why $\mathbb{A}^1$-homotopy theory? 12/21, L. Hesselholt

$(d = \dim X)$

Recall: $H^p(X, \mathbb{Z}(q)) = CH^q(X, 2q-p)$ If, $p = q+d$, then

$$H^{q+d}(X, \mathbb{Z}(q)) = CH^q(X, q-d)$$

$$= \frac{\mathbb{Z}\{\text{closed pts on } X \times \Delta^{q-d}\}}{\text{relation.}}$$

Can sometimes understand this group directly;

Thm (Rost) let $X_a \subseteq \mathbb{P}_k^{2^{r-1}}$ is the quadratic;

$$(x_1^2 - a_1 y_1^2) \otimes \cdots \otimes (x_{r-1}^2 - a_{r-1} y_{r-1}^2) = a_r t^2$$

where $a_1, \cdots, a_r \in k^\times$, then

$$H^{2^r-1}(X_a, \mathbb{Z}(2^{r-1})) \hookrightarrow k^\times \qquad q + d = 2^{r-1} + 2^{r-1} - 1$$
$$= 2^r - 1 = p$$

If $p < q+d$ the group $H^p(X, \mathbb{Z}(q))$ can never be understood directly from the definition

Strategy: Relate $H^p(X, \mathbb{Z}(q))$ to some $H^{r+d}(X, \mathbb{Z}(r))$

Recall; $DM(k) :=$ derived category of mixed motive$/k$
$=$ tensor triangulated category

$$\left.\begin{array}{ll} X \longmapsto X[p] & \text{change degree by } p \\ X \mapsto X(q) & \text{change weight by } q \end{array}\right\} \text{ autom of } DM(k)$$

$$\begin{array}{ccc} Sm/k & \to & DM(k) \\ \cup & & \cup \\ X & \mapsto & M(X) \end{array}$$

$\mathbb{Z} = M(\operatorname{Spec} k) \in DM(k)$
unit for tensor product.

$$H^p(X, \mathbb{Z}(q)) = \hom_{DM(k)}(M(X), \mathbb{Z}(q)[p])$$

$$\mathbb{Z} \xrightarrow{\ell} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/\ell\mathbb{Z} \xrightarrow{\delta} \mathbb{Z}[1]$$

$$\begin{tikzcd}
H^p(X, \mathbb{Z}/\ell\mathbb{Z}(q)) \arrow[r, phantom, "="] \arrow[d, "\beta"'] & \hom_{DM(k)}(M(X), \mathbb{Z}/\ell\mathbb{Z}(q)[p]) \arrow[d, "(\pi(1)\circ\delta)_*"] \\
H^{p+1}(X, \mathbb{Z}/\ell\mathbb{Z}(q)) \arrow[r, phantom, "="] & \hom_{DM(k)}(M(X), \mathbb{Z}/\ell\mathbb{Z}(q)[p+1])
\end{tikzcd}$$

$SH(k)$ = stable homotopy category $/k$

= tensor triangulated category

$$\begin{cases} X \mapsto X[p] : \text{shifts } \underline{\text{degree}} \text{ by } p \\ X \mapsto X(q) : \text{changes } \underline{\text{weight}} \text{ by } q. \end{cases}$$

$$\begin{tikzcd}
Sm/k \arrow[r] & SH(k) \\
X \arrow[u, phantom, "\cup" rotate=90] \arrow[r, mapsto] & S(X) \arrow[u, phantom, "\cup" rotate=90] \arrow[d, equal] \\
& \left(\Sigma^\infty_{\mathbb{P}^1}(X \sqcup \operatorname{Spec} k)\right)
\end{tikzcd}$$

$\mathbb{S} = S(\operatorname{Spec} k) \in SH(k)$: unit for tensor prod.

$\mathbb{Z} \in SH(k)$: Eilenberg–Maclane object.

= monoid for tensor product.

$$\mathbb{S} \to \mathbb{Z} \qquad \mathbb{Z} \otimes_{\mathbb{S}} \mathbb{Z} \xrightarrow{\mu} \mathbb{Z}$$

$$\begin{tikzcd}
\mathbb{Z} \otimes_{\mathbb{S}} \mathbb{Z} \otimes_{\mathbb{S}} \mathbb{Z} \arrow[r, "\mathrm{id}\otimes\mu"] \arrow[d, "\mu\otimes\mathrm{id}"'] & \mathbb{Z} \otimes_{\mathbb{S}} \mathbb{Z} \arrow[d, "\mu"] \\
\mathbb{Z} \otimes_{\mathbb{S}} \mathbb{Z} \arrow[r] & \mathbb{Z}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{S} \otimes_{\mathbb{S}} \mathbb{Z} \arrow[r] \arrow[dr, "\sim", "\mathrm{can}"'] & \mathbb{Z} \otimes_{\mathbb{S}} \mathbb{Z} \arrow[d, "\mu"] & \mathbb{Z} \otimes_{\mathbb{S}} \mathbb{S} \arrow[l] \arrow[dl, "\sim"', "\mathrm{can}"] \\
& \mathbb{Z} &
\end{tikzcd}$$

