k: perfect field $\quad D^- \overset{\mathrm{dfn}}{:=} D^-(NSwT/k) \supseteq DM^{\mathrm{eff}}_-(k)$: full subcategory consisting of cpx's whose cohomology sheaves are h.i. No. 1

4. categories and 4 cohomologies $\quad$ 12/21. Hagihara. K

$$DM^{\mathrm{eff}}_{gm}(k) \subseteq DM^{\mathrm{eff}}_-(k) \left(\subseteq D^-\right)$$

ll are ensor triangulated ctors are mpatible

$$\begin{array}{ccc} DM^{\mathrm{eff}}_{gm}(k) & \subseteq & DM^{\mathrm{eff}}_-(k) \\ \downarrow & & \downarrow \\ DM_{gm}(k) & \subseteq & DM_-(k) \end{array} \quad \text{invert } \otimes \mathbb{Z}(1)$$

$M(X) := C_*(\mathbb{Z}_{tr}(X)) \in DM^{\mathrm{eff}}_-(k)$

DM^{eff}_{gm}: the minimal full subcategory

- containing $M(X)$ $(X \in Sm/k)$
- closed under direct summand.
- — — taking cones, shifts.

$DM(k)$: the category obtained from $DM^{\mathrm{eff}}_-(k)$ by inverting $-\otimes \mathbb{Z}(1)$

$DM_{gm}(k)$: ——— ———, $DM^{\mathrm{eff}}_{gm}(k)$ ———

More precisely,

$$\mathrm{Obj}(DM_-(k)) := \left\{ (M, m) \;\middle|\; \begin{array}{l} M \in DM^{\mathrm{eff}}_-(k) \\ m \in \mathbb{Z} \end{array} \right\}$$

$$\mathrm{Hom}_{DM_-(k)}((M,m),(N,n)) := \operatorname*{colim}_{k \geq -m,-n} \mathrm{Hom}_{DM^{\mathrm{eff}}_-(k)}(M(m+k), N(n+k))$$

we denote $(M,m) = M(m)$

Rem: (1): In fact, we have

$$\mathrm{Hom}_{DM^{\mathrm{eff}}_-}(A, B) \xrightarrow{\simeq} \mathrm{Hom}_{DM^{\mathrm{eff}}_-}(A(1), B(1))$$

(Cancellation Thm)

So the vertical fct's are in fact full faithful

(2): In order for DM_-, DM_{gm} to become symm. tensor cat, we have to check 「the cyclic permutation on $\mathbb{Z}(1)^{\otimes 3}$ $\mathfrak{S}_3 \curvearrowright$ is the identity」

(3): We easily see that DM_- also has $\underline{Hom}(M(X), -)$ but the two internal hom in DM_-, DM^{eff}_- does not coincide in gen. No.2

(ex): $\underline{Hom}_{DM_-}(\mathbb{Z}(1), \mathbb{Z}) =: \mathbb{Z}(-1) \notin DM^{eff}_-$

$\underline{Hom}_{DM^{eff}_-}(\mathbb{Z}(1), \mathbb{Z}) = 0.$

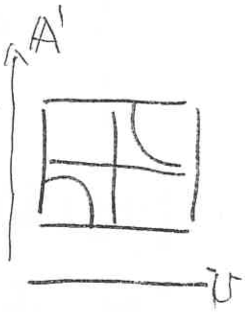
(4): Later we prove that DM_{gm} has $\underline{Hom}$ & "rigid"

For, $X \in Sch/k$ we construct $M(X)$, $M^c(X)$

$$\mathbb{Z}_{tr}(X) : U \longmapsto \left\langle \begin{array}{c} Z \subseteq U \times X : \text{closed integral} \\ \downarrow \\ U \end{array} \right\rangle_{\mathbb{Z}}$$

finite surj. connected comp.

