An Introduction to Lawson Homology

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Defining Chow groups via Chow varieties

Assume $X \subset \mathbb{P}^n$ is projective$/k$.

$$\begin{aligned}\mathcal{C}_{r,e}(X) &= \text{Chow variety of effective cycles} \\ &\quad \text{of dimension } r \text{ and degree } e \text{ on } X\end{aligned}$$

$$\mathcal{C}_r(X) = \coprod_{e \geq 0} \mathcal{C}_{r,e}(X).$$

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$$\mathcal{C}_r(X) = \coprod_{e \geq 0} \mathcal{C}_{r,e}(X).$$

For example,

$$\mathcal{C}_{0,e}(X) = Symm^e(X) := X^{\times e}/\Sigma_e$$

$$\mathcal{C}_{n-1,e}(\mathbb{P}^n) = \mathbb{P}(\text{degree } e \text{ part of } k[x_0, \ldots, x_n])$$

For most other cases, structure of $\mathcal{C}_r(X)$ is very complicated.

The functor represented by Chow varieties

For a smooth T, we have

$$\begin{aligned}\mathrm{Hom}(T, \mathcal{C}_r(X)) = \{ &\text{effective cycles } \gamma = \sum_i n_i W_i \text{ on } T \times X : \\ &\text{each } W_i \to T \text{ is equi-dimensional} \\ &\text{of relative dimension } r \}\end{aligned}$$

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$\mathrm{Hom}(T, \mathcal{C}_r(X))$ is monoid under addition of cycles, and

$$\begin{aligned}\mathrm{Hom}(T, \mathcal{C}_r(X))^+ &:= \text{group completion of } \mathrm{Hom}(T, \mathcal{C}(X)) \\ &= \text{free abelian group on integral } W \subset T \times X \\ &\quad \text{such that } W \to T \text{ is equi-dimensional} \\ &\quad \text{of relative dimension } r\end{aligned}$$

Chow varieties lead to rational equivalence

$$\mathcal{C}_r(X)(k)^+ = \operatorname{Hom}(\operatorname{Spec} k, \mathcal{C}_r(X))^+ = \{r\text{-cycles on } X\}$$

$\operatorname{Hom}(\mathbb{A}^1, \mathcal{C}_r(X))^+$ is the free abelian group on:

$$\begin{tikzcd} \Gamma \arrow[r, hook] \arrow[dr, "\text{rel. dim. r}"'] & \mathbb{A}^1 \times X \arrow[d] \\ & \mathbb{A}^1 \end{tikzcd}$$

Let $i_P : \operatorname{Spec} k \rightarrowtail \mathbb{A}^1$ be inclusion at a point $P \in \mathbb{A}^1(k)$. Then

$$\begin{aligned} i_P^* : \operatorname{Hom}(\mathbb{A}^1, \mathcal{C}_r(X))^+ &\to \mathcal{C}_r(X)^+ \\ \gamma \qquad &\mapsto \gamma \cap (\{P\} \times X) \end{aligned}$$

Classical Chow group via Chow varieties

We get a presentation of the classical Chow group:

$$\mathrm{Hom}(\mathbb{A}^1, \mathcal{C}_r(X))^+ \xrightarrow{i_0^* - i_1^*} \mathcal{C}_r(X)^+ \longrightarrow CH_r(X) \longrightarrow 0$$

Or, in other words, $CH_r(X)$ is "π_0^{alg}" of $\mathcal{C}_r(X)^+$.

Definition of higher Chow groups via Chow varieties

Define the "algebraic n-simplex"

$$\Delta^n = \operatorname{Spec} k[x_0, \dots, x_n]/(\sum_i x_i - 1) \cong \mathbb{A}^n$$

Then $[n] \mapsto \mathrm{Hom}(\Delta^n, \mathcal{C}_r(X))^+$ is a simplicial abelian group.

Definition

The higher Chow groups of a projective variety X are

$$\begin{aligned} CH_r(X,n) &:= \pi_n\left(Hom(\Delta^{\cdot}, \mathcal{C}_r(X))^+\right) \\ &= \text{“}\pi_n^{alg}(\mathcal{C}_r(X)^+)\text{''}. \end{aligned}$$

Note that $CH_r(X,0) = CH_r(X)$.

