k : field

$$Sch/_k = \{\text{scheme sep. of f.t}\}/_k \supseteq Sm/_k$$

$$Sm/_k \longrightarrow \underset{\substack{\uparrow \\ \text{additive cat.}}}{SmCor(k)} \qquad \text{Obj : Same} \qquad Hom(X,Y) = \left\{ \begin{array}{c} \overset{\text{closed. int.}}{Z} \subset X \times Y \\ {}_{\text{finite surj.}} \searrow \ \downarrow \\ X \end{array} \right\}_{\mathbb{Z}}$$

$$PSwT/_k \supseteq NSwT/_k$$

$$PSwT/_k = \left\{\begin{array}{l}\text{add. contr.} \\ \text{fct. } SmCor(k) \to Ab\end{array}\right\} \qquad NSwT/_k = \{ F \mid F|_{Sm/_k} \text{ is a Nisnevich shf} \}$$

Abelian cat. w enough proj. and inj.

$\mathbb{Z}_{tr}(X) \in PSwT/_k$ rep. by $X \in SmCor(k)$

In fact, a Nisnevich sf.

$$M(X) \overset{dfn}{:=} C_*(\mathbb{Z}_{tr}(X)) \in D^- := D^-(NSwT/_k)$$

$DM^{eff}_- \subseteq D^-$; cpx. with homotopy inv. shf cohomology

$$\mathbb{Z}(q) := C^*_*(\underbrace{\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q})}_{\substack{\uparrow \\ \text{a direct summand of } \mathbb{Z}_{tr}(\mathbb{G}_m^{\times q})}})[-q] \in D^-$$

$$= [\to \underset{q-1}{C^{-1}_*(\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q}))} \to \underset{\dot{q}}{C^0_*(\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q}))} \to]$$

<u>ex</u> : $\mathbb{Z}(1) = C^*_*(Coker(\mathbb{Z} \to \mathbb{Z}_{tr}(\mathbb{G}_m)))[-1]$

<u>Thm</u> : <u>0)</u> : $M(X), \mathbb{Z}(q) \in DM^{eff}_-(k)$

<u>1)</u> : $Hom_{DM^{eff}_-(k)}(M(X), \mathbb{Z}(q)[p])$

$$\simeq H^p_{Nis}(X, \mathbb{Z}(q)) \simeq H^p_{Zar}(X, \mathbb{Z}(q))$$

No.

2): $\mathbb{Z}(1) \simeq \mathcal{O}^\times[-1]$

(Rem: $\mathcal{O}^\times \in \mathrm{NSW}$)

Similar Prop holds for étale NOT for Zariski

$\downarrow$

The category NSWT

Notation: C: site T: topology

$C^\wedge$: the category of Abel. presheaves on C

$C^\sim_T$: the category of Abel. T-sheaves on C

Prop: $F \in \mathrm{PSWT}/k$ $\quad F_{Nis}$: the Nisnevich shf associated to F regarded as an object in $(Sm/k)^\wedge$.

1): F_{Nis} has a unique str. of PSWT s.t.

$$F \to F_{Nis} \text{ is a morph in PSWT!}$$

2): $G \in \mathrm{NSWT}/k$ $\quad F \to G$: morph in PSWT

$\Rightarrow$ $F_{Nis} \to G$: induced map in $(Sm/k)^\sim_{Nis}$

induce

$$\begin{tikzcd} & F_{Nis} \arrow[r] & G \\ & F \arrow[u] \arrow[ur] & \end{tikzcd}$$

· Results

$$\begin{tikzcd} \mathrm{NSWT}/k \arrow[d, "\tilde{\varphi}"'] \arrow[r, hook, "\tilde{i}"', shift right] & \mathrm{PSWT}/k \arrow[l, "\tilde{a}"', shift right] \arrow[d, "\hat{\varphi}"] \\ (Sm/k)^\sim_{Nis} \arrow[r, hook, "i"', shift right] & (Sm/k)^\wedge \arrow[l, "a"', shift right] \end{tikzcd}$$

($\hat{\varphi}, \tilde{\varphi}$: forget.)

enough inj.

