cat. of mixed Tate motive, $MT(\mathbb{Z})$ 12/21. Yamashita. G

MZV's $\longleftrightarrow$ $\pi_1^{\mathcal{M}}(\mathbb{P}^1 - \{0,1,\infty\}, \vec{01})$: motivic fundamental group

multiple zeta value.

$k_1, \cdots, k_d \geq 1$

$$\zeta(k_1, \cdots, k_d) \overset{\text{dfn}}{:=} \sum_{n_1 < \cdots < n_d} \frac{1}{n_1^{k_1} \cdots \cdots n_d^{k_d}} \in \mathbb{R}$$

converge $\Leftrightarrow$ $k_d \geq 2$.

$$\zeta(3) = \zeta(1,2) \quad \text{(Euler)}$$

$$\zeta(4) = \zeta(1,1,2) = 4\zeta(1,3) = \frac{4}{3}\zeta(2,2) = \frac{\pi^4}{90} \quad \text{(Euler)}$$

d : depth $\quad k_1 + \cdots + k_d$: weight

$$\underset{n>0}{Z_n} := \left\langle \zeta(k_1, \cdots, k_d) \;\middle|\; \begin{array}{c} k_1 + \cdots + k_d = n \\ k_d \geq 2 \end{array} \right\rangle \subseteq \mathbb{R}.$$

$n > 0$ $\quad Z_0 := \mathbb{Q}.\ , \ Z_1 = 0, \quad Z_2 = \zeta(2)\mathbb{Q} = \pi^2\mathbb{Q}.$

$$Z_3 = \zeta(3)\mathbb{Q} + \zeta(1,2)\mathbb{Q} = \zeta(3)\mathbb{Q}$$

$$Z_4 = \zeta(4)\mathbb{Q} + \zeta(1,1,2)\mathbb{Q} + \zeta(1,3)\mathbb{Q}$$

$$+ \zeta(2,2)\mathbb{Q} = \pi^4\mathbb{Q}.$$

$Z_5 = \langle \zeta(5), \pi^2\zeta(3) \rangle_{\mathbb{Q}}$ $\quad \dim \leq 2$ $\quad k$: field

$\vdots$

Z_{10} $\quad$ 256 induce $\quad \dim \leq 7$ $\quad$ Conj (Beilinson–Soulé vanishing Conj.)

$\vdots$

Z_{20} $\quad$ 262144 $\quad \dim \leq 114$

$K_{2q-p}(k)_{\mathbb{Q}}^{(q)} = 0$ for $p < 0$

strong version $K_{2q-p}(k)^{(q)}_{\mathbb{Q}} = 0$ for $p \le 0$, $q > 0$

cf: $H^p_{\mathcal{M}}(k, \mathbb{Q}(q)) \simeq K_{2q-p}(k)^{(q)}_{\mathbb{Q}}$

$DM_{gm}(k) \otimes \mathbb{Q} \supseteq DMT(k)$: full sub triangulated cat. generated by $\mathbb{Q}(n)$ $n \in \mathbb{Z}$

"weight str."

$$W_{n-1} \to W_n \to Gr^W_n \to W_{n-1}[1]$$

Def: $X \in DMT(k)^{\ge 0} \underset{\text{def.}}{\Longleftrightarrow} gr^W_a(X) \simeq \bigoplus_{n \le 0} \mathbb{Q}(-\frac{a}{2})^{m_n}[n]$ $\forall a$

$X \in DMT(k)^{\le 0}$

$\underset{\text{def.}}{\Longleftrightarrow} gr^W_a(X) \simeq \bigoplus_{n \ge 0} \mathbb{Q}(-\frac{a}{2})^{m_n}[n]$ $\forall a$

Thm (Levine) Assume strong version of BS vanishing holds for k

①: $\{DMT(k)^{\ge 0}, DMT(k)^{\le 0}\}$: t-structure.

($\Rightarrow$ Def: $MT(k) := DMT(k)^{\ge 0} \cap DMT(k)^{\le 0}$: abelian cat.)

②:

$$\operatorname{Ext}^p_{MT(k)}(M, N) \to \hom_{DMT(k)}(M, N[p])$$

is isom. for $p=1$

inj for $p=2$

k : number ~~fd.~~ field $\Rightarrow$ holds $\forall p$.

