Program:

17/Mar(Mon)	
15:30-17:00	Go Yamashita (RIMS)
	“Review of Taylor-Wiles system”
17:30-19:00	Go Yamashita (RIMS)
	“Galois representations associated to Hilbert modular forms via congruence after Taylor”, and “Global-local compatibility after Carayol.”
18/Mar(Tue)	
9:00-10:30	Go Yamashita (RIMS)
	“Modularity lifting for potentially Barsotti-Tate deformations after Kisin I.”
11:00-12:30	Seidai Yasuda (RIMS)
	“Base change argument of Skinner-Wiles”, and “Integral p-adic Hodge theory after Breuil and Kisin.”
14:30-16:00	Seidai Yasuda (RIMS)
	“Modularity lifting for potentially Barsotti-Tate deformations after Kisin II.”
16:30-18:00	Go Yamashita (RIMS)
	“Modularity lifting for crystalline deformations of intermediate weights after Kisin.”
19/Mar(Wed)	
9:00-10:30	Seidai Yasuda (RIMS)
	“p-adic local Langlands correspondence and mod p reduction of crystalline representations after Berger, Breuil, and Colmez.”
11:00-12:30	Go Yamashita (RIMS)
	“Modularity lifting of residually reducible case after Skinner-Wiles.”
14:30-16:00	Seidai Yasuda (RIMS)
	“Potential modularity after Taylor.”
16:30-18:00	Go Yamashita (RIMS)
	“Taylor-Wiles system for unitary groups after Clozel-Harris-Taylor I.”
18:30–	Party
20/Mar(Thu)	
9:00-10:30	Seidai Yasuda (RIMS)
	“Taylor-Wiles system for unitary groups after Clozel-Harris-Taylor II.”
11:00-12:30	Seidai Yasuda (RIMS)
	“Proof of Sato-Tate conjecture after Taylor et al.”
14:30-16:00	Go Yamashita (RIMS)
	“First step of the induction of the proof of Serre’s conjecture after Tate, Serre, and Schoof.”
16:30-18:00	Go Yamashita (RIMS)
	“Proof of Serre’s conjecture of level one case after Khare.”
21/Mar(Fri)	
9:00-10:30	Seidai Yasuda (RIMS)
	“Proof of Serre’s conjecture after Khare-Wintenberger.”
11:00-12:30	Seidai Yasuda (RIMS)
	“Breuil-Mézard conjecture and modularity lifting for potentially semistable deformations after Kisin.”

Talks [1]:

(1) "Review of Taylor-Wiles system."
We will give a review of the method of Taylor-Wiles system in [TW], and [D1]. We also explain how the method of Taylor-Wiles system developed until now.

(2) "Galois representations associated to Hilbert modular forms via congruence after Taylor."
We explain the construction of Galois representations associated to Hilbert modular forms in the case of $2 \mid [F : \mathbb{Q}]$ via congruences after Taylor [T1].

(3) "Global-local compatibility after Carayol."
We explain the global-local compatibility of Langlands correspondence for Hilbert modular forms in $\ell \neq p$ after Carayol [Ca1].

(4) "Modularity lifting for potentially Barsotti-Tate deformations after Kisin I."
We explain axiomatically Kisin's technique of $R^{\mathrm{red}} = T$ in [K1]. We study global deformation rings over local ones, and a moduli of finite flat group schemes to get informations about local deformation rings in [K1]. We can use this technique in the non-minimal cases too.

(5) "Base change argument of Skinner-Wiles."
We explain Skinner-Wiles level lowering technique allowing solvable field extensions in Kisin's paper [K1].

(6) "Integral p-adic Hodge theory after Breuil and Kisin."
We prepare the tools of integral p-adic Hodge theory used in [K1]. We can consider them as variants of Berger's theory.

(7) "Modularity lifting for potentially Barsotti-Tate deformations after Kisin II."
The sequel to the previous talk.

(8) "Modularity lifting for crystalline deformations of intermediate weights after Kisin."
We show Kisin's modularity lifting theorem for crystalline deformations of intermediate weights [K3]. We use results of Berger-Li-Zhu [BLZ] and Berger-Breuil [BB1] about mod p reduction of crystalline representations of intermediate weights.

(9) "p-adic local Langlands correspondence and mod p reduction of crystalline representations after Berger, Breuil, and Colmez."
We explain results of Berger-Li-Zhu and Berger-Breuil about mod p reduction of crystalline representations of intermediate weights [BLZ], [BB1]. We use p-adic local Langlands ([C1], [C2], [BB2]) in the latter case.

(10) "Modularity lifting of residually reducible case after Skinner-Wiles."
We explain Skinner-Wiles' modularity lifting theorem for residually reducible representations [SW1].

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[1]written by Go Yamashita (gokun@kurims.kyoto-u.ac.jp)

(11) "Potential modularity after Taylor."
We explain Taylor's potential modularity [T2], [T3]. This is a variant of Wiles' $(3,5)$-trick replaced by Hilbert-Blumenthal abelian varieties.

(12) "Taylor-Wiles system for unitary groups after Clozel-Harris-Taylor I."
We explain Clozel-Harris-Taylor's Taylor-Wiles system for unitary groups [CHT], and Taylor's improvement for non-minimal case by using Kisin's arguments [T4].

(13) "Taylor-Wiles system for unitary groups after Clozel-Harris-Taylor II."
The sequel to the previous talk.

(14) "Proof of Sato-Tate conjecture after Taylor et al."
We show Sato-Tate conjecture after Taylor et al. under mild conditions. We use a variant of $(3,5)$-trick replaced by a family of Calabi-Yau varieties [HSBT].

(15) "First step of the induction of the proof of Serre's conjecture after Tate, Serre, and Schoof."
We show the first step of the proof of Serre's conjecture, that is, $p = 2$ [Ta2], $p = 3$ [Se2], and $p = 5$ [Sc]. We use Odlyzko's discriminant bound, and Fontaine's discriminant bound.

(16) "Proof of Serre's conjecture of level one case after Khare."
We explain Khare-Wintenberger's constuction of compatible systems by using Taylor's potential modularity [T2], [T3] and Böckle's technique of lower bound of the dimension of global deformation rings [Bo]. We show Serre's conjecture of level one case after Khare [Kh1].

(17) "Proof of Serre's conjecture after Khare-Wintenberger."
We prove Serre's conjecture after Khare-Wintenberger [KW2], [KW3].

(18) "Breuil-Mézard conjecture and modularity lifting for potentially semistable deformations after Kisin."
We explain Breuil-Mézard conjecture, and Kisin's approach of modularity lifting theorem for potentially semistable deformations via Breuil-Mézard conjecture [K6].

