

$$\begin{array}{ccc}
 \downarrow & & \downarrow \\
 DM_{-} & \supseteq & DM_{gm} \\
 \end{array}
 \quad
 \begin{array}{l}
 M(X) \simeq M(X \times \mathbb{A}^1) \\
 X = UUV \\
 M(U \cup V) \rightarrow M(U) \oplus M(V) \rightarrow M(X) \xrightarrow{+1}
 \end{array}$$

Using cdh-topology to prove.
 "localization", "blow-up seq"

$$\left(\text{Sch}/\mathbb{k} \right)_{cdh} \xrightleftharpoons[\pi_*]{\pi^*} \left(\text{Sm}/\mathbb{k} \right)_{Nis}$$

Thm: \mathbb{k} : RS $F \in \text{PSWT}(\mathbb{k})$
 $F_{cdh} = 0 \Leftrightarrow C_*(F)_{Nis}$; acyclique

Rmk:
 This also implies $C_*(F)_{Zar}$: acyclic

\underline{x} : $p: Y \hookrightarrow X$: cl. imm. in Sch/\mathbb{k}

$$\Rightarrow M^c(Y) \rightarrow M^c(X) \rightarrow M^c(X|Y) \xrightarrow{+1}$$

: distinguished triangle

$M^c(X) := C_*(\mathcal{D}_{tr}^c(X))$

☺ $0 \rightarrow \mathcal{D}_{tr}^c(Y) \rightarrow \mathcal{D}_{tr}^c(X) \xrightarrow{(*)} \mathcal{D}_{tr}^c(X|Y)$: easy to prove surjective

Suffice to prove (*) is surjective in $(\text{Sch}/\mathbb{k})_{cdh}$

$\forall U \in \text{Sm}/\mathbb{k}, \forall a \in \mathcal{D}_{tr}^c(X|Y)(U) \exists U' \rightarrow U$: proper cdh-cov.

s.t. $\mathcal{D}_{tr}^c(X|Y)(U) \rightarrow \mathcal{D}_{tr}^c(X|Y)(U') \xrightarrow{\text{Sm}/\mathbb{k}} \mathcal{D}_{tr}^c(X|Y)(U)$

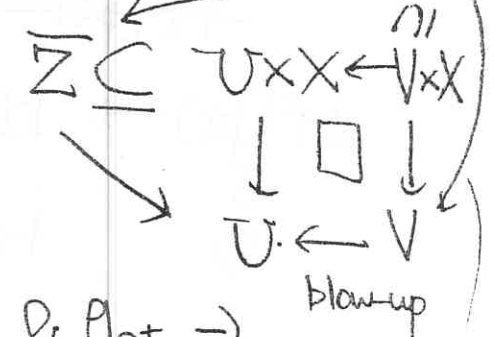
Σ : closure of Σ in $U \times X$

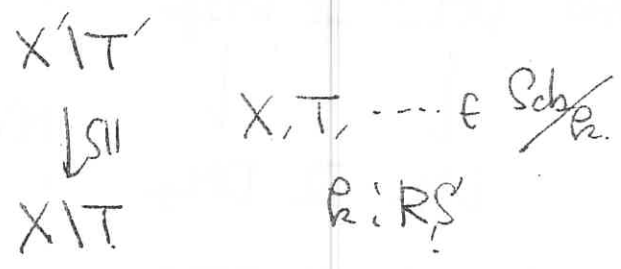
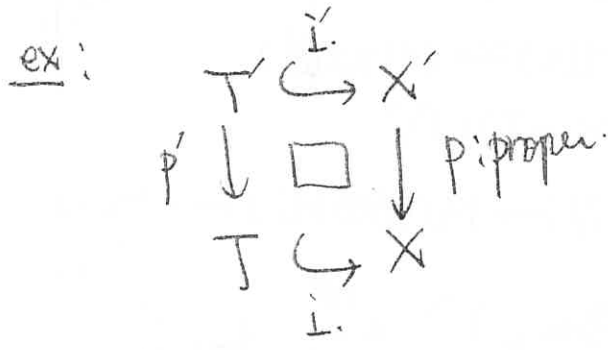
$\exists V \rightarrow U$: blow-up s.t. $\mathcal{D}_{tr}^c(X)(U) \rightarrow \mathcal{D}_{tr}^c(X)(V) \rightarrow \mathcal{D}_{tr}^c(X|Y)(U)$

Σ : strict trans. of Σ is flat.