Some properties of the higher Chow groups

- Localization sequence: For $Y \subset X$ closed and $U := X \setminus Y$

$$\cdots \to CH_r(Y,1) \to CH_r(X,1) \to CH_r(U,1) \to CH_r(Y,0) \\ \to CH_r(X,0) \to CH_r(U,0) \to 0.$$

- Homotopy invariance:

$$CH_r(X,n) \cong CH_{r+m}(X \times \mathbb{A}^m, n)$$

- There is a generalized cycle class map

$$CH_r(X,n) \to H^{\mathrm{BM}}_{2r+n}(X(\mathbb{C})).$$

Here, H^{BM} = Borel-Moore homology. $H^{\mathrm{BM}} = H^{\mathrm{sing}}$ for compact spaces.
This is a map of "Borel-Moore homology" theories.

Motivic homology notation

Let us re-index to get the cycle map to look nicer:

$$\text{motivic homology } = H_n^{\mathcal{M}}(X, \mathbb{Z}(r)) := CH_r(X, n-2r)$$

With this notation, thus classical Chow groups are

$$CH_r(X) = H_{2r}^{\mathcal{M}}(X, \mathbb{Z}(r))$$

and the generalized cycle map looks like

$$H_n^{\mathcal{M}}(X, \mathbb{Z}(r)) \to H_n^{\mathsf{BM}}(X(\mathbb{C})).$$

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In fact, this notation is even more pleasing if we take into account mixed Hodge structures. We get a map of MHS:

$$H_n^{\mathcal{M}}(X, \mathbb{Z}(r)) \to H_n^{\mathsf{BM}}(X(\mathbb{C}), \mathbb{Z}(r)),$$

where $\mathbb{Z}(r)$ on the right refers to shifting the MHS.

Motivic cohomology

I won't define it carefully here, but motivic cohomology is

$$H^n_{\mathcal{M}}(X, \mathbb{Z}(t)) := \mathbb{H}^n_{Zar}(X, \mathbb{Z}_{\mathcal{M}}(t)).$$

The pair

$$H_*^{\mathcal{M}}(X, \mathbb{Z}(*)) \text{ and } H^*_{\mathcal{M}}(X, \mathbb{Z}(*))$$

form a "Bloch-Ogus" duality theory. In particular, Poincaré duality holds:

$$H^n_{\mathcal{M}}(X, \mathbb{Z}(t)) \cong H^{\mathcal{M}}_{2d-n}(X, \mathbb{Z}(d-t)),$$

for X smooth.

From now on, all varieties are assumed to be quasi-projective varieties over $\mathbb{C}$.

Algebraic equivalence

Two r-cycles γ_1, γ_2 on X are algebraically equivalent if they are members of a family of cycles parametrized by some smooth curve C.

In detail, $\gamma_1 \sim_{alg} \gamma_2$ if there is a cycle

$$\begin{tikzcd} \Gamma \arrow[r, hook] \arrow[dr, "\text{rel. dim. } r"'] & C \times X \arrow[d] \\ & C \end{tikzcd}$$

and points $c_1, c_2 \in C$ such that the fiber of Γ over c_i is γ_i:

$$\Gamma \cap (\{c_i\} \times X) = \gamma_i, \quad i = 1, 2.$$

The Chow group modulo algebraic equivalence

Define

$$CH_r(X)_{alg.\sim 0} \subset CH_r(X)$$

to be the subgroup generated by $\gamma_1 - \gamma_2$ for $\gamma_1 \sim_{\text{alg. equiv.}} \gamma_2$.
Let

$$CH_r(X)/(\text{alg. equiv.}) := CH_r(X)/CH_r(X)_{alg.\sim 0}$$

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$$CH_r(X)/(\text{alg. equiv.}) := CH_r(X)/CH_r(X)_{alg.\sim 0}$$

For codimension one cycles on a smooth, projective X:

$$CH_{\dim(X)-1}(X)_{alg.\sim 0} = \underline{\mathrm{Pic}}^0(X) = \text{abelian variety.}$$

The group $CH_{\dim(X)-1}(X)/(\text{alg. equiv.}) = NS(X)$.