References

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[Ta1] Tate, J. *Algebraic cycles and poles of zeta functions.* Arithmetic Algebraic Geometry, Proc. of Purdue Univ. Conf. 1963, New York, (1965) 93–110.

[FM] Fontaine, J.-M., Mazur, B. *Geometric Galois representations.* Elliptic Curves, Modular Forms, and Fermat's last Theorem (Hong Kong 1993), Internat. Press, Cambridge, MA, 1995, 190–227.

[Se1]: Serre's conjecture. [Ta1]: Sato-Tate conjecture. [FM]: Fontaine-Mazur conjecture.

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[S2] Saito, T. *Fermat conjecture II.* Iwanami publisher, 2008.
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[D2] Diamond, F. *On deformation rings and Hecke rings.* Ann. of Math. **144** (1996), 137–166.
[CDT] Conrad, B., Diamond, F., Taylor, R. *Modularity of certain potentially Barsotti-Tate Galois representations.* J. Amer. Math. Soc. **12**(2) (1999), 521–567.
[BCDT] Breuil, C., Conrad, B., Diamond, F., Taylor, R. *On the modularity of elliptic curves over* $\mathbb{Q}$*: wild 3-adic exercises.* J. Amer. Math. Soc. **14**(4) (2001), 843–939.
[F] Fujiwara, K. *Deformation rings and Hecke algebras for totally real fields.* preprint.

[W1]: Fermat's last theorem. [TW]: Taylor-Wiles system. [DDT]: Survey of the proof of Fermat's last theorem. [S1], [S2]: Books about Fermat's last theorem. [D1]: Axiomization and improvement of Taylor-Wiles system. The freeness of Hecke modules became the output from the input. [D2]: Shimura-Taniyama conjecture for elliptic curves, which are semistable at 3 and 5. [CDT]: Shimura-Taniyama conjecture for elliptic curves, whose conductor is not divisible by 27. [BCDT]: Shimura-Taniyama conjecture in full generality. [F]: $R = T$ in totally real case.

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[H] Hida, H. *On p-adic Hecke algebras for* GL_2 *over totally real fields.* Ann. of Math. (2) **128**(2) (1988), 295–384.

[W2]: Construction of Galois representations associated to Hilbert modular forms in the $2 \mid [F : \mathbb{Q}]$ and nearly ordinary case (including parallel weight 1) by using Hida theory. [T1]: Construction of Galois representations associated to Hilbert modular forms in the $2 \mid [F : \mathbb{Q}]$ case by the congruences. [H]: GL_2 Hida theory for totally real case.

[Ca1] Carayol, H. *Sur les représentations ℓ-adiques associées aux formes modulaires de Hilbert.* Ann. Sci. École Norm. Sup. (4) **19**(3) (1986), 409–468.
[S3] Saito, T. *Modular forms and and p-adic Hodge theory.* Invent. Math. **129**(3) (1997), 607–620.
[S4] Saito, T. *Hilbert modular forms and and p-adic Hodge theory.* preprint.

[Ca1]: Global-local compatibility for $\ell \neq p$ for totally real case. [S3]: Global-local compatibility for $\ell = p$ for $\mathbb{Q}$. [S4]: Global-local compatibility for $\ell = p$ for totally real case.

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[SW2] Skinner, C., Wiles, A. *Nearly ordinary deformations of irreducible residual representations.* Ann. Fac. Sci. Toulouse Math. (6) **10**(1) (2001), 185–215.

[SW3] Skinner, C., Wiles, A. *Base change and a problem of Serre.* Duke Math. J. **107**(1) (2001), 15–25.

[SW1]: Modularity lifting in the residulally reducible case. Taylor-Wiles arguments in the Hida theoretic situations. [SW2]: Modularity lifting for the nearly ordinary deformations in the residually irreducible case by the method of [SW1]. Minor remark: we do not need to assume that $\overline{\rho}|_{\mathrm{Gal}(\overline{F}/F(\zeta_p))}$ is irreducible. [SW3]: Level lowering technique allowing solvable field extensions.

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[PR] Pappas, G., Rapoport, M. *Local models in the ramified case. I. The EL-case.* J. Algebraic Geom. **12** (2003), 107–145.
[G] Gee, T., *A modularity lifting theorem for weight two Hilbert modular forms.* Math. Res. Lett. **13** (2006), no. 5, 805–811.
[I] Imai, N., *On the connected components of moduli spaces of finite flat models.* preprint.
[B1] Breuil, C., *Integral p-adic Hodge theory.* Algebraic Geometry 2000, Azumino, Adv. Studies in Pure Math. **36** (2002), 51–80.
[K2] Kisin, M., *Crystalline representations and F-crystals.* Algebraic geometry and number theory, Progr. Math. **253**, Volume in honor of Drinfeld's 50th birthday, Birkhäuser, Boston (2006), 459–496.
[K3] Kisin, M., *Modularity for some geometric Galois representations.* preprint.
[BLZ] Berger, L., Li, H., Zhu, H. J. *Construction of some families of* 2*-dimensional crystalline representations.* Math. Ann. **329**(2) (2004), 365–377.
[BB1] Berger, L., Breuil, C., *Sur la réduction des représentations cristallines de dimension* 2 *en poid moyens.* preprint.
[K4] Kisin, M., *Potentially semi-stable deformation rings.* preprint.
[K5] Kisin, M., *Modularity of* 2*-adic Barsotti-Tate representations.* preprint.
[K6] Kisin, M., *The Fontaine-Mazur conjecture for* GL_2. preprint.
[K7] Kisin, M., *Modularity of potentially Barsotti-Tate Galois representations.* preprint.

[K1]: Furthur improvement of $R = T$ for potentially Barsotti-Tate representations studying global deformation rings over local ones. We study a moduli of finite flat group schemes to get informations of local deformation rings. We can also use this technique in non-minimal case. [PR]: Used in [K1] to get informations of a moduli of finite flat group schemes. [G]: Connectedness of the moduli of finite flat models considered in [K1] in the case where the residue field is not $\mathbb{F}_p$ and the residual representation is trivial. [I]: Connectedness of the moduli of finite flat models considered in [K1] in the case where the residue field is not $\mathbb{F}_p$ and the residual representation is not trivial. [B1]: Used in [K1] to study a moduli of finite flat group schemes in terms of linear algebra. [K2]: Generalization of [B1], which is a variant of Berger's theory too. [K3]: Modularity lifting for crystalline representations of intermediate weights by the method of [K1]. [BLZ]: Explicite construction of a family of Wach modules. The determination of the mod p reduction of crystalline representations of intermediate weights is used in [K3], and [KW1]. [BB1]: By using p-adic local Langlands ([C1], [C2], and [BB2]), we determine the mod p reduction of crystalline representations of intermediate weights, which are not treated in [BLZ]. This is used in [K3]. [K4]: Construction of potentially semistable deformation rings. [K5]: $p = 2$ version of [K1]. Used in [KW2] and [KW3]. [K6]: Proof of many cases of Breuil-Mézard conjecture by using p-adic local Langlands ([C1], [C2], and [BB2]), and deduce a modularity lifting theorem in a high generality from this. [K7]: Survey of [K1], [T2], [T3], and [KW1].