In the following, assume k: number field.

$$\mathrm{Ext}^1_{MT(k)}(\mathbb{Q}(0),\mathbb{Q}(n)) \simeq K_{2n-1}(k)^{(n)}_{\mathbb{Q}}$$

$$\dim_{\mathbb{Q}} = \begin{cases} \infty & n=1 \\ r_2 & n = \text{even} \\ r_1+r_2 & n:\text{odd} \neq 1 \end{cases}$$

want to construct $MT(\mathcal{O}_S)$ $\quad \mathcal{O}_S$: ring of S-integer

$$\mathrm{Ext}^1_{MT(\mathcal{O}_S)}(\mathbb{Q}(0),\mathbb{Q}(n)) \simeq K_{2n-1}(\mathcal{O}_S)^{(n)}_{\mathbb{Q}}$$

Deligne-Goncharov constructed $\underset{MT(k)}{\overset{MT(\mathcal{O}_S)}{\cap}}$ not geometrically

$k=\mathbb{Q}$, $\mathcal{O}_S=\mathbb{Z}$ $\quad MT(\mathbb{Z})$

$$\dim_{\mathbb{Q}} \mathrm{Ext}^1_{MT(\mathbb{Z})}(\mathbb{Q}(0),\mathbb{Q}(n)) = \begin{cases} 0 & n:\text{even or } 1 \\ 1 & n:\text{odd} \neq 1 \end{cases}$$

$$\omega = \bigoplus_n \omega_n$$

$$\begin{array}{ccc} \omega: MT(\mathbb{Z}) & \longrightarrow & \mathrm{Vect}_{\mathbb{Q}} \\ \cup & & \cup \\ M & \longmapsto & \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}(\mathbb{Q}(n), \mathrm{gr}^W_{2n} M) \end{array}$$

theory of Tannakian cat. $\rightsquigarrow$ G_ω: proalg. group $= \mathrm{Aut}^{\otimes}(\omega)$

Rem):

$$\begin{array}{ccc} \mathrm{Rep}\, G_\omega & \overset{\sim}{\leftarrow} & MT(\mathbb{Z}) \\ \cup & & \cup \\ \omega(M) & \longleftrightarrow & M \end{array}$$

Deligne:

$G \in \text{Pro-}MT(\mathbb{Z})$: gp scheme obj. s.t. $\forall F$: fibre functor $F(G) = \mathrm{Aut}^{\otimes}(F)$

$$\begin{array}{ccc} MT(\mathbb{Z}) & \xrightarrow{F} & \mathrm{Vect}_{\mathbb{Q}} \\ \cup & & \cup \\ G & \longmapsto & G_F = \mathrm{Aut}^{\otimes}(F) \end{array}$$

Thm: (Deligne–Goncharov)

$\exists\ \pi_1^{\mathcal{M}}(\mathbb{P}^1_{\mathbb{Q}}-\{0,1,\infty\},\ \overrightarrow{01}) \in$ pro-$\mathcal{M}T(\mathbb{Z})$ ← pro-unipotent group

s.t. Hodge real. $= \pi_1^{\mathrm{Hodge}}(\mathbb{P}^1_{\mathbb{Q}}-\{0,1,\infty\},\ \overrightarrow{01})$

also ℓ-adic.

$G_\omega \curvearrowright \omega(\mathbb{Q}(1))$

$G_\omega \xrightarrow{\tau} \mathbb{G}_m$ ker $=: U_\omega$

splitting

$\mathbb{G}_m \curvearrowright \omega_n$

$\lambda \quad \lambda^n$

$G_\omega = \mathbb{G}_m \ltimes U_\omega$.

$U_\omega \curvearrowright \omega(\mathbb{Q}(n))$ triv.

↑ pro-Unipotent

$\left(\pi_1^{B}(\mathbb{P}^1-\{0,1,\infty\},\ \overrightarrow{01})\right)$

$MT(\mathbb{Z}) \underset{dR}{\overset{B}{\rightrightarrows}} \mathrm{Vect}_{\mathbb{Q}}$

$B\otimes\mathbb{C} \simeq dR\otimes\mathbb{C}$

$\cancel{\mathbb{G}_m}$ $\quad G_\omega(\mathbb{C}) \overset{\exists}{\ni} a$ s.t.

$M_B\otimes\mathbb{C} \simeq M_{dR}\otimes\mathbb{C}$

$a(M_{dR}) = M_B$ for $\forall M$

$$\begin{array}{ccc} M_B\otimes\mathbb{C} & \simeq & M_{dR}\otimes\mathbb{C} \\ \cup\!| & & \cup\!| \\ M_B & & M_{dR} \end{array}$$

$$\begin{array}{ccc} \mathbb{Q}(1)_B\otimes\mathbb{C} & \simeq & \mathbb{Q}(1)_{dR}\otimes\mathbb{C} \\ \cup & & \cup\!| \\ \mathbb{Q} & & \mathbb{Q} \\ 1 & \longmapsto & \overset{\psi}{2\pi i} \end{array}$$

