Classical case. Spec $\mathcal{O}_K$ $r_1 =$ real place. No. 1

12/22. T. Geisser $r_2 =$ complex place.

$\zeta_K(s)$

$$\lim_{s\to 1}(s-1)\zeta_K(s) = \frac{2^n (2\pi)^{r_2} h_K R_K}{w_K \sqrt{|D_K|}} \qquad g_n = \begin{cases} r_1 + r_2 & n:\text{odd} \\ r_2 & n:\text{even}\end{cases}$$

$$\lim_{s\to 0} s^{-r_1-r_2+1}\zeta_K(s) = -\frac{hR}{w}$$

Borel $\rho_n : K_{2n-1}(\mathcal{O}_K)_{\mathbb{R}} \xrightarrow{\sim} \mathbb{R}^{g_n}$ $R_n = |\det \rho_n|$

$$\begin{array}{ccc} K_{2n-1}(\mathcal{O}_K)_{\mathbb{R}} & \xrightarrow{\sim} & \mathbb{R}^{g_n} \\ \cup & & \cup \\ K_{2n-1}(\mathcal{O}_K)/\text{tor} & & \mathbb{Z}^{g_n} \end{array}$$

$$-\ \operatorname{ord}_{s=1-n}\zeta_K(s) = \operatorname{rk} K_{2n-1}(\mathcal{O}_K)$$

$$-\ \lim_{s\to 1-n}(s-(1-n))^{-g_n}\zeta_K(s) = R_n(K) \quad \text{up to a rational number}$$

$$\| \quad \pm R_n(K) \cdot \frac{|K_{2n-2}(\mathcal{O}_K)|}{|K_{2n-1}(\mathcal{O}_K)_{tor}|}$$

Lichtenbaum (LN 342)
1972.

n: even
g. K: totally real field $\Big) \Rightarrow g_n = 0$

$$\zeta_K(1-n) \overset{?}{=} \pm \frac{|K_{2n-2}(\mathcal{O}_K)|}{|K_{2n-1}(\mathcal{O}_K)_{tor}|} 2^{r_1} \left(K_i(\mathbb{F}_\ell) : \ell\text{-torsion free.} \right)$$

1990 $\longrightarrow$ $\|$ $\quad \|?$

Wiles
Iwasawa
Theory

$$\pm \prod_{\ell} \frac{|H^1_{et}(\mathcal{O}_K[\frac{1}{\ell}], \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))|}{|H^0_{et}(\mathcal{O}_K[\frac{1}{\ell}], \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))|}$$

Quillen ICM 74: $K_{2n-1}(\mathcal{O}_S)\otimes\mathbb{Z}_\ell = H^1(\mathcal{O}_S, \mathbb{Z}_\ell(n))$
$K_{2n-2}(\mathcal{O}_S)\otimes\mathbb{Z}_\ell = H^2(\mathcal{O}_S, \mathbb{Z}_\ell(n))$ $\quad \frac{1}{\ell}\in\mathcal{O}_S$

"Lichtenbaum-Quillen conjecture"

Generalize to use. arithmetic schemes. $X/\operatorname{Spec}\mathbb{Z}$: of finite type.

Hasse–Weil – ζ

$$\zeta(X,s)=\prod_{x\in X_0}\frac{1}{1-N_x^{-s}}$$

converges absolutely for $\mathrm{Re}(s)>\dim X$ – conjectured to have meromorphic continuation

$X = A\cup B$ $\quad \zeta(X,s)=\zeta(A,s)\cdot\zeta(B,s)$

$X/\mathbb{F}_q$ $\quad \zeta(X,s)=\prod \det\left(1-F\cdot q^{-s}\mid H^i_c(\overline{X},\mathbb{Q}_\ell)\right)^{(-1)^{i+1}}$

<u>Conj.</u> (Soulé ICM'83) $\quad \mathrm{ord}_{s=n}\,\zeta(X,s)=-\sum_{i\in\mathbb{Z}}(-1)^i\dim_{\mathbb{Q}} K_i(X)_{(n)}$

generalize $X=\mathrm{Spec}\,\mathcal{O}_S$ $\qquad$ $\mathrm{char}\,k=p$ $\quad X$: smooth projective