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[T4] Taylor, R. *Automorphy for some ℓ-adic lifts of automorphic mod ℓ Galois representations II.* preprint.

[HSBT] Harris, M., Shepherd-Barron, N., Taylor, R. *A family of Calabi-Yau varieties and potential automorphy.* preprint.

[T2]: Potential modularity in the ordinary case. Variant of $(3,5)$-trick replaced by Hilbert-Blumenthal abelian variety. [T3]: Potential modularity in the crystalline of lower weights case. [BR]: Motive of Hilbert modular forms. Used in [T2] and [T3]. [HT]: local Langlands for GL_n by the "vanishing cycle side" in the sense of Carayol's program. [CHT]: Taylor-Wiles system for unitary groups. Proof of Sato-Tate conjecture assuming a generalization of Ihara's lemma. [T4]: By using Kisin's modified Taylor-Wiles arguments [K1], improvements are made so that we do not need level raising arguments and the generalization of Ihara's lemma. [HSBT]: Proof of Sato-Tate conjecture under mild conditions. Variant of $(3,5)$-trick replaced by a family of Calabi-Yau varieties.

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[KW3] Khare, C., Wintenberger, J.-P. *Serre's modularity conjecture (II).* preprint.

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[KW1]: Constuction of compatible system of minimally ramified lifts by using Taylor's potential modularity ([T2] and [T3]) and Böckle's technique. Starting point of [Kh1], [KW2], and [KW3]. [Kh1]: Proof of Serre's conjecture for level one case. Construct more general compatible systems than [KW1]. [KW2]: Proof of Serre's conjecture Part 1. [KW3]: Proof of Serre's conjecture Part 2. [Kh2]: Survey of [Kh1]. [Kh3]: Serre's conjecture implies Artin's conjecture for two dimensional odd representations. [Ca2]: Carayol's lemma used in [KW1], and [KW2]. [Di1]: Existence of compatible system. [Di2]: Another proof of Serre's conjecture of level one case, not using the distribution of Fermat primes. [Bo]: The technique of the lower bound of the dimension of global deformation rings by using local deformation rings used in [KW1], and [Kh1]. [Sa]: Non-vanishing of certain local deformation rings and some calculations of strongly divisible modules are used in [Kh1], [KW2], and [KW3]. [Sc]: Non-existence of certain abelian varieties by using Fontaine's technique and Odlyzko's bound. Used in [KW1] to show Serre's conjecture for $p = 5$. [Ta2]: Proof of Serre's conjecture for $p = 2$. Minkowski's bound is used. [Se2]: Proof of Serre's conjecture for $p = 3$. Odlyzko's bound is used.

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[C1] Colmez, P. *Série principale unitaire pour* $\mathrm{GL}_2(\mathbb{Q}_p)$ *et représentations triangulines de dimension* 2. preprint.

[C2] Colmez, P. *Une correspondance de Langlands locale p-adique pour les représentations semi-stable de dimension* 2. preprint.

[BB2] Berger, L., Breuil, C. *Sur quelques représentations potentiellement cristallines de* $\mathrm{GL}_2(\mathbb{Q}_p)$. preprint.

[BM]: Breuil-Mézard conjecture, which says Hilbert-Samuel multiplicity of universal deformation rings is explicitly described by the terms of automorphic side. [B2]: Conjecture about mod p reduction of crystalline representations of intermediate weights, which is partially proved in [BLZ] and [BB1]. This conjecture comes from the insight of "mod p reduction" of p-adic local Langlands. Used in [BB1], and [K6]. [C1]: p-adic local Langlands. Construction of a bijection between trianguline irreducible two dimensional representations of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ between "unitary principal series" of $\mathrm{GL}_2(\mathbb{Q}_p)$. Used in [BB1], and [K6]. [C2]: p-adic local Langlands. By using (φ, Γ)-modules, we construct a correspondence between two dimensional irreducible semistable representations of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ between unitary representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. Used in [BB1], and [K6]. [BB2]: p-adic local Langlands. We associate Banach representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ to two dimensional potentially crystalline representations of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$. Used in [BB1], and [K6].

Un fil d'Ariane pour ce workshop [2]

(Main Tools)

- Modularity Lifting Theorems
  - MLT for residually reducible representations [SW1],
  - MLT for potentially Barsotti-Tate deformations [K1],
  - (MLT for crystalline deformations of intermediate weights [K3]),
  - MLT for unitary groups [CHT], etc.
- Potential Modularity Theorems
  - PMT by using Hilbert-Blumenthal modular varieties
    [T2] (ordinary case), [T3] (crystalline of low weight case),
  - PMT by using a family of Calabi-Yau varieties [HSBT] (GSp case).
- existence of Strictly Compatible Systems
  [KW1], [Kh1], and [KW3]
  - crystalline liftings of low weights,
  - weight 2 liftings etc.

(Conjectures)

- MLT for unitary groups & PMT by using Calabi-Yau family
  $\rightsquigarrow$Sato-Tate conjecture (under mild condition),
- MLT's & the existence of several kinds of SCS's
  $\rightsquigarrow$Serre's conjecture.

(Influence, or logical dependence)

- Wiles' $(3, 5)$-trick
  $\rightsquigarrow$PMT,
- Kisin's modified TW argument in non-minimal cases
  $\rightsquigarrow$MLT for unitary groups in non-minimal cases,
- PMT & MLT
  $\rightsquigarrow$the existence of SCS's,
- (p-adic local Langlands
  $\rightsquigarrow$MLT for crystalline deformations of intermediate weights),
- (p-adic local Langlands
  $\rightsquigarrow$Breuil-Mézard conjecture
  $\rightsquigarrow$MLT for potentially semistabe deformations of arbitrary weights).

