

$\Rightarrow M^c(X) \to M^c(Y)$

$j: U \hookrightarrow X$: open

$\Rightarrow M^c(X) \to M^c(U)$

natural trans : $M(X) \longrightarrow M^c(X)$

X : proper $\Longrightarrow$ $M(X) \xrightarrow{\simeq} M^c(X)$

Def: $X \in Sch/k$: $H^i(X, \mathbb{Z}(j)) = \mathrm{Hom}_{DM_-^{eff}}(M(X), \mathbb{Z}(j)[i])$

$H_i(X, \mathbb{Z}(j)) = \mathrm{Hom}_{DM_-^{eff}}(\mathbb{Z}(j)[i], M(X))$

$H_c^i(X, \mathbb{Z}(j)) = \mathrm{Hom}_{DM_-}(M^c(X), \mathbb{Z}(j)[i])$

$H_i^{BM}(X, \mathbb{Z}(j)) = \mathrm{Hom}_{DM_-}(\mathbb{Z}(j)[i], M^c(X))$

('Borel–Moore homology')

ex: $H^i(X, \mathbb{Z}) = \hom_{DM_-}(M(X), \mathbb{Z}[i])$

$H_i(X, \mathbb{Z}) = \hom_{DM_-}(\mathbb{Z}[i], M(X))$ $\;\; = H_{Nis}^{-i}(\mathrm{Spec}\,k, C_*(\mathbb{Z}_{tr}(X)))$ — Nis. coh. dim $= 0$

$= \hom_{DM_-}(\mathbb{Z}, M(X)$

$= H^{-i}(C_*(\mathbb{Z}_{tr}(X))(\mathrm{Spec}\,k))$

$= H_i\left(\to \underset{2}{\mathbb{Z}_{tr}(X)(\Delta^2)} \to \underset{1}{\mathbb{Z}_{tr}(X)(\Delta^1)} \to \underset{0}{\mathbb{Z}_{tr}(X)(\Delta^0)}\right)$

(Suslin homology)

properties:

i) (homotopy invariant) $X \in Sm/k$, $M(X) \xleftarrow{\simeq} M(X \times \mathbb{A}^1)$

(☺ $\forall F \in DM_-^{eff}(k)$, $H_{Nis}^i(X \times \mathbb{A}^1, F) \simeq H_{Nis}^i(X, F)$

X : smooth $\qquad \hom(M(X \times \mathbb{A}^1), F[i]) \qquad \hom(M(X), F[i])$)

ii) : (Mayer–Vietorius) $X \in Sch/k$, $X = U \cup V$: open cov.

$\Rightarrow M(U \cap V) \to M(U) \oplus M(V) \to M(U \cup V) \xrightarrow{+1}$

$M(U \cup V) = X$

(☺ Follows from

$$0 \to \mathbb{Z}_{tr}(U \cap V) \to \mathbb{Z}_{tr}(U) \oplus \mathbb{Z}_{tr}(V) \to \mathbb{Z}_{tr}(X) \to 0$$

exact in $(Sm/k)^{\sim}_{Nis}$)

<u>Cor</u>: $X \in Sm/k$ $E \to X$; v.b. $\Rightarrow M(E) \xrightarrow{\cong} M(X)$

$$M(\mathbb{A}^n \setminus \{0\}) \cong \mathbb{Z} \oplus \mathbb{Z}(n)[2n-1]$$

$$M(\mathbb{P}^n) \simeq \bigoplus_{i=0}^{n} \mathbb{Z}(i)[2i]$$

where ☺ $\mathcal{O}_{\mathbb{P}^n}(1) \in \mathrm{Pic}(\mathbb{P}^n) \simeq H^2(\mathbb{P}^n, \mathbb{Z}(1))$

$= \hom(M(\mathbb{P}^n), \mathbb{Z}(1)[2])$

$\quad\quad \Psi$

$\to (\tau : M(\mathbb{P}^n) \to \mathbb{Z}(1)[2])$

$$\begin{tikzcd} M(\mathbb{P}^n) \arrow[r, "\Delta"] \arrow[rr, bend right, "\tau^k"'] & M(\mathbb{P}^n)^{\otimes k} \arrow[r, "\tau^{\otimes k}"] & \mathbb{Z}(k)[2k] \end{tikzcd}$$

<u>iv)</u>: (projective bundle formula) $E \to X$: v.b. of $rk = r$

$$P = \mathbb{P}(E) \xrightarrow{p} X \Rightarrow M(P) \xrightarrow{\sim} \bigoplus_{i=0}^{r-1} M(X)(i)[2i]$$

<u>cdh-topology</u>:

<u>Def</u>: k : field k : admits resolution of singularities

$\overset{\Longleftrightarrow}{\text{def}}$ <u>(i)</u>: $X \in Sch/k$: int $\exists Y \to X$: proper birational
$\quad Y \in Sm/k$, $X \in Sch/k$.

(ii): $X \in Sch/k$: int. $\forall Y \to X$ $\Rightarrow$ $X_n \to X_{n-1} \to \cdots \to X_0 = X$: a seq of
proper birat. smooth center

Rem: $\mathrm{char}\, k = 0 \Rightarrow k$: admits RS
$\Rightarrow k$: perfect

Def: cdh-topology on (Sch/k) = the weakest Grothendieck top. s.t. Nisnevich cov. is cdh-cov.

$X' \xrightarrow{p} X$: proper, $Z \hookrightarrow X$: closed imm.
s.t. $p^{-1}(X \setminus Z) \xrightarrow{\sim} X \setminus Z$

$\Rightarrow Z \sqcup X' \to X$: cdh-covering

ex:
- covering by irreducible comp. $\Rightarrow$ cdh.
- $X_{red} \to X$ $\Rightarrow$ cdh.
- $X' \to X$: blow up smooth center $\Rightarrow$ cdh.

Using this idea we have

Lem: $X' \xrightarrow{p} X$: proper s.t. $\forall x \in X \ \exists x' \in X'$ s.t. $p(x') = x$, $k(x) \xrightarrow{\sim} k(x')$

$\Rightarrow$ cdh-covering.

Lem: X : int $\in Sch^{m}/k$ (k : R.S.) $X' \to X$: proper birat.

$\Rightarrow$ cdh-cov.

$Sm/k \hookrightarrow Sch/k$: induces an adjoint pair

$$(Sch/k)^{\sim}_{cdh} \underset{\pi_*}{\overset{\pi^*}{\rightleftarrows}} (Sm/k)^{\sim}_{Nis}.$$

Rem: The exactness of π^* is non-trivial
($Sm/k \hookrightarrow Sch/k$: does not pull back.)

Prop: F: Nisnevich shf. on Sm/k

$$F_{cdh}=0 \iff \forall X\in Sm/k \quad s\in F(X)$$

$$\exists p: X'\to X : \text{proper cdh-covering}$$

s.t. $p^*(s)=0$.

$$F_{cdh}=(\pi^*F)_{\cancel{cdh}}$$

why $\mathbb{A}^1$-homotopy theory? (d = dim X) 12/21, L. Hesselholt

Recall: $H^p(X,\mathbb{Z}(q)) = CH^q(X, 2q-p)$ If, $p=q+d$, then

$$H^{q+d}(X,\mathbb{Z}(q)) = CH^q(X, q-d)$$

$$= \frac{\mathbb{Z}\{\text{closed pts on } X\times\Delta^{q-d}\}}{\text{relation.}}$$

Can sometimes understand this group directly;

Thm (Rost) let $X_a \subseteq \mathbb{P}_k^{2^{r-1}}$ is the quadratic;

$$(x_1^2 - a_1 y_1^2)\otimes \cdots \otimes (x_{r-1}^2 - a_{r-1} y_{r-1}^2) = a_r t^2$$

where $a_1, \cdots, a_r \in k^\times$, then

$$H^{2^r-1}(X_a, \mathbb{Z}(2^{r-1})) \hookrightarrow k^\times \quad q+d = 2^{r-1}+2^{r-1}-1 = 2^r-1 = p$$

If $p < q+d$ the group $H^p(X,\mathbb{Z}(q))$ can never be understood directly from the definition

Strategy: Relate $H^p(X,\mathbb{Z}(q))$ to some $H^{r+d}(X,\mathbb{Z}(r))$

Recall; $DM(k) :=$ derived category of mixed motive/k
= tensor triangulated category

$X \longmapsto X[p]$ change degree by p
$X \mapsto X(q)$ change weight by q
} autom of $DM(k)$

$Sm/k \to DM(k)$
$X \longmapsto M(X)$

$\mathbb{Z} = M(\operatorname{Spec} k) \in DM(k)$
unit for tensor product.

