山下剛

$$B\mu_\ell \rightarrowtail \mathcal{O}(-\ell)_{\mathbb{P}^\infty} \twoheadrightarrow Th_{\mathbb{P}^\infty}(\mathcal{O}(-\ell))$$

$\rightsquigarrow$ $\mathbb{Z}/\ell$ - coef

$$0 \to \tilde{H}^{*,*}(k)[\![\sigma]\!] \to H^{*,*}(B\mu_\ell) \to (\tilde{H}(k)[\![\sigma]\!])^{*-1,*-1} \to 0$$

$$\sigma \longmapsto v$$

$$\sigma \in [\mathcal{O}(-1)] \in H^{2,1}(\mathbb{P}^\infty)$$

$v := c(\text{taut. line bundle on } B\mu_\ell) \in H^{2d}$

Lem $\quad H^{1,1}(B\mu_\ell, \mathbb{Z}/\ell) \xrightarrow{\delta} H^{2,1}(B\mu_\ell, \mathbb{Z})$

$\exists!\, u \mapsto v \qquad u|_* = 0$

Prop $\quad v^i, uv^i \ (i \geq 0)$ forms a basis of $\tilde{H}^{*,*}(F_\cdot \wedge (B\mu_\ell)_+, \mathbb{Z}/\ell)$ over $\tilde{H}^{*,*}(F_\cdot, \mathbb{Z}/\ell)$ $\square$

$u^2 = ? \qquad \ell > 2 \Rightarrow$ graded commutativity $\quad u^2 = 0$

$\ell = 2$ $\quad H^{2,2}(B\mu_\ell, \mathbb{Z}/2) = H^{0,1}(k, \mathbb{Z}/2)v \oplus H^{1,1}(k, \mathbb{Z}/2)u \oplus H^{2,2}(k, \mathbb{Z}/2)$

$(2,1) \qquad (1,1) \qquad u|_* = 0$

$$u^2 = xv + yu + 0$$

Fact $\quad x = \tau \in H^{0,1} \simeq \mu_2 \qquad y = \rho \in H^{1,1} \simeq k^\times/(k^\times)^2$

Prop $\quad \tilde{H}^{*,*}(F_\cdot \wedge (B\mu_\ell)_+, \mathbb{Z}/\ell)$

$$\simeq \begin{cases} \tilde{H}^{*,*}(F_\cdot, \mathbb{Z}/\ell)[\![u, v]\!]/(u^2) & \ell > 2 \\ \tilde{H}^{*,*}(F_\cdot, \mathbb{Z}/\ell)[\![u, v]\!]/(u^2 - \tau v - \rho u) \end{cases}$$

$\square$

$\rightsquigarrow B\mathfrak{S}_\ell \qquad \xi_\ell$: taut. vect. bundle on $B\mathfrak{S}_\ell$

$(\mathfrak{S}_\ell \xrightarrow{id} \mathfrak{S}_\ell)$

$d := c(\xi_\ell/\mathcal{O}) \in H^{2(\ell-1),\ell-1}(B\mathfrak{S}_\ell, \mathbb{Z}/\ell)$

Thm $\exists! \; c \in H^{2\ell-3,\ell-1}(B\mathfrak{S}_\ell) \quad \delta(c) = d, \quad c|_* = 0$ □

graded commutativity

$\ell > 2 \Rightarrow c^2 = 0$

$\ell = 2 \Rightarrow B\mu_2 = B\mathfrak{S}_2$

$\left(k \ni \zeta \qquad B\mu_\ell \xrightarrow[p_\xi]{} B\mathfrak{S}_\ell \qquad p_\xi^*(d) = \prod_{i=1}^{\ell-1}(iv) = -v^{\ell-1}, \quad p_\xi^*(c) = -uv^{\ell-2} \right)$