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Taylor-Wiles system ([W1], [TW], [D1])

- $H^1_{\mathcal{L}}(\mathbb{Q}, \mathrm{ad}^0\overline{\rho}) \cong \mathrm{Hom}_k(\mathfrak{m}_{R_{\mathcal{L}}}/(\lambda, \mathfrak{m}^2_{R_{\mathcal{L}}}), k)$,
  its dim.= (the number of topological generators of the corresponding universal deformation ring).

- $\dim H^1_{\mathcal{L}}(\mathbb{Q}, \mathrm{ad}^0\overline{\rho}) = \dim H^1_{\mathcal{L}^*}(\mathbb{Q}, \mathrm{ad}^0\overline{\rho}(1)) +$ (sum of local terms)
  by global Tate-Poitou duality, and
  - (local term at ∞)$= -1$,
  - (local term at p in the minimal case)≤ 1 by Fontaine-Laffaille theory,
  - (local terms ($\neq p$) at "minimally ramified" deformations)$= 0$,
  - (local terms at TW-type deformations)$= 1$,
  - $\dim H^1_{\mathcal{L}^*_{\emptyset}}(\mathbb{Q}, \mathrm{ad}^0\overline{\rho}(1)) - \dim H^1_{\mathcal{L}^*_{Q_n}}(\mathbb{Q}, \mathrm{ad}^0\overline{\rho}(1)) = \#Q_n$,
  - $H^1_{\mathcal{L}^*_{Q_n}}(\mathbb{Q}, \mathrm{ad}^0\overline{\rho}(1)) = 0$ by Cebotarev arguments,

  $\rightsquigarrow$
  (the number of topological generators of n-th level TW-type universal deformation ring)
  $= \dim H^1_{\mathcal{L}_{Q_n}}(\mathbb{Q}, \mathrm{ad}^0\overline{\rho}) \leq \dim H^1_{\mathcal{L}^*_{Q_n}}(\mathbb{Q}, \mathrm{ad}^0\overline{\rho}(1)) + \#Q_n$
  $= \#Q_n = \dim H^1_{\mathcal{L}^*_{\emptyset}}(\mathbb{Q}, \mathrm{ad}^0\overline{\rho}(1))$ (independent of n).

- $\mathbb{T}_{Q_n}$ is free over $\mathcal{O}[\Delta_{Q_n}]$ by de Shalit's argument (need of mod p multiplicity one) (resp. H_{Q_n} is free over $\mathcal{O}[\Delta_{Q_n}]$ by the argument in [D1] (no need of mod p multiplicity one)).

- TW system is *not* a compatible system with respect to n. We make a compatible system from TW system by using the argument of "finite isomorphism classes", and take a projective limit. In the limit level, the situation is simple. So, we get $R_\infty \xrightarrow{\sim} \mathbb{T}_\infty$ in the limit level. We deduce $R_\emptyset \xrightarrow{\sim} \mathbb{T}_\emptyset$ & l.c.i. (resp. + freeness of $H_\emptyset$ over $\mathbb{T}_\emptyset$ [D1]) in the finite level from the limit level.

- $\mathbb{T}_\Sigma$ is reduced $\rightsquigarrow \#(\mathcal{O}/\eta_\Sigma) < \infty$. (cf. we do not know *a priori* $\#(\mathfrak{p}_\Sigma/\mathfrak{p}_\Sigma^2) < \infty$).

- Ihara' lemma and its generalization +Gorenstein-ness of $\mathbb{T}_{\Sigma'}$ and $\mathbb{T}_\Sigma$ (resp. no need of Gorenstein-ness [D1])
  $\rightsquigarrow$calculation of $\#(\eta_\Sigma/\eta_{\Sigma'})$ (resp. $\mathrm{length}_{\mathcal{O}}\Omega_{\Sigma'}/\Omega_\Sigma$, where $\Omega_\Sigma := H_\Sigma/(H_\Sigma[\mathfrak{p}_\Sigma] + H_\Sigma[I_\Sigma])$)
  $\rightsquigarrow \#(\mathfrak{p}_{\Sigma'}/\mathfrak{p}^2_{\Sigma'})/(\mathfrak{p}_\Sigma/\mathfrak{p}^2_\Sigma) \leq \#(\eta_\Sigma/\eta_{\Sigma'})$
  (resp. $\mathrm{length}_{\mathcal{O}}(\mathfrak{p}_{\Sigma'}/\mathfrak{p}^2_{\Sigma'})/(\mathfrak{p}_\Sigma/\mathfrak{p}^2_\Sigma) \leq \mathrm{length}_{\mathcal{O}}\Omega_{\Sigma'}/\Omega_\Sigma$)
  $\rightsquigarrow$
  $R_\Sigma \xrightarrow{\sim} \mathbb{T}_\Sigma$ & l.c.i. (resp. + freeness of H_Σ over $\mathbb{T}_\Sigma$) implies
  $R_{\Sigma'} \xrightarrow{\sim} \mathbb{T}_{\Sigma'}$ & l.c.i. (resp. + freeness of $H_{\Sigma'}$ over $\mathbb{T}_{\Sigma'}$ [D1]).

Kisin's modification of TW argument ([K1], [K3], [K7])

- We study a global deformation ring over local deformation rings
  $\rightsquigarrow$
  - we can show $R^{\text{red}} = T$ even if the local deformation rings at the places dividing p have complicated singularity, and
  - we can show $R^{\text{red}} = T$ without level raising in non-minimal cases.

- We consider framed deformations
  $\rightsquigarrow$we can study local framed deformation rings even if $\overline{\rho}|_{G_{\mathbb{Q}_v}}$ is not irreducible.

- dim. of Selmer group + local contributions
  $\rightsquigarrow$the number of topological generators of R over $\widehat{\otimes}_{v\in\Sigma}R_v$.

- We have to study the following things about local framed deformation rings to apply Kisin's modified TW argument:
  (1) calculation of the dimensions of the local deformation rings,
  (2) to show that the local deformation rings are formally smooth after inverting p, and
  (3) to show that the local deformation rings are domains.