$a = a^\circ\,\tau(2\pi i)$

$a^\circ \in U_a(\mathbb{C})$

$$\pi_1^{B}(\mathbb{P}^1-\{0,1,\infty\}, \overrightarrow{01}, \overrightarrow{10})(\mathbb{C}) \cong \pi_1^{dR}(\text{---})(\mathbb{C})$$

$$\begin{array}{ccc} \Psi & & \Psi \quad \wr\!\wr \\ (0 \to 1) & \longmapsto & \Phi_{dR} \in \mathbb{C}\langle\langle e_0, e_1\rangle\rangle \end{array}$$

Drinfeld's associater

Φ_{dR} coeffi. of

$$e_0^{k_d-1} e_1 \cdots e_0^{k_1-1} e_1$$

is $(-1)^d \zeta(k_1, \cdots, k_d)$

$$H_w := \text{"}\mathrm{Aut}\{\pi_1^w(\mathbb{P}^1-\{0,1,\infty\}, \overrightarrow{0P}\, a, b)\}_{a,b}\text{"}$$

$a, b \in \{\overrightarrow{01}, \overrightarrow{10}, \overrightarrow{0\infty}, \overrightarrow{\infty 0}, \overrightarrow{1\infty}, \overrightarrow{\infty 1}\}$

$\| $

$\mathbb{G}_m \ltimes V_w$ ← prounipotent $/\mathbb{Q}$

$G_w \curvearrowright \{\pi_1^w(\cdots, a, b)\}_{a,b \in \cdots}$

factored through

$$\begin{tikzcd} G_w \arrow[r] & H_w \\ \mathbb{G}_m \arrow[u, "\tau"] \arrow[ur, "\tau"'] & \end{tikzcd}$$

$U_w \longrightarrow V_w$

small enough, too big.

$V_w(\mathbb{C}) \xrightarrow{\sim} \mathbb{C}\langle\langle e_0, e_1\rangle\rangle$

Prop $\langle \Phi_{DR}\rangle \tau(2\pi i) \in H_w(\mathbb{C})$ sends

$\pi_1^w(\mathbb{P}^1-\{0,1,\infty\}, \overrightarrow{01})$, $\pi_1^w(\mathbb{P}^1-\{0,1,\infty\}, \overrightarrow{01}, \overrightarrow{10})$

to $\pi_1^B(\text{---}, \overrightarrow{01})$, $\pi_1^B(\text{---}, \overrightarrow{01}, \overrightarrow{10})$

$\Rightarrow \exists\, v \in H(\mathcal{O})$ s.t. $\langle \Phi_{DR}\rangle \tau(2\pi i) = \iota(a) v$

$$\Rightarrow \langle \Phi_{DR} \rangle = \iota(a^0)\cdot\left(\tau(2\pi i)\cdot w\tau(2\pi i)^{-1}\right)$$

$$\begin{array}{ccc} D & & t \longmapsto \tau(t)\, v\tau(t^{-1}) \\ \wr\wr & & \\ (\mathbb{A}^1) & & \langle \Phi_{DR} \rangle \in \left(\iota(v_w)\times D\right)(\mathbb{C}) \end{array}$$

$\iota(v_w)\times D = \mathrm{Spec}\, A$ $\quad$ A: graded $A = \bigoplus_n A_n$.

$$\langle \Phi_{DR} \rangle \in (\mathrm{Spec} A)(\mathbb{C})$$

$$\Downarrow$$

$$\varphi : A \to \mathbb{C} \qquad \varphi(t') = \pi^2$$

$$Z_n \subseteq \varphi(A) \qquad \sum_{n=0}^{\infty} (\dim A_n) t^n$$

$$\sum_{n=0}^{\infty} (\dim A_n)\, t^n = \frac{1}{1-t^2}\cdot\frac{1}{1-(t^3+t^5+t^7+\cdots)}$$

$$\dim K_{2n-1}(\mathbb{Z})_{\mathbb{Q}} = \begin{cases} 0 & n:\text{even or } n=1 \\ 1 & n:\text{odd} \neq 1. \end{cases}$$

$$= \frac{1}{1-t^2}\,\frac{1}{1-\frac{t^3}{1-t^2}} = \frac{1}{1-t^2-t^3} = \sum_{n=0}^{\infty} D_n t^n.$$

$$\Rightarrow \begin{cases} D_0=1,\ D_1=0,\ D_2=1. \\ D_{n+3} = D_{n+1}+D_n \end{cases} \xRightarrow{\text{Thm}} \left(\begin{array}{c}\text{Goncharov,} \\ \text{Deligne,} \\ \text{Terasoma.}\end{array}\right)$$

$$\dim_{\mathbb{Q}} Z_n \leq D_n \quad (\forall n)$$

Conj: (Grothendieck)

$a^0 \in U_n(\mathbb{C})$

$\mathbb{Q}$-Zariski dense.

Zagier: $\dim_{\mathbb{Q}} Z_n = D_n$.