homotopy inv. 12721. Hagihara. K **No.1**

$$\begin{tikzcd}
k : \text{perfect} & DM^{eff}_{-} \arrow[r, phantom, "\supseteq"] \arrow[d] & DM^{eff}_{gm} \arrow[d] \\
& DM_{-} \arrow[r, phantom, "\supseteq"] & DM_{gm}
\end{tikzcd}$$

homotopy inv. $M(X) \xrightarrow{\sim} M(X \times \mathbb{A}^1)$

MV $X = U \cup V$

: $M(U \cap V) \to M(U) \oplus M(V) \to M(X) \xrightarrow{+1}$

Using cdh-topology to prove "localization", "blow-up seq"

$$\begin{tikzcd}
(Sch/k)^{\sim}_{cdh} \arrow[r, shift right, "\pi_*"'] & (Sm/k)^{\sim}_{Nis} \arrow[l, shift right, "\pi^*"']
\end{tikzcd}$$

Thm: k : RS $F \in PSwT(k)$

$F_{cdh} = 0 \iff C_*(F)_{Nis}$; acyclique

Rmk: This also implies $C_*(F)_{Zar}$: acyclic

ex: $p : Y \hookrightarrow X$: cl. imm. in Sch/k

$\Rightarrow M^c(Y) \to M^c(X) \to M^c(X\backslash Y) \xrightarrow{+1}$: distinguished triangle

in DM^{eff}_{-} $M^c(X) \overset{dfn}{:=} C_*(\mathbb{Z}^c_{tr}(X))$

☹ $0 \to \mathbb{Z}^c_{tr}(Y) \to \mathbb{Z}^c_{tr}(X) \xrightarrow{(*)} \mathbb{Z}^c_{tr}(X\backslash Y)$: easy to prove surjective

Suffice to prove $(*)$ is surjective in $(Sch/k)^{\sim}_{cdh}$

$^{\forall}U \in Sm/k$, $^{\forall}a \in \mathbb{Z}^c_{tr}(X\backslash Y)(U)$ $^{\exists}U' \to U$: proper cdh-cov.

$\cap$ Sm/k

s.t.

$$\begin{tikzcd}
\mathbb{Z}^c_{tr}(X\backslash Y)(U) \arrow[r] & \mathbb{Z}^c_{tr}(X\backslash Y)(U') \\
a \arrow[u, phantom, sloped, "\in"] \arrow[ur, mapsto] & \mathbb{Z}^c_{tr}(X)(U') \arrow[u] \\
& a' \arrow[u, phantom, sloped, "\in"] \arrow[uu, mapsto, bend left]
\end{tikzcd}$$

$\overline{Z}$: closure of Z in $U \times X$

$^{\exists}V \to U$: blow-up s.t.

Z' : strict trans. of $\overline{Z}$ is flat.

(Raynaud-Gruson platification)

$$\begin{tikzcd}
& & Z' \arrow[dll] \arrow[d, phantom, sloped, "\subseteq"] \arrow[dd, bend left=60, "flat"] \\
\overline{Z} \arrow[r, phantom, "\subseteq"] \arrow[dr] & U \times X \arrow[d] \arrow[dr, phantom, "\square"] & V \times X \arrow[l] \arrow[d] \\
& U & V \arrow[l] \arrow[l, phantom, ""] \\
& & \text{blow-up}
\end{tikzcd}$$

May Assume $V \in Sm/k$. $Z' \to V$: gen fin & flat $\Rightarrow$ q-finite.

ex:

$$\begin{tikzcd} T' \arrow[r, hook, "i'"] \arrow[d, "p'"'] \arrow[dr, phantom, "\square"] & X' \arrow[d, "p\text{: proper}"] \\ T \arrow[r, hook, "i"'] & X \end{tikzcd}$$

$X' \setminus T' \xrightarrow{\simeq} X \setminus T$ $\quad X, T, \cdots \in Sch/k$

k: RS

$\Rightarrow$ $M(T') \to M(T) \oplus M(X') \to M(X) \xrightarrow{+1}$; distinguished triangle.