$\Cap$ Sm/k



$$\mathbb{Z}^c_{tr}(X) : U \longmapsto \left\langle \begin{array}{c} Z \subseteq U \times X : \text{closed int.} \\ \downarrow \\ U \end{array} \right\rangle_{\mathbb{Z}}$$

quasi-finite dominant a comp.com

$\mathbb{Z}_{tr}(X)$, $\mathbb{Z}^c_{tr}(X) \in NSwT/k$

$M(X) \overset{dfn}{:=} C_*(\mathbb{Z}_{tr}(X)) \in DM^{eff}_-$

$M^c(X) := C_*(\mathbb{Z}^c_{tr}(X)) \in$

functoriality:

$f : X \to Y$

$\Rightarrow M(X) \to M(Y)$

$f : X \to Y$: proper

$\Rightarrow M^c(X) \to M^c(Y)$

$j : U \hookrightarrow X$: open

$\Rightarrow M^c(X) \to M^c(U)$

natural trans : $M(X) \longrightarrow M^c(X)$

X : proper $\Rightarrow$ $M(X) \xrightarrow{\simeq} M^c(X)$

Def: $X \in Sch/k$: $H^i(X, \mathbb{Z}(j)) = \mathrm{Hom}_{DM^{eff}_-}(M(X), \mathbb{Z}(j)[i])$

$H_i(X, \mathbb{Z}(j)) = \mathrm{Hom}_{DM^{eff}_-}(\mathbb{Z}(j)[i], M(X))$

$H^i_c(X, \mathbb{Z}(j)) = \mathrm{Hom}_{DM_-}(M^c(X), \mathbb{Z}(j)[i])$

$H^{BM}_i(X, \mathbb{Z}(j)) = \mathrm{Hom}_{DM_-}(\mathbb{Z}(j)[i], M^c(X))$

('Borel–Moore homology')

ex : $H^i(X, \mathbb{Z}) = \hom_{DM_-}(M(X), \mathbb{Z}[i])$

$H_i(X, \mathbb{Z}) = \hom_{DM_-}(\mathbb{Z}[i], M(X))$

Nis. coh. dim $= 0$

$= \hom_{DM_-}(\mathbb{Z}, M(X)$ ~~hom~~ $H^{-i}_{Nis}(\mathrm{Spec}\, k, C_*(\mathbb{Z}_{tr}(X)))$

$= H^{-i}(C_*(\mathbb{Z}_{tr}(X))(\mathrm{Spec}\, k))$

$$= H_i\left(\underset{2}{\longrightarrow} \mathbb{Z}_{tr}(X)(\Delta^2) \underset{1}{\longrightarrow} \mathbb{Z}_{tr}(X)(\Delta^1) \longrightarrow \underset{0}{\mathbb{Z}_{tr}(X)(\Delta^0)} \right)$$

(Suslin homology)

properties :

i) (homotopy invariant) $X \in Sm/k$, $M(X) \xleftarrow{\simeq} M(X \times \mathbb{A}^1)$

(☺ $\forall F \in DM^{eff}_-(k)$, $H^i_{Nis}(X \times \mathbb{A}^1, F) \simeq H^i_{Nis}(X, F)$

X : smooth $\quad \hom(M(X \times \mathbb{A}^1), F[i]) \quad \hom(M(X), F[i])$)

ii) : (Mayer–Vietorius) $X \in Sch/k$, $X = U \cup V$: open cov.

$\Rightarrow M(U \cap V) \longrightarrow M(U) \oplus M(V) \longrightarrow \underset{X}{M(U \cup V)} \xrightarrow{+1}$

(☺ Follows from

$$0 \to \mathbb{Z}_{tr}(U \cap V) \to \mathbb{Z}_{tr}(U) \oplus \mathbb{Z}_{tr}(V) \to \mathbb{Z}_{tr}(X) \to 0$$

exact in $(Sm/k)^{\sim}_{Nis}$)

