

why A^1 -homotopy theory? (d = dim X) ^{12/21, L. Hesselholt}

No. 1

Recall: $H^p(X, \mathbb{Z}(q)) = CH^q(X, 2q-p)$ If, $p = q + d$, then

$$H^{q+d}(X, \mathbb{Z}(q)) = CH^q(X, q-d)$$

= \mathbb{Z} {closed pts on $X \times \Delta^{q-d}$ } relation. Can sometimes understand this group directly;

Thm (Rost) let $X_a \subseteq \mathbb{P}_k^{2^{r-1}}$ is the quadratic;

$$(x_1^2 - a_1 y_1^2) \otimes \dots \otimes (x_{r-1}^2 - a_{r-1} y_{r-1}^2) = a_r t^2$$

where $a_1, \dots, a_r \in k^x$, then

$$H^{2^{r-1}}(X_a, \mathbb{Z}(2^{r-1})) \hookrightarrow k^x \quad q+d = 2^{r-1} + 2^{r-1} - 1 = 2^r - 1 = p$$

If. $p < q+d$ the group $H^p(X, \mathbb{Z}(q))$ can never be understood directly from the definition

Strategy: Relate $H^p(X, \mathbb{Z}(q))$ to some $H^{r+d}(X, \mathbb{Z}(r))$

Recall; $DM(k) :=$ derived category of mixed motive / \mathbb{Z}
= tensor triangulated category

$X \mapsto X[p]$ change degree by p } autom of $DM(k)$
 $X \mapsto X(q)$ change weight by q }

$\text{Sm}_k / \mathbb{Z} \rightarrow DM(k)$ $\mathbb{Z} = M(\text{Spec } k) \in DM(k)$
 $\cup \quad \cup$ unit for tensor product -

$$X \mapsto M(X)$$

$$H^p(X, \mathbb{Z}(q)) = \text{hom}_{DM(k)}(M(X), \mathbb{Z}(q)[p])$$

$$\mathbb{Z} \xrightarrow{\ell} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/\ell\mathbb{Z} \xrightarrow{\delta} \mathbb{Z}[1]$$

$$H^p(X, \mathbb{Z}/\ell\mathbb{Z}(q)) \cong \text{Hom DM}(\mathbb{k})(M(X), \mathbb{Z}/\ell\mathbb{Z}(q)[p])$$

$$\beta \downarrow \qquad \qquad \qquad \downarrow (\pi(1) \circ \delta)_*$$

$$H^{p+1}(X, \mathbb{Z}/\ell\mathbb{Z}(q)) = \text{Hom DM}(\mathbb{k})(M(X), \mathbb{Z}/\ell\mathbb{Z}(q)[p+1])$$

SH(k) = stable homotopy category / k
 = tensor triangulated category

- $\left\{ \begin{array}{l} X \mapsto X[p] : \text{shifts degree by } p \\ X \mapsto X(q) : \text{changes weight by } q. \end{array} \right.$

$$\begin{array}{ccc} \mathbb{S}_{\mathbb{k}} & \rightarrow & \text{SH}(\mathbb{k}) \\ \cup & & \cup \\ X & \mapsto & \mathbb{S}(X) \\ & & \parallel \end{array}$$

$$\mathbb{S} = \mathbb{S}(\text{Spec } \mathbb{k}) \in \text{SH}(\mathbb{k}) : \text{unit for tensor prod.} \quad \left(\sum_{i=0}^{\infty} \mathbb{P}^i(X \amalg \text{Spec } \mathbb{k}) \right)$$

$\mathbb{Z} \in \text{SH}(\mathbb{k})$: Eilenberg-MacLane object.
 = monoid for tensor product

$$\mathbb{S} \rightarrow \mathbb{Z} \qquad \mathbb{Z} \otimes_{\mathbb{S}} \mathbb{Z} \xrightarrow{\mu} \mathbb{Z}$$