$$H^p(X,\mathbb{Z}(q)) = \hom_{DM(k)}(M(X), \mathbb{Z}(q)[p])$$

$$\mathbb{Z} \xrightarrow{\ell} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/\ell\mathbb{Z} \xrightarrow{\delta} \mathbb{Z}[1]$$

$$\begin{array}{ccc}
H^p(X, \mathbb{Z}/\ell\mathbb{Z}(q)) & = & \hom_{DM(k)}(M(X), \mathbb{Z}/\ell\mathbb{Z}(q)[p]) \\
\beta \downarrow & & \downarrow (\pi(1)\circ\delta)_* \\
H^{p+1}(X, \mathbb{Z}/\ell\mathbb{Z}(q)) & = & \hom_{DM(k)}(M(X), \mathbb{Z}/\ell\mathbb{Z}(q)[p+1])
\end{array}$$

$SH(k)$ = stable homotopy category $/k$

= tensor triangulated category

$\left\{ \begin{array}{l} X \mapsto X[p] : \text{shifts } \underline{\text{degree}} \text{ by } p \\ X \mapsto X(q) : \text{changes } \underline{\text{weight}} \text{ by } q. \end{array} \right.$

$$\begin{array}{ccc}
Sm/k & \longrightarrow & SH(k) \\
\cup & & \cup \\
X & \longmapsto & S(X) \\
 & & \| \\
 & & \left(\sum_{\mathbb{P}^1}^{\infty}(X \amalg \operatorname{Spec} k)\right)
\end{array}$$

$\mathbb{S} = S(\operatorname{Spec} k) \in SH(k)$: unit for tensor prod.

$\mathbb{Z} \in SH(k)$: Eilenberg–Maclane object.

= monoid for tensor product.

$\mathbb{S} \to \mathbb{Z}$ $\qquad$ $\mathbb{Z} \otimes_{\mathbb{S}} \mathbb{Z} \xrightarrow{\mu} \mathbb{Z}$

$$\begin{array}{ccc}
\mathbb{Z} \otimes_{\mathbb{S}} \mathbb{Z} \otimes_{\mathbb{S}} \mathbb{Z} & \xrightarrow{id\otimes\mu} & \mathbb{Z} \otimes_{\mathbb{S}} \mathbb{Z} \\
\mu\otimes id \downarrow & & \downarrow \mu \\
\mathbb{Z} \otimes_{\mathbb{S}} \mathbb{Z} & \longrightarrow & \mathbb{Z} \\
 & \mu &
\end{array}$$

$$\begin{array}{ccccc}
\mathbb{S} \otimes_{\mathbb{S}} \mathbb{Z} & \longrightarrow & \mathbb{Z} \otimes_{\mathbb{S}} \mathbb{Z} & \longleftarrow & \mathbb{Z} \otimes_{\mathbb{S}} \mathbb{S} \\
 & \overset{\sim}{\underset{can}{\searrow}} & \downarrow \mu & \overset{\sim}{\underset{can.}{\swarrow}} & \\
 & & \mathbb{Z} & &
\end{array}$$

$\underline{\mathbb{Z}\text{-modules}}$:

(M, α), $M \in SH(k)$

$\mathbb{Z} \otimes_{\mathbb{S}} M \xrightarrow[\alpha]{} M$

$$\begin{array}{ccc}
\mathbb{Z} \otimes_{\mathbb{S}} \mathbb{Z} \otimes_{\mathbb{S}} M & \xrightarrow{\mu\otimes id} & \mathbb{Z} \otimes_{\mathbb{S}} M \\
id\otimes\alpha \downarrow & & \downarrow \alpha \\
\mathbb{Z} \otimes_{\mathbb{S}} M & \xrightarrow[\alpha]{} & M
\end{array}$$

$$\begin{tikzcd} M \arrow[r, "\eta"] \arrow[rr, bend right, "\mathrm{id}"'] & \mathbb{Z}\otimes_{\mathbb{S}} M \arrow[r, "\alpha"] & M \end{tikzcd}$$

$\mathbb{Z}$-linear map ;

$f : M \to M'$

$$\begin{tikzcd} \mathbb{Z}\otimes_{\mathbb{S}} M \arrow[r, "\mathrm{id}\otimes f"] \arrow[d, "\alpha"] & \mathbb{Z}\otimes_{\mathbb{S}} M' \arrow[d, "\alpha'"] \\ M \arrow[r, "f"'] & M' \end{tikzcd}$$

$\mathrm{Mod}_{\mathbb{Z}}(k)$

= category of $\mathbb{Z}$-modules and $\mathbb{Z}$-linear map.

<u>Prop</u> : The functor $\mathrm{Sm}/k \to \mathrm{Mod}_{\mathbb{Z}}(k)$

$$X \mapsto \mathbb{Z}\otimes_{\mathbb{S}} S(X)$$

extends to an equivalence of categories ;

$$DM(k) \xrightarrow{\sim} \mathrm{Mod}_{\mathbb{Z}}(k)$$

So $H^p(X, \mathbb{Z}(q)) = \hom_{DM(k)}(M(X), \mathbb{Z}(q)[p])$

$= \hom_{\mathbb{Z}}(\mathbb{Z}\otimes_{\mathbb{S}} S(X), \mathbb{Z}(q)[p])$ $\quad \theta \in \mathrm{Hom}_{\mathbb{S}}(\mathbb{Z}, \mathbb{Z}(t)[s])$

$= \hom_{SH(k)}(S(X), \mathbb{Z}(q)[p])$ $\quad \hom_{\mathbb{S}} = \hom_{SH(k)}$

gives rise to a natural transformation

$$H^p(X, \mathbb{Z}(q)) = \hom_{DM(k)}(M(X), \mathbb{Z}(q)[p])$$

$$\simeq \hom_{\mathbb{S}}(S(X), \mathbb{Z}(q)[p])$$

$$\downarrow$$

$$H^{p+s}(X, \mathbb{Z}(q+t)) \simeq \hom_{\mathbb{S}}(S(X), \mathbb{Z}(q+t)[p+s])$$

Point

$$\mathrm{Hom}_{\Delta}(\mathbb{Z}, \mathbb{Z}(t)[s]) \subseteq \mathrm{Hom}_{\mathcal{S}}(\mathbb{Z}, \mathbb{Z}(t)[s])$$

is a proper inclusion This way one obtains natural transf.

$$H^p(X, \mathbb{Z}/\ell\mathbb{Z}(q)) \xrightarrow{Q_i} H^{p+2\ell^i-1}(X, \mathbb{Z}/\ell\mathbb{Z}(q+\ell^i-1))$$

for all $i \geq 0$ of which $Q_0 = \beta$.

Note; $2\ell^i - 1 > \ell^i - 1$

$\mathcal{K} \in SH(k)$: represents algebraic $\mathcal{K}$-theory

$$\mathcal{K}^{p,q}(X) = \mathrm{hom}_{\mathcal{S}}(S(X), \mathcal{K}(q)[p])$$

$\mathcal{K} \xrightarrow{\sim} \mathcal{K}(1)[2]$: Bott Periodicity

<u>Thm</u> (Hopkins–Morel) There is a descending "filtration"

$$\cdots \to \mathrm{Fil}^2\mathcal{K} \to \mathrm{Fil}^1\mathcal{K} \to \mathrm{Fil}^0\mathcal{K} = \mathcal{K}$$

$\rightsquigarrow$

$$\mathrm{Fil}^{s+1}\mathcal{K} \to \mathrm{Fil}^s\mathcal{K} \to \mathbb{Z}(s)[2s] \to \mathrm{Fil}^{s+1}\mathcal{K}[1] \quad //$$

This Gives a Spectral Sequence

$$E_1^{s,t} = \mathrm{hom}_{\mathcal{S}}(S(X), \mathbb{Z}(s)[2s][s+t]) = H^{3s+t}(X, \mathbb{Z}(s))$$

$$\Rightarrow \mathrm{hom}_{\mathcal{S}}(S(X), \mathcal{K}[s+t])$$

$$\overset{\text{dfn}}{=} \mathcal{K}^{s+t}(X) = K_{-s-t}(X)$$

Re-index the Spectral seq. s.t. E_1 becomes E_2

, get $E_2^{s,t} = H^{s-t}(X, \mathbb{Z}(-t)) \Rightarrow K_{-s-t}(X)$

$$K \xrightarrow{\ell} K \to K/\ell K \to K[1]$$

$$K/\ell = \bigoplus_{i=0}^{\ell-2} L[2i]$$

$$\begin{array}{ccc}
\tau_{\geq 4\ell-4} L & & \\
\downarrow & & \\
\tau_{\geq 2\ell-2} L & \xrightarrow{\pi} & \mathbb{Z}/\ell\mathbb{Z}(\ell-1)[2\ell-2] \\
\downarrow & & \\
L = \tau_{\geq 0} L \to \mathbb{Z}/\ell\mathbb{Z} & \xrightarrow{\partial} & \tau_{\geq 2\ell-2} L
\end{array}$$

$$\begin{tikzcd}
\mathbb{Z}/\ell\mathbb{Z} \arrow[r] \arrow[rr, bend right] & \tau_{\geq 2\ell-2} L[1] \arrow[r, "\pi"] & \mathbb{Z}/\ell\mathbb{Z}(\ell-1)[2\ell-1]
\end{tikzcd}$$

Basis Model str.