- The above (1), (2), and (3) are easy in the case of $v \nmid p$. In the case of $v \mid p$:
  (1) Calculation of the dimension is easy,
  (2) Formally smooth after inverting p:
  Breuil's theorem (crystalline representations of HT weights in {0,1} come from p-divisible groups)
  $\rightsquigarrow D^{\text{fl}}_{V_{\mathbb{F}},(\xi)} \overset{\sim}{\to} D^{\text{crys}}_{V_\xi}$
  $\rightsquigarrow$check explicitly the formally smoothness by constructing a lifting,
  (3) Domain: Consider a moduli of finite flat models $\mathcal{GR}^{\mathbf{v}}_{V_{\mathbb{F}}}$,
    (a) Tate's theorem
    $\rightsquigarrow\mathcal{GR}^{\mathbf{v}}_{V_{\mathbb{F}}}$ is isomorphic to Spec$R^{\mathbf{v}}$ after inverting p,
    (b) comparing $\mathcal{GR}^{\mathbf{v}}_{V_{\mathbb{F}}}$ with a complete local ring of a Hilbert modular variety
    $\rightsquigarrow\mathcal{GR}^{\mathbf{v}}_{V_{\mathbb{F}}} \otimes \mathbb{F}$ is normal, in particular, reduced,
    (c) Kisin's theory of $\mathfrak{S}$-modules in the integral p-adic Hodge theory
    $\rightsquigarrow$the special fiber $\mathcal{GR}^{\mathbf{v},\text{non-ord}}_{V_{\mathbb{F}},0}$ is connected by explicit linear algebra calculations (repeat connecting a point to another point by $\mathbb{P}^1$),
    (d) $H_0(\text{Spec}R^{\mathbf{v},\text{non-ord}}[\frac{1}{p}]) \cong H_0(\mathcal{GR}^{\mathbf{v},\text{non-ord}}_{V_{\mathbb{F}}} \otimes \mathbb{Q}_p)$ by (a)
    $\cong H_0(\mathcal{GR}^{\mathbf{v},\text{non-ord}}_{V_{\mathbb{F}}})$ by (b) $\cong H_0(\widehat{\mathcal{GR}^{\mathbf{v},\text{non-ord}}_{V_{\mathbb{F}}}})$ by formal GAGA
    $\cong H_0(\mathcal{GR}^{\mathbf{v},\text{non-ord}}_{V_{\mathbb{F}},0}) \cong \{*\}$ by (c).

Potential Modularity Theorems ([T2], [T3], [HSBT])

GL_2 case ([T2], [T3]):

- We want to find a Hilbert-Blumenthal abelian variety A such that
$$\overline{\rho} \cong A[\lambda] \longleftarrow T_\lambda A \leftrightsquigarrow T_\wp A \longrightarrow A[\wp] \cong \mathrm{Ind}\overline{\psi}.$$
- We consider Hilbert-Blumenthal modular varieties, and try to find such an abelian variety as a rational point of this modular variety.
- (We allow "potentiality")
  Moret-Bailly's theorem
  $\rightsquigarrow$it suffices to find local points (at λ, $\wp$, and ∞) to get such an abelian variety.
- Ordinary case ([T2]):
  - Honda-Tate theory
    $\rightsquigarrow$find an abelian variety over a finite field,
  - Serre-Tate theory
    $\rightsquigarrow$find an abelian variety over a local field.
- Crystalline of low weight case ([T3]):
  - We consider a twist of the modular variety, which is isomorphic over $\mathbb{Q}_\ell$, $\mathbb{Q}_{p_1}$, $\mathbb{Q}_{p_2}$, and $\mathbb{R}$,
  - CM theory
    $\rightsquigarrow$find a $\mathbb{Q}$ rational point on the twisted variety
    $\rightsquigarrow$find local points on the original variety,
  - studying mod ℓ representations of $\mathrm{GL}_2(O_{F_\lambda})$
    $\rightsquigarrow$change of weights.

GSp_n case ([HSBT]):

- We use Calabi-Yau varieties instead of abelian varieties, and a Calabi-Yau family instead of Hilbert-Blumenthal modular variety.
- The condition of the relation with $\overline{\rho}$
  $\rightsquigarrow$we have to consider a covering of the Calabi-Yau family.
- The Calabi-Yau family has big monodromy
  $\rightsquigarrow$the covering is geometrically connected
  $\rightsquigarrow$we can apply Moret-Bailly's theorem.
- trivial reason, or Fontaine-Laffaille theory, or Serre-Tate theory
  $\rightsquigarrow$find local points.

existence of Strict Compatible Systems ([KW1], [Kh1], and [KW3])

- Savitt's study of local deformation rings
  $\rightsquigarrow$local deformation rings we are considering are not zero.

- (Böckle's method) For $\theta^i : H^i(G_{\mathbb{Q},S}, \mathrm{ad}^0\overline{\rho}) \to \oplus_{v\in\Sigma} H^i(\mathbb{Q}_v, \mathrm{ad}^0\overline{\rho})$,
  - calculation of $\dim \ker\theta^1$
    $\rightsquigarrow$the number of topological generators of R over $\widehat{\otimes}_{v\in\Sigma} R_v$, and
  - calculation of $\dim \mathrm{coker}\theta^1 + \dim \ker\theta^2$
    $\rightsquigarrow$the number of relations of R over $\widehat{\otimes}_{v\in\Sigma} R_v$

  $\rightsquigarrow\dim R \leq 1$.

- PMT
  $\rightsquigarrow$global deformation ring R_F of $\overline{\rho}|_{G_F}$ is flat over $\mathcal{O}$ by MLT
  $\rightsquigarrow R/(p)$ is finite by de Jong's argument
  $\rightsquigarrow R$ is flat over $\mathcal{O}$
  $\rightsquigarrow$we get a minimally ramified lifting ρ (with some conditions) to characteristic 0.

- PMT
  $\rightsquigarrow\rho|_{G_F}$ arises from an automorphic representaion π of $\mathrm{GL}_2(\mathbb{A}_F)$
  $\rightsquigarrow$we can make ρ a part of SCS's by Brauer's thoerem:
  $\rho_\lambda := \sum_i n_i \mathrm{Ind}_{G_{F_i}}^{G_{\mathbb{Q}}} (\chi_i \otimes \rho_{\pi_{F_i},\lambda})$,
  where $1 = \sum_i n_i \mathrm{Ind}_{G_{F_i}}^{G_{\mathbb{Q}}} \chi_i$ (F/F_i's are elementary, in particular, solvable), and π_{F_i} is an automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{F_i})$ such that $\rho_{\pi_{F_i},\wp} \cong \rho|_{G_{F_i}}$ (we can check that ρ_λ is a true representation).

COMPREHENSION CHECK [3]

Taylor-Wiles system ([W], [TW], and [D1]).

$\boxed{1}$-1. Explain which part of the axioms of Taylor-Wiles system means "R is enough small", and which part means "T is enough large".

$\boxed{1}$-2. We need some numerical coincidence to make Taylor-Wiles system. Explain this.

$\boxed{1}$-3. How do we kill dual Selmer groups?

$\boxed{1}$-4. Explain how Auslander-Buchsbaum theorem was used in Diamond-Fujiwara's improvement of Taylor-Wiles system.