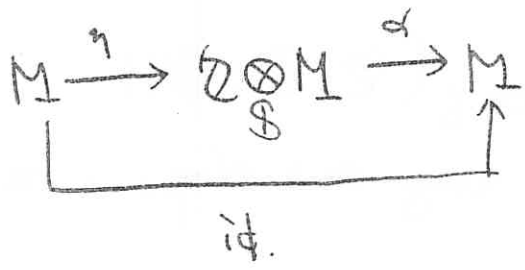
$$\mathbb{Z} \otimes_{\mathbb{S}} \mathbb{Z} \otimes_{\mathbb{S}} \mathbb{Z} \xrightarrow{\text{id} \otimes \mu} \mathbb{Z} \otimes_{\mathbb{S}} \mathbb{Z} \xrightarrow{\mu} \mathbb{Z}$$

$$\begin{array}{ccc} \mathbb{Z} \otimes_{\mathbb{S}} \mathbb{Z} \otimes_{\mathbb{S}} \mathbb{Z} & \xrightarrow{\text{id} \otimes \mu} & \mathbb{Z} \otimes_{\mathbb{S}} \mathbb{Z} \\ \downarrow \text{id} \otimes \mu & & \downarrow \mu \\ \mathbb{Z} \otimes_{\mathbb{S}} \mathbb{Z} & \xrightarrow{\mu} & \mathbb{Z} \end{array}$$

\mathbb{Z} -modules:
 $(M, \alpha), M \in \text{SH}(\mathbb{k})$

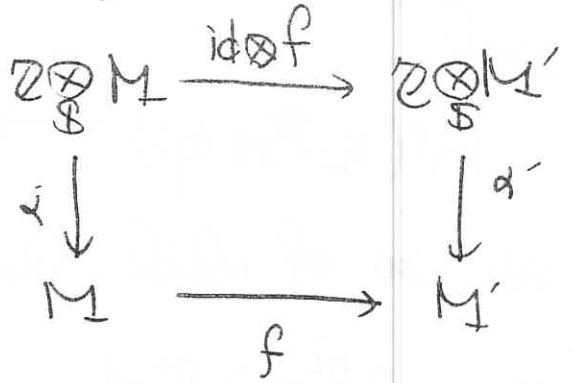
$$\begin{array}{ccc} \mathbb{Z} \otimes_{\mathbb{S}} M & \xrightarrow{\alpha} & M \\ \mathbb{Z} \otimes_{\mathbb{S}} \mathbb{Z} \otimes_{\mathbb{S}} M & \xrightarrow{\text{id} \otimes \mu} & \mathbb{Z} \otimes_{\mathbb{S}} M \\ \downarrow \text{id} \otimes \alpha & & \downarrow \alpha \\ \mathbb{Z} \otimes_{\mathbb{S}} M & \xrightarrow{\alpha} & M \end{array}$$

$$\begin{array}{ccc} \mathbb{S} \otimes_{\mathbb{S}} \mathbb{Z} & \rightarrow & \mathbb{Z} \otimes_{\mathbb{S}} \mathbb{Z} \leftarrow \mathbb{Z} \otimes_{\mathbb{S}} \mathbb{S} \\ \downarrow \text{can} & & \downarrow \mu \\ & & \mathbb{Z} \end{array}$$



\mathbb{Z} -linear map;
 $f: M \rightarrow M'$

$\text{Mod}_{\mathbb{Z}}(\mathbb{Z})$
 = category of \mathbb{Z} -modules
 and \mathbb{Z} -linear map.



Prop: The functor $\frac{S_M}{\mathbb{Z}} \rightarrow \text{Mod}_{\mathbb{Z}}(\mathbb{Z})$
 $X \mapsto \mathbb{Z} \otimes_{\mathbb{Z}} S(X)$

extends to an equivalence of categories;

$$DM(\mathbb{Z}) \xrightarrow{\sim} \text{Mod}_{\mathbb{Z}}(\mathbb{Z})$$

$$\begin{aligned}
 \text{So } H^p(X, \mathbb{Z}(q)) &= \text{hom}_{DM(\mathbb{Z})}(M(X), \mathbb{Z}(q)[p]) \\
 &= \text{hom}_{\mathbb{Z}}(\mathbb{Z} \otimes_{\mathbb{Z}} S(X), \mathbb{Z}(q)[p]) \quad \text{in } \text{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}(t)[s]) \\
 &= \text{hom}_{SH(\mathbb{Z})}(S(X), \mathbb{Z}(q)[p]) \quad \text{hom}_{\mathbb{Z}} = \text{hom}_{SH(\mathbb{Z})}
 \end{aligned}$$

