

[Batyrev] Birational C-ymfds have same. betti number

(pf): p-adic intoger + Weil Conj.

12/20 Yasuda.T

Kontsevich introduces the motivic integration as a complex analogue. of p-adic units.

developped by Denef & Loeser & others

§. 1, Grothendieck ring of varieties  $\underline{C}X$  closed.

$\mathbb{R}$ : field  $\equiv e_{top}(X \setminus I) + e_{top}(I)$

- $e_{top}(X)$
- $e_{top}(X \times I) = e_{top}(X) e_{top}(I)$

$\# X(\mathbb{F}_q) = \#(X \setminus I)(\mathbb{F}_q) + \# I(\mathbb{F}_q)$

$\#(X \times I)(\mathbb{F}_q) = \# X(\mathbb{F}_q) \cdot \# I(\mathbb{F}_q)$

Some properties for other invariants  $\downarrow$  the universal one.

$X = \bigsqcup X_i$ : stratification by locally closed subvarieties

$[\cdot] \in K_0(\text{Var})$

$\Rightarrow [X] = \sum [X_i] \in K_0(\text{Var})$

$X \supseteq C = \bigsqcup_i C_i, C_i \subseteq X$ : loc. closed.  $\rightsquigarrow [C] \in K_0(\text{Var})$

§. 2. Motivic Measure.

$\sum [C_i]$ : well-defind.

$X$ : variety  $\mathbb{R}$

$C = \bigsqcup C_i$

$\mathcal{C}(X) := \{ \text{constructible subsets of } X \}$   $\nu_X(C) = \sum_i \nu_X(C_i)$

$\nu_X: \mathcal{C}(X) \rightarrow K_0(\text{Var})$

$\nu_X$  is a "measure" in a broad sense.

$C \mapsto [C]$

$F: X \rightarrow k_0(\text{Var})$ : constructible function.

( $\stackrel{\text{def}}{=} \Rightarrow$ ) every fibre  $F^{-1}(a)$  is constructible.  
 $\# F(X) < \infty$  有限.

$$\implies \int F d\nu_X = \sum_{a \in k_0(\text{Var})} \nu_X(F^{-1}(a)) \cdot a \in k_0(\text{Var}_e)$$

Ex:  $X$ : smooth var.

$\uparrow$   
 $Y$ : smooth closed subvariety of  $\text{codim} = r$ .

$$F(x) := \begin{cases} 1 = [1 \text{ pt}] & (x \in X \setminus Y) \\ [\mathbb{P}^r] & (x \in Y) \end{cases} \int F d\nu_X$$

$$\int F d\nu_X = [X \setminus Y] + [\mathbb{P}^r][Y] = [B_{\mathbb{P}^r} \times Y]$$

### § 3. Jet & Arcs

$X^d$ : smooth var /  $\mathbb{P} = \mathbb{R}$

Zariski Tangent vector.

$$\text{Spec} \left( \frac{\mathbb{R}[T]}{T^2} \right) \rightarrow X$$

$TX = \{ \text{tangent vectors} \}$   
 $\uparrow$   
 tangent bundle.

$TX \rightarrow X$ : projection.  
 v.b. of rank = d.

$n \in \mathbb{Z} \geq 0$  m-jet on X:  $\text{Spec} \frac{\mathbb{R}[T]}{T^{n+1}} \rightarrow X$

$J_n X := \{ n\text{-jets on } X \}$   
 $\uparrow$   
 smooth scheme.

$$\text{Spec} \frac{\mathbb{R}[T]}{T^{m+1}} \hookrightarrow \text{Spec} \frac{\mathbb{R}[T]}{T^{n+1}} \rightarrow X$$