For zero cycles on a connected X:

$$CH_0(X)/(\text{alg. equiv.}) \cong \mathbb{Z} \text{ via degree map.}$$

Inspiration of Lawson homology

Build a theory like higher Chow groups (motivic homology), but start with $CH_*(-)/(\text{alg. equiv.})$ in place of $CH_*(-)$.

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Build a theory like higher Chow groups (motivic homology), but start with $CH_*(-)/$(alg. equiv.) in place of $CH_*(-)$.

Two points in $\mathcal{C}_{r,e}(X)$ lie in the same connected component iff they are joined by a chain of "paths" of the form $C \to \mathcal{C}_{r,e}(X)$ with C a smooth curve. Each path $C \to \mathcal{C}_{r,e}(X)$ gives a family of cycles indexed by C. We get:

Proposition

For a complex, projective variety X

$$\pi_0 \mathcal{C}_r(X)(\mathbb{C})^+ \cong CH_r(X)/\textit{(alg. equiv.)}.$$

This suggests replacing "π_*^{alg}" with actual homotopy groups...

Definition of Lawson homology

For X projective, let $\mathcal{C}(X)(\mathbb{C})$ be complex points of the Chow variety, equipped with the analytic topology. Define

$$\mathcal{Z}_r(X) = \mathcal{C}_r(X)(\mathbb{C})^+$$

to be the "naive" topological group completion of $\mathcal{C}_r(X)(\mathbb{C})$:

$$\mathcal{Z}_r(X) = \mathcal{C}_r(X)(\mathbb{C}) \times \mathcal{C}_r(X)(\mathbb{C}) / \{(\alpha, \beta) \sim (\alpha + \gamma, \beta + \gamma)\}$$

Definition

The Lawson homology groups of a complex projective variety X are

$$L_r H_m(X) = \pi_{m-2r}\left(\mathcal{Z}_r(X)\right).$$

For example, by the proposition above,

$$L_r H_{2r}(X) := \pi_0 \mathcal{Z}_r(X) \cong CH_r(X)/(\text{alg. equiv.}).$$

There is a map from motivic to Lawson homology

$$H_m^{\mathcal{M}}(X, \mathbb{Z}(r)) \to L_r H_m(X)$$

given by "applying $\pi_*^{alg} \to \pi_*$ to $\mathcal{Z}_r(X)$".

For $m = 2r$, this is the evident map:

$$H_{2r}^{\mathcal{M}}(X, \mathbb{Z}(r)) = CH_r(X) \twoheadrightarrow CH_r(X)/(\text{alg. equiv.}) = L_r H_{2r}(X).$$

Remark

The map $H_m^{\mathcal{M}}(X, \mathbb{Z}(r)) \to L_r H_m(X)$ is certainly not onto in general. In fact, for $m > 2r$ and X smooth, projective, it's reasonable to conjecture the map is torsion.

An element of $L_rH_m(X)$ is a "families of cycles parametrized by a topological sphere"

An element of $L_rH_m(X)$ is given by a continuous map

$$S^{m-2r} \to \mathcal{Z}_r(X)$$

which can be visualized as:

$$\begin{tikzcd} \Gamma \arrow[r, hook] \arrow[dr, "\text{rel. dim. } r"'] & S^{m-2r} \times X \arrow[d] \\ & S^{m-2r} \end{tikzcd}$$

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Similarly, an element in $H_m^{\mathcal{M}}(X, \mathbb{Z}(r))$ may be visualized as a family of r-cycles parametrized by the "algebraic sphere"

$$S_{alg}^{m-2r} := \partial \Delta^{m-2r+1}.$$

The map $H_m^{\mathcal{M}}(X, \mathbb{Z}(r)) \to L_r H_m(X)$ is pull-back along $S^{m-2r} \to S_{alg}^{m-2r}(\mathbb{C})$.

Visualizing elements of $\pi_1 \mathcal{Z}_r(X)$

An element of $L_r H_{2r+1}(X)$ is a family parametrized by S^1.

