$X/\mathbb{Z}$: of finite type. 12/20 Geisser :

$$\zeta(X,s) = \prod_{x\in|X|} \frac{1}{1-Nx^{-s}}$$ want to understand this.
Next time, special values

$X/\mathbb{F}_q$: finite field X ; smooth, projective

Grothendieck ; Use $\boxed{\text{pure motive}}$ to prove / understand Weil Conjecture

arbitrary $X \rightsquigarrow$ mixed motive.

k : field V_k : smooth projective ~~variety~~ k-schemes

$Z^d(X)$: free Abelian group on irred. subvar. of codim $= d$.

those equivalence relations $\sim$ s.t. pull-back, push-forward.

conjecturally equal.

$$\text{rat. eq.} \subseteq \cdots \subseteq \text{alg. eq.} \subseteq \text{arith. eq.}$$

Chow Motives. very good eq. (Jannsen)

$A^d(X) = Z^d(X)/\sim$ Remark : Don't read $\sim$ in Voevodsky's situation

$\phi : X \longrightarrow Y$ induces $\phi^* : A^*(Y) \longrightarrow A^*(X)$

$\phi_* : A^*(X) \longrightarrow A^{*+\dim Y - \dim X}(Y)$

product : $A^d(X) \otimes A^e(X) \longrightarrow A^{d+e}(X)$

if, $\dim X = d$ define $\mathrm{Corr}^r(X, Y) = A^{d+r}(X \times Y)$

extend linearly to $X = \coprod_{i\in I} X_i$. get

$$\mathrm{Corr}^r(X,Y) \times \mathrm{Corr}^s(Y,S) \longrightarrow \mathrm{Corr}^{r+s}(X,S)$$

$$f \otimes g \longmapsto g \circ f = \underbrace{}_{(p_{13})_*}(p_{12}^* f \cdot p_{23}^* g)$$

$$\begin{tikzcd}
 & X \times Y \times S \arrow[dl, "p_{12}"'] \arrow[d, "p_{13}"] \arrow[dr, "p_{23}"] & \\
X \times Y & X \times S & Y \times S
\end{tikzcd}$$

$\mathcal{M}_k$: cat. objects (X, p, m) $X \in Var_k$, $m \in \mathbb{Z}$. $p = p^2 \in Cor^0(X, X)$

$\mathrm{Hom}((X, p, m), (Y, q, n)) = q \cdot Cor^{n-m}(X, Y) \circ p$

Thm: $\mathcal{M}_k$ is additive $\mathbb{Q}$-linear category, which is pseudo-Abelian

Pf. $M = (X, p, m)$ and $f = pfp \in \mathrm{End}(M)$ then

$$M = (X, p-pfp, m) \oplus (X, pfp, m) \qquad (X, p, m) \oplus (Y, q, m) = (X \sqcup Y, p \oplus q, m)$$

In general, $\mathcal{M}_k$ is not Abelian, but if $F : M \to N$ has a left inverse $g \circ f = id_M$, then $F : M \xrightarrow{\sim} fg N \subseteq N$

right $f \circ g = id_N$ then $M \supseteq gfN \xleftarrow{\sim} N : g$

There is a functor $h : V_k^{opp} \longrightarrow \mathcal{M}_k$ $\qquad \phi : Y \to X$

$$X \longmapsto (X, id, 0) = h(X) \qquad h(\phi) = \phi^* = [\Gamma_\phi] \in Corr^0(X, Y) = \mathrm{Hom}_{\mathcal{M}_k}(h(X), h(Y))$$

Tensor Product

$$(X, p, n) \otimes (Y, q, m) = (X \times Y, p \times q, m+n)$$

on morph. $q_1 f_1 p_1 \otimes q_2 f_2 p_2 = (q_1 \otimes q_2)(f_1 \otimes f_2)(p_1 \otimes p_2) \in Corr(X_1 \times X_2, Y_1 \times Y_2)$

$1 = (\mathrm{Spec}\, k, id, 0)$ identity for $\otimes$

$\mathcal{L} = (\mathrm{Spec}\, k, id, -1)$ $\qquad ((X, p, m) = ph(X) \otimes \mathcal{L}^{-m} \subseteq h(X) \otimes \mathcal{L}^{-m})$