$\boxed{1}$-5. Explain $\mathcal{O}[\Delta_Q]$-structure of universal deformation rings.

$\boxed{1}$-6. Explain $\mathcal{O}[\Delta_Q]$-structure and its properties of Hecke modules.

$\boxed{1}$-7. Why is the Taylor-Wiles system applicable only for "minimal case"?

$\boxed{1}$-8. The theory of Taylor-Wiles system was improved, by Faltings ([TW, appendix]), and Diamond-Fujiwara ([D1]). Explain how the inputs and the outputs were changed about the following things.
  (a) $\mathbb{T}$ is locally complete intersection,
  (b) $R \xrightarrow{\sim} \mathbb{T}$, and
  (c) freeness of Hecke modules.

$\boxed{1}$-9. The Gorenstein-ness of Hecke algebras was used in three ways in the original arguments of [W] and [TW]. They are used in minimal case, non-minimal case, and a ring theoretic proposition [4]. Explain these.

$\boxed{1}$-10. What did we deduce the Gorenstein-ness of Hecke algebras from?

$\boxed{1}$-11. Now, we do not need to show the Gorenstein-ness of Hecke algebras to use Taylor-Wiles sytem. How was it improved?

$\boxed{1}$-12. How do we use the assumption that "$\overline{\rho}|_{\mathbb{Q}(\sqrt{(-1)^{(p-1)/2}p})}$ is absolutely irreducible"? Explain at least two usage concretely.

$\boxed{1}$-13. Explain Ihara's lemma.

$\boxed{1}$-14. Explain $(3,5)$-trick.

$\boxed{1}$-15. Which does not exist?
  (a) elliptic curve,
  (b) modular curve,
  (c) Shimura curve,
  (d) Frey curve, or
  (e) Fermat curve.

[3]written by Go Yamashita (gokun@kurims.kyoto-u.ac.jp)

[4]The assumption of Gorenstein-ness was removed soon by Lenstra about this ring theoretic proposition.

COMPREHENSION CHECK

Galois representations associated to Hilbert modular forms, and congruences ([T1]).

$\boxed{2}$-1. How do we use Shimura curves and the Jacquet-Langlands correspondence to construct Galois representations associated to Hilbert modular forms in the case where $[F : \mathbb{Q}]$ is odd or π is special or supercuspidal at some finite place?

$\boxed{2}$-2. How do we construct "congruences between old forms and new forms" to construct Galois representations associated to Hilbert modular forms in the case where $[F : \mathbb{Q}]$ is even?

$\boxed{2}$-3. How is the "error term" of congruences controllable?

$\boxed{2}$-4. Explain how we show there are enough congruences by using a Hilbert modular variety (not Shimura curve).

$\boxed{2}$-5. Explain in the point of view of applications to $R = T$ why it is useful to construct Galois representations associated to Hilbert modular form in the case of even degree.

COMPREHENSION CHECK

Global-local compatibility ([Ca1], [S3], and [S4]).

$\ell \neq p$ case:

$\boxed{3}$-1. Express supersingular loci adelically.

$\boxed{3}$-2. Explain how principal series representations appear in the cohomology of the normalization of the special fiber of Shimura curves.

$\boxed{3}$-3. Explain how (a part of) special representations appear in the cohomology of supersingular loci of the special fiber of Shimura curves.

$\boxed{3}$-4. Explain how supercuspidal representations appear in the cohomology of vanishing cycle sheaves of Shimura curves, and how we can show that $\sigma_{\mathfrak{p}}(\pi)$ depends only on $\pi_{\mathfrak{p}}$.

$\boxed{3}$-5. Explain the relation between the cohomology of vanishing cycle sheaves of Shimura curves, and (GL_2-case of) "vanishing cycle side" of Carayol's program.

$\boxed{3}$-6. Explain how we used the "congruence relation" in $\boxed{3}$-2 and $\boxed{3}$-3.

$\boxed{3}$-7. Explain how we can calculate the monodromy operator by using Picard-Lefschetz formula.

$\boxed{3}$-8. Explain how we can show the global-local compatibility for extraordinary representations.

$\ell = p$ case:

$\boxed{3}$-9 Explain how we use Lefschetz trace formulae and weight spectral sequences to compare ℓ-adic side and p-adic side of Weil-Deligne representations.

$\boxed{3}$-10 Explain how we use the weight-monodromy conjecture to compare ℓ-adic side and p-adic side of monodromy operators.

$\boxed{3}$-11 The only way to compare ℓ-adic side and p-adic side is to use geometry. However, there is not moduli interpretation of Shimura curves, so we cannot consider "Kuga-Sato variety" in a naive way. Explain how we overcome this difficulty.

COMPREHENSION CHECK

Kisin's modified Taylor-Wiles system ([K1]).

$\boxed{4}$-1. What is the merit (for Taylor-Wiles system) of using automorphic forms on a quaternion, which ramifies at all archemedian places?

$\boxed{4}$-2. When we consider deformations, which are Barsotti-Tate at p only after wildly ramified extensions, then the universal deformation ring has bad singularity, and we cannot expect that it is locally complete intersection. How did Kisin's modified Taylor-Wiles system overcome this difficulty?

$\boxed{4}$-3. Explain how Kisin's modified Talyor-Wiles system can treat "non-minimal case".

$\boxed{4}$-4. The properties of local deformation rings are important for Kisin's modified Taylor-Wiles system. Explain how the following things are used in the "$R^{\mathrm{red}} = \mathbb{T}$" theorem.
- (a) local deformation rings are domain,
- (b) local deformation rings are formally smooth after inverting p, and
- (c) calculations of the dimensions of local deformation rings.

$\boxed{4}$-5. Calculate the dimension of
- (a) local deformation rings for $v \in \Sigma$,
- (b) local deformation rings for $v \mid p$, and
- (c) global deformation rings.

$\boxed{4}$-6. Explain the technique of investigating stalks of points in generic fiber of local deformation rings.

$\boxed{4}$-7. How did we get the needed information about local deformation rings from the moduli of finite flat group schemes?

$\boxed{4}$-8. Explain the following things, which we did to get the needed information about the moduli of finite flat group schemes:
- (a) To relate it with complete local rings of a Hilbert modular variety.
- (b) To calculate linear algebraic data.

$\boxed{4}$-9.
- (a) Where is a "geometric incarnation" of Tate's theorem $\{p\text{-divisible groups}/O_K\} \hookrightarrow \underline{\mathrm{Rep}}_{\mathbb{Z}_p}(G_K)$ in [K1]?
- (b) Where did we use Breuil's theorem $\underline{\mathrm{Rep}}^{\mathrm{fl}}_{\mathrm{tor}}(G_K) \hookrightarrow \underline{\mathrm{Rep}}_{\mathrm{tor}}(G_{K_\infty})$ in [K1]?
- (c) Where did we use Breuil's theorem "crystalline representations of Hodge-Tate weights $\subset \{0,1\}$ come from p-divisible groups" in [K1]?