<u>Finite Fields</u>: Leading coefficient of $\zeta(X,s)$ at $s=0$ can be expressed in terms of $H^*(X_{\acute{e}t},\mathbb{Z})$ $\quad$ Milne '84

$\mathbb{Z}=K_0$

$\mathbb{G}_m=K_1$

<u>$s=1$</u> : $\quad H^*(X_{\acute{e}t},\mathbb{G}_m)$

At $s=n$ $\quad H^*(X_{\acute{e}t},K_n)$? no.

Lichtenbaum LN.1068. $\qquad K_i \qquad \zeta$ at $s=i$.

$\cdot K_0 \qquad \zeta(0)$

<u>Ex</u>, $X=\mathrm{Spec}\,\mathbb{Z}$, $n=2$. $\qquad \mathbb{Z}(0) \quad \mathbb{Z}(1) \quad \cdot K_1 \qquad \zeta(1)$

$\zeta(2)\longleftrightarrow\zeta(-1)=-\frac{1}{12}$ $\quad \mathbb{Z}(2) \quad \cdot\, K_3 \xrightarrow{\text{ind}} K_2 \qquad \zeta(2)$

$\mathbb{Z}(3) \longrightarrow \cdot\, K_3^M \qquad \zeta(3)$

$$\frac{|K_2(\mathbb{Z})|}{|K_3(\mathbb{Z})|}=\frac{2}{48}=\frac{1}{24}\cdot \longrightarrow \cdot\, K_4^M \quad \cdot\, K_3 \qquad \zeta(4)$$

K_4

$$K_3^M(\mathbb{Z})=\mathbb{Z}/2 \qquad \frac{|K_2(\mathbb{Z})|}{|K_3^{\text{ind}}(\mathbb{Z})|}=\frac{1}{12}$$

Expect a <u>complex</u> of étale sheaves $\mathbb{Z}(n)$ s.t.
$\zeta(X,s)$ can be expressed in terms of étale coh. $\mathbb{Z}(n)$

Beilinson (Letter to Soulé '82) Complex of Zariski ~~of simplicial pre~~ sheaves with

<u>0)</u>: $\mathbb{Z}(0)=\mathbb{Z}$, $\mathbb{Z}(1)=\mathbb{G}_m[-1]$ ✓ Bloch

<u>1)</u>: $\mathbb{Z}(r)$ is acyclic outside of $[1, r]$ unknown!

<u>2)</u>: $H^{r+1}_{et}(k, \mathbb{Z}(r))=0$ $\ \underset{k}{\ }$ "Hilbert 90" Beilinson–Soulé vanishing Conj. Suslin–Voevodsky

<u>3)</u>: $\mathbb{Z}/m(n) = \mu_m^{\otimes n}$ if $\frac{1}{m}\in\mathcal{O}_X^\times$ $\left(\mathbb{Z}/p^r(n)_{et} = W_r\Omega^n_{X,\log}[-n]\right)$ Gr–Levine Bloch–Kato

<u>4)</u>: product $\mathbb{Z}(r)\otimes\mathbb{Z}(s)\longrightarrow\mathbb{Z}(r+s)$ ✓ Bloch

<u>4)</u>: $H^r_{et}(k, \mathbb{Z}(r)) = K^M_r(k)$ Nesterenko–Suslin, Totaro

<u>5)</u>: $H^i(X, \mathbb{Z}(r)) \sim gr^r_\gamma K_{2r-i}(X)$ isom up to small fail. Grayson–Suslin, Levine

(SS. $E_2^{s,t} = H^{s-t}(X, \mathbb{Z}(-t)) \Rightarrow K_{-s-t}(X)$) dege up to small torsion

In particular, $K_i(X)^{(n)}_{\mathbb{Q}} = H^{2n-i}(X, \mathbb{Q}(n))$

Connection to Beilinson: $\mathbb{Z}(n)_{Zar} \xrightarrow{\sim} \tau_{\le n} R\varepsilon_* \mathbb{Z}(n)_{et}$ Voevodsky(?)

