12/19 Sato. K.

B-K conj. & B-L conj. (cycle class maps)

important notation

§.1. Cycle Class. (complement)

§.2. higher Chow Groups. & B-L-conj.

§.3. B-K conj.

§.4. examples in arithmetic situation

we like consider under the case where $\exists X_0/\mathbb{R}$ s.t. $X_0 \otimes_{\mathbb{R}} \mathbb{C} \cong X$

§.1. Cycle Class.

X: smooth alg. var/$\mathbb{C}$

$$CH^c(X) \longrightarrow H^{2c}_{an}(X(\mathbb{C})^{an}, \mathbb{Z})$$ not compatible with complex conj.

Case $n=1$ $\quad 0 \to 2\pi i\mathbb{Z} \to \mathcal{O}_{X^{an}} \to \mathcal{O}^{\times}_{X^{an}} \to 0$ (exact) on $X(\mathbb{C})^{an}$.

標準的

$$H^1(X(\mathbb{C})^{an}, \mathcal{O}^{\times}_{X^{an}}) \longrightarrow H^2_{an}(X(\mathbb{C})^{an}, 2\pi i\mathbb{Z})$$

Isom class gp of complex line b'dle

$cl_X(\mathfrak{Z})$ complex conj.

$H^1_{Zar}(X, \mathcal{O}^*_X)$ isom class gr of algebraic line b'dle on X

$\operatorname{Pic} X$

$CH^1(X)$

$$CH^1(X) \longrightarrow H^2_{an}(X(\mathbb{C})^{an}, 2\pi i\mathbb{Z})$$

compatible with complex conj.

Case. $n \geq 2$. For $\mathfrak{Z} \subseteq X$: integral closed alg. subvar. of codim $=c$ on X

$$cl_X(\mathfrak{Z}) \in H^{2c}_{an}(X(\mathbb{C})^{an}, \underbrace{(2\pi i)^c\mathbb{Z}})$$

complex conj. acts by $(-1)^c$

<u>Ex</u>: $c=2 \quad \mathfrak{Z} = D_1 \cap D_2$: smooth divisors intersecting transversally.

$$cl_X(\mathfrak{Z}) := cl_X(D_1) \cap cl_X(D_2)$$

$$cl_X(D_1) \in H^2(X, (2\pi i)\mathbb{Z}) \qquad cl_X(D_2) \in H^2(X, (2\pi i)\mathbb{Z})$$

cycle class gives

$$\begin{pmatrix}\text{rational} \\ \text{equivalence} \\ \text{to } 0\end{pmatrix} \subseteq Z^c(X) \longrightarrow H^{2c}_{an}(X(\mathbb{C})^{an}, (2\pi i)^c \mathbb{Z})$$

$Z^c(X)$ = gp of alg. cycles on X of codim $= c$ $\quad \cup\!\!\!\mid \quad \{0\}$?

これは,

$Z \subseteq X \times \mathbb{A}^1$

① : homotopy invariance of $H^{2c}_{an}(X(\mathbb{C})^{an}, (2\pi i)^c \mathbb{Z})$

② : intersection product と cup 積 の compatibility

Algebraic Version

k: field X: smooth algebraic variety $/k$

$n \in \mathbb{N}_{\geq 2}$ $\quad c \in \mathbb{Z}_{\geq 0}$

$\frac{1}{n} \in k$ (assumption .

μ_n : étale sheaf of n-th roots of unity on X.

$(U \xrightarrow{\text{étale}} X) \longmapsto \{ x \in \Gamma(U, \mathcal{O}_U) \mid x^n = 1 \}$ という sheaf.

Rem: $k = \mathbb{C}$ の場合,

$$\begin{array}{ccc} ((2\pi i)^c \mathbb{Z}) \otimes \mathbb{Z}/n\mathbb{Z} & \longrightarrow & \mu_n^{\otimes c} \\ \Psi & & \Psi \\ (2\pi i)^c \otimes \bar{1} & \longmapsto & \underbrace{\exp\left(\frac{2\pi i}{n}\right) \otimes \cdots \otimes \exp\left(\frac{2\pi i}{n}\right)}_{c\text{ コ}} \end{array}$$

$\mathbb{Z}/n\mathbb{Z}$ 一加群.

Returning to the algebraic situation we have,

$$\begin{tikzcd} Z^c(X) \arrow[r] \arrow[d] & H^{2c}_{et}(X, \mu_n^{\otimes c}) \\ CH^c(X)/n \arrow[ur, dashed] & \end{tikzcd}$$

in a similar way.

Rem: $k = \mathbb{C}$ の場合.

exponential map

$$\begin{tikzcd} CH^c(X) \arrow[r] \arrow[d] & H^{2c}_{an}(X(\mathbb{C})^{an}, (2\pi i)^c \mathbb{Z}) \arrow[d] \\ H^{2c}_{et}(X, \mu_n^{\otimes c}) \arrow[r, "\sim"] & H^{2c}(X(\mathbb{C})^{an}, \mu_n^{\otimes c}) \end{tikzcd}$$

§. 2. higher Chow Groups.

X : alg. var. smooth.

$$\bigoplus_{i=0}^{\dim X} CH^i(X)_{\mathbb{Q}} \xrightarrow{\cong} K_0(X)_{\mathbb{Q}}$$

$$[Z] \longmapsto [\text{proj. res. of } \mathcal{O}_Z]$$

Def (Bloch):

k, X, c algebraic situation $\downarrow$ higher Chow Group. cycle.

$\downarrow$ $K_q(X)$: higher K 群 homotopy theoretic.

