

k : field

12/20 Hagiwara, K

No. 1

$Sch/k = \{ \text{scheme sep. of } f + \mathbb{1}/k \} \supseteq Sm/k$ closed int.

$$Sm/k \rightarrow SmCor(k)$$

additive cat.

Obj: Same

$$Hom(X, Y) = \left\{ \begin{array}{c} \Sigma C \times X \times Y \\ \text{finite surj.} \downarrow \\ X \end{array} \right\} \mathbb{Z}$$

$$PSWT/k \supseteq NSWT/k$$

add. contr.

$$fct. SmCor(k) \rightarrow Ab$$

$F/Sm/k$ is a Nisnevich shaf?

$$\mathcal{Z}_{tr}(X) \in PSWT/k$$

rep. by $X \in SmCor(k)$

Abelian cat. w enough proj. and inj.

In fact, a Nisnevich sf.

$$M(X) := C_*^{dfn}(\mathcal{Z}_{tr}(X))$$

$$DM_{-}^{eff} \subseteq D^{-}; \text{ cpx. with.}$$

$$\in D^{-} := D^{-}(NSWT/k)$$

homotopy inv. shaf
cohomology

$$\mathcal{Z}(q) := C_*^*(\mathcal{Z}_{tr}(\mathbb{B}_m^{nq})) [q] \in D^{-}$$

a direct summand of $\mathcal{Z}_{tr}(\mathbb{B}_m^{nq})$

$$= [\rightarrow C_*^{-1}(\mathcal{Z}_{tr}(\mathbb{B}_m^{nq})) \rightarrow C_*^0(\mathcal{Z}_{tr}(\mathbb{B}_m^{nq})) \rightarrow]$$

$$\text{ex: } \mathcal{Z}(1) = C_*^*(\text{Coker}(\mathcal{Z} \rightarrow \mathcal{Z}_{tr}(\mathbb{B}_m))) [-1]$$

Thm: 0): $M(X), \mathcal{Z}(q) \in DM_{-}^{eff}(k)$

$$\underline{1}): \text{Hom}_{DM_{-}^{eff}(k)}(M(X), \mathcal{Z}(q)[p])$$

$$\simeq H_{Nis}^p(X, \mathcal{Z}(q)) \simeq H_{Zar}^p(X, \mathcal{Z}(q))$$

2): $\mathbb{Z}(1) \simeq \mathcal{O}^x[-1]$

The category NSWT

(Rem: $\mathcal{O}^x \in \text{NSW}$)

Notation: C: site T: topology

Similar Prop holds for étale NOT for Zariski

C^\wedge : the category of Abel. presheaves on C

C_T^\sim : the category of Abel. T-sheaves on C



Prop: $F \in \text{PSWT}/\mathbb{R}$ F_{Nis} : the Nisnevich sheaf associated to F regarded as an object in $(\text{Sm}/\mathbb{R})^\wedge$.

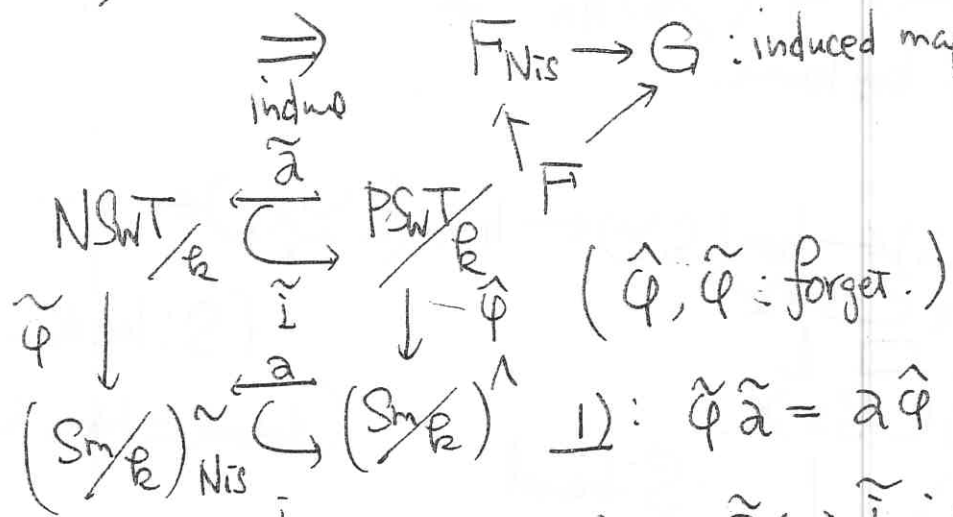
1): F_{Nis} has a unique str. of PSWT s.t.

$F \rightarrow F_{\text{Nis}}$ is a morphi in PSWT!