From the surjective of cdh-sheaves

$$\mathbb{Z}_{tr}(T)_{cdh} \oplus \mathbb{Z}_{tr}(X')_{cdh} \to \mathbb{Z}_{tr}(X)_{cdh} :$$ ~~[illegible]~~

similarly

Cor: k: RS, $X \in Sch/k$. $M(X), M^c(X) \in DM^{eff}_{gm} \subseteq DM^{eff}_{-}$.

$\uparrow$ generated by $M(X)$ $(X \in Sm/k)$

Rem: the minimal full subset of DM^{eff}_{-} containing $M(X)$ X: proj. smooth. closed undertaking direct summands, cones, shifts, $\oplus$

Cor: $X, Y \in Sch/k$. k: RS $M(X) \otimes M(Y) \xrightarrow{\simeq} M(X \times Y)$

$M^c(X) \otimes M^c(Y) \xrightarrow{\simeq} M^c(X \times Y)$

In particular

$$M^c(X \times \mathbb{A}^1) \simeq M^c(X)(1)[2]$$

($\because$ reduced to the case X, Y: smooth proj.

~~$M^c(pt) \to M^c(\mathbb{A}^1) \to M^c(\mathbb{P}$~~

$$\begin{tikzcd} M^c(pt) \arrow[r] \arrow[d, phantom, "\simeq" rotate=90] & M^c(\mathbb{P}^1) \arrow[r] \arrow[d, phantom, "\simeq" rotate=90] & M^c(\mathbb{A}^1) \simeq \mathbb{Z}(1)[2] \\ \mathbb{Z} & M(\mathbb{P}^1) \arrow[d, phantom, "\simeq" rotate=90] & \\ & \mathbb{Z} \oplus \mathbb{Z}(1)[2] & \end{tikzcd}$$

equidimensional cycles & two moving lemma:

$\mathrm{Sm}/k \ni U$ closed integral

Def: $X \in \mathrm{Sch}/k$, $r \geq 0$. $Z_{equi.}(X, r) : U \mapsto$

$$\begin{tikzcd} \Big\langle Z \arrow[r, phantom, "\subseteq"] \arrow[dr, "\text{equidim of dim} = r"'] & X \times U \arrow[d] \\ & X \Big\rangle \end{tikzcd}$$

$p : X \to S$: equidimensional of dim $= r$.

$\underset{\text{dfn}}{\iff}$ 1): of finite type. 2) $\forall$ irred comp of X dominates an irreducible comp. of S

3): $\forall x \in X$ $\dim_x(p^{-1}(p(x))) = r$.

- $Z_{equi}(X, r)$: NSwT/k. $\cdot$ $Z_{equi}(X, 0) = Z^c_{tr}(X)$ k : RS
- localizing sequence. $Y \hookrightarrow X$: closed immersion $r \geq 0$

$$\Rightarrow \quad C_*(Z_{equi}(Y, r)) \to C_*(Z_{equi}(X, r)) \to C_*(Z_{equi}(X \setminus Y, r)) \xrightarrow{+1} \text{ in } DM^{eff}_-$$

(Similar to the case $r=0$)

Rem:

$$Z_{equi}(X, r)(\Delta^m) \subseteq Z^{n-r}(X, m) \quad \text{(Bloch's cycle complex)}$$

$(X : \text{equidimensional } \dim = n/k)$

Thm: (Suslin) X : affine, equidim of dim n/k $r \geq 0$

k: any field $\quad C_*(Z_{equi.}(X, r))(\mathrm{Spec}\, k) \underset{\text{q-isom}}{\xrightarrow{\sim}} Z^{n-r}(X, *)$

Thm (Friedlander–Lawson–Voevodsky)

$X \in \mathrm{Sch}/k$, $U \in \mathrm{Sm}/k$, U : quasi-proj. equidim

k : RS. $\quad C_*(Z_{equi}(X, r))(U) \underset{\text{q-isom}}{\xrightarrow{\sim}} C_*\, Z_{equi}(U \times X, r + \dim U)(\mathrm{Spec}\, k)$

$$\begin{tikzcd} & Z' \arrow[dl] \arrow[r, phantom, "\sim_k \text{ moving}"] & Z \arrow[ddll] \\ \Delta^r \times U \arrow[d] & & \\ \Delta^r & & \end{tikzcd}$$