$\ell > 2$

$$\tilde{H}(F_\cdot)[\![c,d]\!]/(c^2) \longrightarrow \tilde{H}^{*,*}(F_\cdot \wedge B\mathfrak{S}_\ell)_+, \mathbb{Z}/\ell)$$

$$\xhookrightarrow{p_\xi^*} \tilde{H}^{*,*}(F_\cdot \wedge (B\mu_\ell)_+, \mathbb{Z}/\ell)^{Aut(\mu_\ell)}$$

$$\simeq \tilde{H}^{*,*}(F_\cdot)[\![x,y]\!]/(x^2)$$

$$x = uv^{\ell-2}, \quad y = v^{\ell-1}$$

Thm $\tilde{H}^{*,*}(F_\cdot \wedge (B\mathfrak{S}_\ell)_+, \mathbb{Z}/\ell)$

$$\begin{cases} \tilde{H}^{*,*}(F_\cdot, \mathbb{Z}/\ell)[\![c,d]\!]/(c^2) & \ell > 2 \\ \tilde{H}^{*,*}(F_\cdot, \mathbb{Z}/\ell)[\![c,d]\!]/(c^2 - \tau d - \rho c) & \ell = 2 \end{cases}$$

$\deg c = (2\ell-3, \ell-1)$

$\deg d = (2\ell-2, \ell-1)$

□

total power operation

$$K_{n,R} = K(\mathbb{Z}(n)\otimes R, 2n) = R\otimes \mathbb{Z}_{tr}((\mathbb{P}^1,\infty)^{\wedge n})$$
$$= R\otimes \mathbb{Z}_{tr}(\mathbb{A}^n/(\mathbb{A}^n-\{0\}))$$

construction ①

$X \in Sm/k$, R: commutative ring

$$\left[\begin{array}{l} \text{input } E:\text{ vect. bundle} \\ \text{cycle } \begin{tikzcd} Z \arrow[dr, "\text{fin. equidim}"'] & \subset E \arrow[d] \\ & X \end{tikzcd} \end{array}\right] \rightsquigarrow \left[\begin{array}{l} \text{output} \\ \chi(Z)\in H^{2rk(E),\,rk(E)}(X,R) \\ \text{Not depend on } L,\phi \end{array}\right]$$

$$\Big\downarrow \qquad\qquad\qquad\qquad \wr\ \text{Thom isom.}$$

$$\left[\begin{array}{l} \text{auxiliary} \\ L:\text{v.b on } X \\ \phi: E\oplus L \xrightarrow{\sim} \mathcal{O}^N \end{array}\right] \longrightarrow \left[Th_X(L) \xrightarrow[\chi(Z)]{} K_{N,R} \right]$$

$$\begin{tikzcd} Z \arrow[d, "\text{fin. equidim}"'] & \subset E \arrow[dl] & \{(l',\text{over } x), Z_x, l\} \arrow[d] & \cong_{\phi}\ \mathbb{A}^N_X \\ X & & (l' \text{ over } x),\ x\in X & \subset L\times_X E\times_X L \arrow[dl, "\text{fin. equidim}"'] \arrow[d] \\ & & & L \end{tikzcd}$$

$$\rightsquigarrow \quad \begin{array}{ccc} L & \to & \mathbb{Z}_{tr}(\mathbb{A}^N_X) \\ \cup & & \cup \\ L-X & \to & \mathbb{Z}_{tr}(\mathbb{A}^N_X-\{0\}) \end{array}$$

$$Th_X(L) \xrightarrow[\chi(Z)]{} K_{N,R} \quad \longleftarrow$$

input G: fin. grp.

$\left\{ \begin{array}{l} r : G \longrightarrow \mathfrak{S}_n \ \text{hom} \\ U \in Sm/k \\ \circlearrowleft_G \text{ freely} \end{array} \right.$

$(\xi_n$ vector bundle on $U/G)$

$\rightsquigarrow$

output

$P : K_{iR} \wedge (U/G)_+ \longrightarrow K_{in,R}$

Not depend on ϕ L

Thom iso.

$\left(\begin{array}{lll} G = \mathfrak{S}_\ell & n = \ell & r = id \\ U = \mathbb{A}^{m\ell} - 0 & R = \mathbb{Z}/\ell & U/G \sim B\mathfrak{S}_\ell \end{array} \right)$

auxiliary.