$X \xrightarrow{\sim} X'$
$\Updownarrow$
$X(C) \xrightarrow{\sim} X'(C)$

$X \rightarrowtail X'$
$\Updownarrow$
$X(C) \rightarrowtail X'(C)$
$\forall C \in Sm/k$

$$\begin{tikzcd}
\forall A \arrow[r] \arrow[d, tail] & X \arrow[d, two heads] \\
B \arrow[r] \arrow[ur, dashed, "\exists"] & X'
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Preshv}(Sm/k)^{\Delta^{op}} \arrow[r, shift left, "a"] \arrow[d, phantom, "\rotatebox{90}{=}"] \arrow[dr, squiggly, "\text{localize w.r.t. stalkwise weak equivalence}"] & \mathrm{Shv}_{NB}(Sm/k)^{\Delta^{op}} \arrow[l, shift left] \arrow[r, shift left, "|-|"] \arrow[dr, squiggly, "Sin"] & \mathrm{Shv}_{NB}(Sm/k) \arrow[l, shift left] \arrow[d, squiggly, "\Delta^{op}"] \\
(Set^{\Delta^{op}})^{(Sm/k)^{op}} & \mathrm{Preshv}(Sm/k)^{\Delta^{op}} \arrow[r, shift left, "a"] \arrow[d, squiggly, "\text{localize w.r.t. } X \times \mathbb{A}^1 \xrightarrow{pr} X"] & \mathrm{Shv}_{Nis}(Sm/k) \arrow[l, shift left, "\text{Quillen equiv}"] \arrow[d, squiggly, "\Delta^{op}"] \\
& \mathrm{Preshv}(Sm/k)^{\Delta^{op}} \arrow[r, shift left, "a"] & \mathrm{Shv}_{Nis}(Sm/k) \arrow[l, shift left]
\end{tikzcd}$$

$$
\begin{tikzcd}
\mathrm{Shv}_{NB}(Sm/k)^{\Delta^{op}} \arrow[r, shift left, "|-|"] & \mathrm{Shv}_{NB}(Sm/k) \arrow[l, shift left] \arrow[d, squiggly] \\
& \mathrm{Shv}_{Nis}(Sm/k) \arrow[d, squiggly] \\
& \mathrm{Shv}_{Nis}(Sm/k) \\
& \boxed{\mathrm{Spc}(k)} \arrow[u]
\end{tikzcd}
$$

$\boxed{\text{Voevodsky ICM Berlin}}$

cat. of mixed Tate motive, $MT(\mathbb{Z})$ 12/21. Yamashita. G No.1

$$MZV's \longleftrightarrow \pi_1^{M}(\mathbb{P}^1-\{0,1,\infty\},\overrightarrow{01}) \text{ : motivic fundamental group}$$

multiple zeta value.

$k_1,\cdots,k_d \geq 1$

$$\zeta(k_1,\cdots,k_d) \overset{\text{dfn}}{:=} \sum_{n_1<\cdots<n_d} \frac{1}{n_1^{k_1}\cdots\cdots n_d^{k_d}} \in \mathbb{R}$$

converge $\Leftrightarrow k_d \geq 2$.

$\zeta(3)=\zeta(1,2)$ (Euler)

$\zeta(4)=\zeta(1,1,2)=4\zeta(1,3)=\frac{4}{3}\zeta(2,2)=\frac{\pi^4}{90}$ (Euler)

d : depth $k_1+\cdots+k_d$: weight

$$\underset{n>0}{Z_n} := \left\langle \zeta(k_1,\cdots,k_d) \,\middle|\, \begin{matrix} k_1+\cdots+k_d=n \\ k_d\geq 2 \end{matrix} \right\rangle \subseteq \mathbb{R}.$$

$n>0$ $Z_0:=\mathbb{Q}$. , $Z_1=0$, $Z_2=\zeta(2)\mathbb{Q}=\pi^2\mathbb{Q}$.

$Z_3=\zeta(3)\mathbb{Q}+\zeta(1,2)\mathbb{Q}=\zeta(3)\mathbb{Q}$

$Z_4=\zeta(4)\mathbb{Q}+\zeta(1,1,2)\mathbb{Q}+\zeta(1,3)\mathbb{Q}$

$+\zeta(2,2)\mathbb{Q}=\pi^4\mathbb{Q}$.

$Z_5=\langle\zeta(5),\pi^2\zeta(3)\rangle_{\mathbb{Q}}$ dim ≤ 2 k : field

⋮

Z_{10} 256 induce dim ≤ 7 Conj (Beilinson-Soulé vanishing Conj.)

⋮

Z_{20} 262144 dim ≤ 114

$\mathbb{K}$. $K_{2q-p}(k)_{\mathbb{Q}}^{(q)}=0$

for $p<0$

strong version $\quad K_{2q-p}(k)^{(q)}_{\mathbb{Q}} = 0$ for $p \leq 0,\ q > 0$

cf: $\quad H^p_M(k, \mathbb{Q}(q)) \simeq K_{2q-p}(k)^{(q)}_{\mathbb{Q}}$

$DM_{gm}(k) \otimes \mathbb{Q} \supseteq DMT(k)$: full sub triangulated cat. generated by $\mathbb{Q}(n)$, $n \in \mathbb{Z}$

"weight str."

$$W_{n-1} \to W_n \to Gr^W_n \to W_{n-1}[1]$$

Def: $X \in DMT(k)^{\geq 0} \underset{\text{def.}}{\Longleftrightarrow} gr^W_a(X) \simeq \bigoplus_{n \leq 0} \mathbb{Q}(-\frac{a}{2})^{m_n}[n] \quad \forall a$

$X \in DMT(k)^{\leq 0}$

$\underset{\text{def.}}{\Longleftrightarrow} gr^W_a(X) \simeq \bigoplus_{n \geq 0} \mathbb{Q}(-\frac{a}{2})^{m_n}[n] \quad \forall a$

Thm (Levine) Assume strong version of BS vanishing holds for k

①: $\{DMT(k)^{\geq 0}, DMT(k)^{\leq 0}\}$: t-structure.

($\Rightarrow$ Def: $MT(k) := DMT(k)^{\geq 0} \cap DMT(k)^{\leq 0}$: abelian cat.)

②: $\operatorname{Ext}^p_{MT(k)}(M, N) \to \hom_{DMT(k)}(M, N[p])$

is isom. for $p = 1$

inj for $p = 2$

k : number ~~fiel~~ field $\Rightarrow$ holds $\forall p$.

In the following, assume k: number field.

$$\mathrm{Ext}^1_{MT(k)}(\mathbb{Q}(0),\mathbb{Q}(n)) \simeq K_{2n-1}(k)^{(n)}_{\mathbb{Q}}$$

$$\dim_{\mathbb{Q}} = \begin{cases} \infty & n=1 \\ r_2 & n = \text{even} \\ r_1+r_2 & n : \text{odd} \neq 1 \end{cases}$$

want to construct $MT(\mathcal{O}_S)$ $\mathcal{O}_S$: ring of S-integer

$$\mathrm{Ext}^1_{MT(\mathcal{O}_S)}(\mathbb{Q}(0),\mathbb{Q}(n)) \simeq K_{2n-1}(\mathcal{O}_S)^{(n)}_{\mathbb{Q}}$$

Deligne-Goncharov constructed $\leftarrow$ $MT(\mathcal{O}_S)$ not geometrically

$$MT(\mathcal{O}_S) \subset MT(k)$$

$k=\mathbb{Q}$, $\mathcal{O}_S=\mathbb{Z}$ $MT(\mathbb{Z})$

$$\dim_{\mathbb{Q}} \mathrm{Ext}^1_{MT(\mathbb{Z})}(\mathbb{Q}(0),\mathbb{Q}(n)) = \begin{cases} 0 & n : \text{even or } 1 \\ 1 & n : \text{odd} \neq 1 \end{cases}$$

$$\omega = \bigoplus_n \omega_n$$

$$\omega : MT(\mathbb{Z}) \longrightarrow \mathrm{Vect}_{\mathbb{Q}}$$

$$M \longmapsto \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}(\mathbb{Q}(n), gr^W_{2n} M)$$

theory of Tannakian cat. $\rightsquigarrow$ G_ω: proalg. group $= \mathrm{Aut}^{\otimes}(\omega)$

