The Category of mixed Tate Motives. 記:佐. No.1

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$MTM(k)$:

$\otimes$

$\mathbb{Q}$-linear Bloch - kriz $\mathbb{Z}$.

Properties: ①: Abelian category.

②: semi-simple obj.; $\mathbb{Q}(n)$

③: $\forall M \in MTM(k)$ $\exists$ weight filtration

$gr^W_{2n+1} M = 0.$

$gr^W_{2n} M = \bigoplus \mathbb{Q}(-n)$

finite length.

④: $\hom(\mathbb{Q}(p), \mathbb{Q}(r))$

$$= \begin{cases} \mathbb{Q} & p = r \\ 0 & p \neq r \end{cases}$$

⑤ $Ext^i(\mathbb{Q}, \mathbb{Q}(p)) = 0$ $\cancel{i \geq p}$. $i \leq 0$ $(i,p) \neq (0,0)$

$\phi : MTM(k) \longrightarrow \{\mathbb{Z}\text{-graded. vector sp}/\mathbb{Q}\}$

$\phi_{-n}(M) = \hom(\mathbb{Q}(n), gr^W_{2n} M) \Longrightarrow$ $\mathrm{Rep}\, G \simeq MTM(k)$

Tannakian category. $G \simeq \mathbb{G}_m \ltimes U$

property $Ext^i_{MTM(k)}(\mathbb{Q}, \mathbb{Q}(p)) \simeq CH^p(k, 2p-i)_{\mathbb{Q}}$.

$\exists \mathcal{L}$: Graded Pro Lie -alg$/\mathbb{Q}$. grading

$MTM(k) \xrightarrow{\sim} \{\text{rep of } \mathcal{L} \text{ finite dim}/\mathbb{Q}\}$

constructing $\mathcal{L}$. explicitly using alg. cycle.

"de Rham Complex"/ $BG/B\mathbb{G}_m$. (using rational homotopy theory.)

Cube version ← Bloch complex

alg cycle $\rightsquigarrow$ $N^\cdot$. DGA

$\mathcal{L}$ $\rightsquigarrow$ $BN^\cdot \rightsquigarrow \chi_{mot} := H^0 BN^\cdot$

Bar construction.

$$\mathcal{L} = \left(\chi^+_{mot} / (\chi^+_{mot})^2 \right)^*$$

§. cubical obj. $\longleftrightarrow$ (simplicial. obj. みたいなもの.) No2

$\square \hookrightarrow$ (finite set の category)

sub cat. objeet: $\square^n$ $\square^0 = 0$

$\square^1 = \{0,1\}$

$\mathfrak{S}_n = \mathrm{Aut}_{\square}\, \square^n$

Morphism: $\square^n = (\square^1)^{\times n}$

$\mathrm{hom}(\square^i, \square^j) = \{\text{injection}\} = \mathfrak{S}_n \ltimes \{\pm 1\}^n$

$c = 0, 1$

closed under finite product & their morphisms $\delta^n_{p,c}$

$: \square^n \longrightarrow \square^{n+1}$

$(i_1, \cdots, i_n) \longmapsto (i_1, \cdots, c, \cdots i_n)$

p-番目

$\mathcal{C}$: permutative ~~&~~ cat.

$\left\{ \begin{array}{l} \text{Strict tensor category. with unit.} \\ A \otimes (B \otimes C) = (A \otimes B) \otimes C \end{array} \right.$

$\times : \square \times \square \longrightarrow \square$

$(\square^n, \square^m) \longmapsto \square^{n+m}$

$\mathcal{A} : \square^{op} \to \mathcal{C}$. $m : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$

: associative
commutative.

$\mathcal{A}^p = \mathcal{A}(\square^p)$

$\mathcal{A}^p \longrightarrow \mathcal{A}^p$

$\mathrm{alt.} = \frac{1}{p!} \sum$, $\quad : \mathcal{A}^{p+1} \longrightarrow \mathcal{A}^p$

$\mathrm{alt} = \frac{1}{2^p p!} \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma) \cdot \sigma$.