L: vect b. on U/G

$\phi : \xi_n \oplus L \xrightarrow{\sim} \mathcal{O}^N$

$\tilde{P} : K_{iR} \wedge Th_{U/G}(L^i) \to K_{iN,R}$

auxiliary

$$\begin{tikzcd} Z \arrow[r, phantom, "\subset"] \arrow[dr, "\substack{\text{fin} \\ \text{equi dim}}"'] & X \times \mathbb{A}^i \arrow[d] \\ & X \end{tikzcd}$$

$\xrightarrow{\Theta}$

$X_+ \wedge Th(L^i) \xrightarrow[a(Z)]{} K_{iN,R}$

$$\begin{tikzcd} Z_1 \arrow[r, phantom, "\subset"] \arrow[dr, "\substack{\text{fin} \\ \text{equi dim}}"'] & X \times \mathbb{A}^i \arrow[d] \\ & X \end{tikzcd}$$

$$\begin{tikzcd} p^*(Z^{\times n}) \arrow[r, phantom, "\subset"] \arrow[d] & (X \times \mathbb{A}^i)^n \times U \arrow[d] \\ Z' \arrow[r, phantom, "\subset"] \arrow[dr, "\substack{\text{fin} \\ \text{equi dim}}"'] & ((X \times \mathbb{A}^i)^n \times U)/G \arrow[d] \\ & (X^n \times U)/G \end{tikzcd}$$

$X \times \xi_n^{\oplus i}$

$\parallel$

$$\begin{tikzcd} Z'' \arrow[r, phantom, "\subset"] \arrow[dr, "\substack{\text{fin} \\ \text{equi dim}}"'] & X \times (\mathbb{A}^{in} \times U)/G \arrow[d] \\ & X \times U/G \end{tikzcd}$$

Δ^*

$(\Delta : X \to X^n)$

$a(Z'') : X_+ \wedge Th_{U/G}(L^i) \to K_{iN,R}$

Construction (?)

$G = \mathfrak{S}_\ell \quad n = \ell \quad r = id \quad U = \mathbb{A}^{\ell m} - \Delta, \; R = \mathbb{Z}/\ell$
$(m \to \infty)$

$P : \tilde{H}^{2d,d}(F_\cdot, \mathbb{Z}/\ell) \longrightarrow \tilde{H}^{2d\ell, d\ell}((F_\cdot \wedge B\mathfrak{S}_\ell)_+, \mathbb{Z}/\ell)$
total power operation

$\rightsquigarrow P^i, B^i$

property of $P \rightsquigarrow$ property of P^i, B^i

$P(a \wedge b)$
$= \Delta^*(P(a) \wedge P(b)) \rightsquigarrow$ Cartan formula

symm. thm $\rightsquigarrow$ Adem relation

$\beta P = 0 \rightsquigarrow \beta B^i = 0, \; \beta P^i = B^i$

$m < n \Rightarrow \tilde{H}^{*m}(K_{n,\Lambda}, B) = 0 \rightsquigarrow P^i = 0 \quad B^i = 0 \quad i < 0$

$\mathcal{A}^{*,*} := \langle \beta, P^i \, (i \geq 0), \, a \in H^{*,*}(k) \rangle_{\text{left } H^{*,*}(k)\text{-alg}}$
$\subset$ (bistable coh. operation)
$\overset{?}{=} \mathrm{Hom}_{\mathcal{SH}}(H_{\mathbb{Z}/\ell}, \Sigma_s^* \Sigma_t^* H_{\mathbb{Z}/\ell})$

$\mathcal{A}_{*,*} = \mathrm{Hom}_{\text{left } H^{*,*}(k)\text{-mod}}(\mathcal{A}^{*,*}, H^{*,*}(k))$

$H^{*,*}$ cup prod $\rightsquigarrow \mathcal{A}^{*,*}$: coprod. $\rightsquigarrow \mathcal{A}_{*,*}$. prod.