In fact, we may think of classes in $L_r H_{2r+1}(X)$ as giving "two proofs that a pair of cycles γ_1, γ_2 are algebraically equivalent": Suppose

$$\Gamma \subset C \times X \text{ and } \Gamma' \subset C' \times X$$

and we have points $c_1, c_2 \in C$ and $c_1', c_2' \in C'$ such that

$$\gamma_1 = \Gamma_{c_1} = \Gamma_{c_1'} \text{ and } \gamma_2 = \Gamma_{c_2} = \Gamma_{c_2'}.$$

Pick paths $I \to C, I \to C'$ joining c_1 to c_2 and c_1' to c_2'. Let Y be the singular curve obtained by gluing these smooth curves together:

$$Y = C \cup C' = C \amalg C' / (c_1 \sim c_1', c_2 \sim c_2')$$

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Visualizing elements of $\pi_1 \mathcal{Z}_r(X)$, cont.

This data determines a loop

$$S^1 \to Y = C \cup C'$$

and a family of r-cycles on X

$$\begin{tikzcd} \Gamma \arrow[r, hook] \arrow[dr, "\text{rel. dim. } r"'] & Y \times X \arrow[d] \\ & Y \end{tikzcd}$$

This gives a map

$$S^1 \to Y \xrightarrow{\Gamma_*} \mathcal{Z}_r(X),$$

where $\Gamma_* : y \mapsto \Gamma_y \in \mathcal{Z}_r(X)$, and hence an element of

$$\pi_1 \mathcal{Z}_r(X) = L_r H_{2r+1}(X).$$

Every element of $L_r H_{2r+1}(X)$ is constructed in this manner.

Lawson homology for quasi-projective varieties

For U quasi-projective, $U \subset X$ open with X projective, and $Y := X \setminus U$.

$$\mathcal{Z}_r(U) := \mathcal{Z}_r(X)/\mathcal{Z}_r(Y)$$

We define the Lawson homology groups of U as

$$L_r H_m(U) := \pi_{m-2r}\mathcal{Z}_r(U).$$

$\mathcal{Z}_r(Y) \rightarrowtail \mathcal{Z}_r(X) \twoheadrightarrow \mathcal{Z}_r(U)$ is a fibration sequence. We get a localization long exact sequence:

$$\cdots \to L_r H_{2r+1}(Y) \to L_r H_{2r+1}(X) \to L_r H_{2r+1}(U)$$
$$\to L_r H_{2r}(Y) \to L_r H_{2r}(X) \to L_r H_{2r}(U) \to 0.$$

Other Basic properties of Lawson homology

- "Base" group is what we wanted:

$$L_r H_{2r}(U) = CH_r(U)/(\text{alg. equiv.}).$$

- there is a natural map $H^{\mathcal{M}}_m(U, \mathbb{Z}(r)) \to L_r H_m(U)$.
- Homotopy invariance:

$$L_r H_n(U) \cong L_{r+m} H_{n+2m}(U \times \mathbb{A}^m).$$

- There is a companion cohomology theory, morphic cohomology, written $L^t H^n(-)$. The pair $L_* H_*, L^* H^*$ form a Bloch-Ogus duality theory.
- In particular, Poincaré duality holds:

$$L^t H^n(U) \cong L_{d-t} H_{2d-t}(U) \text{ for } U \text{ smooth of dim. } d.$$

Zero cycles: Computing $L_0 H_m$

Recall $\mathcal{C}_0(X)(\mathbb{C}) = \coprod_{e \geq 0} Symm^e(X(\mathbb{C}))$ for X projective.