<u>Cor</u>: $X \in Sm/k$ $\quad E \to X$; v.b. $\Rightarrow M(E) \xrightarrow{\simeq} M(X)$

$$M(\mathbb{A}^n \setminus \{0\}) \cong \mathbb{Z} \oplus \mathbb{Z}(n)[2n-1]$$

$$M(\mathbb{P}^n) \simeq \bigoplus_{i=0}^{n} \mathbb{Z}(i)[2i]$$

☺ where $\mathcal{O}_{\mathbb{P}^n}(1) \in \mathrm{Pic}(\mathbb{P}^n) \simeq H^2(\mathbb{P}^n, \mathbb{Z}(1))$

$$= \hom(M(\mathbb{P}^n), \mathbb{Z}(1)[2])$$

$$\ni (\tau : M(\mathbb{P}^n) \to \mathbb{Z}(1)[2])$$

$$\begin{tikzcd} M(\mathbb{P}^n) \arrow[r, "\Delta"] \arrow[rr, bend right, "\tau^k"'] & M(\mathbb{P}^n)^{\otimes k} \arrow[r, "\tau^{\otimes k}"] & \mathbb{Z}(k)[2k] \end{tikzcd}$$

<u>iv)</u>: (projective bundle formula) $\quad E \to X$: v.b. of $rk = r$

$$P = \mathbb{P}(E) \xrightarrow{p} X \Rightarrow M(P) \xrightarrow{\sim} \bigoplus_{i=0}^{r-1} M(X)(i)[2i]$$

<u>cdh-topology</u>:

<u>Def</u>: k : field $\quad k$: admits resolution of singularities

$\overset{\Leftrightarrow}{\text{def}}$ <u>(i)</u>: $X \in Sm/k$: int $\quad \exists\, Y \to X$: proper birational, $Y \in Sm/k$, $X \in Sm/k$.

<u>(ii)</u>: $X \in Sm/k$: int. $\forall\, Y \to X$ $\Rightarrow$ $X_n \to X_{n-1} \to \cdots \to X_0 = X$: a seq of smooth center

: proper birat.

<u>Rem</u>: $\mathrm{char}\, k = 0 \Rightarrow k$: admits RS
$\Rightarrow k$: perfect

<u>Def</u>: cdh-topology on (Sch/k) = the weakest Grothendieck top. s.t. Nisnevich cov. is cdh-cov.

$X' \xrightarrow{p} X$: proper, $Z \hookrightarrow X$: closed imm. s.t. $p^{-1}(X \setminus Z) \xrightarrow{\sim} X \setminus Z$

$\Rightarrow Z \sqcup X' \to X$: cdh-covering

<u>ex</u>:
- covering by irreducible comp. $\Rightarrow$ cdh.
- $X_{red} \to X$ $\Rightarrow$ cdh.
- $X' \to X$: blow up smooth center $\Rightarrow$ cdh.

Using this idea we have

<u>Lem</u>: $X' \xrightarrow{p} X$: proper s.t. $\forall x \in X \; \exists x' \in X'$ s.t. $p(x') = x$, $k(x) \xrightarrow{\sim} k(x')$

$\Rightarrow$ cdh-covering.

<u>Lem</u>: X : int $\in Sch^m/k$ (k : R.S.) $X' \to X$: proper birat.

$\Rightarrow$ cdh-cov.

$Sm/k \hookrightarrow Sch/k$: induces an adjoint pair

$$(Sch/k)^{\sim}_{cdh} \underset{\pi_*}{\overset{\pi^*}{\rightleftarrows}} (Sm/k)^{\sim}_{Nis}.$$

<u>Rem</u>: The exactness of π^* is non-trivial
($Sm/k \hookrightarrow Sch/k$: does not pull back.)

Prop: F: Nisnevich sheaf on Sm/k

$$F_{cdh}=0 \iff \forall X \in Sm/k \quad s \in F(X)$$

$$\exists\, p: X' \to X : \text{proper cdh-covering}$$

s.t. $p^*(s)=0$.

$$\bar{F}_{cdh} = (\pi^* F)_{\cancel{cdh}}$$