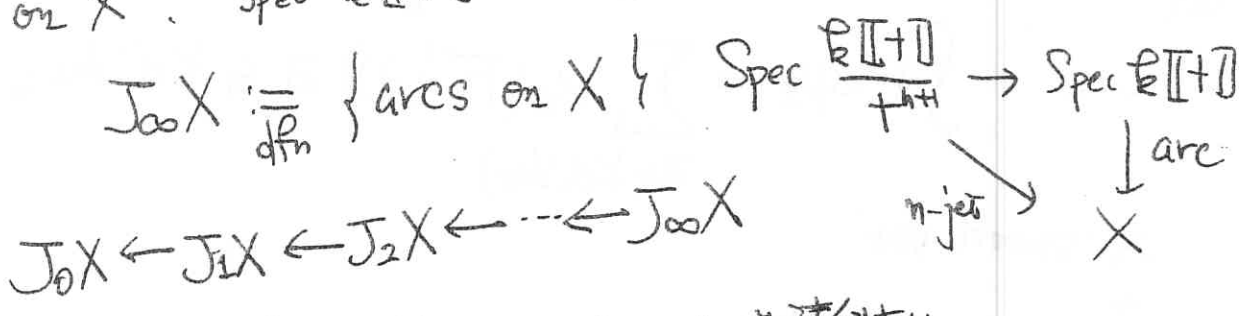
$\downarrow$  m-jet

$J_m X \rightarrow J_n X$   $\exists \pi_n^m$

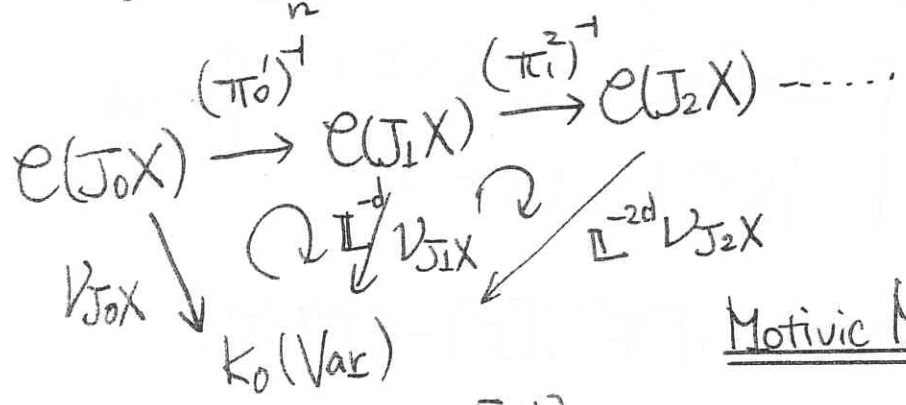
n-jet  $\searrow$

Fact:  $\pi_n^{n+1}$  is a Zariski Local trivial  $\mathbb{A}^1$ -fibration

are on  $X$ :  $\text{Spec } \mathbb{C}[[t]] \rightarrow X$



$J_{\infty} X = \varinjlim_n J_n X$  ← この空間の上で積分したい。



Motivic Measure on  $J_{\infty} X$

$\mathbb{L} := [\mathbb{A}^1]$

$:= \lim_{\text{roughly } n \rightarrow \infty} \mathbb{L}^{-nd} \nu_{J_n X}$

$\mathcal{M} := K_0(\text{Var})[\mathbb{L}^{-1}]$

$F_m \subseteq \mathcal{M}$ : subgroup generated by  
 $[V][\mathbb{L}^i]$  with  
 $i \in \mathbb{Z}$   $\dim V + i \leq -m$ .  
 $\uparrow$   
 not ideal.

$\hat{\mathcal{M}}$ : dimensional completion

$\{F_m\}_{m \in \mathbb{N}}$ : descending filtration of  $\mathcal{M}$

$\hat{\mathcal{M}} = \varprojlim_{F_m} \mathcal{M} / F_m$

Ex:  $\sum_{i=0}^{\infty} [V_i][\mathbb{L}^{a_i}]$   $\dim V_i + a_i \rightarrow -\infty$   
 $\uparrow$   
 well-defined.

motivic measure  $\mu_X$  on  $J_{\infty} X$

take values in  $\hat{\mathcal{M}}$   $F: J_{\infty} X \rightarrow \hat{\mathcal{M}}$  "measurable" fcn.

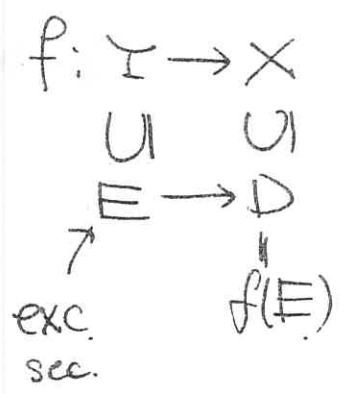
Rem:  $\hat{\mathcal{M}} \rightarrow \hat{K}_0(\text{HS}) \rightsquigarrow \int F d\mu_X \in \hat{\mathcal{M}} \cup \{\infty\}$

$\uparrow$   
weight completion

ex:  $F \equiv 1 = [1 \text{ pt}]$

$\int 1 d\mu_X = \mu_X(\text{Jac } X) = [X]$

Birational Morphism & Transformation rule.



proper birational

一般  $f: Y \rightarrow X$  : morphism  $\rightsquigarrow 0 \leq n \leq \infty$

$f_n: J_n Y \rightarrow J_n X$

: natural morphism

Fact ①  $\swarrow$  can neglect

$J_n(E) \subseteq J_n(Y)$

$J_n(D) \subseteq J_n(X)$

infinite codimensional of measure zero.

②: valuation criterion

$f_n: J_n Y | J_n E$

$\longleftrightarrow J_n X | J_n D$

1:1 formula

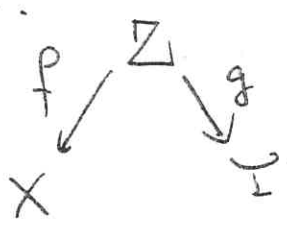
Th: (Transformation rule) Change of variables

$\int F d\mu_X = \int (F \circ f_n) \mathbb{L}^{-\text{ord } J_n f} d\mu_Y \in \hat{M}U(\infty)$

$J_n f \subseteq \mathcal{O}_X$  : Jacobian ideal of  $f$

$\uparrow$   $X, Y$  smooth,  $J_n f$  is the ideal of  $K_X/Y$

Cor:  $X, Y, Z$  smooth projective  $\mathbb{C}$



$f, g$  : birational

$K_{Z/X} = K_{Z/Y}$

$\cong K_Y - f^* K_X$

relative can. div. discrepancy div.

OK, where  $X, Y: \mathbb{C}Y$

$\Rightarrow R^{p,q}(X) = R^{p,q}(Y)$

(pf):  $[X] = \int 1 d\mu_X = \int \mathbb{L}^{-\text{ord } J_n f} d\mu_Z = \int 1 d\mu_Y = [Y] \neq$

Problem:  $\exists$  natural  
isom.

$$H^i(X) \xrightarrow{\sim} H^i(\mathbb{A}^1)?$$

No 5

- Construct a "factorial" motivic integration!