Rem):

$$\mathrm{Rep}\, G_\omega \xleftarrow{\sim} MT(\mathbb{Z})$$

$$\omega(M) \longleftarrow M$$

Deligne:

$G \in \mathrm{Pro}\text{-}MT(\mathbb{Z})$: gp scheme obj. s.t. $\forall F$: fibre functor $F(G) = \mathrm{Aut}^{\otimes}(F)$

$$MT(\mathbb{Z}) \xrightarrow{F} \mathrm{Vect}_{\mathbb{Q}}$$

$$G \longmapsto G_F = \mathrm{Aut}^{\otimes}(F)$$

Thm: (Deligne–Goncharov)

pro-unipotent group

$\exists\ \pi_1^{\mathcal{M}}(\mathbb{P}^1_{\mathbb{Q}} - \{0,1,\infty\}, \vec{01}) \in$ pro-$\mathcal{MT}(\mathbb{Z})$

s.t. Hodge real. $= \pi_1^{\mathrm{Hodge}}(\mathbb{P}^1_{\mathbb{Q}} - \{0,1,\infty\}, \vec{01})$

also l-adic.

$G_\omega \curvearrowright \omega(\mathbb{Q}(1))$

$G_\omega \xrightarrow{\tau} \mathbb{G}_m \quad \ker =: U_\omega$

splitting

$\mathbb{G}_m \curvearrowright \omega_n$

$\lambda \quad \lambda^n$

$G_\omega = \mathbb{G}_m \ltimes U_\omega.$

$(\pi_1^B(\mathbb{P}^1 - \{0,1,\infty\}, \vec{01}))$

$U_\omega \curvearrowright \omega(\mathbb{Q}(n))$

trivi.

pro-Unipotent.

$$MT(\mathbb{Z}) \overset{B}{\underset{dR}{\rightrightarrows}} \mathrm{Vect}_{\mathbb{Q}}$$

$B \otimes \mathbb{C} \simeq dR \otimes \mathbb{C}$

$\cancel{\mathbb{G}_m}$

$G_\omega(\mathbb{C}) \ni \overset{\exists}{a}$ s.t.

$$\begin{array}{ccc} M_B \otimes \mathbb{C} & \simeq & M_{dR} \otimes \mathbb{C} \\ \cup\!\! \mid & & \cup\!\! \mid \\ M_B & & M_{dR} \end{array}$$

$a(M_{dR})$
$= M_B$ for $\forall M$

$$\begin{array}{ccc} \mathbb{Q}(1)_B \otimes \mathbb{C} & \simeq & \mathbb{Q}(1)_{dR} \otimes \mathbb{C} \\ \cup & & \cup\!\! \mid \\ \mathbb{Q} & & \mathbb{Q} \\ 1 & \longmapsto & \frac{1}{2\pi i} \end{array}$$

$a = a^0\, \tau(2\pi i)$

$a^0 \in U_a(\mathbb{C})$

$$\pi_1^B(\mathbb{P}^1-\{0,1,\infty\}, \vec{01}, \vec{10})(\mathbb{C}) \cong \pi_1^{dR}(\text{—})(\mathbb{C})$$

$$\begin{array}{ccc} \psi & & \psi \quad \cong \\ (0 \to 1) & \longmapsto & \Phi_{dR} \in \mathbb{C}\langle\langle e_0, e_1 \rangle\rangle \end{array}$$

Drinfeld's associater

Φ_{dR} coeffi. of

$$e_0^{k_d-1}e_1 \cdots e_0^{k_1-1}e_1$$

is $(-1)^d \zeta(k_1, \cdots, k_d)$

$$H_W := \text{"}\mathrm{Aut}\{\pi_1^W(\mathbb{P}^1-\{0,1,\infty\}, \vec{0?}\, a, b)\}_{a,b}\text{"}$$

$a,b \in \{\vec{01}, \vec{10}, \vec{0\infty}, \vec{\infty 0}, \vec{1\infty}, \vec{\infty 1}\}$

$\|$

$\mathbb{G}_m \ltimes V_W$ ← prounipotent$/\mathbb{Q}$

$G_W \curvearrowright \{\pi_1^W(\cdots, a, b)\}_{a,b\in\cdots}$

factored through

$$\begin{array}{ccc} G_W & \longrightarrow & H_W \\ \tau \uparrow & \nearrow \tau & \\ \mathbb{G}_m & & \end{array} \qquad U_W \longrightarrow V_W$$

small enough — too big

$V_W(\mathbb{C}) \xrightarrow{\sim} \mathbb{C}\langle\langle e_0, e_1\rangle\rangle$

<u>Prop</u> $\langle \Phi_{DR} \rangle \tau(2\pi i) \in H_W(\mathbb{C})$ sends

$\pi_1^W(\mathbb{P}^1-\{0,1,\infty\}, \vec{01})$, $\pi_1^W(\mathbb{P}^1-\{0,1,\infty\}, \vec{01}, \vec{10})$

to $\pi_1^B(\text{——}, \vec{01})$, $\pi_1^B(\text{—}, \vec{01}, \vec{10})$

$\Rightarrow \exists\, v \in H(\mathbb{Q})$ s.t. $\langle \Phi_{DR} \rangle \tau(2\pi i) = \iota(a) v$

$$\Rightarrow \langle \Phi_{DR} \rangle = i(a^0)\cdot\left(\tau(2\pi i)\cdot w\tau(2\pi i)^{-1}\right)$$

$$D \qquad t \longmapsto \tau(t)\, v\tau(t^{-1})$$

$$\wr$$

$$(\mathbb{A}') \qquad \langle \Phi_{DR} \rangle \in \left(i(U_w)\times D\right)(\mathbb{C})$$

$$i(U_w)\times D = \operatorname{Spec} A \qquad A : \text{graded} \quad A = \bigoplus_n A_n.$$

$$\langle \Phi_{DR} \rangle \in (\operatorname{Spec} A)(\mathbb{C})$$

$$\Downarrow$$

$$\varphi : A \to \mathbb{C} \qquad \varphi(t') = \pi^2$$

$$Z_n \subseteq \varphi(A) \qquad \sum_{n=0}^{\infty}(\dim A_n)t^n$$

$$\sum_{n=0}^{\infty}(\dim A_n)\, t^n = \frac{1}{1-t^2}\cdot\frac{1}{1-(t^3+t^5+t^7+\cdots)}$$

$$\dim K_{2n-1}(\mathbb{Z})_{\mathbb{Q}} = \begin{cases} 0 & n : \text{even or } n=1 \\ 1 & n : \text{odd} \neq 1. \end{cases}$$

$$= \frac{1}{1-t^2}\,\frac{1}{1-\frac{t^3}{1-t^2}} = \frac{1}{1-t^2-t^3} = \sum_{n=0}^{\infty} D_n t^n.$$

$$\Rightarrow \begin{cases} D_0=1,\ D_1=0,\ D_2=1. \\ D_{n+3} = D_{n+1}+D_n \end{cases} \xRightarrow{\text{Thm}} \left(\begin{array}{c}\text{Goncharov,} \\ \text{Deligne,} \\ \text{Terasoma.}\end{array}\right)$$

$$\dim_{\mathbb{Q}} Z_n \le D_n \quad (\forall n)$$

Conj: (Grothendieck)

$a^0 \in U_n(\mathbb{C})$

$\mathbb{Q}$-Zariski dense.

Zagier:

$$\dim_{\mathbb{Q}} Z_n = D_n.$$

Classical case. Spec $\mathcal{O}_K$ r_1 = real place. No. 1
$\frac{12}{22}$. T. Geisser r_2 = complex place.