$\Big($ $\cdot$ (X, U) : smooth proj. (F–L) : geometrically

$\cdot$ (X, U) : as above (F–V) : cdh-topology. k : (RS) $\Big)$

<u>Cor</u> (of FLV) $k; X, U$: as above.

$$\Rightarrow H_i(C_* \mathrm{Zequi}(X,\underline{r})(U)) \xrightarrow{\sim} H^{-i}_{Zar}(U, C_*(\mathrm{Zequi}(X,\underline{r})))$$

$$\uparrow$$

$$\xrightarrow{\sim} H^{-i}_{Nis}$$

$$\uparrow$$

The Theory of. PSWT.

(☺ : MV-distinguished triangle ; $X' = U' \cup V'$

$$C_*(\mathrm{Zequi}(X',\underline{r}'))(\mathrm{Spec}\,k) \to C_*(\mathrm{Zequi}(U',\underline{r}'))(\mathrm{Spec}\,k) \oplus C_*(\mathrm{Zequi}(V',\underline{r}))(\mathrm{Spec}\,k)$$

$$\to C_*(\mathrm{Zequi}(U' \cap V', \underline{r}))(\mathrm{Spec}\,k) \xrightarrow{+1} \quad)$$

+FLV thm §4.

~~$C_*(\mathrm{Zequi}(Y,\underline{r}))(X) \to C_*(\mathrm{Zequi}(I,\underline{r}))$~~

~~$X = U \cup V$~~

~~smooth~~.

$$\Rightarrow C_*(\mathrm{Zequi}(X,\underline{r}))(U) \to C_*(\mathrm{Zequi}(X,\underline{r}))(U_1) \oplus C_*(\mathrm{Zequi}(X,\underline{r}))(U_2)$$

$$U = U_1 \cup U_2$$

$$\to C_* \mathrm{Zequi}(X,\underline{r})(U_1 \cap U_2) \xrightarrow{+1}$$

This Property implies Cor. //

In particular,

$$H^{-i}_{Nis}(U, C_*(\mathrm{Zequi}(X,\underline{r})))$$

$$\xrightarrow{\sim} H_i(C_*(\mathrm{Zequi}(U \times X, d_i U + \underline{r}))(\mathrm{Spec}\,k))$$

In the following (k: RS)

Cor: $X \in Sch/k$, $U \in Sm/k$, $r \geq 0$

$$\Rightarrow \mathrm{Hom}_{DM_-^{eff}}(M(U)(r)[2r+i], M^c(X))$$

$$\cong H^{-i}_{Nis}(U, C_*(z_{equi}(X, r)))$$

∵ $r=0$: clear $r=1$.

$$H^{-i}_{Nis}(\cancel{U, C_*(z_{equi}(X,0))})$$

$$/\!/$$

Projective Bundle formula

$$\mathrm{Hom}(M(U)[i], M^c(X)) \oplus \mathrm{Hom}(M(U)(1)[2+i], M^c(X))$$

$$\simeq$$

$$\mathrm{Hom}(M(U \times \mathbb{P}^1)[i], M^c(X))$$

$$\simeq H^{-i}_{Nis}(U \times \mathbb{P}^1, C_*(z_{equi}(X, 0))) \overset{FLV}{\simeq} H^{-i}_{Nis}(U, C_*(z_{equi}(X \times \mathbb{P}^1, 1)))$$

$$\simeq H^{-i}_{Nis}(U, C_*(z_{equi}(X, 1))) \oplus H^{-i}_{Nis}(U, C_*(z_{equi}(X \times \mathbb{A}^1, 1)))$$