(Raynaud-Gruson platification)

May Assume $V \in \text{Sm}/\mathbb{k}$. $\Sigma' \rightarrow V$: gen fin & flat \Rightarrow φ finite. flat.





$$\Rightarrow M(T') \rightarrow M(T) \oplus M(X') \rightarrow M(X) \xrightarrow{+1} : \text{distinguished triangle.}$$

From the surjectivity of cdh-sheaves

$$\mathcal{Z}_{\text{tr}}(T)_{\text{cdh}} \oplus \mathcal{Z}_{\text{tr}}(X')_{\text{cdh}} \rightarrow \mathcal{Z}_{\text{tr}}(X)_{\text{cdh}} \quad \text{similarly}$$

Cor: $R: RS, X \in \text{Sch}/\mathbb{R}. M(X), M^c(X) \in DM_{\text{gm}}^{\text{eff}} \subset DM_{-}^{\text{eff}}$

Rem: the minimal full subset of DM_{-}^{eff} containing $M(X)$ $X: \text{proj. smooth.}$ generated by $M(X)$ ($X \in \text{Sm}/\mathbb{R}$)
 closed under taking direct summands, cones, shifts, \oplus

Cor: $X, Y \in \text{Sch}/\mathbb{R}. R: RS$

$$\begin{aligned}
 M(X) \otimes M(Y) &\simeq M(X \times Y) \\
 M^c(X) \otimes M^c(Y) &\simeq M^c(X \times Y)
 \end{aligned}$$

In particular

$$M^c(X \times \mathbb{A}^1) \simeq M^c(X)(1)[2] \quad \left(\begin{array}{l} \text{reduced to the case} \\ X, Y: \text{smooth} \\ \text{proj.} \end{array} \right)$$

$$\begin{array}{ccccc}
 \cancel{M^c(\text{pt})} & \rightarrow & \cancel{M^c(\mathbb{A}^1)} & \rightarrow & \cancel{M^c(\mathbb{P}^1)} \\
 M^c(\text{pt}) & \rightarrow & M^c(\mathbb{P}^1) & \rightarrow & M^c(\mathbb{A}^1) \simeq \mathcal{O}(1)[2] \\
 \downarrow \cong & & \downarrow \cong & & \\
 \mathcal{O} & & M^c(\mathbb{P}^1) & & \\
 \downarrow \cong & & \downarrow \cong & & \\
 \mathcal{O} & & \mathcal{O} \oplus \mathcal{O}(1)[2] & &
 \end{array}$$

equidimensional cycles: \mathbb{Q} two moving lemma: $\frac{Sm}{k} \dashrightarrow$ closed integral

Def: $X \in \text{Sch}/k, r \geq 0. \quad Z_{\text{equi.}}(X, r) : U \mapsto \left\langle \begin{array}{c} Z \subset X \times U \\ \downarrow \\ X \end{array} \right\rangle$
 equidi- of dim = r.

$p: X \rightarrow S$: equidimensional of dim = r.

(\Rightarrow) 1): of finite type. 2): \forall irred comp of X dominates an irreducible comp. of S

3): $\forall x \in X \quad \dim_x(p^{-1}(p(x))) \leq r.$

$\cdot Z_{\text{equi}}(X, r) : \text{NswT}/k. \quad \cdot Z_{\text{equi}}(X, 0) = Z_{\text{tr}}(X) \quad \mathbb{R}: \mathbb{R}S$

\cdot localizing sequence. $Y \hookrightarrow X$: closed immersion $r \geq 0.$

$\Rightarrow C_*(Z_{\text{equi}}(Y, r)) \rightarrow C_*(Z_{\text{equi}}(X, r)) \rightarrow C_*(Z_{\text{equi}}(X|Y, r))$
 Similar to the case $r=0$ Rem: $\xrightarrow{+1}$ in DM_{-}^{eff}

$Z_{\text{equi}}(X, r)(\Delta^m) \subseteq Z^{n-r}(X, m)$ (Bloch's cycle complex)
 (X : equidimensional $\dim = n/k$)

Thm: (Suslin) X : affine, equidin of dim $n/k. \quad r \geq 0.$

\mathbb{R} : any field $C_*(Z_{\text{equi.}}(X, r)(\text{Spec } k)) \xrightarrow[\text{q-isom}]{\sim} Z^{n-r}(X, *)$

Thm (Friedlander-Lawson-Voevodsky)