The Dold-Thom Theorem states that for a compact CW complex T,

$$\pi_q\left(\left(\coprod_e Symm^e(T)\right)^+\right) \cong H_q^{\text{sing}}(T, \mathbb{Z})$$

Proposition

For X projective,

$$\pi_n \mathcal{Z}_0(X) =: L_0 H_n(X) \cong H_n^{\mathit{sing}}(X(\mathbb{C})).$$

More generally for U quasi-projective,

$$L_0 H_n(U) \cong H_n^{BM}(U(\mathbb{C})).$$

The s-map on cycle spaces

Define

$$\begin{array}{rcl} \mathbb{P}^1(\mathbb{C}) \times \mathcal{Z}_r(X) & \to & \mathcal{Z}_r(\mathbb{P}^1 \times X) \\ (P, \gamma) & \mapsto & \{P\} \times \gamma. \end{array}$$

by

Compose with

$$\mathcal{Z}_r(\mathbb{P}^1 \times X) \twoheadrightarrow \mathcal{Z}_r(\mathbb{A}^1 \times X) := \mathcal{Z}_r(\mathbb{P}^1 \times X)/\mathcal{Z}_r(\{*\} \times X)$$

to get

$$\mathbb{P}^1(\mathbb{C}) \wedge \mathcal{Z}_r(X) \to \mathcal{Z}_r(\mathbb{A}^1 \times X).$$

By choosing an inverse of the homotopy equivalence $\mathcal{Z}_{r-1}(X) \xrightarrow{\sim} \mathcal{Z}_r(\mathbb{A}^1 \times X)$, we get:

$$S^2 \wedge \mathcal{Z}_r(X) = \mathbb{P}^1(\mathbb{C}) \wedge \mathcal{Z}_r(X) \to \mathcal{Z}_{r-1}(X).$$

The adjoint of this is the s-map on cycle spaces:

$$s : \mathcal{Z}_r(X) \to \Omega^2 \mathcal{Z}_{r-1}(X).$$

The s-map in Lawson homology

Taking homotopy groups of

$$s : \mathcal{Z}_r(X) \to \Omega^2 \mathcal{Z}_{r-1}(X).$$

gives the s-map on Lawson homology:

$$s : L_r H_n(X) \to L_{r-1} H_n(X).$$

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The s-map can also be defined as multiplication by the s-element in morphic cohomology:

$$s \in L^1 H^0(\operatorname{Spec} \mathbb{C}) \cong \mathbb{Z}.$$

This is the integral analogue of multiplication by the "Bott element" in motivic cohomology:

$$\beta \in H^0_{\mathcal{M}}(\operatorname{Spec} k, \mathbb{Z}/n(1)) \cong \mu_n(k).$$

The map from Lawson to singular homology

We get a sequence of maps

$$L_rH_n(X) \xrightarrow{s} \cdots \xrightarrow{s} L_0H_n(X) \cong H_n^{\text{sing}}(X(\mathbb{C})).$$

whose composition is the generalized cycle map from Lawson homology to singular homology, for X projective:

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$$L_r H_n(X) \to H_n^{\mathsf{sing}}(X(\mathbb{C}))$$

This generalizes to quasi-projective varieties U:

$$L_r H_n(U) \xrightarrow{s} \cdots \xrightarrow{s} L_0 H_n(U) \cong H_n^{\mathsf{BM}}(U(\mathbb{C}))$$

$$L_r H_n(U) \to H_n^{\mathsf{BM}}(U(\mathbb{C}))$$

The topological filtration

$$F_r^{top} H_m^{\text{sing}}(X) := \operatorname{im}\left(L_r H_m(X) \to H_m^{\text{sing}}(X(\mathbb{C}), \mathbb{Z})\right)$$

$$F_r^{top} H_m^{\text{sing}}(X) \subset F_{r-1}^{top} H_m^{\text{sing}}(X) \subset \cdots \subset F_0^{top} H_m^{\text{sing}}(X) = H_m^{\text{sing}}(X(\mathbb{C}))$$

Conjecture (Friedlander-Mazur)

The topological filtration coincides (rationally) with the filtration by dimension of support ("niveau" filtration) of H_*^{sing}:

$$F_r^{top} H_m^{sing}(X)_{\mathbb{Q}} = N_{m-r} H_m^{sing}(X(\mathbb{C}), \mathbb{Q}).$$

In particular, this conjecture predicts that

$$L_r H_m(X) \to H_m^{\text{sing}}(X(\mathbb{C}))$$

is onto for $m \geq \dim(X) + r$. This is also known as the "Weak Suslin Conjecture".