$\mathcal{L}^n \overset{df}{:=} \mathcal{L}^{\otimes n} = (\mathrm{Spec}\, k, id, -n)$

$\phi : Y \to X$ $\dim X = d$, $\dim Y = e$. $\qquad ({}^t\Gamma_\phi) \in A^d(Y \times X) = Corr^{d-e}(Y, X)$

$\rightsquigarrow \quad \phi_* : h(Y) \longrightarrow h(X) \otimes \mathcal{L}^{e-d}$ $\quad X$: irred. $\dim X = d$, $x \in X(k)$

$\alpha : X \to \operatorname{Spec} k \qquad x^* \alpha^* = \mathrm{id}$ so $\alpha^* : \mathbb{1} \hookrightarrow h(X)$: subobject $h^0(X)$

$\alpha_* x_* = \mathrm{id}$ so $\alpha_* : h(X) \longrightarrow \mathcal{L}^d$ quotient $h^{2d}(X)$

in fact $\quad h^0(X) = (X, \{x\} \times X, 0) = \mathbb{1}$

$\qquad h^{2d}(X) = (X, X \times \{x\}, 0) = \mathcal{L}^d \qquad$ diagonal $\Delta \sim X \times \{x\} + \{x\} \times X$

e.g. $\quad h(\mathbb{P}^1) = h^0(\mathbb{P}^1) \oplus h^2(\mathbb{P}^1) = \mathbb{1} \oplus \underline{\mathcal{L}}$

<u>direct sum</u>: $\quad M = (X, p, m) \quad N = (Y, q, n)$ if $m < n$

then $M \doteq (X, p, n) \otimes \mathcal{L}^{n-m} = (X, p, n) \otimes h^2(\mathbb{P}^1)^{n-m}$

$\qquad = \left(X \times (\mathbb{P}^1)^{n-m}, p', n\right)$

$M \oplus N = \left(X \times (\mathbb{P}^1)^{n-m}, p' \oplus q, n\right)$

$\vee : \mathcal{M}_k^{\mathrm{opp}} \longrightarrow \mathcal{M}_k \qquad (X, p, m)^\vee = (X, {}^t p, d-m) \qquad d = \dim X$

transpose on morph.

$h(X)^\vee = h(X) \otimes \mathcal{L}^{-d} \qquad$ "Poincaré Duality" $\quad M^{\vee\vee} = M$.

$\operatorname{Hom}(M \otimes N, P) = \operatorname{Hom}(M, N^\vee \otimes P)$ so $\underline{\operatorname{Hom}}(M, N) = M^\vee \otimes N$

$\rightsquigarrow$ "rigid additive tensor category"

<u>Manin's identity principal</u>:

$A^n(X) = \hom(\mathcal{L}^n, h(X)) \qquad \xi^* : h(X) \longrightarrow \mathcal{L}^{\dim X - n}$; transpose.

$\xi \longmapsto \xi_* \qquad \bar{\xi} : h(X) \otimes \mathcal{L}^d \longrightarrow h(X) \otimes h(X) = h(X \times X) \xrightarrow{\Delta^*} h(X)$

so defined $\quad A^n(M) = \hom(\mathcal{L}^n, M)$

$\mathcal{M}_k \to \mathrm{Fct}(\mathcal{M}_k, \mathrm{Vect}_{\mathbb{Q}})$ $\quad A^0(M \otimes N^\vee)$

$M \longmapsto A^0(M \otimes -^\vee)$ $\quad = \mathrm{Hom}(1, M \otimes N^\vee)$

$= \mathrm{Hom}(N, M)$

: fully faithful

every $N \in \mathcal{M}_k$ is a summand of $h(Y) \otimes \mathcal{L}^n$

$A^0(M \otimes h(Y) \otimes \mathcal{L}^n) = A^{-n}(M \otimes h(Y))$ so $\mathcal{M}_k \to \mathrm{Fct}(\mathcal{V}_k^{opp}, \mathrm{Vect}_{\mathbb{Q}})$

$M \mapsto \omega_M$

is fully faithful

$\omega_M(Y) = A^*(M \otimes h(Y))$

MIP $\underline{i)}$: $f: M \to N$ is an isom $\Longleftrightarrow$ $\omega(Y): A^*(M \otimes h(Y)) \longrightarrow A^*(N \otimes h(Y))$ is an isom for all $Y \in \mathcal{V}_k$

$\underline{ii)}$: $f, g: M \to N$ are equal

$\Longleftrightarrow \omega_f(Y) = \omega_g(Y) \quad \forall Y \in \mathcal{V}_k$.

