[Batyrev] Birational CY mfds have same betti number.

12/20 Yasuda. T

(pf): p-adic integer + Weil Conj.

Kontsevich introduces the motivic integration as a complex analogue of p-adic units.
developped by Denef & Loeser & others

§. 1, Grothendieck ring of varieties $Y \underset{\text{closed}}{\subset} X$

/ k : field

$e_{top}(X) = e_{top}(X \setminus Y) + e_{top}(Y)$

- $e_{top}(X)$
- $e_{top}(X \times Y) = e_{top}(X)\, e_{top}(Y)$

- $\#X(\mathbb{F}_q) = \#(X \setminus Y)(\mathbb{F}_q) + \#Y(\mathbb{F}_q)$
- $\#(X \times Y)(\mathbb{F}_q) = \#X(\mathbb{F}_q) \cdot \#Y(\mathbb{F}_q)$

Some properties for other invariants

$\downarrow$

the universal one

$[\cdot] \in K_0(Var)$

$X = \coprod X_i$: stratification by locally closed subvarieties

$\Rightarrow [X] = \sum [X_i] \in K_0(Var)$

$X \supseteq C = \coprod_i C_i$, $C_i \subseteq X$: loc. closed

C: constructible subset

$\rightsquigarrow [C] \in K_0(Var)$

$\quad\quad \| $

$\sum_i [C_i]$: well-defined.

§. 2. Motivic Measure.

X : variety / k

$\mathcal{C}(X) := \{ \text{constructible subsets of } X \}$

$\nu_X : \mathcal{C}(X) \longrightarrow K_0(Var)$

$\quad C \longmapsto [C]$

$C = \coprod C_i$

$\nu_X(C) = \sum \nu_X(C_i)$

ν_X is a "measure" in a broad sense.

$F : X \to K_0(\mathrm{Var})$: constructible function.

$\left(\overset{\text{def}}{\Longleftrightarrow} \right.$ every fibre $F^{-1}(a)$ が constructible. $\left. \right)$

$\# F(X) < \infty$ 有限和.

$$\rightsquigarrow \int F \, d\nu_X = \sum_{a \in K_0(\mathrm{Var})} \nu_X(F^{-1}(a)) \cdot a \in K_0(\mathrm{Var}_k)$$

<u>Ex</u>: X : smooth var.

$\circlearrowleft$

Y : smooth closed subvariety of codim $= r$.

$$F(x) := \begin{cases} 1 = [1pt] & (x \in X \setminus Y) \\ [\mathbb{P}^{r-1}] & (x \in Y) \end{cases} \qquad \int F \, d\nu_X$$

$$\int F \, d\nu_X = [X \setminus Y] + [\mathbb{P}^{r-1}][Y] = [\mathrm{Bl}_X Y]$$

§3. Jet & Arcs

X^d : smooth var $/ k = \bar{k}$

Zariski Tangent vector.

$$\mathrm{Spec}(k[t]/t^2) \to X$$

$TX = \{\text{tangent vectors}\}$

$\uparrow$

tangent bundle.

$TX \to X$: projection.

v.b. of rank $= d$.

$n \in \mathbb{Z}_{\geq 0}$ $\boxed{n\text{-jet on } X}$: $\mathrm{Spec} \dfrac{k[t]}{t^{n+1}} \to X$

$J_n X := \{n\text{-jets on } X\}$

$\uparrow$

smooth scheme.

$m \geq n$

$$\begin{tikzcd} \mathrm{Spec} \dfrac{k[t]}{t^{n+1}} \arrow[r, hook] \arrow[dr, "n\text{-jet}"'] & \mathrm{Spec} \dfrac{k[t]}{t^{m+1}} \arrow[d, "m\text{-jet}"] \\ & X \end{tikzcd}$$

$J_m X \to J_n X$ が存在する.

