良 体　　12/19 Iamashita. G　　No.1

goal: to construct Voevodsky's tensor triangulated category of mixed motive

properties $(n \geq 0)$

Tate object. $\mathbb{Z}(n) \leftarrow$ complex of Zariski sheaves. degree $\leq n$

$(Sm/k)_{Zar}$

Beilinson:

①: $\mathbb{Z}(0) = \mathbb{Z}$

②: $\mathbb{Z}(1) = \mathcal{O}^{\times}[-1]$

③: F: field/k

$\mathbb{H}^{n}_{Zar}(\mathrm{Spec} F, \mathbb{Z}(n)) \simeq K^{M}_{n}(F) \leftarrow$ Milnor K-group

④: $\mathbb{H}^{2n}_{Zar}(X, \mathbb{Z}(n)) \simeq CH^{n}(X)$

$\left(\mathbb{H}^{p}_{Zar}(X, \mathbb{Z}(q)) \simeq CH^{q}(X, 2q-p) \right)$

↑

higher Chow 群.

⑤: $X \in Sm/k$

$\Rightarrow$ spectral sequence. $E_2^{p,q} = \mathbb{H}^{p-q}_{Zar}(X, \mathbb{Z}(-q)) \Rightarrow K_{-p-q}(X)$

: Bloch-Litchtenbaum, Friedlander-Suslin.

: Voevodsky, Levine

: Grayson-Suslin.

$\mathbb{H}^{p}_{Zar}(X, \mathbb{Z}(q)) \otimes \mathbb{Q}$

$\simeq gr^{q}_{\gamma} K_{2q-p}(X) \otimes \mathbb{Q}$

Beilinson-Litchtenbaum Conj:

F: field/k, l: prime $\neq$ char k

$$\mathbb{H}^{p}_{Zar}(F, \mathbb{Z}(q) \otimes \mathbb{Z}/l) \simeq \begin{cases} H^{p}_{et}(F, \mu_l^{\otimes q}) & p \neq q \\ 0 & p > q. \end{cases}$$

$\Longleftrightarrow$ $\mathbb{Z}(q) \otimes \mathbb{Z}/l \xrightarrow{qis} \tau_{\leq q} R\alpha_* \mu_l^{\otimes q}$

(Suslin-Voevodsky) (Bloch-Kato Conj. generalized Hilbert 90) $\alpha: (Sm/k)_{et} \longrightarrow (Sm/k)_{Zar}$

$\left((\mathbb{Z}(q) \otimes \mathbb{Z}/l)_{et} \simeq \mu_l^{\otimes q} \right)$

Beilinson–Soulé: Conj. $\mathbb{H}^p_{Zar}(X, \mathbb{Z}(q)) \overset{df}{=}: H^p_{\mathcal{M}}(X, \mathbb{Z}(q))$

Vanishing : motivic coh.

$X \in Sm/k \Rightarrow \mathbb{H}^i_{Zar}(X, \mathbb{Z}(n)) = 0$ for $i < 0$.

$$DM_{gm}(k) \quad M : Sm/k \longrightarrow DM_{gm}(k)$$

$$X \longmapsto M(X) \quad \text{motive of } X$$

$$H^p_{\mathcal{M}}(X, \mathbb{Z}(q)) \cong \hom_{DM_{gm}(k)}(M(X), \mathbb{Z}(q)[p])$$

$$DM_{gm}(k) \rightsquigarrow DM^{eff}_{gm}(k) \subseteq DM^{eff}_{-}(k)$$

"invert $\mathbb{Z}(1)$"

$\uparrow$ bounded above complexes of Nisnevich shf with transfers with homotopy invariant cohomologies.

three key words:

- homotopy invariant
- Nisnevich shf.
- with transfers.

$$Sm/k \rightsquigarrow SmCor/k \rightsquigarrow D^-(Shv_{Nis}(SmCor(k)))$$

$$\rightsquigarrow DM^{eff}_{-}(k)$$

Def: $SmCor(k)$

Object: X: smooth$/k$ Mor: $\hom_{SmCor(k)}(X, Y)$

$$Sm/k \longrightarrow SmCor/k$$

$$X \longmapsto X$$

$$f \longmapsto \Gamma_f : \text{graph}.$$

$$= \left\langle Z \;\middle|\; \begin{array}{l} Z \subseteq X \times Y : \text{closed subscheme. integral} \\ Z \to X \text{ finite surj over irred. comp. of } X \end{array} \right\rangle_{\mathbb{Z}}$$

Def: $F : SmCor/k \to Ab$: additive contravariant functor.
presheaf with transfer.

(= pretheory)

Def: F : presheaf homotopy invariant
$\underset{\text{def}}{\iff} \forall X \in Sm/k \quad F(X) \xrightarrow[\sim]{pr_1^*} F(X \times \mathbb{A}^1)$: isom.