$X \in \text{Sch}/k, U \in \text{Sm}/k, U$: quasi-proj. equidin

$\mathbb{R}: \mathbb{R}S. \quad C_*(Z_{\text{equi}}(X, r)(U)) \xrightarrow[\text{q-isom}]{\sim} C_*(Z_{\text{equi}}(U \times X, r + \dim U))$
 (Spec k)

$\Delta^r \times U \xleftarrow{Z_{\text{tr}} \text{ moving}} Z \xleftarrow{\Delta^r} \Delta^r$
 ((X, U) : smooth proj. (F-L): geometrically)
 ((X, U) : as above (F-V): cde topology)
 $\mathbb{R}: \mathbb{R}C$

Cor (of FLV) $k; X, U$: as above.

No. 4.

$$\Rightarrow H_i(C_* \text{Zequi}(X, r)(U)) \xrightarrow{\sim} H_{\text{Zar}}^{-i}(U, C_*(\text{Zequi}(X, r)))$$

$$\xrightarrow{\sim} H_{\text{Nis}}^{-i}$$

The Theory of
PSWT.

(☺: MV-distinguished triangle; $X = U \cup V$)

$$C_*(\text{Zequi}(X, r))(\text{Spec } k) \rightarrow C_*(\text{Zequi}(U, r))(\text{Spec } k) \oplus C_*(\text{Zequi}(V, r))(\text{Spec } k)$$

$$\oplus C_*(\text{Zequi}(V, r))(\text{Spec } k)$$

$$\rightarrow C_*(\text{Zequi}(U \cap V, r))(\text{Spec } k) \xrightarrow{+1}$$

+ FLV thm #4.

~~$$C_*(\text{Zequi}(Y, r))(X) \rightarrow C_*(\text{Zequi}(Y, r))$$~~

~~$$X = U \cup V$$~~

smooth.

$$\Rightarrow C_*(\text{Zequi}(X, r))(U) \rightarrow C_*(\text{Zequi}(X, r))(U_1) \oplus C_*(\text{Zequi}(X, r))(U_2)$$

$$U = U_1 \cup U_2$$

$$\rightarrow C_*(\text{Zequi}(X, r))(U_1 \cap U_2) \xrightarrow{+1}$$

This Property implies Cor. #.

In particular,

$$H_{\text{Nis}}^{-i}(U, C_*(\text{Zequi}(X, r)))$$

$$\xrightarrow{\sim} H_i(C_*(\text{Zequi}(U \times X, d_U + r))(\text{Spec } k))$$

In the following $(B:RS)$

Cor: $X \in \text{Proj } \mathbb{P}^n, U \in \text{Sm}_\mathbb{P}^n, r \geq 0$

$$\Rightarrow \text{Hom}_{DM_{\text{eff}}} (M(U)(r)[2r+1], M^c(X))$$

$$\cong H_{Nis}^{-i}(U, C_*(Z_{\text{equi}}(X, r)))$$

~~$$H_{Nis}^{-i}(U, C_*(Z_{\text{equi}}(X, 0)))$$~~

⊙ $r=0$: clear $r=1$.

Properive Bundle formula. $\text{Hom}(M(U)[1], M^c(X))$

$$\cong \text{Hom}(M(U)(1)[2+1], M^c(X))$$

$$\text{Hom}(M(U \times \mathbb{P}^1)[1], M^c(X))$$

$$\cong H_{Nis}^{-i}(U \times \mathbb{P}^1, C_*(Z_{\text{equi}}(X, 0))) \stackrel{FLV}{\cong} H_{Nis}^{-i}(U, C_*(Z_{\text{equi}}(X \times \mathbb{P}^1, 1)))$$

$$\cong H_{Nis}^{-i}(U, C_*(Z_{\text{equi}}(X, 1))) \oplus H_{Nis}^{-i}(U, C_*(Z_{\text{equi}}(X \times \mathbb{A}^1, 1)))$$

(Localization for Zariski)

$$H_{Nis}^{-i}(U \times \mathbb{A}^1, \rightarrow)$$

Cor (Cancellation Thm)