gives rise to a natural transformation

$$\begin{aligned}
 H^p(X, \mathbb{Z}(q)) &= \text{hom}_{DM(\mathbb{Z})}(M(X), \mathbb{Z}(q)[p]) \\
 &\simeq \text{hom}_{\mathbb{Z}}(S(X), \mathbb{Z}(q)[p])
 \end{aligned}$$

$$\begin{array}{c}
 \downarrow \\
 H^{p+s}(X, \mathbb{Z}(q+t)) \simeq \text{hom}_{\mathbb{Z}}(S(X), \mathbb{Z}(q+t)[p+s])
 \end{array}$$

Point

N64

$$\text{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}(t)[s]) \subseteq \text{Hom}_{\mathbb{S}}(\mathbb{Z}, \mathbb{Z}(t)[s])$$

is a proper inclusion. This way one obtains natural transf.

$$H^p(X, \mathbb{Z}_\ell \mathbb{Z}(q)) \xrightarrow{Q_i} H^{p+2\ell^i-1}(X, \mathbb{Z}_\ell \mathbb{Z}(q+\ell^i))$$

for all $i \geq 0$ of which $Q_0 = \beta$.

Note; $2\ell^i - 1 > \ell^i - 1$.

$\mathcal{K} \in \text{SH}(k)$: represents algebraic \mathcal{K} -theory

$$\mathcal{K}^{p,q}(X) = \text{hom}_{\mathbb{S}}(S(X), \mathcal{K}(q)[p])$$

$\mathcal{K} \simeq \mathcal{K}(1)[2]$: Bott Periodicity

Thm (Hopkins-Morel) There is a descending

"filtration"

$$\begin{aligned} & \dots \rightarrow \text{Fil}^2 \mathcal{K} \rightarrow \text{Fil}^1 \mathcal{K} \rightarrow \text{Fil}^0 \mathcal{K} = \mathcal{K} \\ \rightsquigarrow & \text{Fil}^s \mathcal{K} \rightarrow \text{Fil}^{s+1} \mathcal{K} \rightarrow \mathbb{Z}(s)[2s] \rightarrow \text{Fil}^{s+1} \mathcal{K}[1] // \end{aligned}$$

This Gives a Spectral Sequence.

$$E_1^{s,t} = \text{hom}_{\mathbb{S}}(S(X), \mathbb{Z}(s)[2s][s+t]) = H^{s+t}(X, \mathbb{Z}(s))$$

$$\Rightarrow \text{hom}_{\mathbb{S}}(S(X), \mathcal{K}[s+t])$$

Re-index the Spectral

seq. s.t. E_1 becomes E_2 $\mathcal{K}^{s+t}(X) = \mathcal{K}_{-s-t}(X)$

, get $F_2^{s,t} = H^{s-t}(X, \mathbb{Z}(\frac{t}{2})) \Rightarrow \mathcal{K}_{-s-t}(X)$