Def: X : scheme. $\{U_i \to X\}$ が Nisnevich covering
$\underset{\text{def}}{\iff} \underbrace{\forall x \in X \text{ について}}_{\text{étale covering}}$, $\exists i$. $\exists u_i \in U_i$ $k(x) \xrightarrow{\sim} k(u_i)$: isom.
であって,

Zar Nis. ét
coarse $\longleftrightarrow$ fine $\leadsto (Sm/k)_{Nis}$.
talk loc. ~~hensel~~. strict hensel

Def:
F : Nis. sheaf with transfer
F : presheaf with transfer

Nis sheaf on $(Sm/k)_{Nis}$. $\nearrow$ $SmCor/k \to Ab$.
Sm/k

$Shv_{Nis}(SmCor(k))$

$D^-(Shv_{Nis}(SmCor(k)))$: derived category of Nisnevich sheaf with transfers.

$\cup$

$DM^{eff}_-(k)$: bounded above ~~x~~ cpxes of.
cohomology が, homotopy invariant.

Def: $X \in Sm/k$

$\mathbb{Z}_{tr}(X)$: representable presheaf by X on $SmCor(k)$

$$Y \mapsto \hom_{SmCor(k)}(Y, X) \overset{\text{df}}{=}: \mathbb{Z}_{tr}(X)(Y)$$

Def: (Suslin complex)

F: preshf $C_*(F)$: Suslin complex

$\Delta^n := \operatorname{Spec} k[T_0, \dots, T_n] / \left(\sum_{i=0}^{n} T_i - 1\right)$ $\quad C_n(F) := F(\Delta^n \times -)$

associated cpx.

$\longrightarrow C_*(F)$: has homotopy invariant cohomologies

$$\begin{array}{c} D^-(Shv_{Nis}(SmCor(k))) \\ \cup \\ DM_-^{eff}(k) \end{array}$$

RC_*

각 H, Nisnevich shf.

Def: $C_*(\mathbb{Z}_{tr}(X)) \in DM_-^{eff}(k)$ eff.

$\quad \overset{df}{\parallel}$

$M(X)$ motive of X

tensor str: $\mathbb{Z}_{tr}(X) \otimes \mathbb{Z}_{tr}(Y) := \mathbb{Z}_{tr}(X \times Y)$

F: preshf with transfer.

take res. of F, G by "$\mathbb{Z}_{tr}(X)$" $\rightsquigarrow F \otimes G$

$\rightsquigarrow$

$DM_-^{eff}(k) \ni M, M'$

$M \otimes M' \overset{dfn}{:=} RC_*(M \overset{\mathbb{L}}{\otimes} M')$ $\qquad M(X) \otimes M(Y) \simeq M(X \times Y)$

internal hom: F, G: preshf with transfer.

$\underline{Hom}(F, G)(X) :\underset{df}{=} Hom(F \otimes \mathbb{Z}_{tr}(X), G)$

$\Rightarrow \quad Hom(F, \underline{Hom}(G, H)) \simeq Hom(F \otimes G, H)$

$$\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n}) := \operatorname{coker}\left(\bigoplus_{i=0}^{n-1} \mathbb{Z}_{tr}(\mathbb{G}_m^{\times n-1}) \to \mathbb{Z}_{tr}(\mathbb{G}_m^{\times n})\right)$$

$$\mathbb{Z}(n) \overset{\text{dfn}}{:=} C_*\left(\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n})\right)[-n]$$

$$\begin{array}{ccc} Sm/k & \to & DM_-^{eff}(k) \\ \Psi & & \Psi \\ X & \mapsto & M(X) \end{array}$$

an algebraic ~~corr~~

char $k=0$

$$M(X)(\operatorname{Spec} k) = \left\langle \begin{tikzcd} Z \subseteq \Delta^{\cdot} \times X \arrow[dr, "\text{fini. surj.}"'] \arrow[d] \\ \Delta^{\cdot} \end{tikzcd} \right\rangle_{\mathbb{Z}}$$

a

$$\simeq \hom\left(\Delta^{\cdot}, \coprod_{d=0}^{\infty} \operatorname{Sym}^d(X)\right)^+ \nwarrow \text{gr. completion.}$$

$$\operatorname{Hom}_{\substack{\text{conti.}\\ \text{map.}}}\left(\Delta^{\cdot}_{top}, \coprod_{d=0}^{\infty} \operatorname{Sym}^d(X(\mathbb{C}))\right)^+$$

$$\xleftarrow[\text{q-isom}]{} \mathbb{Z}\left(\operatorname{Hom}_{\substack{\text{conti.}\\ \text{map}}}\left(\Delta^{\cdot}_{top}, X(\mathbb{C})\right)\right)$$