$\varepsilon: X_{et}\longrightarrow X_{Zar}$ "Beilinson–Litchtenbaum Conj"

Parshin-Conj ⇒ Beilinson–Soulé vanishing Conj.

<u>Rem</u>: If one wants to consider non-smooth, non-projective scheme.

$\mathbb{Q}(r)$ is acyclic except at r

one needs eithers

- cohom. with compact support, ~~finer topology~~.
- Borel-Moore homology (= higher Chow Group) ← open problem.

} localizat

Combinate: Bloch's higher Chow Groups, X: smooth (then ~~BM hom.~~ = coh.) '86

— Voevodsky's CM '90

Thm (S-V) equal for X: smooth

Rem: The spectral sequence is from higher Chow groups to K'-Theory, there cannot be a SS from MC to K-theory.

Soulé Conj: $\mathrm{ord}_{s=n}\zeta(X, s) = -\sum_{i\in\mathbb{Z}}(-1)^i \dim_{\mathbb{Q}} CH_n(X, i)_{\mathbb{Q}}$ $\ X/\mathrm{Spec}\,\mathbb{Z}$: f.t. $\ \dim X = d$

- $CH_n(X, i)$: f.t (Bass Conj). $\ CH^{d-n}(X, i)$

$CH_i(X, i)$

Ex: $X = \mathrm{Spec}\, \mathcal{O}_K$,

$$\mathrm{ord}_{s=1-n} \zeta(s) = \dim_{\mathbb{Q}} CH_{1-n}(\mathcal{O}_K, 2n-1)$$
$$= \dim K_{2n-1}((\mathcal{O}_K))^{(n)}_{\mathbb{Q}} \quad \checkmark \text{ Borel.}$$

$H^{2n+2}(X_{\acute{e}t}, \mathbb{Z}(n))$: is not fin gen.

Lithtenbaum's conj. (including L'03, G'04)

Conj: Let $X/\mathbb{F}_q$: of finite type fix.

1): $H^i_c(X_{ar}, \mathbb{Z}(n))$ fin. gen. vanish for allmost all

2): $H^i_c(X_{ar}, \mathbb{Z}(n)) \otimes \mathbb{Z}_\ell \simeq H^i_c(X_{\acute{e}t}, \mathbb{Z}_\ell(n))$

3): $\mathrm{ord}_{s=n} \zeta(X,s) = \sum_i (-1)^i i \cdot \mathrm{rk}\, H^i_c(X_{ar}, \mathbb{Z}(n))$

4): $\zeta(X,s) \sim \pm(1-q^{-n})^{\rho_n} \chi(H^\cdot_c(X_{ar}, \mathbb{Z}(n))) \cdot q^{?}$ as $s \to n$

5): $H^{i-1}_c(X_{ar}, \mathbb{Z}(n)) \xrightarrow{ve} H^i_c(X_{ar}, \mathbb{Z}(n)) \xrightarrow{ve} H^{i+1}_c(X_{ar}, \mathbb{Z}(n))$

Rem: X : smooth, proj. all groups. torsion expect

$$H^{2n}(X_{ar}, \mathbb{Z}(n)) \to H^{2n+1}(X_{ar}, \mathbb{Z}(n)) : \text{"regulator"}$$

Known in many cases ($\Leftrightarrow$ Tate's Conj. Beilinson $\sim_{rat} = \sim_{hom}$)

Ex: ①: $n=1$, X : smooth, proj. $H^3(X_{ar}, \mathbb{Z}(1)) \supseteq Br(X)$: conj. o.k.

$\Rightarrow Br X$: finite

②: X : flat$/\mathbb{Z}$ $\quad X/\mathbb{F}_q \to$ only rational numbers.

transcendental number. (Borel-regulators)

'83: Beilinson's Conj. on special values of $\mathcal{L}$.

Bl... ... for the special value