$Z^c(X,*)$: $\longrightarrow Z^c(X,q) \xrightarrow{d} \cdots \longrightarrow Z^c(X,1) \xrightarrow{d} Z^c(X,0) \longrightarrow 0 \rightarrow$

cpx of abelian gps. determined by c

$$Z^c(X,q) \underset{\text{dfn}}{:=} \left\{ I \overset{c}{\subset} (X\times\Delta^q) \;\middle|\; \begin{array}{l} I \text{ meets properly} \\ \text{with all faces on } X\times\Delta^q \end{array} \right\}$$

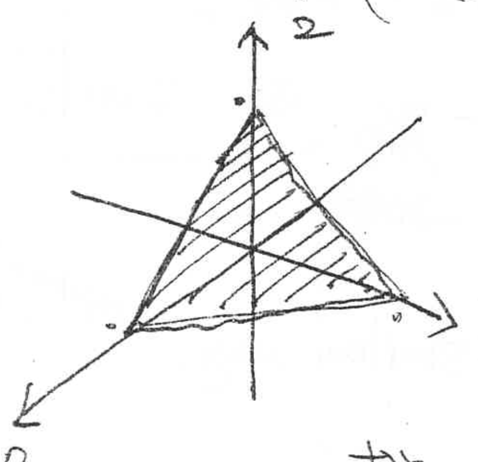
closed integral subvariety codim $= c$.

$$\Delta^q = \operatorname{Spec} k[T_0,\cdots,T_q]/(T_0+\cdots+T_q-1) \xrightarrow{\sim} \mathbb{A}^q_k$$

$$\Delta^q \text{の faces} \overset{\text{dfn}}{:=} \left\{ \begin{array}{l} T_{i_1} = \cdots = T_{i_r} = 0 \\ \text{で, def される } \Delta^q \text{ の} \\ \text{closed subvariety.} \end{array} \right\}$$

$q=2$

$$\Delta^2 = \operatorname{Spec}\left(k[T_0,T_1,T_2]/(T_0+T_1+T_2-1) \right)$$



face of $X\times\Delta^q = X\times(\text{face of } \Delta^q)$

$d :=$ alternate sum of pull-backs

$$CH^c(X,q) \overset{\text{dfn}}{=} H_q(Z^c(X,*)) \text{ とおく。}$$

(Bloch)

すると、

$$\bigoplus_{q=0} CH^{2m-q}(X,q)_{\mathbb{Q}} \simeq K_m(X)_{\mathbb{Q}}$$

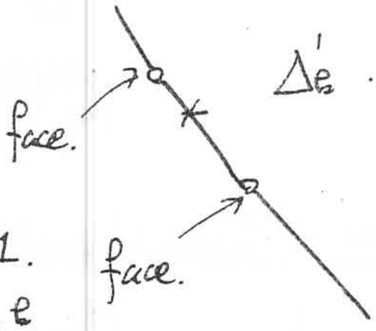
Ex.1: $CH^1(\mathrm{Spec}\,k, q) \xrightarrow{\simeq} \begin{cases} k^\times & (q=1) \\ 0 & \text{otherwise.} \end{cases}$

(Bloch's Rersults)

Nesterenko - Suslin - Totaro

$$\begin{array}{ccc} k^\times & \longrightarrow & CH^1(\mathrm{Spec}\,k, 1) \\ \Psi & & \Psi \\ a & \longmapsto & \left(\frac{1}{1-a}, \frac{-a}{1-a}\right) \in \Delta^1_k \\ 1 & \longmapsto & 0. \end{array}$$

face. face. Δ^1_k.



Ex 2: $(q \geq 2)$

$$\underbrace{k^\times \otimes \cdots \otimes k^\times}_{q} \xrightarrow{\cong} CH^1(k, 1) \otimes \cdots \otimes CH^1(k, 1) \xrightarrow{\text{product}} CH^q(k, q)$$

is surjective and the kernel is generated by Steinberg Symbol.

Steinberg's Symbol Symbol of the form.

$\{a_1, \cdots, a_q\}$ $\exists i\ \exists j$ $a_i + a_j = 1$ $a_j, a_i \in k \setminus \{0, 1\}$

X: smooth$/k$ $\Longrightarrow$ we have canonical map.

$$CH^c(X, q) \longrightarrow H_{\text{ét}}^{2c-q}(X, \mu_n^{\otimes c})$$

$$CH^c(X, q; \mathbb{Z}/n\mathbb{Z}) = H_q\left(Z^c(X, *) \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}\right) \xrightarrow{\alpha^{c,q}} H_{\text{ét}}^{2c-q}(X, \mu_n^{\otimes c})$$

Conj. (B-L):

$$CH^c(X, q; \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\alpha^{c,q}} H_{\text{ét}}^{2c-q}(X, \mu_n^{\otimes c})$$

$\alpha^{c,q}$: bijective for any c, q with $q \geq c$.

Case: $c = q$ $\quad CH^q(k, q) \xrightarrow{\alpha^{q,q}} H^q_{et}(X, \mu_n^{\otimes q})$

$\quad \quad \quad \quad \quad \quad \quad \wr\mathrel{\|} \; \otimes \mathbb{Z}/n\mathbb{Z} \qquad$: Bloch-Kato Conjecture

$\quad \quad \quad \quad \quad \quad K_q^M(k)/n$

Thm (Suslin-Voevodsky / Geisser-Levine)

Assume. Conj. (B-K) holds for any finitely generated fields$/k$ $\Rightarrow$ (B. L)-conj. holds for any smooth var.$/k$.