1): $\tilde{\varphi}\tilde{a} = a\hat{\varphi}$

2): $\tilde{a} \leftrightarrow \tilde{i}$: adjoint

$\tilde{a} \circ \tilde{i} \cong \mathrm{id}$

3): NSWT: Abel. $\tilde{a}$: exact.

4): $E = [0 \to F_1 \to F_2 \to F_3 \to 0]$ in NSWT/k.

E: exact $\Leftrightarrow$ $\tilde{\varphi}E$: exact

<u>Thm</u>: $F \in NSWT$, $X \in Sm/k$

$$\mathrm{EXT}^{i}_{NSWT/k}(\mathbb{Z}_{tr}(X), F) \simeq H^{i}_{Nis}(X, \tilde{\varphi}F)$$

precisely

In particular $F \in DM^{eff}_{-}$, $X \in Sm/k$.

$$\Rightarrow \hom_{D^-}(\mathbb{Z}_{tr}(X), F[i]) \simeq H^{i}_{Nis}(X, F)$$

$\because$ Suffice to Prove 「I : inj obj. in $NSWT \Rightarrow H^{i}_{Nis}(X, I) = 0 \ (i > 0)$」

This follows from.

<u>Key lemma</u>: $Y \xrightarrow{f} X$: Nisnevich cov.

$$\Rightarrow 0 \leftarrow \mathbb{Z}_{tr}(X) \leftarrow \mathbb{Z}_{tr}(Y) \leftarrow \mathbb{Z}_{tr}(Y \times_X Y) \leftarrow \cdots$$

(exact) in $(Sm/k)^{\sim}_{Nis}$ //

ideal of pf of key lemma.

$$0 \leftarrow \hom(S, X) \leftarrow \hom(S, Y) \leftarrow \hom(S, Y \times_X Y) \leftarrow \cdots$$

(S : loc. hen.)

(Hom = Hom in SmCor)

generator

$$\begin{tikzcd} Z_Y \arrow[d] \arrow[r, phantom, "\subset"] \arrow[dr, phantom, "\square"] & S \times Y \arrow[d] \\ Z \arrow[r, phantom, "\subset"] \arrow[dr, "\text{finite surj.}"'] & S \times X \arrow[d] \\ & S \end{tikzcd}$$

S : hensel $\Rightarrow Z$: hensel $\Rightarrow Z_Y \to Z$: split : Nisnevich covering

<u>homotopy inv pre.shf w. T.</u>:

<u>Thm</u>: k : perfect $F \in PSWT$; homotopy inv.

<u>(1)</u>: F_{Nis} is also h.i. &

$X \mapsto H^{i}_{Nis}(X, F_{Nis})$ has a str. of

P : pr. w. tr.

(2): $H^i_{Zar}(X, F_{Zar}) \simeq H^i_{Nis}(X, F_{Nis})$ $(i \geq 0)$ No.

In particular $F_{Zar} (\simeq F_{Nis})$ has a str. of pres. w. tr.

これから、以下全て k: perfect とする

Cor: $HI(k) (\subseteq NSwT)$ is closed under taking ker, Coker, extension

$\uparrow$ the full subcategory of h.i. NSwT

In particular $\begin{cases} \cdot\ HI(k) : \text{Abel \& the inclusion is exact.} \\ \cdot\ DM^{eff}_{-}(k) : \text{subtri. cat.} \end{cases}$

Cor: $F \in NSwT \Rightarrow C_*(F) \in DM^{eff}_{-}(k)$

$\left(H_i(C_*(F)) \text{ is h.i. so is } H_i(C_*(F))_{Nis}. \right)$

Ex: $M(X), \mathbb{Z}(q) \in DM^{eff}_{-}(k)$ #

Cor: $F \in DM^{eff}_{-}(k)$, $X \in Sm/k$

$$H^i_{Nis}(X, F) \simeq H^i_{Zar}(X, F)$$

$$RC^*;\ D^- \longrightarrow DM^{eff}_{-};\ F^\cdot \longmapsto Tot(C^*(F^\cdot))$$

$$DM^{eff}_{-} \overset{i}{\hookrightarrow} D^-$$

Thm: $F \in D^-$, $G \in DM^{eff}_{-}$

$$\Rightarrow \mathrm{Hom}_{DM^{eff}_{-}}(RC^*(F), G) \simeq \mathrm{Hom}_{D^-}(F, G)$$

(i.e. RC^* は、i の left adj.)