Natural maps between theories

The map from motivic homology to singular homology factors through Lawson homology. For X projective:

$$H_m^{\mathcal{M}}(X, \mathbb{Z}(r)) \to L_r H_m(X) \to H_m^{\text{sing}}(X(\mathbb{C}), \mathbb{Z}(r))$$

(These maps are maps of mixed Hodge structures.)

When $m = 2r$, the above sequence is

$$CH_r(X) \twoheadrightarrow CH_r(X)/(\text{alg. equiv.}) \to H_r^{\text{sing}}(X(\mathbb{C}), \mathbb{Z}(2r)).$$

(Recall $H_m^{\mathcal{M}}(X, \mathbb{Z}(r)) \to L_r H_m(X)$ is usually not onto.)

Natural maps for quasi-projective varieties

This factorization extends to quasi-projective varieties U, and has the form

$$H_m^{\mathcal{M}}(U, \mathbb{Z}(r)) \to L_r H_m(U) \to H_m^{\mathsf{BM}}(U(\mathbb{C}), \mathbb{Z}(r)).$$

(Recall H^{BM} denotes Borel-Moore singular homology and $H^{\mathsf{BM}} = H^{\mathsf{sing}}$ for compact spaces.)

There are also maps on the corresponding cohomology theories:

$$H^n_{\mathcal{M}}(U, \mathbb{Z}(t)) \to L^t H^n(U) \to H^n_{\mathsf{sing}}(U(\mathbb{C}), \mathbb{Z}(t))$$

and, together with the maps above, they give maps of Bloch-Ogus duality theories.

Known calculations: Codimension one cycles

For X connected, smooth, projective of dimension d we have a fibration sequence

$$\mathbb{P}^\infty(\mathbb{C}) \rightarrowtail \mathcal{Z}_{d-1}(X) \twoheadrightarrow \underline{\mathrm{Pic}}(X)(\mathbb{C}).$$

Known calculations: Codimension one cycles

For X connected, smooth, projective of dimension d we have a fibration sequence

$$\mathbb{P}^\infty(\mathbb{C}) \rightarrowtail \mathcal{Z}_{d-1}(X) \twoheadrightarrow \underline{\mathrm{Pic}}(X)(\mathbb{C}).$$

Moreover,

$$\underline{\mathrm{Pic}}(X)(\mathbb{C}) = NS(X) \times \underline{\mathrm{Pic}}^0(X)(\mathbb{C})$$

and $\underline{\mathrm{Pic}}^0(X)(\mathbb{C})$ (a torus) is the classifying space of the free abelian group $H^1_{\mathsf{sing}}(X(\mathbb{C}), \mathbb{Z})$. Also, $\mathbb{P}^\infty(\mathbb{C}) = K(\mathbb{Z}, 2)$.

$$\begin{aligned} \pi_1 \mathcal{Z}_{d-1}(X) &\cong \pi_1 \underline{\mathrm{Pic}}^0(X)(\mathbb{C}) \\ &\cong H^1_{\mathsf{sing}}(X(\mathbb{C})) \\ &\cong H^{\mathsf{sing}}_{2d-1}(X(\mathbb{C})). \end{aligned}$$

$$\mathbb{P}^\infty(\mathbb{C}) \rightarrowtail \mathcal{Z}_{d-1}(X) \twoheadrightarrow \underline{\mathrm{Pic}}(X)(\mathbb{C})$$

also implies:

$$\pi_0 \mathcal{Z}_{d-1}(X) = NS(X) \subset H^2_{\mathsf{sing}}(X(\mathbb{C})) \cong H^{\mathsf{sing}}_{2d-2}(X(\mathbb{C})).$$

and

$$\pi_2 \mathcal{Z}_{d-1}(X) = \mathbb{Z} = H^{\mathsf{sing}}_{2d}(X(\mathbb{C})).$$

For all smooth, quasi-projective varieties U:

$$L_{d-1}H_m(U) = \begin{cases} H^{\mathsf{sing}}_m(U(\mathbb{C}), \mathbb{Z}) & m \geq 2d-1 \\ NS(U) \subset H^{\mathsf{BM}}_m(U(\mathbb{C}), \mathbb{Z}) & n = 2d-2 \\ 0 & n < 2d-2 \end{cases}$$