$\boxed{4}$-10. Explain Kisin's generalization or another proof of the above $\boxed{4}$-9 (b) and (c), in terms of modules with connection on open unit disk.

$\boxed{4}$-11. Explain Skinner-Wiles' base change arguments [SW3].

COMPREHENSION CHECK

Modularity lifting theorem for crystalline representasions of intermediate weights ([K3], [BLZ], [BB1], [C1], and [C2]).

$\boxed{5}$-1. What do we use in the case of crystalline deformations of intermediate weights instead of the moduli of finit flat group schemes?

$\boxed{5}$-2. How did we use the information of mod p reduction of crystalline representations of intermediate weights to show the modularity lifting?

$\boxed{5}$-3. Explain the relation between the category of Wach modules (lattices) and the category of crystalline representations (and their lattices).

$\boxed{5}$-4. Explain the following two methods of determing mod p reduction of crystalline representations of intermediate weights:

(a) the method of Berger-Li-Zhu ([BLZ]), and

(b) the method of Berger-Breuil ([BB1]).

$\boxed{5}$-5. Explain how the p-adic local Langlands correspondence was used in [BB1].

$\boxed{5}$-6. Explain the compatibility of p-adic local Langlands correspondence and mod p local Langlands correspondence.

$\boxed{5}$-7. Explain how the technique of "flatening" was used in Gabber's appendix in [K3].

COMPREHENSION CHECK

Modularity lifting theorem for residually reducible representations ([SW1]).

$\boxed{6}$-1. Explain pseudo representaions.

$\boxed{6}$-2. We do not have deformations to Hecke algebras in the residually reducible case, and only have pseudo deformations to Hecke algebras. This makes arguments complicated. Explain this, and how we solved this technical problem.

$\boxed{6}$-3. In Skinner-Wiles' technique, we have to make base changes to make codimension of loci of reducible representaion larger in the spectrum of a universal deformation ring. Explain this.

$\boxed{6}$-4. We use Hida theoretic Hecke algebras in the Skinner-Wiles case. So, the quotient algebras modulo prime ideals do not have finite cardinality in general. Thus, we have to modify Taylor-Wiles' patching arguments. Explain this.

$\boxed{6}$-5. We do not have $\overline{\rho}|_{F(\zeta_p)}$ is absolutely irreducible in the residually reducible case [SW1], and we do not need to assume it even in the residually irreducible and nearly ordinary deformation case [SW2]. How do we modify the Taylor-Wiles arguments, especially killing dual Selmer groups?

$\boxed{6}$-6. How do we show the existence of Eisenstein ideals by using p-adic L-function in [SW1]?

$\boxed{6}$-7. How do we construct "nice" primes in [SW1]?

$\boxed{6}$-8. How do we use Washington's theorem about p-rank of ideal class groups of cyclotomic $\mathbb{Z}_\ell$-extensions in [SW1]?

$\boxed{6}$-9. How do we use Mme Raynaud's theorem in [SW1]?

$\boxed{6}$-10. Explain how we show pro-modularity of other irreducible components from pro-modularity of an irreducible component.

$\boxed{6}$-11. How do we use the (technical) concept of "nice" prime?

COMPREHENSION CHECK

Taylor's potential modularity theorem ([T2], and [T3]).

$\boxed{7}$-1. How do we use a Hilbert-Blumenthal modular variety to do "(ℓ, ℓ')-trick"?.
$\boxed{7}$-2. How do we use Moret-Bailly's theorem to do "(ℓ, ℓ')-trick"?.
$\boxed{7}$-3. Explain the following things to construct "local points":
  (a) construction of Hilbert-Blumenthal abelian varieties over a finite field, and Honda-Tate theory, and
  (b) construction of a lifting of it to a local field, and Serre-Tate theory.
$\boxed{7}$-4. Explain the arguments of raising level, and congruences between different weights.
$\boxed{7}$-5. How do we use principal ideal theorem in [T3]?
$\boxed{7}$-6. Explain the following applications:
  (a) Fontaine-Mazur conjecture of degree 2, and
  (b) meromorphic continuation and functional equation of L-functions for (strongly compatible system of) ℓ-adic representations of degree 2.
$\boxed{7}$-7. Explain how we use these potential modularity theorems for Khare-Wintenberger's proof of Serre's conjecture.

COMPREHENSION CHECK

Modularity lifting theorem for unitary groups, and proof of Sato-Tate conjecture ([CHT], [T4], and [HSBT]).

$\boxed{8}$-1. Explain why we use unitary groups to consider GL_n.
$\boxed{8}$-2. We need some numerical coincidence to use Taylor-Wiles system. Explain this coincidence for unitary groups and essentially self-dual representations.
$\boxed{8}$-3. Explain Taylor-Wiles type deformations and $\mathcal{O}[\Delta_Q]$-structure of universal deformation rings for unitary groups.
$\boxed{8}$-4. Explain $\mathcal{O}[\Delta_Q]$-structure and its properties of Hecke modules for unitary groups.
$\boxed{8}$-5. Explain how we used "Ramakrishna deformations" and "one more deformations" in [CHT].
$\boxed{8}$-6. How do we avoid using Ihara's lemma, and make the modularity lifting theorem for non-minimal case unconditional?
$\boxed{8}$-7. We have to know some information about the action of local deformation rings on Hecke modules to use Kisin's modified Taylor-Wiles arguments. How do we get the information about it?
$\boxed{8}$-8. How do we use the fact that the Calabi-Yau family we are considering has big monodromy to show the potential modularity?
$\boxed{8}$-9. Explain how the potential modularity of $\mathrm{Sym}^{\mathrm{n}} H^1(E)$ deduce from the modularity lifting thoerem for unitary groups, and a Calabi-Yau family.
$\boxed{8}$-10. Explain how we deduce Sato-Tate conjecture from the potential modularity of $\mathrm{Sym}^{\mathrm{n}} H^1(E)$ for odd n's.

COMPREHENSION CHECK

Serre's conjecture for $p = 2, 3$, and 5 ([Ta2], [Se2], and [Sc]).

$\boxed{9}$-1. How do we use Minkowski's (root) discriminant bound and global class field theory in Tate's proof of Serre's conjecture for $p = 2$?