$\underline{\mathbb{Z}\text{-modules}}$:

(M, α), $M \in SH(k)$

$$\mathbb{Z} \otimes_{\mathbb{S}} M \xrightarrow{\alpha} M$$

$$\begin{tikzcd}
\mathbb{Z} \otimes_{\mathbb{S}} \mathbb{Z} \otimes_{\mathbb{S}} M \arrow[r, "\mu\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\alpha"'] & \mathbb{Z} \otimes_{\mathbb{S}} M \arrow[d, "\alpha"] \\
\mathbb{Z} \otimes_{\mathbb{S}} M \arrow[r, "\alpha"] & M
\end{tikzcd}$$

$$\begin{tikzcd} M \arrow[r, "\eta"] \arrow[rr, bend right, "id"'] & \mathbb{Z}\otimes_{\mathbb{S}} M \arrow[r, "\alpha"] & M \end{tikzcd}$$

$\mathbb{Z}$-linear map; $f: M \to M'$

$$\begin{tikzcd} \mathbb{Z}\otimes_{\mathbb{S}} M \arrow[r, "id\otimes f"] \arrow[d, "\alpha"'] & \mathbb{Z}\otimes_{\mathbb{S}} M' \arrow[d, "\alpha'"] \\ M \arrow[r, "f"'] & M' \end{tikzcd}$$

$Mod_{\mathbb{Z}}(k)$

= category of $\mathbb{Z}$-modules and $\mathbb{Z}$-linear map.

<u>Prop</u>: The functor $Sm/k \to Mod_{\mathbb{Z}}(k)$

$$X \mapsto \mathbb{Z}\otimes_{\mathbb{S}} S(X)$$

extends to an equivalence of categories;

$$DM(k) \xrightarrow{\sim} Mod_{\mathbb{Z}}(k)$$

So $H^p(X, \mathbb{Z}(q)) = \hom_{DM(k)}(M(X), \mathbb{Z}(q)[p])$

$= \hom_{\mathbb{Z}}(\mathbb{Z}\otimes_{\mathbb{S}} S(X), \mathbb{Z}(q)[p])$ $\quad \theta \in \mathrm{Hom}_{\mathbb{S}}(\mathbb{Z}, \mathbb{Z}(t)[s])$

$= \hom_{SH(k)}(S(X), \mathbb{Z}(q)[p])$ $\quad \hom_{\mathbb{S}} = \hom_{SH(k)}$

gives rise to a natural transformation

$$H^p(X, \mathbb{Z}(q)) = \hom_{DM(k)}(M(X), \mathbb{Z}(q)[p])$$
$$\simeq \hom_{\mathbb{S}}(S(X), \mathbb{Z}(q)[p])$$
$$\downarrow$$
$$H^{p+s}(X, \mathbb{Z}(q+t)) \simeq \hom_{\mathbb{S}}(S(X), \mathbb{Z}(q+t)[p+s])$$

Point

$$\mathrm{Hom}_{\mathcal{DA}}(\mathbb{Z}, \mathbb{Z}(t)[s]) \subseteq \mathrm{Hom}_{\mathcal{S}}(\mathbb{Z}, \mathbb{Z}(t)[s])$$

is a proper inclusion. This way one obtains natural transf.

$$H^p(X, \mathbb{Z}/\ell\mathbb{Z}(q)) \xrightarrow{Q_i} H^{p+2\ell^i-1}(X, \mathbb{Z}/\ell\mathbb{Z}(q+\ell^i-1))$$

for all $i \geq 0$ of which $Q_0 = \beta$.