(Localization for Zariski)

$$\wr\!\!\text{S}\ FLV$$

$$H^{-i}_{Nis}(U \times \mathbb{A}^1, -)$$

$$\wr\!\!\text{S}$$

$$H^{-2}_{Nis}(\cancel{U, C_*(z}_{equi}(X, 0)))$$

Cor (Cancellation Thm)

$A, B \in DM_-^{eff}$

$$\mathrm{hom}_{DM_-^{eff}}(A, B) \simeq \mathrm{hom}_{DM_-^{eff}}(A(1), B(1))$$

∵ $A = M(X)[i]$ とにおい。

$B = M(Y)$

Y, X: proj smooth $i \in \mathbb{Z}$

$$\mathrm{hom}(M(X)[i], M(Y))$$

$\mathrm{Hom}(A(1), B(1)) = \mathrm{hom}\left(M(X)(1)[i], M(Y)(1)\right)$

$= \mathrm{Hom}\left(M(X)(1)[i+2], M(Y)(1)[2]\right)$

$= \mathrm{Hom}\left(M(X)(1)[i+2], M^c(Y \times \mathbb{A}^1)\right)$

$= H^{-i}_{Nis}\left(X, C_*(Z_{equi}(Y \times \mathbb{A}^1, 1))\right) \overset{FLT}{\simeq} H^{-i}_{Nis}\left(X, C_*(Z_{equi}(Y, 0))\right)$

$\simeq \mathrm{Hom}\left(\underbrace{M(X)[i]}_{\overset{\shortparallel}{A}}, \underbrace{M(Y)}_{\overset{\shortparallel}{B}}\right)$

<u>Cor</u>: X: q-proj. equidi. of di = n

$CH_i^{\cancel{n-i}}(X, j) \xrightarrow{\simeq} H^{BM}_{2i+j}(X, \mathbb{Z}(i))$

∵ <u>$i \geq 0$</u>: X: affine

$\Rightarrow CH_i(X, j) \simeq H_j\left(C_*(Z_{equi}(X, i))(\mathrm{Spec}\, k)\right)$

$\simeq H^{-j}_{Nis}\left(\mathrm{Spec}\, k, C_*(Z_{equi}(X, i))\right)$

$\cong \mathrm{Hom}\left(\mathbb{Z}(i)[2i+j], M^c(X)\right)$

X: general: Use Localization seq. both for

$\begin{cases} CH_*(-, *) \ (\text{Bloch}) \\ H_*^{BM}(-, *) \end{cases}$

<u>$i < 0$</u>: use h.i. for higher Chow & $M^c(X)(i)[2i]$

$= M^c(X \times \mathbb{A}^i)$

Duality: **Def**: $A \in DM_{gm}(k)$, $A^* \overset{df}{:=} \underline{Hom}_{DM_{gm}}(A, k)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \shortparallel$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad M(\mathrm{Spec}\, k)$

Thm: $A, B \in DM_{gm}$

$\underline{1)}$: $A \simeq (A^*)^*$ $\quad \underline{2)}$: $(A \otimes B)^* \simeq A^* \otimes B^*$

$\underline{Hom}(A, B) \simeq A^* \otimes B$.

$\underline{3)}$: X : Sm/k

: equidimensional $\quad \underline{i)}$: $M(X)^* \simeq M^c(X)(-n)[-2n]$

$\underline{ii)}$: $M^c(X)^* \simeq M(X)(-n)[-2n]$

Cor: DM_{gm} : closed under $\underline{Hom}$ & rigid.

ex): $CH_i(X, j) \simeq H^{BM}_{2i+j}(X, \mathbb{Q}(i)) \simeq H^{2n-(2i+j)}(X, \mathbb{Q}(n-i))$

X : smooth of $\dim X = n$.

$\rightsquigarrow$

$$\begin{tikzcd}
V_k \arrow[r, hook] \arrow[d] & Sch/k \arrow[d, "M"] \\
\mathcal{M}_k^{opp} \arrow[r, dashed] & DM_{gm}
\end{tikzcd}$$

$(X, id, m) \longmapsto M(X)(-m)[-2m]$

Rem: DM_-^{eff} : pseudo-Abel.