$\partial = \sum_k (-1)^k \{ \mathcal{A}(\delta^p_{k,1}) - \mathcal{A}(\delta^p_{k,0}) \}$.

$\mathrm{alt}\, \mathcal{A}^*$: DGA.

$\mathcal{N}(*) = \bigoplus_{r \geq 0} \mathcal{N}(r)$ $\quad \mathcal{N}(r) = \mathrm{Alt}(\mathrm{Cycl}^r \langle n \rangle)[2r]$

$\Delta^n = \mathrm{Spec}\, k[T_0, \cdots, T_n] / (\sum T_i - 1)$ $\quad \square^n = (\mathbb{P}^1 - \{1\})^n$

(face, $(\cdots 0 \cdots)$
$(\cdots \infty \cdots)$)

$$\mathrm{cycl}^r\langle n\rangle = \left\langle V \subset \square^n \;\middle|\; \begin{array}{l} V : \mathrm{codim} = r \\ \text{integral closed subscheme} \\ \text{meet all faces properly} \end{array} \right\rangle_{\mathbb{Q}}$$

$$H^n(\mathcal{N}(r)) \cong CH^r(k, 2r-n)_{\mathbb{Q}}$$

$$A^+ = \ker(A \to \mathbb{Q})$$

§ ~~Bar~~ Construction.

$A^\cdot$: augumented DGA

$$BA^\cdot = \cancel{T(\mathbb{Q}, A, \mathbb{Q})} / D(\mathbb{Q}, A, \mathbb{Q}) \simeq T(\mathbb{Q}, A^+, \mathbb{Q})$$

$$T(\mathbb{Q}, A, \mathbb{Q}) = \bigoplus_{n \geq 0} A^{\otimes n}$$

$$[a_1|a_2|\cdots|a_n] \overset{df}{:=} a_1 \otimes \cdots \otimes a_n$$

$$\deg[a_1|\cdots|a_n] = \sum \deg a_i - n.$$

$$\delta_p^{n+1} : A^{\otimes(n+1)} \longrightarrow A^{\otimes(n-1)}$$

$$[a_1|\cdots|a_{n+1}] \longmapsto [a_1|\cdots|a_p a_{p+1}|\cdots|a_{n+1}]$$

$1 \leq p \leq n$

$$s_p^n : A^{\otimes n} \longrightarrow A^{\otimes(n+1)}$$

$$[a_1|\cdots|a_n] \longmapsto [a_1|\cdots|\underset{p\text{-th}}{1}|\cdots|a_n]$$

$1 \leq p \leq n+1$

$$D(\mathbb{Q}, A, \mathbb{Q}) = \left\langle \bigcup s(T(\mathbb{Q}, A, \mathbb{Q})) \right\rangle$$

$$\partial = \sum_p (-1)^{n+1} \delta_p^{n+1}$$

$$\begin{array}{ccccc} & & \uparrow & & A^+ \\ & & & & \uparrow \\ A^{+\otimes 3} & \xrightarrow{\partial} & A^{+\otimes 2} & \xrightarrow{\partial} & A^+ \\ \uparrow & & \uparrow & & \uparrow \\ A^{+\otimes 3} & \xrightarrow{\partial} & A^{+\otimes 2} & \xrightarrow{\partial} & A^{\oplus} \end{array}$$

$\Psi : BA \to BA \otimes BA$: comulti.

$$\begin{array}{c} \Psi \\ \downarrow \end{array}$$

$$[a_1 | \cdots | a_n] \mapsto \sum (-1)^p [a_1 | \cdots | a_p] \cdot [a_{p+1} | \cdots | a_n]$$

$$\underset{(\mathbb{k})}{\chi_{mot}} \overset{\text{dfn.}}{:=} H^0 BN(*) \qquad \underset{(\mathbb{k})}{\mathcal{L}} = \left(\chi^+_{mot} \Big/ (\chi^+_{mot})^2 \right)^*$$