Note; $2\ell^i - 1 > \ell^i - 1$

$\mathcal{K} \in SH(k)$: represents algebraic K-theory

$$\mathcal{K}^{p,q}(X) = \mathrm{hom}_{\mathcal{S}}(S(X), \mathcal{K}(q)[p])$$

$\mathcal{K} \xrightarrow{\sim} \mathcal{K}(1)[2]$: Bott Periodicity

Thm (Hopkins–Morel) There is a descending "filtration"

$$\cdots \to \mathrm{Fil}^2\mathcal{K} \to \mathrm{Fil}^1\mathcal{K} \to \mathrm{Fil}^0\mathcal{K} = \mathcal{K}$$

$\rightsquigarrow$

$$\mathrm{Fil}^{s+1}\mathcal{K} \to \mathrm{Fil}^s\mathcal{K} \to \mathbb{Z}(s)[2s] \to \mathrm{Fil}^{s+1}\mathcal{K}[1] \ //$$

This Gives a Spectral Sequence

$$E_1^{s,t} = \mathrm{hom}_{\mathcal{S}}(S(X), \mathbb{Z}(s)[2s][s+t]) = H^{3s+t}(X, \mathbb{Z}(s))$$

$$\Rightarrow \mathrm{hom}_{\mathcal{S}}(S(X), \mathcal{K}[s+t])$$

$$\overset{\text{dfn}}{=} \mathcal{K}^{s+t}(X) = K_{-s-t}(X)$$

Re-index the Spectral seq. s.t. E_1 becomes E_2

, get $E_2^{s,t} = H^{s-t}(X, \mathbb{Z}(-t)) \Rightarrow K_{-s-t}(X)$

$$K \xrightarrow{l} K \to K/lK \to K[1]$$

$$K/l = \bigoplus_{i=0}^{l-2} L[2i]$$

$$\begin{tikzcd}
\tau_{\geq 4l-4} L \arrow[d] & \\
\tau_{\geq 2l-2} L \arrow[r, "\pi"] \arrow[d] & \mathbb{Z}/l\mathbb{Z}(l-1)[2l-2] \\
L = \tau_{\geq 0} L
\end{tikzcd}$$

$$L = \tau_{\geq 0} L \to \mathbb{Z}/l\mathbb{Z} \xrightarrow{\partial} \tau_{\geq 2l-2} L$$

$$\begin{tikzcd}
\mathbb{Z}/l\mathbb{Z} \arrow[r] \arrow[rr, bend right] & \tau_{\geq 2l-2} L[1] \arrow[r, "\pi"] & \mathbb{Z}/l\mathbb{Z}(l-1)[2l-1]
\end{tikzcd}$$

Basis Model. str.

$X \xrightarrow{\sim} X'$

$\Updownarrow$

$X(C) \xrightarrow{\sim} X'(C)$

$X \rightarrowtail X'$

$\Updownarrow$

$X(C) \rightarrowtail X'(C)$

$\forall C \in Sm/k$

$$\begin{tikzcd}
\mathrm{Preshv}(Sm/k)^{\Delta^{op}} \arrow[r, shift left, "a"] & \mathrm{Shv}_{NB}(Sm/k)^{\Delta^{op}} \arrow[l, shift left, "\text{localize}"] \arrow[r, shift left, "|-|"] & \mathrm{Shv}_{NB}(Sm/k) \arrow[l, shift left, "\mathrm{Sin}"]
\end{tikzcd}$$

Q_1

$\mathrm{Preshv}(Sm/k)^{\Delta^{op}} = (\mathrm{Set}^{\Delta^{op}})^{(Sm/k)^{op}}$

localize w.r.t. stalkwise weak equivalence

$$\begin{tikzcd}
\forall A \arrow[r] \arrow[d, tail] & X \arrow[d, two heads] \\
B \arrow[r] \arrow[ur, dashed, "\exists"] & X'
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Preshv}(Sm/k)^{\Delta^{op}} \arrow[r, shift left, "a"] \arrow[d, squiggly] & \mathrm{Shv}_{Nis}(Sm/k)^{\Delta^{op}} \arrow[l, shift left, "\text{Quillen equiv.}"] \arrow[d, squiggly] \\
\mathrm{Preshv}(Sm/k)^{\Delta^{op}} \arrow[r, shift left, "a"] & \mathrm{Shv}_{Nis}(Sm/k) \arrow[l, shift left]
\end{tikzcd}$$

localize w.r.t. $X \times \mathbb{A}^1 \xrightarrow{pr} X$

Δ^{op} $\quad \Delta^{0}$

$$
\begin{tikzcd}
\mathrm{Shv}_{NB}(\mathrm{Sm}/k)^{\Delta^{op}} \arrow[r, shift left, "|-|"] & \mathrm{Shv}_{NB}(\mathrm{Sm}/k) \arrow[l, shift left] \arrow[d, squiggly] \\
& \mathrm{Shv}_{Nis}(\mathrm{Sm}/k) \arrow[d, squiggly] \\
& \mathrm{Shv}_{Nis}(\mathrm{Sm}/k) \\
& \boxed{\mathrm{Spc}(k)} \arrow[u]
\end{tikzcd}
$$

$\boxed{\text{Voevodsky ICM Berlin}}$