2): $G \in \text{NSWT}/\mathbb{R}$ $F \rightarrow G$: morphi in PSWT

$F_{\text{Nis}} \rightarrow G$: induced map in $(\text{Sm}/\mathbb{R})_{\text{Nis}}^\sim$

Results



($\hat{\varphi}, \tilde{\varphi}$: forget.)

1): $\tilde{\varphi} \tilde{a} = \tilde{a} \hat{\varphi}$

2): $\tilde{a} \leftrightarrow \tilde{\tilde{a}}$: adjoint

$\tilde{a} \circ \tilde{\tilde{a}} \simeq \text{id}$

3): NSWT: Abel \tilde{a} : exact.

4): $E = [0 \rightarrow F_1 \rightarrow F_2 \rightarrow F_3 \rightarrow 0]$ in NSWT/\mathbb{R}

E : exact $\Leftrightarrow \tilde{\varphi} E$: exact

Thm: $F \in \text{NSWT}$ $X \in \text{Sm}/\mathbb{E}$ precisely

$$\text{Ext}_{\text{NSWT}/\mathbb{E}}^i(\mathcal{D}_{\text{tr}}(X), F) \simeq H_{\text{Nis}}^i(X, \overleftarrow{F}) \cong F.$$

In particular $F \in \text{DM}_{-}^{\text{eff}}$, $X \in \text{Sm}/\mathbb{E}$.

$$\Rightarrow \text{hom}_{\mathcal{D}}(\mathcal{D}_{\text{tr}}(X), F) \simeq H_{\text{Nis}}^i(X, F)$$

☺ Suffice to Prove. $\Gamma: \text{inj obj. in NSWT} \Rightarrow$

$$H_{\text{Nis}}^i(X, \Gamma) = 0 (i > 0)$$

This follows from.

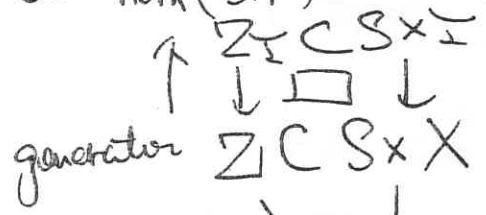
Key lemma: $\Gamma \xrightarrow{f} X: \text{Nisnevich cov.}$

$$\Rightarrow 0 \leftarrow \mathcal{D}_{\text{tr}}(X) \leftarrow \mathcal{D}_{\text{tr}}(\Gamma) \leftarrow \mathcal{D}_{\text{tr}}(\Gamma \times_X \Gamma) \leftarrow \dots$$

(exact) in $(\text{Sm}/\mathbb{E})_{\text{Nis}} \neq$

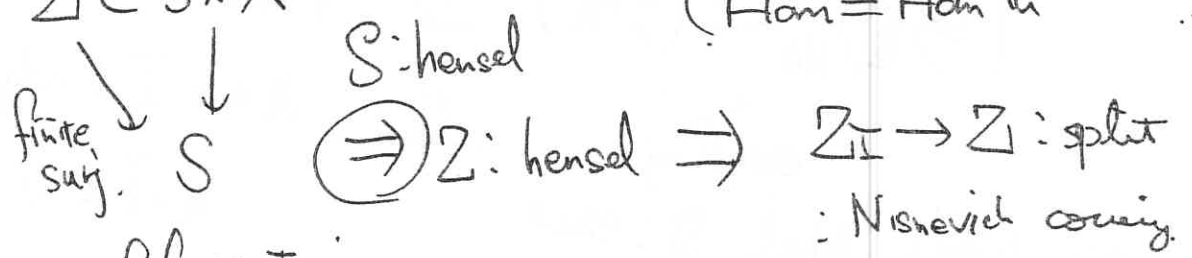
Ideal of pf of key lemma.

$$0 \leftarrow \text{hom}(S, X) \leftarrow \text{hom}(S, \Gamma) \leftarrow \text{hom}(S, \Gamma \times_X \Gamma) \leftarrow \dots$$



(S: loc. hem.)

(Hom = Hom in SmCor)



homotopy inv. pre. ref. w. T.:

Thm: \mathbb{E} : perfect $F \in \text{PSWT}$; homotopy inv.

(1): F_{Nis} is also h.i. &

$X \mapsto H_{\text{Nis}}^i(X, F_{\text{Nis}})$ has a str. of.

ρ : on w. tr.