In particular,

$$\mathrm{Hom}_{DM^{eff}_{-}}(M(X), F) \simeq \mathrm{Hom}_{D^-}(\mathbb{Z}_{tr}(X), F)$$

$$(F \in DM^{eff}_{-})$$

Key lem's for pf:

Lem: $F, G \in \mathrm{NSwT}/k$ F: contractible G: h.i.

$\Rightarrow \mathrm{Ext}^i_{\mathrm{NSwT}}(F, G) = 0 \ (\forall i)$

Lem: $\mathrm{Cohen}(F \to C_1(F))$: contractible.

F: contractible $\overset{\text{def.}}{\Leftrightarrow}$

$$\begin{tikzcd} & & F \\ F \arrow[rru, bend left, "0"] \arrow[r, dashed, "\exists"] \arrow[rrd, bend right, "\mathrm{id}"'] & C_1F \arrow[ru, "\partial_0"] \arrow[rd, "\partial_1"'] & \\ & & F \end{tikzcd}$$

$C_1F : X \mapsto F(X \times \Delta^1)$

Sketch of prf (2): $C_*(\mathbb{Z}_{tr}(\mathbb{G}_m)) \simeq \mathbb{Z} \oplus \mathcal{O}^\times$

$\mathcal{M}^*(\mathbb{P}^1; 0, \infty) ; U \mapsto \left\{ f \in k(U \times \mathbb{P}^1) \,\middle|\, \begin{array}{l} f \text{ equals to } \\ \text{on } U \times \{0, \infty\} \end{array} \right\}$

$\mathrm{Pic}(\mathbb{P}^1; 0, \infty) : U \mapsto \mathrm{Pic}(U \times \mathbb{P}^1, U \times \{0, \infty\})$

$\overset{\text{df}}{:=} \left\{ (L, \phi) \,\middle|\, \begin{array}{l} L: \text{line bdle on } U \times \mathbb{P}^1 \\ L|_{U \times \{0,\infty\}} \underset{\phi}{\simeq} \mathcal{O} \end{array} \right\} \Big/ \mathrm{Isom}.$

Then for $U \in \mathrm{Sm}/k$.

$$\begin{tikzcd} & & \mathbb{Z} \arrow[r] & \mathcal{O}(\mathbb{Z}) & \\ 0 \arrow[r] & \mathcal{M}^*(\mathbb{P}^1, 0, \infty)(U) \arrow[r] & \mathbb{Z}_{tr}(\mathbb{G}_m)(U) \arrow[r] & \mathrm{Pic}(\mathbb{P}^1, 0, \infty)(U) \arrow[u] \arrow[r] & 0 \end{tikzcd}$$

U: affine

$\uparrow$ Ts?

$\mathbb{Z}_{tr}(\mathbb{G}_m)(U) = \left\{ \begin{array}{l} \text{divisors on } U \times \mathbb{P}^1. \\ \text{supp} \cap (U \times \{0, \infty\}) \\ = \phi \end{array} \right\}$

$f \mapsto \mathrm{div}(f)$

$\rightsquigarrow$ $(C_i = C^{-i})$

$$0 \to C_*\left(\mathcal{M}^*(\mathbb{P}^1, 0, \infty)\right) \longrightarrow C_*\left(\mathbb{Z}_{tr}(\mathbb{G}_m)\right) \to C_*\left(\mathrm{Pic}(\mathbb{P}^1, 0, \infty)\right) \longrightarrow 0$$

q.i.s. $\mathrm{S}\!\!\parallel$ h.i. of Pic

In fact, acyclic for each U.

(can construct a homotopy explicitly)

$\mathrm{Pic}(\mathbb{P}^1, 0, \infty)[0]$

IS

$\mathbb{Z} \oplus \mathcal{O}^\times$