$\zeta_K(s)$

$$\lim_{s\to 1}(s-1)\zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2} h_K R_K}{w_K\sqrt{|D_K|}} \qquad g_n = \begin{cases} r_1+r_2 & n:\text{odd} \\ r_2 & n:\text{even}\end{cases}$$

$$\lim_{s\to 0} s^{-r_1-r_2+1}\zeta_K(s) = -\frac{hR}{w}$$

Borel

$$\begin{array}{ccc} \rho_n : K_{2n-1}(\mathcal{O}_K)_{\mathbb{R}} & \xrightarrow{\sim} & \mathbb{R}^{g_n} \\ \cup\!\!| & & \cup\!\!| \\ K_{2n-1}(\mathcal{O}_K)/_{\text{tor}} & & \mathbb{Z}^{g_n} \end{array} \qquad R_n = |\det \rho_n|$$

- $\operatorname{ord}_{s=1-n}\zeta_K(s) = \operatorname{rk} K_{2n-1}(\mathcal{O}_K)$
- $\lim_{s\to 1-n}(s-(1-n))^{-g_n}\zeta_K(s) = R_n(K)$ up to a rational number

$$\| \pm R_n(K)\cdot\frac{|K_{2n-2}(\mathcal{O}_K)|}{|K_{2n-1}(\mathcal{O}_K)_{\text{tor}}|}$$

Lichtenbaum (LN 342)
1972.

$\left.\begin{array}{l} n:\text{even} \\ \text{g. } K:\text{totally real field} \end{array}\right) \Rightarrow g_n = 0$

$$\zeta_K(1-n) \overset{?}{=} \pm \frac{|K_{2n-2}(\mathcal{O}_K)|}{|K_{2n-1}(\mathcal{O}_K)_{\text{tor}}|}\, 2^{r_1} \left(\begin{array}{l} K_i(\mathbb{F}_\ell) : \ell\text{-torsion} \\ \text{free}. \end{array}\right)$$

1990 Wiles Iwasawa Theory $\longrightarrow$ $\| \quad \|?$

$$\pm\prod_{\ell} \frac{|H^1_{et}(\mathcal{O}_K[\frac{1}{\ell}], \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))|}{|H^0_{et}(\mathcal{O}_K[\frac{1}{\ell}], \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))|}$$

Quillen ICM 74:
$$K_{2n-1}(\mathcal{O}_S)\otimes\mathbb{Z}_\ell = H^1(\mathcal{O}_S, \mathbb{Z}_\ell(n))$$
$$K_{2n-2}(\mathcal{O}_S)\otimes\mathbb{Z}_\ell = H^2(\mathcal{O}_S, \mathbb{Z}_\ell(n)) \qquad \frac{1}{\ell}\in\mathcal{O}_S$$

"Lichtenbaum-Quillen conjecture"

Generalize to use. arithmetic schemes. $X/\operatorname{Spec}\mathbb{Z}$: of finite type.

No2

Hasse–Weil – ζ

$$\zeta(X,s)=\prod_{x\in X_0}\frac{1}{1-N_x^{-s}}$$

converges absolutely for $Re(s) > \dim X$ — conjectured to have meromorphic continuation

$X = A\cup B$ $\quad \zeta(X,s)=\zeta(A,s)\cdot\zeta(B,s)$

$X/\mathbb{F}_q$ $\quad \zeta(X,s)=\prod \det\left(1-F\cdot q^{-s}\mid H^i_c(\bar X,\mathbb{Q}_\ell)\right)^{(-1)^{i+1}}$

<u>Conj</u> (Soulé ICM'83) $\quad \operatorname{ord}_{s=n}\zeta(X,s)=-\sum_{i\in\mathbb{Z}}(-1)^i\dim_{\mathbb{Q}} K_i'(X)_{(n)}$

generalize $X=\operatorname{Spec}\mathcal{O}_S$ $\quad$ char $k=p$ $\quad X$: smooth projective

<u>Finite Fields</u>: Leading coefficient of $\zeta(X,s)$ at $s=0$ can be expressed in terms of $H^*(X_{et},\mathbb{Z})$ $\quad$ Milne '84

$\mathbb{Z}=K_0$

<u>$s=1$</u>: $\quad H^*(X_{et},\mathbb{G}_m)$ $\quad \mathbb{G}_m=K_1$

At $s=n$ $\quad H^*(X_{et},K_n)$? no.

Lichtenbaum LN. 1068. $\quad K_i \quad \zeta$ at $s=i$.

<u>Ex</u>, $X=\operatorname{Spec}\mathbb{Z}$, $n=2$.

$\mathbb{Z}(0)$	$\cdot K_0$	$\zeta(0)$	
$\mathbb{Z}(1)$	$\cdot K_1$	$\zeta(1)$	
$\zeta(2)\longleftrightarrow\zeta(-1)=-\frac{1}{12}$ $\mathbb{Z}(2)$	$K_3^{ind}\to K_2$	$\zeta(2)$	
$\mathbb{Z}(3)$	K_3^M	$\zeta(3)$	
	K_4^M	K_3	$\zeta(4)$
		K_4	

$$\frac{|K_2(\mathbb{Z})|}{|K_3(\mathbb{Z})|}=\frac{2}{48}=\frac{1}{24}.$$

$K_3^M(\mathbb{Z})=\mathbb{Z}/2$ $\qquad \dfrac{|K_2(\mathbb{Z})|}{|K_3^{ind}(\mathbb{Z})|}=\dfrac{1}{12}$

Expect a <u>complex</u> of étale sheaves $\mathbb{Z}(n)$ s.t. $\zeta(X,s)$ can be expressed in terms of étale coh. $\mathbb{Z}(n)$

Beilinson (Letter to Soulé '82) Complex of Zariski of ~~simplicial pre~~ sheaves with

0): $\mathbb{Z}(0)=\mathbb{Z}$, $\mathbb{Z}(1)=\mathbb{G}_m[-1]$ ✓ Bloch

1): $\mathbb{Z}(r)$ is acyclic outside of $[1,r]$ unknown!

2): $H^{r+1}_{et}(k,\mathbb{Z}(r))=0$ $\underset{k}{\text{"}}$Hilbert 90" Beilinson-Soulé vanishing Conj, Suslin-Voevodsky

3): $\mathbb{Z}/m(n)=\mu_m^{\otimes n}$ if $\frac{1}{m}\in\mathcal{O}_X^\times$ ($\mathbb{Z}/p^r(n)_{et}=W_r\Omega^n_{X,\log}[-n]$) Gr-Levine Bloch-Kato

4): product $\mathbb{Z}(r)\otimes\mathbb{Z}(s)\to\mathbb{Z}(r+s)$ ✓ Bloch

4'): $H^r_{et}(k,\mathbb{Z}(r))=K^M_r(k)$ Nesterenko-Suslin, Totaro

5): $H^i(X,\mathbb{Z}(r))\sim gr^r_\gamma K_{2r-i}(X)$ isom up to small fail Grayson-Suslin, Levine

(SS $E_2^{s,t}=H^{s-t}(X,\mathbb{Z}(-t))\Rightarrow K_{-s-t}(X)$) dege up to small torsion

In particular, $K_i(X)^{(n)}_{\mathbb{Q}}=H^{2n-i}(X,\mathbb{Q}(n))$

Connection to Beilinson: $\mathbb{Z}(n)_{Zar}\xrightarrow{\sim}\tau_{\le n}R\varepsilon_*\mathbb{Z}(n)_{et}$ Voevodsky(?)

$\varepsilon: X_{et}\to X_{Zar}$ "Beilinson-Lichtenbaum Conj"

Parshin-Conj ⇒ Beilinson-Soulé vanishing Conj

Rem: If one wants to consider non-smooth, non-projective scheme.

$\mathbb{Q}(r)$ is acyclic except at r

one needs eithers

- cohom. with compact support, ~~finer topology~~.
- Borel-Moore homology (= higher Chow Group) ← open problem.

} localization

Combinate: Bloch's higher Chow Groups, X: smooth ('86)
(then ~~BM hom.~~ = coh.)

— Voevodsky's CM '90

Thm (S-V) equal for X: smooth

Rem: The spectral sequence is from higher Chow groups to K'-Theory, there cannot be a SS from MC to K-theory.

Soulé Conj: $\mathrm{ord}_{s=n}\zeta(X,s)=-\sum_{i\in\mathbb{Z}}(-1)^i\dim_{\mathbb{Q}}CH_n(X,i)_{\mathbb{Q}}$ $X/\mathrm{Spec}\,\mathbb{Z}$: f.t. $\dim X=d$

- $CH_n(X,i)$: f.t (Bass Conj). $CH^{d-n}(X,i)$

$CH_1(X,i)$

Ex: $X = \mathrm{Spec}\, \mathcal{O}_K$,

$$\mathrm{ord}_{s=1-n} \zeta(s) = \dim_{\mathbb{Q}} CH_{1-n}(\mathcal{O}_K, 2n-1)$$
$$= \dim K_{2n-1}(\mathcal{O}_K)^{(n)}_{\mathbb{Q}} \quad \checkmark \text{ Borel.}$$

$H^{2n+2}(X_{\acute{e}t}, \mathbb{Z}(n))$: is not fin gen.

Lichtenbaum's conj. (including L'03, G'04)

Conj: Let $X/\mathbb{F}_q$: of finite type fix.