$\boxed{9}$-2. How do we use Odlyzko's (root) discriminant bound and global class field theory in Serre's proof of Serre's conjecture for $p = 3$?

$\boxed{9}$-3. How do we use Odlyzko's (root) discriminant bound, Fontaine's (root) discriminant bound, and global class field theory in Schoof's proof of Serre's conjecture for $p = 5$ [1]?

$\boxed{9}$-4. How do we show Odlyzko's (root) discriminant bound by using Dedekind zeta function?

$\boxed{9}$-5. How was the PD-structure used for Fontaine's (root) discriminant bound?

$\boxed{9}$-6. Explain the following things in Schoof's proof [Sc]:

- (a) How do we use Herbrand's theorem and p-adic L-function when we consider extensions of μ_p by $\mathbb{Z}/p\mathbb{Z}$ over Spec $\mathbb{Z}[1/\ell]$?
- (b) How do we use Burnside's basis theorem?
- (c) How do we use Taussky's theorem about 2-groups?

$\boxed{9}$-7. Show that any group of order less than 60 is solvable as follows [2]:

- (a) Show that any p-group is nilpotent.
- (b) Show that any group G of order $p^n q$ (p, q: primes) is solvable as follows:
  - (i) Assume that G does not have non-trivial normal subgroup. Show that the number of p-Sylow subgroups is q.
  - (ii) Let D be one of the maximal subgroups, which can be expressed as the intersection of two p-Sylow subgroups. Put H be the normalizer of D in G. Then, show that H has at least two p-Sylow subgroups. (Hint: any p-Sylow subgroup of H can be expressed as the intersection of a p-Sylow subgroup with H.)
  - (iii) Show that H has exactly q p-Sylow subgroups, and all of them contain D.
  - (iv) Show that $D = \{1\}$.
  - (v) Show that any intersection of different two p-Sylow groups is $\{1\}$.
  - (vi) Show that by counting elements, G has unique q-Sylow subgroup, so it is normal.
- (c) Show that any group of order $p^n q^2$ ($p > q$ primes) is solvable by similar arguments as above.
- (d) Show that groups of order $30 = 2.3.5$ and of order $42 = 2.3.7$ are solvable.

[1] Precisely speaking, he proved non-existence of certain abelian varieties having restricted reduction conditions.

[2] If we use Feit-Thompson's theorem (any group of odd order is solvable) and Burnside's theorem (any group of order $p^n q^m$ (p, q: primes) is solvable), then it is easy. However, these theorems are very difficult.

COMPREHENSION CHECK

Existence of strictly compatible system, and proof of Serre's conjecture ([KW1], [Kh1], [KW2], and [KW3]).

$\boxed{10}$-1. Prove Fermat's Last Theorem without "Langlands-Tunnell's theorem" and "Ribet's level lowering" by the method of strictly compatible systems.

$\boxed{10}$-2. How do we use Taylor's potential modularity (twice) for the proof of the existence of strictly compatible systems?

$\boxed{10}$-3. For the existence of strictly compatible systems, we show the following things about global deformation rings. Explain the proof of them.
- (a) Lower bound of the dimension, and
- (b) Flatness over $\mathcal{O}$.

$\boxed{10}$-4. Explain kinds of the strictly compatible systems (3 kinds in the level one case, 4 kinds in general).

$\boxed{10}$-5. Explain Dieulefait's another proof of Serre's conjecture of level one case, which do not use a distribution of Fermat primes [Di2].

$\boxed{10}$-6. Write down how the inductions are arranged in [Kh1], and [KW2].

$\boxed{10}$-7. Explain the following induction steps:
- (a) "add a good dihedral prime",
- (b) "killing ramification", and
- (c) "weight reduction".

$\boxed{10}$-8. Explain applications of Serre's conjecture about:
- (a) some Fermat-like diophantine equations,
- (b) non-existence of some finite flat group schemes,
- (c) the finiteness of isomorphism classes of mod p representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ of bounded ramifications,
- (d) modularity of abelian varieties of GL_2-type,
- (e) Artin's conjecture of degree 2, and
- (f) Fontaine-Mazur conjecture of degree 2.

COMPREHENSION CHECK

Modularity lifting theorem and potential modularity for $p = 2$ ([Kh1], [KW3], and [K5]).

$\boxed{11}$-1. We have $(k \cdot \mathrm{id}) \subset \mathrm{ad}^0\overline{\rho}$ in the case of $p = 2$. So, we do not have $(\mathrm{ad}^0\overline{\rho})^* \cong \mathrm{ad}^0\overline{\rho}$ in this case. How do we modify the usual Galois cohomology calculations in this case ([KW3])?

$\boxed{11}$-2. How do we modify the arguments of killing dual Selmer groups in the case of $p = 2, 3$ in [Kh1] ($p = 3$) and [KW3] ($p = 2, 3$) [1]?

$\boxed{11}$-3. How do we treat the "neatness problem" in the case of $p = 2, 3$ in [Kh1] ($p = 3$) and [KW3] ($p = 2, 3$)?

$\boxed{11}$-4. We cannot take a finite place v above an odd prime such that

$$(1 - Nv)((1 + Nv)^2 \det \overline{\rho}(\mathrm{Fr}_v) - Nv(\mathrm{tr}\overline{\rho}(\mathrm{Fr}_v))^2) \in \mathbb{F}^\times$$

in the case of $p = 2$. How do we overcome this difficulty in [KW3] and [K5]?

$\boxed{11}$-5. We cannot use Breuil's theory in the case of $p = 2$. So, we use Zink's theory of displays and windows in [K5]. How do we use this to overcome the difficulty?

[1] In [T3], the case $p = 2, 3$ are excluded.

COMPREHENSION CHECK

Modularity lifting theorem and Breuil-Mézard conjecture ([K6], [BM], [C1], and [C2]).

$\boxed{12}$-1. Explain Breuil-Mézard conjecture.

$\boxed{12}$-2. How do we use Breuil-Mézard conjecture to show the modularity lifting theorem for potentially semistable deformations? In particular, how do we overcome the difficulty that we do not know that the local deformation rings are domains?

$\boxed{12}$-3. How do we use the p-adic local Langlands correspondence in Kisin's proof of many cases of Breuil-Mézard conjecture.

$\boxed{12}$-4. Explain Colmez' functor in the p-adic local Langlands correspondence.

$\boxed{12}$-5. Explain the compatibility of the p-adic local Langlands correspondence and classical local Langlands correspondence.

$\boxed{12}$-6. Explain how we use the compatibility of p-adic local Langlands correspondence and classical local Langlands correspondence in Kisin's proof of many cases of Breuil-Mézard conjecture.