$\underline{iii)}$: $0 \to M' \to M \to M'' \to 0$

$\Longleftrightarrow 0 \to A^0(M' \otimes h(Y)) \to A^0(M \otimes h(Y)) \to A^0(M'' \otimes h(Y)) \to 0$ (exact)

$\underline{\text{Thm}}$: $S \in \mathcal{V}_k$ $\mathcal{E}$: locally free shf. $X = \mathbb{P}(\mathcal{E}) \xrightarrow[\pi]{} S$

$\xi = c_1(\mathcal{O}_X) \in A^1(X)$; then

$\sum \xi^i \pi^*: \bigoplus_{i=0}^{r} h(S) \otimes \mathcal{L}^i \xrightarrow{\sim} h(X)$ is isom.

$\underline{\text{Thm}}$: $Y \hookrightarrow X$ codim > 1 $\quad X'$: X の Y における blow-up.

$$\begin{array}{ccc} Y & \hookrightarrow & X \\ \uparrow & & \uparrow \\ Y' & \hookrightarrow & X' \end{array}$$

$Y' \to Y$: projective budle.

$0 \to h(Y) \otimes \mathcal{L}^{r+1} \xrightarrow{(i_*,\ )} h(X) \otimes h(Y') \otimes \mathcal{L} \longrightarrow h(X') \to 0$ (exact)

Curves:

$$h(X) = h^0(X) \oplus h^1(X) \oplus h^2(X) \quad h^1(X) = (X, id - p_1 - p_2, 0)$$

non-canonically. X, X': curves.

$$\mathrm{Hom}(h^1(X), h^1(X')) = \mathrm{Hom}(\mathcal{Y}, \mathcal{Y}')_{\mathbb{Q}}$$

if, $\mathcal{Y}, \mathcal{Y}'$ ~~and~~ are Jacobians of X, X'.

$$\mathrm{Hom}(\mathcal{L}, h^1(X)) = \begin{cases} 0 & \text{if } \sim \text{ is num.} \\ \mathcal{Y}(k)_{\mathbb{Q}} & \text{if } \sim \text{ is rat.} \end{cases}$$

$$\mathrm{Hom}(\mathcal{L}, h(X)) = A^1(X)$$

$\mathrm{hom}(\mathcal{L}, h^0(X)) = 0$, $\mathrm{Hom}(\mathcal{L}, h^2(X)) = \mathbb{Q}$

so $\mathrm{Hom}(\mathcal{L}, h^1(X)) = \cancel{\mathrm{hom}(\mathcal{L}, h^1(X))}$

$= \ker(\deg : A^1(X) \to \mathbb{Q}.)$

<u>Prop</u>: If k is not contained in $\overline{\mathbb{F}_q}$ then $\mathcal{M}_k^{rat}$ is not Abelian cat

<u>Prop</u>: (Jannsen) $\mathcal{M}_k^{num}$ is Abelian.

(<u>Proof</u>): There is an elliptic curve E/k. $P \in E(k)$

$\xi = (P) - (0) \in A^1(E)$ $\quad \xi_* : \mathcal{L} \to A^1(E)$ is non-zero.

by $\otimes$ $\quad \underset{\underset{0}{\shortparallel}}{\xi_* \circ \xi^*} : h^1(E) \otimes \mathcal{L} \to h^1(E) \rightsquigarrow \xi_*$ is not.

If $\mathcal{M}_k^{rat}$ were abelian, then $\ker \xi_*$ is a proper subobject of $\mathcal{L}$ ~~have~~ hence 1. $\quad \mathrm{Hom}(1,1) = \mathbb{Q}$.

~~and~~ has a proper subobject