1): $H^i_c(X_{ar}, \mathbb{Z}(n))$ fin. gen. vanish for allmost all

2): $H^i_c(X_{ar}, \mathbb{Z}(n)) \otimes \mathbb{Z}_\ell \simeq H^i_c(X_{\acute{e}t}, \mathbb{Z}_\ell(n))$

3): $\mathrm{ord}_{s=n} \zeta(X,s) = \sum_i (-1)^i i \cdot \mathrm{rk}\, H^i_c(X_{ar}, \mathbb{Z}(n))$

4): $\zeta(X,s) \sim \pm (1-q^{-n})^{\rho_n} \chi(H^{\cdot}_c(X_{ar}, \mathbb{Z}(n)) \cdot q^{?}$ as $s \to n$

5): $H^{i-1}_c(X_{ar}, \mathbb{Z}(n)) \xrightarrow{\vee e} H^i_c(X_{ar}, \mathbb{Z}(n)) \xrightarrow{\vee e} H^{i+1}_c(X_{ar}, \mathbb{Z}(n))$

Rem: X : smooth, proj. all groups. torsion expect

$$H^{2n}(X_{ar}, \mathbb{Z}(n)) \to H^{2n+1}(X_{ar}, \mathbb{Z}(n)) : \text{"regulator"}$$

Known in many cases ($\Leftrightarrow$ Tate's Conj. Beilinson $\sim_{rat} = \sim_{hom}$)

Ex: ①: $n=1$, X : smooth, proj. $H^3(X_{ar}, \mathbb{Z}(1)) \supseteq \mathrm{Br}(X)$: conj. o.k.
$\Rightarrow \mathrm{Br}X$: finite

②: X : flat$/\mathbb{Z}$ $\quad$ $X/\mathbb{F}_q \to$ only rational numbers.

transcendental number. (Borel-regulator)

'83: Beilinson's Conj. on special values of L.

... for the special value

k : perfect
$$\begin{array}{ccc} DM_-^{eff} & \supseteq & DM_{gm}^{eff} \\ \downarrow & & \downarrow \\ DM_- & \supseteq & DM_{gm} \end{array}$$

$M(X) \xrightarrow{\sim} M(X \times \mathbb{A}^1)$

MV $X = U \cup V$: $M(U\cap V) \to M(U) \oplus M(V) \to M(X) \xrightarrow{+1}$

Using cdh-topology to prove "localization", "blow-up seq"

$$\left(Sch/k\right)^{\sim}_{cdh} \underset{\pi_*}{\overset{\pi^*}{\rightleftarrows}} \left(Sm/k\right)^{\sim}_{Nis}$$

Thm: k : RS $F \in PSwT(k)$

$F_{cdh} = 0 \iff C_*(F)_{Nis}$; acyclique

Rmk: This also implies $C_*(F)_{Zar}$: acyclic

Ex: $p : Y \hookrightarrow X$: cl. imm. in Sch/k

$\Rightarrow M^c(Y) \to M^c(X) \to M^c(X \backslash Y) \xrightarrow{+1}$: distinguished triangle in DM_-^{eff}

$M^c(X) \overset{dfn}{:=} C_*(\mathbb{Z}^c_{tr}(X))$

(:) $0 \to \mathbb{Z}^c_{tr}(Y) \to \mathbb{Z}^c_{tr}(X) \xrightarrow{(*)} \mathbb{Z}^c_{tr}(X\backslash Y)$: easy to prove surjective

Suffice to prove $(*)$ is surjective in $\left(Sch/k\right)^{\sim}_{cdh}$

$\forall U \in Sm/k$, $\forall a \in \mathbb{Z}^c_{tr}(X\backslash Y)(U)$ $\exists U' \to U$: proper cdh-cov. $U' \in Sm/k$

s.t.
$$\begin{array}{ccc} \mathbb{Z}^c_{tr}(X\backslash Y)(U) & \to & \mathbb{Z}^c_{tr}(X\backslash Y)(U') \\ \ni a & \longmapsto & \uparrow \\ & & \mathbb{Z}^c_{tr}(X)(U') \ni a' \end{array}$$

$\overline{Z}$: closure of Z in $U \times X$

$\exists V \to U$: blow-up s.t.

Z' : strict trans. of $\overline{Z}$ is flat.

(Raynaud-Gruson platification)

$$\begin{array}{ccccc} \overline{Z} & \subseteq & U \times X & \leftarrow & V \times X \supseteq Z' \\ & & \downarrow & \square & \downarrow \\ & & U & \leftarrow & V \end{array}$$
blow-up

May Assume $V \in Sm/k$. $Z' \to V$: gen fin & flat $\Rightarrow$ finite. flat.

No 2

ex:

$$\begin{tikzcd} T' \arrow[r, hook, "i'"] \arrow[d, "p'"'] \arrow[dr, phantom, "\square"] & X' \arrow[d, "p : \text{proper}"] \\ T \arrow[r, hook, "i"'] & X \end{tikzcd}$$

$X' \setminus T'$ $\downarrow \wr\!\!\!|$ $X \setminus T$ $X, T, \cdots \in Sch/_k$ k : RS

$\Rightarrow$ $M(T') \to M(T) \oplus M(X') \to M(X) \xrightarrow{+1}$; distinguished triangle.

From the surjective of cdh-sheaves

$\mathbb{Z}_{tr}(T)_{cdh} \oplus \mathbb{Z}_{tr}(X')_{cdh} \to \mathbb{Z}_{tr}(X)_{cdh}$: ~~Short~~ similarly

Cor: k : RS, $X \in Sch/_k$. $M(X), M^c(X) \in DM^{eff}_{gm} \subseteq DM^{eff}_-$.

Rem: the minimal full subset of DM^{eff}_- containing $M(X)$ X: proj. smooth. closed undertaking direct summands, cones, shifts, $\oplus$

generated by $M(X)$ $(X \in Sm/_k)$

Cor: $X, Y \in Sch/_k$. k : RS $M(X) \otimes M(Y) \xrightarrow{\sim} M(X \times Y)$

$M^c(X) \otimes M^c(Y) \xrightarrow{\sim} M^c(X \times Y)$

In particular.

$M^c(X \times \mathbb{A}^1) \simeq M^c(X)(1)[2]$ (∵ reduced to the case X, Y ; smooth proj.

~~$M^c(pt) \to M^c(\mathbb{A}^1) \to M^c(\mathbb{P}$~~

$$\begin{array}{ccccc} M^c(pt) & \to & M^c(\mathbb{P}^1) & \to & M^c(\mathbb{A}^1) \simeq \mathbb{Z}(1)[2] \\ \wr\!\!\!| & & \wr\!\!\!| & & \\ \mathbb{Z} & & M(\mathbb{P}^1) & & \\ & & \wr\!\!\!| & & \\ & & \mathbb{Z} \oplus \mathbb{Z}(1)[2] & & \end{array}$$

equidimensional cycles & two moving lemma: $\quad Sm/k \ni U \quad$ closed integral

Def: $X \in Sch/k$, $r \geq 0$. $\quad z_{equi}(X, r) : U \mapsto \left\langle \begin{matrix} Z \subset X \times U \\ \searrow \quad \downarrow \\ X \end{matrix} \right\rangle$ equidim of dim $= r$.

$p : X \to S$; equidimensional of dim $= r$.

$\underset{\text{dfn}}{\Longleftrightarrow}$ 1): of finite type. 2) $\forall$ irred comp of X dominates an irreducible comp. of S

3): $\forall x \in X \quad \dim_x (p^{-1}(p(x))) = r$.

· $z_{equi}(X, r)$: $NSwT/k$. · $z_{equi}(X, 0) = \mathbb{Z}^c_{tr}(X) \quad k$: RS

· localizing sequence. $Y \hookrightarrow X$: closed immersion $\quad r \geq 0$

$$\Rightarrow \quad C_*(z_{equi}(Y, r)) \to C_*(z_{equi}(X, r)) \to C_*(z_{equi}(X \setminus Y, r)) \xrightarrow{+1} \quad \text{in } DM^{eff}_-$$

(Similar to the case $r = 0$)

Rem:

$$z_{equi}(X, r)(\Delta^m) \subseteq Z^{n-r}(X, m) \quad \text{(Bloch's cycle complex)}$$

(X : equidimensional $\dim = n/k$)

Thm: (Suslin) X: affine, equidim of dim n/k $\quad r \geq 0$

k: any field $\quad C_*(z_{equi}(X, r))(\operatorname{Spec} k) \underset{\text{q-isom}}{\xrightarrow{\sim}} Z^{n-r}(X, *)$

Thm (Friedlander–Lawson–Voevodsky)

$X \in Sch/k$, $U \in Sm/k$, U: quasi-proj. equidim

k: RS. $\quad C_*(z_{equi}(X, r))(U) \underset{\text{q-isom}}{\xrightarrow{\sim}} C_* z_{equi}(U \times X, r + \dim U)$ (Spec k)

$$\begin{matrix} & & Z'/k, \text{ moving} \\ \Delta^r \times U & \swarrow & \\ \downarrow & & Z. \\ \Delta^r & \swarrow & \end{matrix}$$

((X, U): smooth proj. (F-L): geometrically

(X, U): as above (F-V): cdh-topology k: (RS))

Cor (of FLV) $k; X, U$: as above.

$$\Rightarrow \quad H_i(C_* Z_{equi}(X, \underline{r})(U)) \xrightarrow{\sim} H_{Zar}^{-i}(U, C_*(Z_{equi}(X, \underline{r})))$$

$$\xrightarrow{\sim} H_{Nis}^{-i}$$

The Theory of PSWT.