~~$$H_{Nis}^{-i}(U, C_*(Z_{\text{equi}}(X, 0)))$$~~

$A, B \in DM_{\text{eff}}$

⊙ $A = M(X)[1] \in \mathbb{Z}[2\pi i]$

$$\text{hom}_{DM_{\text{eff}}}(A, B) \cong \text{hom}_{DM_{\text{eff}}}(A(1), B(1))$$

$$B = M(Y)$$

Y, X : proj smooth $i \in \mathbb{Z}$

$$\text{hom}(M(X)[1], M(Y))$$

$$\text{Hom}(A(i), B(i)) = \text{hom}(M(X)(i)[i], M(Y)(i))$$

$$= \text{Hom}(M(X)(i)[i+2], M(Y)(i)[2])$$

$$= \text{Hom}(M(X)(i)[i+2], M^c(Y \times \mathbb{A}^1)) \quad \text{FLT}$$

$$= H_{\text{Nis}}^{-i}(X, C_*(\text{Zequi}(Y \times \mathbb{A}^1, i))) \cong H_{\text{Nis}}^{-i}(X, C_*(\text{Zequi}(Y, i)))$$

$$\cong \text{Hom}(\underbrace{M(X)[i]}_A, \underbrace{M(Y)}_B) \quad \text{Cor: } X: q\text{-proj. equidi of } d_i = n$$

$$CH_i^*(X, j) \cong H_{2i+j}^{BM}(X, \mathbb{Z}(i))$$

⊙ i=0: X: affine

$$\Rightarrow CH_i(X, j) \cong H_j(C_*(\text{Zequi}(X, i))(\text{Spec } \mathbb{C}))$$

$$\cong H_{\text{Nis}}^{-j}(\text{Spec } \mathbb{C}, C_*(\text{Zequi}(X, i)))$$

$$\cong \text{Hom}(\mathbb{Z}(i)[2i+j], M^c(X))$$

X: general: Use Localization seq. both for.

$$\begin{cases} CH_*(-, *) \text{ (Bloch)} \\ H_*^{BM}(-, *) \end{cases}$$

i < 0: use h.i. for higher Chow & $M^c(X)(i)[2i]$
 $= M^c(X \times \mathbb{A}^i)$

Duality: Def: $A \in \text{DM}_{\text{gm}}(\mathbb{R})$, $A^* := \text{Hom}_{\text{DM}_{\text{gm}}} (A, \mathbb{R})$
 \parallel
 $\text{M}(\text{Spec } \mathbb{R})$

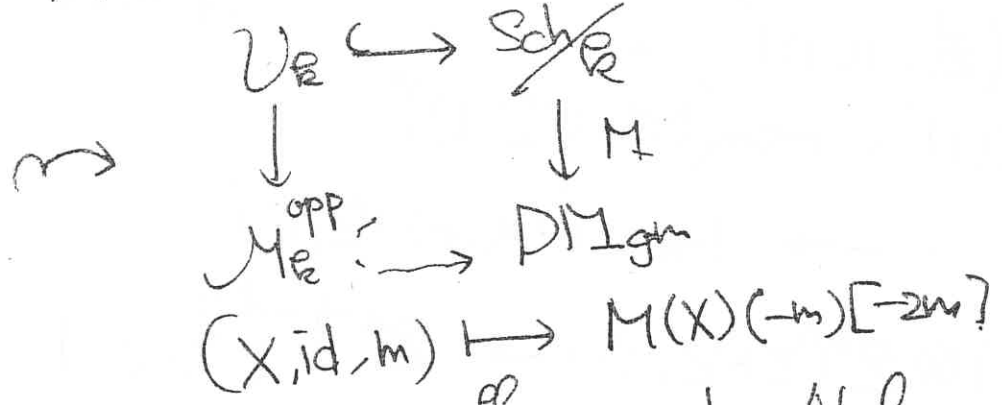
Thm: $A, B \in \text{DM}_{\text{gm}}$
1): $A \simeq (A^*)^*$ 2): $(A \otimes B)^* \simeq A^* \otimes B^*$
3): $X: S_m/\mathbb{R}$ Hom $(A, B) \simeq A^* \otimes B$.

'equidimensional' i): $\text{M}(X)^* \simeq \text{M}^c(X)(-n)[-2n]$
ii): $\text{M}^c(X)^* \simeq \text{M}(X)(-n)[-2n]$

Cor: DM_{gm} : closed under Hom & rigid.

ex: $\text{CH}_i(X, j) \simeq H_{2i+j}^{\text{BM}}(X, \mathbb{Q}(j)) \simeq H^{2n-(2i+j)}(X, \mathbb{Q}(n-i))$

X : smooth of $\dim X = n$.



Rem: $\text{DM}_{-}^{\text{eff}}$: pseudo-Abel.