$$K \xrightarrow{l} K \rightarrow K/lK \rightarrow K[\square]$$

$$K/l = \bigoplus_{i=0}^l L[2i]$$

$$T_{\geq 2l-4}L$$

$$\downarrow$$

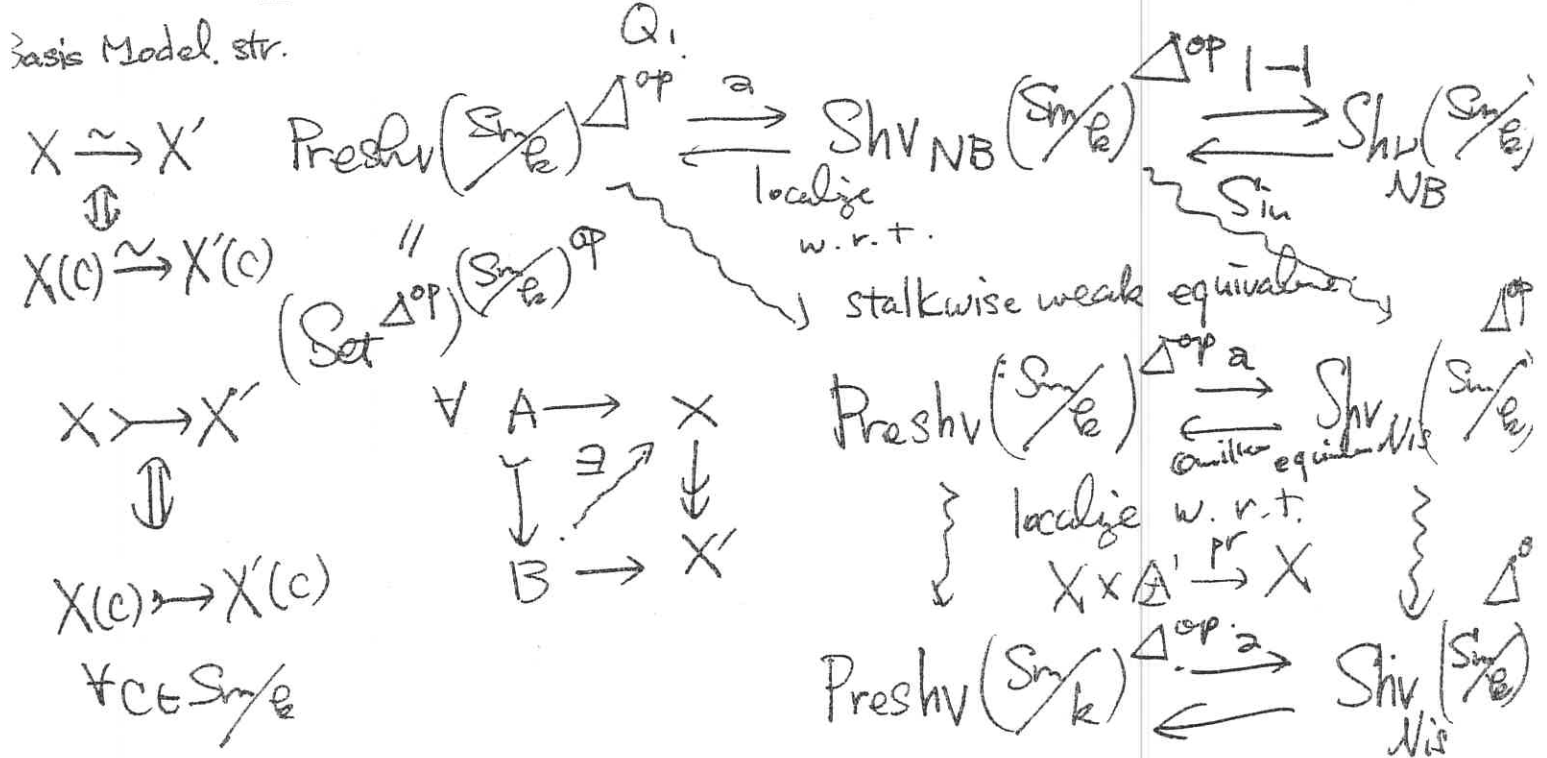
$$T_{\geq 2l-2}L \xrightarrow{\pi} \mathcal{P}/l\mathcal{P}(l-1)[2l-2]$$

$$\downarrow$$

$$L = T_{\geq 0}L \rightarrow \mathcal{P}/l\mathcal{P} \xrightarrow{\alpha} T_{\geq 2l-2}L$$

$$\mathcal{P}/l\mathcal{P} \rightarrow T_{\geq 2l-2}L[\square] \xrightarrow{\pi} \mathcal{P}/l\mathcal{P}(l-1)[2l-1]$$

Basis Model. str.



$$\mathrm{Shv}_{\mathrm{NB}}(\mathcal{S}_m/\mathbb{E}) \xrightleftharpoons{\Delta^{\mathrm{op}} | - |} \mathrm{Shv}_{\mathrm{NB}}(\mathcal{S}_m/\mathbb{E})$$

Voevodsky
ICM Berlin

$$\Downarrow$$
$$\mathrm{Shv}_{\mathrm{NIS}}(\mathcal{S}_m/\mathbb{E})$$

$$\Downarrow$$
$$\mathrm{Shv}_{\mathrm{NIS}}(\mathcal{S}_m/\mathbb{E})$$

$$\uparrow$$
$$\boxed{\mathrm{Spc}(\mathbb{E})}$$