(2): $H_{Zar}^i(X, F_{Zar}) \simeq H_{Nis}^i(X, F_{Nis}) \quad (i \geq 0)$

In particular $F_{Zar} \simeq F_{Nis}$ has a str. of pres. w. tr. inj's, $LX \subseteq \mathbb{A}^2$

Cor: $HI(\mathbb{R})(\subseteq NSWT)$ is closed under taking Per, Cohen, \mathbb{R} perfect extension \uparrow the full subcategory of h.i. NSWT 233

In particular: $\left\{ \begin{array}{l} \cdot HI(\mathbb{R}) : \text{Abel \& the inclusion is exact.} \\ \cdot DM_{-}^{eff}(\mathbb{R}) : \text{subtri. cat.} \end{array} \right.$

Cor: $F \in NSWT \Rightarrow C_*(F) \in DM_{-}^{eff}(\mathbb{R})$
 ($H_{\mathbb{Z}}(C_*(F))$ is h.i. so is $H_i(C_*(F))_{Nis}$.)

Ex: $M(X), \mathbb{Z}(q) \in DM_{-}^{eff}(\mathbb{R}) \neq$

Cor: $F \in DM_{-}^{eff}(\mathbb{R}), X \in Sm/k$

$$H_{Nis}^i(X, F) \simeq H_{Zar}^i(X, F)$$

$$RC^*; D^- \rightarrow DM_{-}^{eff}; F \mapsto Tot(C^*(F)) \xrightarrow{i} RC^*$$

$DM_{-} \hookrightarrow D^-$

Thm: $F \in D^-, G \in DM_{-}^{eff}$
 $\Rightarrow Hom_{DM_{-}^{eff}}(RC^*(F), G) \simeq Hom_{D^-}(F, G)$

(i.e. RC^* is left adj.)

In particular,

$$Hom_{DM_{-}^{eff}}(M(X), F) \simeq Hom_{D^-}(\mathbb{Z}_{tr}(X), F) \quad (F \in DM_{-}^{eff})$$

Key lem's for pf:

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Lem: $F, G \in \text{NSWT}/\mathbb{R}$ F : contractible G : h.i.

$$\Rightarrow \text{Ext}_{\text{NSWT}}^i(F, G) = 0 \quad (\forall i)$$

Lem: $\text{Coker}(F \rightarrow C_2(F))$: contractible.

F : contractible $\stackrel{\text{def.}}{\Rightarrow}$

$$\begin{array}{ccc}
 F & \xrightarrow{\alpha_0} & F \\
 \downarrow \text{id} & \nearrow \alpha_1 & \\
 C_1 F & \xrightarrow{\alpha_2} & F
 \end{array}$$

$C_2 F: X \mapsto F(X \times \Delta^1)$

Sketch of pf(2):

$$C_* \text{Ztr}(B_m) \cong \mathbb{Z} \oplus \mathbb{Q}^x$$

$$\mathcal{M}^*(\mathbb{P}^1; 0, \infty); U \mapsto \left\{ f \in \mathcal{O}(U \times \mathbb{P}^1) \mid f \text{ equals to } 1 \text{ on } U \times]0, \infty[\right\}$$

$$\text{Pic}(\mathbb{P}^1; 0, \infty): U \mapsto \text{Pic}(U \times \mathbb{P}^1, U \times]0, \infty[)$$

$$\begin{array}{l}
 \mathcal{L} \\
 := \left\{ (L, \phi) \mid \begin{array}{l} L: \text{line bundle on } U \times \mathbb{P}^1 \\ L|_{U \times]0, \infty[} \cong \mathcal{O} \end{array} \right\} / \text{isom.}
 \end{array}$$

Then for $U \in \text{Sm}/\mathbb{R}$.

$$\mathbb{Z} \xrightarrow{\quad} \mathcal{O}(\mathbb{Z})$$

$$0 \rightarrow \mathcal{M}^*(\mathbb{P}^1, 0, \infty)(U) \rightarrow \text{Ztr}(\oplus_m)(U) \rightarrow \text{Pic}(\mathbb{P}^1, 0, \infty)(U) \rightarrow 0$$

$$\begin{array}{l}
 \downarrow \\
 f \mapsto \text{div}(f) \left\{ \begin{array}{l} \text{divisors on } U \times \mathbb{P}^1 \\ \text{supp} \cap (U \times]0, \infty[) \\ = \emptyset \end{array} \right\}
 \end{array}$$

U : affine \mathbb{R} .

$\rightsquigarrow (C_i = C^{-i})$

$0 \rightarrow C_* (\mathcal{Y}^*(\mathbb{P}^1, 0, \infty)) \rightarrow C_* (\mathcal{D}tr(\mathfrak{t}_m)) \rightarrow C_* (\text{Pic}(\mathbb{P}^1, 0, \infty)) \rightarrow 0$



q.i.s. Sll h.i. of Pic.

In fact, acyclic for each U .

$\text{Pic}(\mathbb{P}^1, 0, \infty)[0]$

(can construct a homotopy explicitly)

$\cong \mathbb{Z} \oplus \mathbb{C}^*$