$\ni \pi^m_n$

Fact:

π_n^{n+1} is a Zariski Local-trivial $\mathbb{A}^d$-fibration

arc on X : $\operatorname{Spec} k[[t]] \to X$

$$J_\infty X \underset{\text{dfn}}{:=} \{\text{arcs on } X\}$$

$$\begin{tikzcd}
\operatorname{Spec} \frac{k[[t]]}{t^{n+1}} \arrow[r] \arrow[dr, "n\text{-jet}"'] & \operatorname{Spec} k[[t]] \arrow[d, "\text{arc}"] \\
 & X
\end{tikzcd}$$

$$J_0X \leftarrow J_1X \leftarrow J_2X \leftarrow \cdots \leftarrow J_\infty X$$

$J_\infty X = \varprojlim_n J_nX$ ← この空間の上で積分したい。

$$\begin{tikzcd}
\mathcal{C}(J_0X) \arrow[r, "(\pi_0^1)^{-1}"] \arrow[dr, "\nu_{J_0X}"'] & \mathcal{C}(J_1X) \arrow[r, "(\pi_1^2)^{-1}"] \arrow[d, "\mathbb{L}^{-d}\nu_{J_1X}"] & \mathcal{C}(J_2X) \cdots \arrow[dl, "\mathbb{L}^{-2d}\nu_{J_2X}"] \\
 & K_0(Var) &
\end{tikzcd}$$

$\mathbb{L} := [\mathbb{A}^1]$

Motivic Measure on $J_\infty X$ $\underset{\text{roughly}}{:=} \lim_{n\to\infty} \mathbb{L}^{-nd}\nu_{J_nX}$

$\mathcal{M} := K_0(Var)[\mathbb{L}^{-1}]$ ⇝ $\hat{\mathcal{M}}$: dimensional completion

$F_m \subseteq \mathcal{M}$: subgroup generated by $[V]\mathbb{L}^i$ with $\dim V + i \le -m$. $i \in \mathbb{Z}$

↑ not ideal

$\{F_m\}_{m\in\mathbb{N}}$: descending filtration of $\mathcal{M}$

ring. ↗ $\hat{\mathcal{M}} = \varprojlim \mathcal{M}/F_m$

Ex: $\sum_{i=0}^{\infty} [V_i]\mathbb{L}^{a_i}$ $\quad \dim V_i + a_i \to -\infty$ $(i \to \infty)$

$\in \hat{\mathcal{M}}$ well-defined

motivic measure μ_X on $J_\infty X$

take values in $\hat{\mathcal{M}}$ $\quad F : J_\infty X \to \hat{\mathcal{M}}$ "measurable" fcn.

Rem: $\hat{\mathcal{M}} \to \hat{K}_0(HS)$ ⇝ $\int F d\mu_X \in \hat{\mathcal{M}} \cup \{\infty\}$

↑ weight completion

ex: $F \equiv 1 = [1pt]$

$$\int 1 \, d\mu_X = \mu_X(J_\infty X) = [X]$$

Birational Morphism & Transformation rule.

$$\begin{array}{ccc} f: Y & \to & X \\ \cup & & \cup \\ E & \to & D \\ & & \shortparallel \\ & & f(E) \end{array}$$

exc. sec.

: proper birational

一般に, $f: Y \to X$: morphism $\rightsquigarrow$ $0 \le n \le \infty$

$f_n : J_n Y \to J_n X$

: natural morphism

Fact ①

can neglect

$$\begin{cases} J_\infty(E) \subseteq J_\infty(Y) \\ J_\infty(D) \subseteq J_\infty(X) \end{cases}$$

infinite codimensional of measure zero.

②: valuative criterion

$$f_\infty : J_\infty Y \setminus J_\infty E \xleftrightarrow{1:1} J_\infty X \setminus J_\infty D$$

Th: (Transformation rule) Change of variables formula)

$$\int F \, d\mu_X = \int (F \circ f_\infty) \underline{\mathbb{L}^{-\mathrm{ord} J_f}} \, d\mu_Y \in \hat{\mathcal{M}} \cup \{\infty\}$$

$J_f \subseteq \mathcal{O}_X$: Jacobian ideal of f.

X & Y smooth, J_f is the ideal of $K_{Y/X}$ $= K_Y - f^* K_X$

relative can. div.

discrepancy div.

Cor: X, Y, Z smooth projective $/\mathbb{C}$

$$\begin{array}{ccc} & Z & \\ {}^{f}\swarrow & & \searrow^{g} \\ X & & Y \end{array}$$

f, g : birational

$K_{Z/X} = K_{Z/Y}$ (← OK, where X, Y : CY.)

$\Rightarrow$ $e_h^{p,q}(X) = e_h^{p,q}(Y)$

(prf): $[X] = \int 1 \, d\mu_X = \int \mathbb{L}^{-\mathrm{ord} J_f} d\mu_Z = \int 1 \, d\mu_Y = [Y]$ #.

Problem:

- $\exists$ natural isom. $H^i(X) \xrightarrow{\sim} H^i(\Upsilon)$?
- Construct a "factorial" motivic integration!