(☺ : MV-distinguished triangle ; $X' = U' \cup V'$

$$C_*(Z_{equi}(X', \underline{r}))(\mathrm{Spec}\,k) \to C_*(Z_{equi}(U', \underline{r}))(\mathrm{Spec}\,k) \oplus C_*(Z_{equi}(V', \underline{r}))(\mathrm{Spec}\,k)$$

$$\to C_*(Z_{equi}(U' \cap V', \underline{r}))(\mathrm{Spec}\,k) \xrightarrow{+1} \ )$$

+ FLV thm 5.4.

$\cancel{C_*(Z_{equi}(Y, \underline{r}))(X) \to C_*(Z_{equi}(Y, \underline{r}))}$

$\cancel{X = U \cup V}$

~~smooth~~.

$$\Rightarrow \quad C_*(Z_{equi}(X, \underline{r}))(U) \to C_*(Z_{equi}(X, \underline{r}))(U_1) \oplus C_*(Z_{equi}(X, \underline{r}))(U_2)$$

$U = U_1 \cup U_2$

$$\to C_* Z_{equi}(X, \underline{r})(U_1 \cap U_2) \xrightarrow{+1}$$

This Property implies Cor. //

In particular,

$$H_{Nis}^{-i}(U, C_*(Z_{equi}(X, \underline{r})))$$

$$\xrightarrow{\sim} H_i(C_*(Z_{equi}(U \times X, d_i U + \underline{r}))(\mathrm{Spec}\,k)).$$

In the following (k : RS)

Cor: $X \in Sch/k$, $U \in Sm/k$, $r \geq 0$

$$\Rightarrow \mathrm{Hom}_{DM^{eff}_-}(M(U)(r)[2r+i], M^c(X))$$

$$\cong H^{-i}_{Nis}(U, C_*(\mathbb{Z}_{equi}(X, r)))$$

$\because$ $r=0$: clear. $r=1$.

$$H^{-i}_{Nis}(\cancel{U, C_*}(\mathbb{Z}_{equi}(X, 0)))$$

$$//$$

Projective Bundle formula.

$$\mathrm{Hom}(M(U)[i], M^c(X))$$

$$\oplus$$

$$\simeq \quad \mathrm{Hom}(M(U)(1)[2+i], M^c(X))$$

$$\mathrm{Hom}(M(U \times \mathbb{P}^1)[i], M^c(X))$$

$$\simeq H^{-i}_{Nis}(U \times \mathbb{P}^1, C_*(\mathbb{Z}_{equi}(X, 0))) \overset{FLV}{\simeq} H^{-i}_{Nis}(U, C_*(\mathbb{Z}_{equi}(X \times \mathbb{P}^1, 1)))$$

$$\simeq H^{-i}_{Nis}(U, C_*(\mathbb{Z}_{equi}(X, 1))) \oplus H^{-i}_{Nis}(U, C_*(\mathbb{Z}_{equi}(X \times \mathbb{A}^1, 1)))$$

(Localization for Zariski)

$$\wr\!\wr \; FLV$$

$$H^{-i}_{Nis}(U \times \mathbb{A}^1, -)$$

$$\wr\!\wr$$

$$H^{-i}_{Nis}(\cancel{U, C_*}(\mathbb{Z}_{equi}(X, 0)))$$

Cor (Cancellation Thm)

$A, B \in DM^{eff}_-$

$$\mathrm{hom}_{DM^{eff}_-}(A, B) \simeq \mathrm{hom}_{DM^{eff}_-}(A(1), B(1))$$

$\because$ $A = M(X)[i]$ とおけ。

$B = M(Y)$

Y, X : proj smooth $i \in \mathbb{Z}$

$$\mathrm{hom}(M(X)[i], M(Y))$$

$$\mathrm{Hom}(A(1), B(1)) = \mathrm{hom}\left(M(X)(1)[i], M(Y)(1)\right)$$

$$= \mathrm{Hom}\left(M(X)(1)[i+2], M(Y)(1)[2]\right)$$

$$= \mathrm{Hom}\left(M(X)(1)[i+2], M^c(Y \times \mathbb{A}^1)\right)$$

$$= H^{-i}_{Nis}\left(X, C_*(\mathrm{Zequi}(Y \times \mathbb{A}^1, 1))\right) \overset{FLT}{\simeq} H^{-i}_{Nis}\left(X, C_*(\mathrm{Zequi}(Y, 0))\right)$$

$$\simeq \mathrm{Hom}\left(\underbrace{M(X)[i]}_{\overset{\shortparallel}{A}}, \underbrace{M(Y)}_{\overset{\shortparallel}{B}}\right)$$

<u>Cor</u>: X: q-proj. equidi of di $= n$

$$CH_i(X, j) \overset{\simeq}{\Rightarrow} H^{BM}_{2i+j}(X, \mathbb{Z}(i))$$

∵ <u>$i \geq 0$</u>: X: affine

$$\Rightarrow CH_i(X, j) \simeq H_j\left(C_*(\mathrm{Zequi}(X, i))(\mathrm{Spec}\, k)\right)$$

$$\simeq H^{-j}_{Nis}\left(\mathrm{Spec}\, k, C_*(\mathrm{Zequi}(X, i))\right)$$

$$\cong \mathrm{Hom}\left(\mathbb{Z}(i)[2i+j], M^c(X)\right)$$

X: general.: Use Localization seq. both for

$$\begin{cases} CH_*(-, *) \ (\text{Bloch}) \\ H^{BM}_*(-, *) \end{cases}$$

<u>$i < 0$</u>: use h.i. for higher Chow & $M^c(X)(i)[2i]$

$$= M^c(X \times \mathbb{A}^i)$$

<u>Duality</u>: <u>Def</u>: $A \in DM_{gm}(k)$, $A^* \overset{df}{:=} \underline{Hom}_{DM_{gm}}(A, \underset{\underset{M(\mathrm{Spec}\,k)}{\|}}{\mathbb{Z}})$

<u>Thm</u>: $A, B \in DM_{gm}$

$\underline{1)}$: $A \simeq (A^*)^*$ $\quad\underline{2)}$: $(A \otimes B)^* \simeq A^* \otimes B^*$

$\underline{3)}$: X: Sm/k $\quad \underline{Hom}(A,B) \simeq A^* \otimes B$.

: equidimensional $\quad\underline{i)}$: $M(X)^* \simeq M^c(X)(-n)[-2n]$

$\underline{ii)}$: $M^c(X)^* \simeq M(X)(-n)[-2n]$

<u>Cor</u>: DM_{gm}: closed under $\underline{Hom}$ & rigid.

<u>ex)</u>: $CH_i(X,j) \simeq H^{BM}_{2i+j}(X, \mathbb{Z}(i)) \simeq H^{2n-(2i+j)}(X, \mathbb{Z}(n-i))$

X: smooth of $\dim X = n$.

$$\rightsquigarrow \begin{tikzcd} V_k \arrow[r, hook] \arrow[d] & Sch/k \arrow[d, "M"] \\ \mathcal{M}_k^{opp} \arrow[r, dashed] & DM_{gm} \end{tikzcd}$$

$(X, id, m) \longmapsto M(X)(-m)[-2m]$

<u>Rem</u>: DM_-^{eff}: pseudo-Abel.

My Theory. Voevodsky. 12/22. Hanamura. M No.1.

$\mathcal{D}(k)$ $\quad\quad$ $DM_{gm}(k)$

motiviccoh. $\cong$ $*$.

$$DM_{gm}(k) = \left(K^b(SmCor(k)) \big/ T \right)^{\text{pseudo-Abel 化}} + \text{Tate obj. invertible } [\mathbb{Z}(1)^{-1}]$$

<u>$\mathcal{D}(k)$の定義</u>:

$\widetilde{Symb}(k)$ $\quad$ DG-category.