Known calculations: Special varieties

There is a class $\mathcal{S}$ of especially "simple" varieties for which

$$L_r H_m(X) \xrightarrow{\cong} W_{-2r} H_m^{\mathrm{BM}}(X) \subset H_m^{\mathrm{BM}}(X).$$

W_* refers to the (integrally defined) weight filtration on (Borel-Moore) homology. The class $\mathcal{S}$ includes:

- curves
- toric varieties
- cellular varieties
- smooth, projective surfaces S s.t. $CH_2(S) \twoheadrightarrow H^{\mathrm{sing}}(S)$

Finite coefficients

There are variants of motivic homology ($=$ higher Chow groups) and Lawson homology with coefficients in any abelian group A:

$$H_m^{\mathcal{M}}(X, A(r)) \text{ and } L_r H_m(X, A).$$

To define them, take homotopy groups with coefficients in A.

Theorem (Suslin-Voevodsky)

For any quasi-projective variety U,

$$H_m^{\mathcal{M}}(U, \mathbb{Z}/n(r)) \xrightarrow{\cong} L_r H_m(U, \mathbb{Z}/n)$$

for all $n > 0$.

"motivic homology and Lawson homology with finite coefficients coincide"

Illustration of Suslin-Voevodsky theorem for π_0

For example,

$$H^{\mathcal{M}}_{2r}(X, \mathbb{Z}/n(r)) = CH_r(X; \mathbb{Z}/n) = CH_r(X) \otimes \mathbb{Z}/n$$

and

$$L_rH_{2r}(X, \mathbb{Z}/n) = (CH_r(X)/(\text{alg. equiv.})) \otimes \mathbb{Z}/n$$

are isomorphic.

This holds because the kernel $CH_r(X)_{alg.\sim 0}$ of $H^{\mathcal{M}}_{2r}(X, \mathbb{Z}(r)) \twoheadrightarrow L_rH_{2r}(X)$ is divisible.

For example, for codim. one cycles on a smooth, projective X:

$$CH_{\dim(X)-1}(X)_{alg.\sim 0} = \underline{\mathrm{Pic}}^0(X)(\mathbb{C}), \text{ a torus.}$$

Bloch-Kato Theorem in terms of Lawson homology

Theorem (Voevodsky)

For X smooth and $n > 0$, the map

$$L_r H_m(X, \mathbb{Z}/n) \xrightarrow{\cong} H_m^{BM}(X(\mathbb{C}), \mathbb{Z}/n)$$

is an isomorphism provided $m \geq d + r$.
If $m = d + r - 1$, this map is injective.

In terms of morphic cohomology:

$$L^t H^p(X, \mathbb{Z}/n) \cong H^p_{\text{sing}}(X(\mathbb{C}), \mathbb{Z}/n) \quad \text{if } p \leq t.$$

Stronger form: Define $a : (Var/\mathbb{C})_{analytic} \to (Var/\mathbb{C})_{Zar}$.
Then

$$L^t H^p(X, \mathbb{Z}/n) \xrightarrow{\cong} \mathbb{H}^p_{Zar}(X, tr^{\leq t} \mathbb{R}a_* \mathbb{Z}/n)$$

Suslin's Conjecture for Lawson/morphic (co)homology

Conjecture (Suslin's Conjecture — Lawson form)

For a smooth, quasi-projective variety X, the map

$$L_r H_m(X) \to H_m^{sing}(X(\mathbb{C}))$$

is an isomorphism for $m \geq d + r$ and a monomorphism for $m = d + r - 1$.

The cohomological version of Suslin's Conjecture is:

$$L^t H^n(X) \overset{?}{\cong} \mathbb{H}^n_{Zar}(X, tr^{\leq t}\mathbb{R}\pi_*\mathbb{Z}),$$

where $\pi : (Var/\mathbb{C})_{analytic} \to (Var/\mathbb{C})_{Zar}$. Thus,

"Suslin's Conjecture = Bloch-Kato with $\mathbb{Z}$-coefficients (over $\mathbb{C}$)".