<u>Object</u>: $\bigoplus (X_\alpha, r_\alpha)$ $\quad$ $r_\alpha \in \mathbb{Q}$ $\quad$ X_α : proj. smooth $/k$.

$\alpha \leftarrow I$: index set : finite.

<u>Morph</u>: $\mathrm{Hom}((X,r),(Y,s))^{\bullet}$

$$\stackrel{dfn}{:=} Z^{\dim X + s - r}(X \times Y, -\bullet) \text{ cycle cpx of Bloch.}$$

$X \times \square^n$ $\quad\quad$ $\square^n = (\mathbb{A}^1, \{0,1\})^n$

$$\mathrm{Hom}((X,r),(Y,s))^{\bullet} \times \mathrm{Hom}((Y,s),(Z,t))^{\bullet} \dashrightarrow \mathrm{Hom}((X,r),(Z,t))^{\bullet}$$

$$Z^a(X \times Y, \cdot) \otimes Z^b(Y \times Z, \cdot) \dashrightarrow Z^{a+b-d_Y}(X \times Z, \cdot)$$

$\Delta \widetilde{Symb}(k)$: DG-category

Object; C-diagram in $\widetilde{Symb}(k)$

$$(K^m, f^{m,n}) \quad f^{m,n} \in \mathrm{Hom}(K^m, K^n)^{-(n-m-1)} \quad m<n$$

K: symbol.

$$(-1)^n \partial f^{mn} + \sum_{m<l<n} f^{ln} \circ f^{ml} = 0$$

$$K^0 \to K^1 \to K^2 \to$$

No2

$$\mathrm{Hom}(K,L)^{\cdot} = \bigoplus_{m\le n} \mathrm{Hom}(K^m, L^n)^{\cdot}$$

$$D = \sum \partial + \sum (\pm) \cdot f_K^{\cdots} + \sum f_L^{\cdots} (-) = 0.$$

compos def'd $\Rightarrow$ DG-category.

$H_0(\widetilde{\mathrm{Symb}}(k))$: object 同じ. Morph. $\mathrm{Hom}(K,L)$

def // $\Delta\widetilde{\mathrm{Symb}}(k)$ $\mathcal{D}(k)$

$\mathcal{D}_{finite}(k)$ $H^0(\mathrm{Hom}(K,L)^{\cdot})$

$\uparrow$ triangulated $\mathcal{D}(k) = (\mathcal{D}_{finite}(k))^{4}$ 4 : ps-Abel 化

- $\mathcal{D}(k)$ is triangulated category tensor.   • $\mathrm{Hom}_{\mathcal{D}(k)}(h(X), h(Y))$
- $\mathbb{Z}(1) \overset{dfn}{:=} (pt, 1)[-2]$
- $h : (\mathrm{SmProj}/k)^{opp} \longrightarrow \mathcal{D}(k)$ $h(Y) \otimes \mathbb{Z}(r)[2r]$

$$X \longmapsto (X, 0)[0] \qquad = (Y, r)[0]$$

$$\mathrm{Hom}_{\mathcal{D}(k)}(h(X), h(Y) \otimes \mathbb{Z}(r)[2r-n]) \simeq CH^r(X \times Y, n)$$

- $(q\text{-proj}/k)^{opp} \xrightarrow{h} \mathcal{D}(k)$ の存在. (char $k = 0$ なら)
- $H^*_{Bett} : \mathcal{D}(k) \longrightarrow \underline{Vec}_{\mathbb{Q}}$ [00]; My papers Inventions Math

[95]: Res. Math letters

[04]: Invention. Math

Thm : ($\mathrm{ch}\, k=0$) $\exists$ equiv. of triangulated category tensor.

$$DM_{gm}(k) \xrightarrow{\cong} \mathcal{D}(k)$$

X : smooth proj. $/k$ $\quad U$ smooth$/k$.

$$H((X,r),U)^{\bullet}_{\mathcal{D}\alpha} = Z^{\dim X - r}(X\times U, -\bullet)$$

Y : smooth proj. $\quad \mathrm{Hom}((Y,s),(X,r))^{\bullet}_{\mathcal{D}\,V}$ acts from right ;

$$\alpha\circ V \in H((Y,s),U)^{\bullet}$$

$$\begin{array}{ccc} & Y\times X\times U & \to \text{proj.} \\ \swarrow & \downarrow & \searrow \\ Y\times X & Y\times U & X\times U \end{array}$$

U, U' : smooth quasi-proj. $/k$

$Z \subset U\times U'$. $\quad \mathrm{Corr}(U,U')$

$Z \to U$, $Z \to U'$ finite surj. to a component.

acts from left. $Y\times U$

$$U\circ\alpha \in H((X,r),U')^{\bullet}$$

$$(\alpha\circ V)\circ V' = \alpha\circ(V\circ V')$$

$$u'\circ(u\circ\alpha) = (u'\circ u)\circ\alpha.$$

$$u\circ(\alpha\circ v) = (u\circ\alpha)\circ v.$$

$SmCor(k)$:

Object : U : sm-q-proj$/k$.

$\mathrm{Hom}(U,U')$

$= \mathrm{Corr}(U,U')$ $\quad f^{i+1}\circ f^i = 0$

$C^b(SmCor(k))$ $\quad U^{\bullet} = [\to U^0 \xrightarrow{f^0} U^1 \xrightarrow{f^1} U^2 \to \cdots]$

$K^b(SmCor(k))$

T $\quad [U] \longleftrightarrow [U\times \mathbb{A}^1]$

$$0 \to [U\cup V] \to [U \amalg V] \to [U\cap V] \to 0$$

$\parallel$

W : smooth.

$$\left(K^b(SmCor(k))/T\right)^{\natural}[\mathbb{Q}(1)^{-1}]$$

$\mathbb{Z}(-1) = h^1(\mathbb{P}^1)$

Def: [00] $U^\cdot \in K^b(\mathrm{SmCor}(k))$ A left resolution of $U^\cdot$ is an object $L \in \mathcal{D}_{\mathrm{finite}}(k)$

+ $\alpha \in H^0(H(L, U^\cdot)^\cdot)$ satisfying

$K \in \mathcal{D}_{\mathrm{finite}}(k)$ $U^\cdot \in K^b\mathrm{SmCor}(k)$

$H(K, U^\cdot)^\circ = \bigoplus_{m \le n} H(K^m, U^n)^\circ$ $D = \partial + f_K^\cdot + f_{U^\cdot}$

(*): $\mathrm{Hom}_{\mathcal{D}(k)}(K, L) \xrightarrow[\alpha\circ(-)]{\simeq} H^0(H(K, U^\cdot))$

$\in$

$H^0(\mathrm{Hom}(K, L)^\cdot)$

Thm: (1): $\forall U^\cdot \in K^b(\mathrm{SmCor}(k))$ has a left resolution

(char $k=0$) これを、$L(U^\cdot) \in \mathcal{D}_{\mathrm{finite}}(k)$ とかく。

(2): $\exists!$ functor. $L: U^\cdot \longmapsto L(U^\cdot)$ L: triangulated tensor cat の functor

s.t. (*) $U^\cdot$ について functorial.

これで、$L: K^b(\mathrm{SmCor}(k)) \longrightarrow \mathcal{D}_{\mathrm{finite}}(k)$ が def できる。 $L(A) \to L(A \otimes \mathbb{Z}(-1))$ equivalence in $\mathcal{D}(k)$

$\in$

$L(T) = 0$ $\left(K^b(\mathrm{SmCor}(k))/T\right)[\mathbb{Z}(-1)^{-1}] \xrightarrow{\simeq} \mathcal{D}(k)$

generated by $h(X) \otimes \mathbb{Z}(r)[s]$ as triangulated category

• $\hom(h(X), h(Y)(r)[2r-n])$

$$= CH^{\dim X + r}(X \times Y, n)$$

$\therefore$ $DM_{gm}(k) \xrightarrow{\sim} \mathcal{D}(k)$: category equivalence

(prf) sk.

U : sm of q-proj. $L(U)$

$$\begin{array}{l} U \\ \cap \\ X \end{array} \quad \begin{array}{l} Y = X - U \\ = \bigcup_i Y_i \end{array} = \left[\cdots \to (Y_{(2)}, -2) \to (Y_{(1)}, -1) \xrightarrow{\text{inclusion}} (X, 0) \right]$$

: normal crossing divisor

$$Y_{(1)} = \coprod Y_{i_1} \amalg \cdots \amalg Y_{i_p}$$